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EQUIVALENCE CLASSES OF POLARIZATIONS AND MODULI SPACES OF SHEAVES

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Introduction

Let X be a smooth algebraic variety over the complex number field $\mathbf{C}$ with dimension n larger than one. For fixed c_1 in $\mathrm{Pic}(X)$, c_2 in $A^2_{\mathrm{num}}(X)$ which is the Chow group of codimension-two cycles on X modulo numerical equivalence and a polarization L on X, let $\mathscr{M}_L(c_1, c_2)$ be the moduli space of locally free rank-two sheaves stable with respect to L in the sense of Mumford-Takemoto such that their first and second Chern classes are c_1 and c_2 respectively. In this paper, we consider the problem: *what is the difference between* $\mathscr{M}_{L_1}(c_1, c_2)$ *and* $\mathscr{M}_{L_2}(c_1, c_2)$ *where* L_1 *and* L_2 *are two different polarizations?*

The understanding of this problem has two important implications. The first is in algebraic geometry. If one knows the structure of some moduli space $\mathscr{M}_L(c_1, c_2)$, then one will know the structure of any other moduli space $\mathscr{M}_{L'}(c_1, c_2)$ by comparing it with $\mathscr{M}_L(c_1, c_2)$. The author has applied this idea to the case where X is a ruled surface (for instance, see [21]); the results will appear elsewhere. The second implication is in gauge theory where X is an algebraic surface. When the geometric genus p_g of X is positive, the polynomials defined by Donaldson [6] are differential invariants. When p_g is zero, via the results in [4], these polynomials are defined on chambers of certain type (c_1, c_2); from the work of Mong [18] and Kotschick [15], one sees that we need to understand the difference between moduli spaces in order to compute these polynomials.

Our approach to the problem is to develop a theory about equivalence classes, walls and chambers of type (c_1, c_2) for polarizations on X. This is done in Chapter I. Fix c_1 and c_2 as before. Let L_1 and L_2 be two polarizations on X. We say that L_1 and L_2 are *equivalent* if every locally free rank-two sheaf V with first and second Chern classes c_1 and c_2, respectively, is L_1-stable if and only if it is L_2-stable.

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Theorem 1. *Let V be a locally free rank-two sheaf which is L_1-stable and L_2-unstable. Then, there is an invertible sheaf $\mathcal{O}_X(F)$, and integers i, j with $0 \le i \le j \le (n-1)$ such that, S being equal to $L_1^{n-1-j} \cdot L_2^i \cdot (L_1 + L_2)^{j-1-1}$, we have*

(i) $\mathcal{O}_X(F + c_1(V))$ *is divisible by* 2 *in* $\mathrm{Pic}(X)$;
(ii) $[c_1(V)^2 - 4c_2(V)] \cdot S \le F^2 \cdot S < 0$;
(iii) $(F \cdot L_1) \cdot S < 0 < (F \cdot L_2) \cdot S$.

The special case where X is a surface and c_1 is trivial (see Proposition 1.2.1 in Chapter I) is obtained by Mong [19] and Friedman [7]. Using Theorem 1, we can define walls and chambers of type (c_1, c_2). Let Num (X) be the group of divisors on X modulo numerical equivalence relation. Let $\mathbf{C}_X$ be the Kähler cone in Num $(X) \otimes \mathbf{R}$ generated by all ample divisors. Roughly speaking, a *wall* of type (c_1, c_2) is the intersection of $\mathbf{C}_X$ with a set of the form $\{x \in \text{Num }(X) \otimes R | x \cdot \zeta \cdot S = 0\}$ where ζ and S satisfy some conditions. Walls of type (c_1, c_2) cut $\mathbf{C}_X$ into many connected components; each of these components is a *chamber* of type (c_1, c_2), and intersection of a chamber with Num (X) is **Z**-*chamber*. A basic relation among equivalence classes, walls and chambers is that an equivalence class is a union of **Z**-chambers, and possibly some polarizations lying on walls. We will see that the study of moduli spaces of locally free rank-two sheaves stable with respect to polarizations lying on walls can be reduced to the study of moduli spaces of locally free rank-two sheaves stable with respect to polarizations lying in chambers.

In Chapter II, we focus on the case where X is an algebraic surface, and work in the Kähler cone $\mathbf{C}_X$. It turns out that the concepts of walls and chambers here are slight modifications of those used by Donaldson [5] and Friedman and Morgan [8]. Let ζ be any numerical equivalence class which defines a nonempty wall W^ζ of type (c_1, c_2). We introduce the notation $E_\zeta(c_1, c_2)$, which is the set of all locally free rank-two sheaves V given by nontrivial extensions

$$0 \to \mathcal{O}_X(F) \to V \to \mathcal{O}_X(c_1 - F) \otimes I_Z \to 0,$$

where F is some divisor with $(2F - c_1) \equiv \zeta$, and Z is some locally complete intersection 0-cycle with length $l(Z) = c_2 + (\zeta^2 - c_1^2)/4$. Every sheaf V in $E_\zeta(c_1, c_2)$ has two basic properties: (a) the above defining exact sequence for V is canonical; (b) V is L-stable if $L \cdot \zeta$ is negative and L is contained in a chamber $\mathcal{C}$ such that $W^\zeta \cap \text{Closure}(\mathcal{C})$ is nonempty. Our main result is the following.

Theorem 2. *Let $\mathscr{C}$, $\mathscr{C}_1$ be two chambers sharing a common wall. Then, as sets,*

$$\mathscr{M}_{\mathscr{C}}(c_1, c_2) = \left(\mathscr{M}_{\mathscr{C}_1}(c_1, c_2) - \coprod E_{(-\zeta)}(c_1, c_2)\right) \coprod \left(\coprod E_\zeta(c_1, c_2)\right),$$

where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathscr{C}$, and runs over all numerical equivalence classes which define the common wall.

We now explain the geometric meaning of Theorem 2. It is well known that $\mathscr{M}_L(c_1, c_2)$ is quasiprojective (see [16]). Thus, for two polarizations L_1 and L_2, one would expect to perform only a finite surgery operation to pass from $\mathscr{M}_{L_1}(c_1, c_2)$ to $\mathscr{M}_{L_2}(c_1, c_2)$ by removing and adding locally closed subsets. Indeed, $E_\zeta(c_1, c_2)$ is a finite disjoint union of locally closed subsets of $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$ under the conditions in Theorem 2. Therefore, Theorem 2 gives a precise description about what locally closed subsets should be removed or added when L_1 and L_2 can be connected by a path which intersects with only one wall of type (c_1, c_2).

We notice that $E_\zeta(c_1, c_2)$ plays a significant role in comparing different moduli spaces and in determining the equivalence classes of polarizations. In §2 of Chapter II, we discuss the nonemptiness of $E_\zeta(c_1, c_2)$. We use some classical techniques to construct locally free rank-two sheaves. In this direction, we have two results.

Theorem 3. *$E_\zeta(c_1, c_2)$ or $E_{(-\zeta)}(c_1, c_2)$ is nonempty if $c_2 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1.*

Notice that no condition is imposed on the wall W^ζ. If we do make some assumption on W^ζ, we obtain a stronger result. The second is the following.

Theorem 4. *Let $\mathscr{B}$ be a compact subset in the Kähler cone $\mathbf{C}_X$. For any numerical equivalence class ζ with $W^\zeta \cap \mathscr{B}$ to be nonempty, $E_\zeta(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathscr{B})$ where $c(X, c_1, \mathscr{B})$ is a constant depending on X, c_1 and $\mathscr{B}$.*

In [23], we have estimated the dimension of $E_\zeta(c_1, c_2)$ and studied the birational structure of $\mathscr{M}_L(c_1, c_2)$ when $(4c_2 - c_1^2)$ is sufficiently large.

CHAPTER I

1. Stability under different polarizations

1.1. Stability in the sense of Mumford-Takemoto. We begin with several definitions. Let X be a smooth projective variety of dimension n over the complex number field $\mathbf{C}$. From the exponential sequence

$$0 \to \mathbf{Z} \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0,$$

we obtain a map from $\text{Pic}(X) \to H^2(X; \mathbf{Z})$. Let Num (X) be a Pic (X) modulo the kernel of the induced map Pic $(X) \to H^2(X; Z)/$ Torsion. Clearly, Num (X) is a finely generated free abelian group. The images of ample invertible sheaves in Num (X) are called *polarizations* (a word of caution: this definition is different from the standard one). Since we will only need numerical properties, we make no distinctions between a polarization and its inverse images in Pic (X) or Div (X).

Definition 1.1.1. For a polarization L and a torsion free coherent sheaf V, let

$$\mu_L(V) = \frac{1}{\text{rank}(V)}(c_1(V) \cdot L^{n-1}).$$

Definition 1.1.2. Let V be a torsion free coherent sheaf on X. V is *L-stable* (resp. *L-semistable*) if, for all coherent subsheaves W of V with $0 < \text{rank}(W) < \text{rank}(V)$, we have $\mu_L(W) < \mu_L(V)$ (resp. $\mu_L(W) \le \mu_L(V)$). V is said to be *unstable* if it is not semistable, and *strictly semistable* if it is semistable but not stable.

Remark 1.1.3. If $n = 2$ or rank $(V) = 2$, then it is sufficient to check (semi)stability on locally free subsheaves of V such that the quotients are torsion free.

Definition 1.1.4. Fix $c_1 \in \text{Pic}(X)$ and $c_2 \in A^2_{\text{num}}(X)$ where $A^i_{\text{num}}(X)$ is the Chow group of cycles on X of codimension i modulo numerical equivalence. Let L be a polarization on X. We define $\mathcal{M}_L(c_1, c_2)$ to be the moduli space of L-stable locally free rank-two sheaves with fixed c_1 and c_2.

Remark 1.1.5. It is well known that $\mathcal{M}_L(c_1, c_2)$ is quasiprojective (see [16]).

1.2. Stability under different polarizations.

Proposition 1.2.1 (see [19] and [7]). *Let V be a locally free rank-two sheaf on a smooth algebraic surface X with $c_1(V) = 0$. Let L_1 and L_2 be two polarizations on X. Suppose that V is L_1-stable and L_2-unstable. Then, there exists an invertible sheaf $\mathcal{O}_X(F)$ on X with $L_1 \cdot F < 0 < L_2 \cdot F$ and $-c_2(V) \le F^2 < 0$.*

A slight modificaion of the proof in [7] of the proposition above will result in the following which says that $c_1(V) \ne 0$ makes no big difference.

Proposition 1.2.2. *Let V be a locally free rank-two sheaf on a smooth surface X. Let L_1 and L_2 be two polarizations on X. Suppose that V is L_1-stable and L_2-unstable. Then, there exists an invertible sheaf $\mathcal{O}_X(F)$ such that $\mathcal{O}_X(F + c_1(V))$ is divisible by 2 in* Pic(X), *$L_1 \cdot F < 0 < L_2 \cdot F$ and $c_1(V)^2 - 4c_2(V) \le F^2 < 0$.*

Corollary 1.2.3. *Let V be a locally free rank-two sheaf on a smooth surface X. If V is stable with respect to one polarization and unstable with respect to another polarization, then* $c_1(V)^2 - 4c_2(V) < 0$.

Remark 1.2.4. This corollary is obtained by Takemoto [25]. It is a special case of Bogomolov's instability theorem [1]: if $c_1^2 > 4c_2$, then $\mathcal{M}_L(c_1, c_2) = \emptyset$ for any polarization L on any surface X. Donaldson [4] and Kobayashi [13] (see also [17] and [14]) showed that if $c_1(V) = c_2(V) = 0$, then V is stable with respect to some polarization on X if and only if V comes from an irreducible unitary representation of the fundamental group $\pi_1(X)$. In general, the case $(4c_2 - c_1^2) = 0$ has been studied by Takemoto [24, 25]. Therefore, in case of algebraic surfaces, we always assume $(4c_2 - c_1^2) > 0$ unless otherwise specified.

Theorem 1.2.5. *Let V be a locally free rank-two sheaf on a smooth n-dimensional variety X where* $n \geq 2$. *Let* L_1 *and* L_2 *be two polarizations on X. Suppose that V is* L_1*-stable and* L_2*-unstable. Then, there exist an invertible sheaf* $\mathcal{O}_X(F)$ *on X, and integers i, j satisfying* $0 \leq i < j \leq (n-1)$ *such that, S being equal to* $L_1^{n-1-j} \cdot L_2^i \cdot (L_1 + L_2)^{j-i-1}$, *we have*

(i) $\mathcal{O}_X(F + c_1(V))$ *is divisible by* 2 *in* $\mathrm{Pic}(X)$;
(ii) $[c_1(V)^2 - 4c_2(V)] \cdot S \leq F^2 \cdot S < 0$;
(iii) $(F \cdot L_1) \cdot S < 0 < (F \cdot L_2) \cdot S$.

Proof. By assumption, there exists an invertible subsheaf $\mathcal{O}_X(G)$ of V such that $G \cdot L_1^{n-1} < [c_1(V) \cdot L_1^{n-1}]/2$ and $[c_1(V) \cdot L_2^{n-1}]/2 < G \cdot L_2^{n-1}$. By Remark 1.1.3, we may assume that the quotient $V/\mathcal{O}_X(G)$ is torsion free. Put $F = 2G - c_1(V)$. We have $F \cdot L_1^{n-1} < 0 < F \cdot L_2^{n-1}$. Thus, we can choose integers i and j with $0 \leq i < j \leq (n-1)$ such that

(i) $F \cdot L^{(n-1-k)} \cdot L_2^k \leq 0$ for $k < i$;
(ii) $F \cdot L^{(n-1-k)} \cdot L_2^k < 0$ for $k = i$;
(iii) $F \cdot L^{(n-1-k)} \cdot L_2^k = 0$ for $i < k < j$;
(iv) $F \cdot L^{(n-1-k)} \cdot L_2^k > 0$ for $k = j$.

Put $S = L_1^{n-1-j} \cdot L_2^i \cdot (L_1 + L_2)^{j-i-1}$. Then, we have

$$(F \cdot L_1) \cdot S = F \cdot L_1^{n-1-i} \cdot L_2^i < 0 \quad \text{and} \quad (F \cdot L_2) \cdot S = F \cdot L_1^{(n-1-j)} \cdot L_2^k > 0.$$

Therefore, we obtain that $(F \cdot L_1) \cdot S < 0 < (F \cdot L_2) \cdot S$. Note that after a scaling of L_1 and L_2, we may regard S as a smooth algebraic surface in X. Restrict F, L_1 and L_2 to S. Put $H = (F|_S \cdot L_2|_S) \cdot L_1|_S - (F|_S \cdot L_1|_S) \cdot L_2|_S$. Then H is ample on S and $F|_S \cdot H = 0$. By the Hodge Index Theorem [11], either $(F|_S)^2 < 0$ or $F|_S \equiv 0$. But the second case will not happen since $F|_S \cdot L_1|_S < 0$. Thus, $(F|_S)^2 < 0$, i.e., $F^2 \cdot S < 0$. On the other hand, there exists an exact sequence

$$0 \to \mathcal{O}_X(G) \to V \to \mathcal{O}_X(c_1(V) - G) \otimes I_Z \to 0,$$

where Z is a locally complete intersection codimension-two cycle on X. Thus, $c_2(V) = G\cdot(c_1(V)-G)+[Z]$, and $F^2\cdot S = [c_1(V)^2-4c_2(V)]\cdot S+4[Z]\cdot S \geq [c_1(V)^2-4c_2(V)]\cdot S$. Hence, we have $[c_1(V)^2-4c_2(V)]\cdot S \leq F^2\cdot S < 0$.

2. Equivalence classes and chambers

2.1. Equivalence classes of polarizations.

Definition 2.1.1. Let L_1, L_2 be two polarizations on X. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in A^2_{\mathrm{num}}(X)$. We define $L_1 \overset{s}{\geq} L_2$ if every locally free rank-two sheaf with c_1 and c_2 as its first and second Chern classes is L_1-stable whenever it is L_2-stable. We define $L_1 \overset{s}{=} L_2$ if both $L_1 \overset{s}{\geq} L_2$ and $L_2 \overset{s}{\geq} L_1$.

Remark 2.1.2. Fix c_1 and c_2. Then $L_1 \overset{s}{=} L_2$ means that the moduli spaces $\mathcal{M}_{L_1}(c_1, c_2)$ and $\mathcal{M}_{L_2}(c_1, c_2)$ can be naturally identified.

Notation 2.1.3. For a polarization L, we put

(i) $\Delta_L = \{L' | L'$ is a polarization and $L' \overset{s}{\geq} L\}$;

(ii) $\mathcal{E}_L = \{L' | L'$ is a polarization and $L' \overset{s}{=} L\}$.

Proposition 2.1.4. *Let L and L' be two polarizations on X. Then,*

(i) $\Delta_{L'} \subseteq \Delta_L$ *if and only if* $L \overset{s}{\leq} L'$;

(ii) $\Delta_{L'} = \Delta_L$ *if and only if* $L \overset{s}{=} L'$.

Proof. Follows from Definition 2.1.1 and Notation 2.1.3. q.e.d.

We already know that $\mathrm{Num}(X)$ is a finitely generated free abelian group. There is an open cone (called the *Kähler cone*) $\mathbf{C}_X$, in $\mathrm{Num}(X) \otimes \mathbf{R}$ which is spanned by polarizations. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in A^2_{\mathrm{num}}(X)$.

Definition 2.1.5. (i) Let $S \in A^{n-2}_{\mathrm{num}}(X)$, and $\zeta \in \mathrm{Num}(X) \otimes \mathbf{R}$. We define

$$W^{(\zeta, S)} = \mathbf{C}_X \cap \{x \in \mathrm{Num}(X) \otimes \mathbf{R} | x \cdot \zeta \cdot S = 0\}.$$

(ii) We define $\mathcal{W}(c_1, c_2)$ to be the set whose elements consist of $\mathcal{W}^{(\zeta, S)}$, where S is a complete intersection surface in X, and ζ is the numerical equivalence class of a divisor F on X such that $\mathcal{O}_X(F + c_1)$ is divisible by 2 in $\mathrm{Pic}(X)$, and that

$$F^2 \cdot S < 0, \quad c_2 + \frac{F^2 - c_1^2}{4} = [Z]$$

for some locally complete intersection codimension-two cycle Z in X.

(iii) A *wall* of type (c_1, c_2) is an element in $\mathcal{W}(c_1, c_2)$. A *chamber* of type (c_1, c_2) is a connected component of $\mathbf{C}_X - \mathcal{W}(c_1, c_2)$. A *$\mathbb{Z}$-chamber* of type (c_1, c_2) is the intersection of $\mathrm{Num}(X)$ with some chamber of type (c_1, c_2).

We notice that when $\dim(X) = 2$, the above definitions for walls and chambers are slight modifications of those used in [5], [8], [18] and [15]; they also appeared briefly in [7]. As in §1, Chapter II of [8], we can show the following result.

Proposition 2.1.6. *The set of walls of type (c_1, c_2) is locally finite if* $\dim(X) = 2$.

Remark 2.1.7. If $\dim(X) > 2$, Proposition 2.1.6 will not hold anymore (see 2.3 below). This will limit the application of our theory in higher-dimensional cases. Nevertheless, this theory is quite satisfactory in dimension two (see Chapter II).

2.2. Some relations among walls, chambers and equivalence classes.

Proposition 2.2.1. *Let $\mathcal{C}$ be a chamber, and let $L_1, L_2 \in \mathcal{C}$. Then,*

$$L_1 \stackrel{s}{=} L_2 \stackrel{s}{=} (L_1 + L_2).$$

Proof. By assumption, L_1 and L_2 are not separated by any wall. Thus, by the Theorem 1.2.5 in §1, $L_1 \stackrel{s}{=} L_2$. Since $\mathcal{C}$ is convex and closed under the action of $\mathbf{R}^+$, $(L_1 + L_2)$ is also in $\mathcal{C}$.

Corollary 2.2.2. *Each $\mathbf{Z}$-chamber is contained in some equivalence class. Thus, an equivalence class is a union of $\mathbf{Z}$-chambers, and possibly some polarizations lying on walls.*

In the next chapter, we will discuss this corollary for algebraic surfaces in detail. Right now, using Corollary 2.2.2, we can make

Definition 2.2.3. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in A^2_{\mathrm{num}}(X)$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two $\mathbf{Z}$-chambers of type (c_1, c_2), and $L_1 \in \mathcal{C}_1$, $L_2 \in \mathcal{C}_2$.

(i) We define that $\mathcal{C}_1 \stackrel{s}{\geq} \mathcal{C}_2$ if $L_1 \stackrel{s}{\geq} L_2$, and that $\mathcal{C}_1 \stackrel{s}{=} \mathcal{C}_2$ if $L_1 \stackrel{s}{=} L_2$.

(ii) We define that a locally free rank-two sheaf is $\mathcal{C}_1$-stable if it is L_1-stable. Let $\mathcal{M}_{\mathcal{C}_1}(c_1, c_2)$ be $\mathcal{M}_{L_1}(c_1, c_2)$.

(iii) We define $\Delta_{\mathcal{C}_1}$ to be Δ_{L_1}, and $\mathcal{E}_{\mathcal{C}_1}$ to be $\mathcal{E}_{L_1}$.

Proposition 2.2.4. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two chambers having a unique common face which is part of the wall W. Assume that the two $\mathbf{Z}$-chambers* $\mathrm{Num}(X) \cap \mathcal{C}$ *and* $\mathrm{Num}(X) \cap \mathcal{C}'$ *are nonempty. If W is not of the form $W^{(F,S)}$ where S is a smooth complete intersection surface in X and $F = 2G - c_1$ for some invertible subsheaf $\mathcal{O}_X(G)$ of a locally free rank-two sheaf* *which is either* $\mathscr{C}$ *-stable or* $\mathscr{C}'$ *-stable, then*

$$\mathrm{Num}(X) \cap \mathscr{C} \overset{s}{=} \mathrm{Num}(X) \cap \mathscr{C}' .$$

Moreover, if $\dim(X) = 2$, *the converse is also true.*

Proof. The first statement follows from Definition 2.1.1 and Theorem 1.2.5.

Next, suppose $\dim(X) = 2$ and $\mathrm{Num}(X) \cap \mathscr{C} \overset{s}{=} \mathrm{Num}(X) \cap \mathscr{C}'$. If $W = W^F$ where $\mathscr{O}_X(F)$ is a subsheaf of V which is $(\mathrm{Num}(X) \cap \mathscr{C})$-stable (so must be $(\mathrm{Num}(X) \cap \mathscr{C}')$-stable), then by the definition of stability, $F \cdot L < 0$ for any $L \in \mathrm{Num}(X) \cap \mathscr{C}$ or $\mathrm{Num}(X) \cap \mathscr{C}'$. Thus, W cannot separate $\mathscr{C}$ and $\mathscr{C}'$. A contradiction. q.e.d.

In the following, we consider polarizations lying on walls.

Proposition 2.2.5. *Suppose* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *are two chambers having a unique common face which is part of the wall* W. *Let* $\mathscr{F} = W \cap \mathrm{Closure}(\mathscr{C}_i)$ *be the common face. Assume that* $\mathrm{Num}(X) \cap \mathscr{F}$ *is nonempty and* $L \in \mathrm{Num}(X) \cap \mathscr{F}$. *Then,*

(i) $\mathrm{Num}(X) \cap \mathscr{F}$ *is contained in one equivalence class;*

(ii) $\Delta_L \supseteq (\mathrm{Num}(X) \cap \mathscr{C}_1) \cap (\mathrm{Num}(X) \cap \mathscr{C}_2)$;

(iii) $\mathrm{Num}(X) \cap \mathscr{C}_1 \overset{s}{=} \mathrm{Num}(X) \cap \mathscr{C}_2$ *if and only if* $\mathscr{E}_L \supseteq (\mathrm{Num}(X) \cap \mathscr{C}_1) \cup (\mathrm{Num}(X) \cap \mathscr{C}_2)$.

Proof. (i) Follows from the fact that no wall intersects with $\mathscr{F}$ properly.

(ii) We need only to show that $\Delta_L \supseteq (\mathrm{Num}(X) \cap \mathscr{C}_1)$. Assume $\mathrm{Num}(X) \cap \mathscr{C}_1$ is nonempty, and let $L_1 \in \mathrm{Num}(X) \cap \mathscr{C}_1$. Suppose V is L-stable but is not $(\mathrm{Num}(X) \cap \mathscr{C}_1)$-stable. Then there exists a wall $W' = W^{(F,S)}$ such that $F \cdot L \cdot S < 0 \leq F \cdot L_1 \cdot S$ by the Theorem 1.2.5 in §1. This implies that W' separates L and L', and does not contain L. But the only wall separating L and L' is W which contains L. A contradiction. Thus, V is $(\mathrm{Num}(X) \cap \mathscr{C}_1)$-stable and $\Delta_L \supseteq (\mathrm{Num}(X) \cap \mathscr{C}_1)$.

(iii) Clearly, if $\mathscr{E}_L \supseteq (\mathrm{Num}(X) \cap \mathscr{C}_1) \cup (\mathrm{Num}(X) \cap \mathscr{C}_2)$, then $\mathrm{Num}(X) \cap \mathscr{C}_1 \overset{s}{=} \mathrm{Num}(X) \cap \mathscr{C}_2$. Suppose $\mathrm{Num}(X) \cap \mathscr{C}_1 \overset{s}{=} \mathrm{Num}(X) \cap \mathscr{C}_2$, and V is $(\mathrm{Num}(X) \cap \mathscr{C}_i)$-stable $(i = 1, 2)$. If V is not L-stable, then there is a wall $W' = W^{(F,S)}$ such that $F \cdot L \cdot S \geq 0 > F \cdot \mathscr{C}_i \cdot S$. Thus, $W = W'$ and W cannot separate $\mathscr{C}_1$ and $\mathscr{C}_2$. Contradiction! Therefore, V must be L-stable, so $L \overset{s}{\geq} \mathrm{Num}(X) \cap \mathscr{C}_i$. Combining with (ii), we conclude that $L \overset{s}{=} (\mathrm{Num}(X) \cap \mathscr{C}_i)$. So $\mathscr{E}_L \supseteq (\mathrm{Num}(X) \cap \mathscr{C}_1) \cup (\mathrm{Num}(X) \cap \mathscr{C}_2)$.

Remark 2.2.6. From the proof, we see that V is $(\mathrm{Num}(X) \cap \mathscr{F})$- stable if and only if it is both $(\mathrm{Num}(X) \cap \mathscr{C}_1)$-stable and $(\mathrm{Num}(X) \cap \mathscr{C}_2)$-stable. Thus, the study of moduli spaces of locally free rank-two sheaves stable with respect to polarizations lying on walls may be reduced to the study of moduli spaces of locally free rank-two sheaves with respect to polarizations lying in **Z**-chambers.

2.3. An example. Let $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and let p_i be the i-th projection of X to $\mathbf{P}^1$. Put $\mathscr{L}_i = p_i^* \mathscr{O}_{\mathbf{P}^1}(1)$. Let (s_1, s_2, s_3) denote $(s_1 \cdot \mathscr{L}_1 + s_2 \cdot \mathscr{L}_2 + s_3 \cdot \mathscr{L}_3)$. Then, (s_1, s_2, s_3) is a polarization if and only if s_1, s_2, s_3 are positive. The proof of the following is quite elementary (see [21]); hence we simply state the result.

Proposition 2.3.1. *Let* $c_1 = 0$ *and* $c_2 = 2(\mathscr{L}_2 \cdot \mathscr{L}_3 + \mathscr{L}_1 \cdot \mathscr{L}_3)$. *Then,*

(i) *there is no* **Z**-*chamber of type* $(0, c_2)$;

(ii) *chambers of type* $(0, c_2)$ *exist, and each chamber consists of a ray*;

(iii) $(1, 1, m)$ *is equivalent to* $(1, 1, n)$ *if* $m, n > 1$.

CHAPTER II

1. Theory in the case of algebraic surfaces

1.1. Remarks on polarizations. One essential difference between Proposition 1.2.2 and Theorem 1.2.5 in Chapter I is that when $\dim(X) > 2$, we have restrictions on the complete intersection surface S in X, and we have to work with those elements in $\mathrm{Num}(X) \cap \mathbf{C}_X$ in order to guarantee that the S in Theorem 1.2.5 of Chapter I is indeed an algebraic surface up to a scaling. But in the case where X is a surface, S disappears. This implies that we can work with any element in $\mathbf{C}_X$.

Definition 1.1.1. A *polarization* on an algebraic surface X is an element in $\mathbf{C}_X$.

All definitions and conclusions in Chapter I are true if we simply replace **Z**-chambers and polarizations by chambers and generalized polarizations above.

1.2. Introduction of $E_\zeta(c_1, c_2)$. From now on, X is an algebraic surface. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbf{Z}$ such that $(4c_2 - c_1^2) > 0$. Let L_1 and L_2 be two polarizations which are not equivalent. We may assume that there is a locally free rank-two sheaf V which is L_1-stable but not L_2-stable. From the proof of Theorem 1.2.5 in Chapter I, V sits in

$$0 \to \mathscr{O}_X(G) \to V \to \mathscr{O}_X(c_1 - G) \otimes I_Z \to 0,$$

where Z is a locally complete intersection 0-cycle on X, and $(2G - c_1)$ defines a nonempty wall of type (c_1, c_2) with $L_1 \cdot (2G - c_1) < 0 \le L_2 \cdot (2G - c_1)$. Notice that the extension above is nontrivial since a stable sheaf cannot be a direct sum.

We want to study the inverse process. First of all, we introduce $E_\zeta(c_1, c_2)$.

Definition 1.2.1. Let ζ be some numerical equivalence class which defines a nonempty wall of type (c_1, c_2). We define $E_\zeta(c_1, c_2)$ to be the set of all locally free rank-two sheaves V given by nontrivial extensions of $\mathcal{O}_X(c_1 - F) \otimes I_Z$ by $\mathcal{O}_X(F)$

$$0 \to \mathcal{O}_X(F) \to V \to \mathcal{O}_X(c_1 - F) \otimes I_Z \to 0,$$

where F is some divisor with $(2F - c_1) \equiv \zeta$, and Z is some locally complete intersection 0-cycle with length $l(Z) = c_2 + (\zeta^2 - c_1^2)/4$.

The following lemma implies that the above extension is canonical.

Lemma 1.2.2. *Let V be a sheaf in $E_\zeta(c_1, c_2)$, and let $\mathcal{O}_X(F)$ be its subsheaf coming from the extension in Definition* 1.2.1. *Then,* $\operatorname{Hom}(\mathcal{O}_X(F), V) \cong \mathbf{C}$*; moreover, $\mathcal{O}_X(F_1)$ is equal to $\mathcal{O}_X(F)$ if $\mathcal{O}_X(F_1)$ is a subsheaf of V with $(2F_1 - c_1) \equiv \zeta$.*

Proof. Note that $(c_1 - F - F_1) \equiv -\zeta$. Since $\zeta \cdot L > 0$ for some polarization L, neither $(c_1 - 2F)$ nor $(c_1 - F - F_1)$ can be effective. Let $L_0 \in W^\zeta$. Then, V is strictly L_0-semistable. Thus, $V/\mathcal{O}_X(F_1)$ is torsion free. Therefore, the conclusions follow from the standard fact: two invertible subsheaves of a sheaf with torsion free quotients coincide if the map of one to the quotient by the other is zero.

Theorem 1.2.3. *Assume ζ defines a nonempty wall W of type (c_1, c_2). Let $\mathcal{C}$ be a chamber such that $W \cap \operatorname{Closure}(\mathcal{C})$ is nonempty and $L \cdot \zeta < 0$ for $L \in \mathcal{C}$. Let $L_0 \in W$, and let L_1 be a polarization with $L_1 \cdot \zeta > 0$. If V is contained in $E_\zeta(c_1, c_2)$, then V is L_1-unstable, strictly L_0-semistable and L-stable.*

Proof. Obviously, V is L_1-unstable and strictly L_0-semistable. In the following, we show that V is L-stable. By definition, V sits in a nontrivial extension

$$0 \to \mathcal{O}_X(F) \to V \to \mathcal{O}_X(c_1 - F) \otimes I_Z \to 0,$$

where F is some divisor with $(2F - c_1) \equiv \zeta$. Let $\mathcal{O}_X(F_1)$ be any invertible subsheaf of V with torsion free quotient. Then, we have either

$$0 \to \mathcal{O}_X(F_1) \to \mathcal{O}_X(F) \quad \text{or} \quad 0 \to \mathcal{O}_X(F_1) \to \mathcal{O}_X(c_1 - F) \otimes I_Z.$$

If $0 \to \mathcal{O}_X(F_1) \to \mathcal{O}_X(F)$, then $L \cdot F_1 < (L \cdot c_1)/2$. Assume $0 \to \mathcal{O}_X(F_1) \to \mathcal{O}_X(c_1 - F) \otimes I_Z$. Then, $(c_1 - F - F_1)$ is strictly effective. Thus, $L_0 \cdot (2F_1 - c_1) < 0$. We claim that $L \cdot F_1 < (L \cdot c_1)/2$: if $L \cdot F_1 \geq (L \cdot c_1)/2$, then

$$L'_0 \cdot (2F_1 - c_1) < 0 \leq L \cdot (2F_1 - c_1),$$

where we choose $L'_0 \in W \cap \text{Closure}(\mathscr{C})$; on the other hand, V sits in an extension

$$0 \to \mathscr{O}_X(F_1) \to V \to \mathscr{O}_X(c_1 - F_1) \otimes I_{Z_1} \to 0;$$

as in the proof of Theorem 1.2.5 in Chapter I, $-(4c_2 - c_1^2) \leq (2F_1 - c_1)^2 < 0$; combined with $L'_0 \cdot (2F_1 - c_1) < 0 \leq L \cdot (2F_1 - c_1)$, this implies that $(2F_1 - c_1)$ defines a nonempty wall which separates L'_0 and L, and does not contain L'_0; but $L \in \mathscr{C}$ and $L'_0 \in W \cap \text{Closure}(\mathscr{C})$, so any wall which separates L and L'_0 must contain L'_0; we thus obtain a contradiction. In any case, $L \cdot F_1 < (L \cdot c_1)/2$, so V is L-stable.

Corollary 1.2.4. *Let $\mathscr{C}$ be a chamber such that $W^\zeta \cap \text{Closure}(\mathscr{C})$ is nonempty and $L \cdot \zeta < 0$ for $L \in \mathscr{C}$. Then, $E_\zeta(c_1, c_2)$ is embedded in $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$.*

Proof. By Theorem 1.2.3, each sheaf in $E_\zeta(c_1, c_2)$ is stable with respect to $\mathscr{C}$. By Lemma 1.2.2, $E_\zeta(c_1, c_2)$ is embedded in $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$.

Proposition 1.2.5. *Let ζ and η be two different numerical equivalence classes defining nonempty walls of type (c_1, c_2). Then, $E_\zeta(c_1, c_2)$ and $E_\eta(c_1, c_2)$ have no intersection if either of the following is true:*

(i) *both $\zeta \cdot L'$ and $\eta \cdot L'$ are nonnegative for some polarization L';*

(ii) *the two walls W^ζ and W^η are coincident.*

Proof. We may assume that both $E_\zeta(c_1, c_2)$ and $E_\eta(c_1, c_2)$ are nonempty. Suppose $V \in E_\zeta(c_1, c_2) \cap E_\eta(c_1, c_2)$. Let $\mathscr{C}$ be a chamber such that one of its faces is contained in W^ζ. Without loss of generality, let $L \cdot \zeta < 0$ for some $L \in \mathscr{C}$. By Theorem 1.2.3, V is L-stable. Since $V \in E_\eta(c_1, c_2)$, $L \cdot \eta < 0$ by Definition 1.2.1.

(i) By definition, V fits into two extensions

$$0 \to \mathscr{O}_X(F) \to V \to \mathscr{O}_X(c_1 - F) \otimes I_Z \to 0,$$
$$0 \to \mathscr{O}_X(G) \to V \to \mathscr{O}_X(c_1 - G) \otimes I_U \to 0,$$

where $(2F - c_1) \equiv \zeta$ and $(2G - c_1) \equiv \eta$. Note that $(c_1 - F - G) \equiv -(\zeta + \eta)/2$. Since $L \cdot (\zeta + \eta) < 0$, $(c_1 - F - G)$ cannot be trivial; since $L' \cdot (\zeta + \eta) \geq 0$, $(c_1 - F - G)$ cannot be effective. Applying the standard fact in the proof of Lemma 1.2.2, we conclude that $\mathscr{O}_X(F) = \mathscr{O}_X(G)$, so $\zeta = \eta$. A contradiction.

(ii) Since $L \cdot \eta < 0$ and $\eta = a\zeta$ for some number a, $a > 0$. Choose a polarization L' with $\zeta \cdot L' > 0$, and our conclusion follows from (i).

1.3. Set-theoretic comparison of moduli spaces.

Proposition 1.3.1. *Let $\mathscr{C}$ be a chamber, and $\mathscr{F}$ be one of its faces. Then, as sets,*

$$\mathscr{M}_{\mathscr{C}}(c_1, c_2) = \mathscr{M}_{\mathscr{F}}(c_1, c_2) \coprod \left(\coprod_{\zeta} E_{\zeta}(c_1, c_2) \right),$$

where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathscr{C}$, and runs over all numerical equivalence classes which define the wall containing $\mathscr{F}$.

Proof. First of all, by Theorem 1.2.3 and Proposition 1.2.5 (ii), the right-hand side consists of disjoint unions. Next, by Remark 2.2.6 in Chapter I (replace **Z**-chambers by chambers), $\mathscr{M}_{\mathscr{F}}(c_1, c_2)$ is contained in $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$; by Corollary 1.2.4, $E_{\zeta}(c_1, c_2)$ is contained in $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$; thus, the right-hand side is contained in the left-hand side. Finally, let V be $\mathscr{C}$-stable but not $\mathscr{F}$-stable. Then, V sits in

$$0 \to \mathscr{O}_X(F) \to V \to \mathscr{O}_X(c_1 - F) \otimes I_Z \to 0,$$

where $(2F - c_1) \cdot L < 0 \le (2F - c_1) \cdot L_0$ for $L \in \mathscr{C}$ and $L_0 \in \mathscr{F}$. Let $(2F - c_1) \equiv \zeta$. Then, ζ defines a wall of type (c_1, c_2) separating L and L_0. Since $\mathscr{F}$ is a face of $\mathscr{C}$, W^{ζ} must contain $\mathscr{F}$. Since $V \in E_q(c_1, c_2)$, we conclude that the left-hand side is contained in the right-hand side. Hence, the equality holds.

Corollary 1.3.2. *Let $\mathscr{C}$ be a chamber, and let $\mathscr{F}$ be one of its faces. Then, $\mathscr{F} \subseteq \Delta_{\mathscr{C}}$ if and only if $E_{\zeta}(c_1, c_2)$ is empty for any ζ where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathscr{C}$, and defines the wall containing $\mathscr{F}$.*

This gives a necessary and sufficient condition for a face of a chamber to be contained in the equivalence class determined by the chamber. It sharpens Corollary 2.2.2 in Chapter I. We will study the nonemptiness of $E_{\zeta}(c_1, c_2)$ in §2.

Theorem 1.3.3. *Let $\mathscr{C}$, $\mathscr{C}_1$ be chambers sharing a common wall. Then, as sets,*

$$\mathscr{M}_{\mathscr{C}}(c_1, c_2) = \left(\mathscr{M}_{\mathscr{C}_1}(c_1, c_2) - \coprod_{\zeta} E_{(-\zeta)}(c_1, c_2) \right) \coprod \left(\coprod_{\zeta} E_{\zeta}(c_1, c_2) \right),$$

where ζ satisfies $\zeta \cdot L < 0$ for some $L \in \mathscr{C}$, and runs over all numerical equivalence classes which define the common wall.

Proof. Follows immediately from Proposition 1.3.1.

By Proposition 1.3.1, $\coprod_{\zeta} E_{\zeta}(c_1, c_2)$ is the subset of $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$ representing all sheaves which are not $\mathscr{F}$-stable. Since stability is an open property (see [20]), $\coprod_{\zeta} E_{\zeta}(c_1, c_2)$ is closed in $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$. In general,

$E_\zeta(c_1, c_2)$ is neither open nor closed. A subset is defined to be *locally closed* if it is open in its closure, and *constructible* if it is a finite disjoint union of locally closed subsets (see [11]).

Proposition 1.3.4. *Let $\mathscr{C}$ be a chamber such that $W^\zeta \cap \mathrm{Closure}(\mathscr{C})$ is nonempty and $L \cdot \zeta < 0$ for $L \in \mathscr{C}$. Then, $E_\zeta(c_1, c_2)$ is a constructible subset in $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$.*

Proof. Using the standard construction in [12], one sees that $E_\zeta(c_1, c_2)$ is quasiprojective; moreover, its scheme structure coincides with the induced one from $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$. Therefore, $E_\zeta(c_1, c_2)$ is a constructible subset in the moduli space $\mathscr{M}_{\mathscr{C}}(c_1, c_2)$ by a theorem of Chevalley (see [3] and [11]). q.e.d.

Finally, we make some distinctions among different irreducible components.

Definition 1.3.5. Let $\mathscr{M}_L(c_1, c_2)$ be a nonempty moduli space. An irreducible component is defined to be *nontrivial* if it contains an open subset in which no sheaf is stable with respect to some polarization L'.

Proposition 1.3.6. *Let $\mathscr{M}$ be a nontrivial irreducible component in $\mathscr{M}_L(c_1, c_2)$. Then, an open subset of $\mathscr{M}$ is contained in some $E_\zeta(c_1, c_2)$ where ζ defines a nonempty wall of type (c_1, c_2) with $\zeta \cdot L < 0$.*

Proof. Let L' be a polarization with respect to which sheaves in an open subset U of $\mathscr{M}$ are not stable. By Proposition 2.1.6 in Chapter I, the set of walls of type (c_1, c_2) is locally finite. Therefore, there are finitely many walls of type (c_1, c_2) separating L' and L, and there are finitely many ζ's representing each of these walls and satisfying $\zeta \cdot L < 0 \leq \zeta \cdot L'$. Since each sheaf in U must be contained in some $E_\zeta(c_1, c_2)$, an open subset of U is contained in some $E_\zeta(c_1, c_2)$.

Remark 1.3.7. Let L' be as in the above proof. By Proposition 1.2.5, the ζ in Proposition 1.3.6 is unique if we require that $\zeta \cdot L'$ is nonnegative.

2. Nonemptiness of $E_\zeta(c_1, c_2)$

2.1. Classical techniques. Let Z stand for a locally complete intersection 0-cycle on a surface X.

Proposition 2.1.1 (see [10] and [2]). *Suppose Z consists of n distinct points $\{p_1, \cdots, p_n\}$. Then, a locally free extension of $\mathscr{O}_X(L') \otimes I_Z$ by $\mathscr{O}_X(L)$ exists if and only if every section of $\mathscr{O}_X(L' - L + K_X)$ which vanishes at all but one of the p_i vanishes at the remaining point as well where K_X is the canonical divisor of X.*

Corollary 2.1.2. *Let $l > 2h^0(X, \mathscr{O}_X(L' - L + K_X))$. Then, there exists Z with length l such that a locally free extension of $\mathscr{O}_X(L') \otimes I_Z$ by $\mathscr{O}_X(L)$ exists.*

2.2. Nonemptiness of $E_\zeta(c_1, c_2)$.

Definition 2.2.1. For any divisor F with $(2F - c_1) \equiv \zeta$, we define $E_F(c_1, c_2)$ to be the set of those sheaves in $E_\zeta(c_1, c_2)$ for which $\mathscr{O}_X(F)$ is a subsheaf.

Remark 2.2.2. Let F be as in the above definition. Then $E_{(c_1-F)}(c_1, c_2)$ is the set of those sheaves in $E_{(-\zeta)}(c_1, c_2)$ for which $\mathscr{O}_X(c_1 - F)$ is a subsheaf.

In the following, we fix F such that $(2F - c_1) \equiv \zeta$, and study the nonemptiness of $E_F(c_1, c_2)$ and $E_{(c_1-F)}(c_1, c_2)$ since it implies the nonemptiness of $E_\zeta(c_1, c_2)$ and $E_{(-\zeta)}(c_1, c_2)$. First of all, we impose no condition on the wall W^ζ.

Lemma 2.2.3. *If the linear system $|K_X + (2F - c_1)|$ is empty, then $E_F(c_1, c_2)$ or $E_{(c_1-F)}(c_1, c_2)$ is nonempty when $c_2 > c_1^2 + 2\chi(\mathscr{O}_X))/4$.*

Proof. If $c_2 + (\zeta^2 - c_1^2)/4 > 0$, then $E_{(c_1-F)}(c_1, c_2)$ is nonempty by Corollary 2.1.2. If $c_2 + (\zeta^2 - c_1^2)/4 = 0$ (i.e., $\zeta^2 = -(4c_2 - c_1^2)$), then $E_{(c_1-F)}(c_1, c_2)$ is nonempty unless $\mathrm{Ext}^1(\mathscr{O}_X(F), \Omega_X(c_1 - F)) = 0$, that is, $H^1(X, \Omega_X(c_1 - 2F)) = 0$.

Assume both $\zeta^2 = -(4c_2 - c_1^2)$ and $H^1(X, \mathscr{O}_X(c_1 - 2F)) = 0$. Note that $h^2(X, \mathscr{O}_X(c_1 - 2F)) = h^0(X, \mathscr{O}_X(K_X + 2F - c_1)) = 0$ by assumption, and that $h^0(X, \mathscr{O}_X(c_1 - 2F)) = 0$ for $(c_1 - 2F) \equiv -\zeta$. By the Riemann-Roch formula, we have $\chi(\mathscr{O}_X) + \frac{1}{2}\cdot(c_1 - 2F)[(c_1 - 2f) - K_X] = 0$. Thus, $K_X \cdot \zeta = -2\chi(\mathscr{O}_X) - \zeta^2$. We now consider $E_F(c_1, c_2)$. Since $h^0(X, \mathscr{O}_X(2F - c_1)) = 0$, we have

$$h^1(X, \mathscr{O}_X(2F - c_1)) \geq -\chi(\mathscr{O}_X) - \tfrac{1}{2}(2F - c_1)(2F - c_1)[(2F - c_1) - K_X]$$
$$= (4c_2 - c_1^2) - 2\chi(\mathscr{O}_X).$$

Therefore, if $c_2 > (c_1^2 + 2\chi(\mathscr{O}_X))/4$ then $h^1(X, \mathscr{O}_X(2F - c_1)) > 0$; thus, $E_F(c_1, c_2)$ is nonempty since $\mathrm{Ext}^1(\mathscr{O}_X(c_1 - F), \mathscr{O}_X(F))$ is nontrivial.

Lemma 2.2.4. *If $|K_X + (2F - c_1)|$ and $|K_X - (2F - c_1)|$ are nonempty, then $E_F(c_1, c_2)$ and $E_{(c_1-F)}(c_1, c_2)$ are nonempty if $c_2 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1.*

Proof. Fix a polarization $H \in \mathrm{Num}(X)$ with $H^2 = 1$. Let $\{e_1, \cdots, e_r\}$ be a basis for the space $\{x \in \mathrm{Num}(X) | x \cdot H = 0\}$ with $e_i \cdot e_j = -\delta_{ij}$. Choose a positive number a such that $aH + e_i$ is a polarization for any

i. Now, $0 \le H \cdot [K_X \pm (2F - c_1)]$ implies that $|H \cdot \zeta| \le H \cdot K_X$. Put $\zeta = (H \cdot \zeta)H + \sum_i \zeta_i e_i$. Then,

$$\begin{aligned} 0 &\le (aH + e_i) \cdot [K_X \pm (2F - c_1)] = a(H \cdot K_X) \pm a(H \cdot \zeta) + (K_X \cdot e_i) \mp \zeta_i \\ &\le 2a(H \cdot K_X) + |(K_X \cdot e_i)| \mp \zeta_i . \end{aligned}$$

Thus, $|\zeta_i| \le 2a(H \cdot K_X) + b$ for any i where we have put

$$b = \max\{|(K_X \cdot e_i)| i = 1, \cdots, r\}.$$

So, $|\zeta^2| \le (H \cdot \zeta)^2 + \sum_i |\zeta_i|^2 \le b_1$ where b_1 is a constant depending on X (and H). Since $|K_X + (2F - c_1)|$ and $|K_X - (2F - c_1)|$ are nonempty, $h^0(X, \mathscr{O}_X(K_X \pm (2F - c_1)))$ is not greater than $h^0(X, \mathscr{O}_X(2K_X))$. Put

$$c(X, c_1) = 2h^0(X, \mathscr{O}_X(2K_X)) + (b_1 + c_1^2)/4.$$

Assume $c_2 > c(X, c_1)$. Then

$$c_2 + (\zeta^2 - c_1^2)/4 > 2h^0(X, \mathscr{O}_X(K_X \pm (2F - c_1))).$$

By Corollary 2.1.2, both $E_F(c_1, c_2)$ and $E_{(c_1 - F)}(c_1, c_2)$ are nonempty.

Theorem 2.2.5. *For any numerical equivalence class ζ which defines a nonempty wall of type (c_1, c_2), $E_\zeta(c_1, c_2)$ or $E_{(-\zeta)}(c_1, c_2)$ is nonempty if $c_1 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1.*

Proof. Let F be any divisor with $(2F - c_1) \equiv \zeta$. There are two cases:

(i) $|K_X + (2F - c_1)|$ or $|K_X - (2F - c_1)|$ is empty;
(ii) both $|K_X + (2F - c_1)|$ and $|K_X - (2F - c_1)|$ are nonempty.

For case (i), we apply Lemma 2.2.3. For case (ii), we apply Lemma 2.2.4.

Corollary 2.2.6. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two chambers of type (c_1, c_2) sharing a common face which is part of the wall W. Then for any $L_0 \in W$, $L_0 \notin \mathscr{E}_{\mathscr{C}_1}$ or $L_0 \notin \mathscr{E}_{\mathscr{C}_2}$ if $c_2 > c(X, c_1)$ where $c(X, c_1)$ is a constant depending on X and c_1.*

Proof. Let $c(X, c_1)$ be the same as in Theorem 2.2.5. Assume W is represented by ζ. If $c_2 > c(X, c_1)$, then $E_\zeta(c_1, c_2)$ or $E_{(-\zeta)}(c_1, c_2)$ is nonempty by Theorem 2.2.5. We may assume that $E_\zeta(c_1, c_2)$ is nonempty and that $\zeta \cdot L < 0$ for some $L \in \mathscr{C}_1$. Let $V \in E_\zeta(c_1, c_2)$. By Theorem 1.2.3, V is $\mathscr{C}_1$-stable but strictly L_0-semistable. By the definition of equivalence classes, we conclude that $L_0 \notin \mathscr{E}_{\mathscr{C}_1}$.

Remark 2.2.7. By the results in [21] and [22], the condition that $c_2 >> 0$ in Theorem 2.2.5 cannot be weakened, and Theorem 2.2.5 is the sharpest.

Next, let $\mathscr{B}$ be a compact subset in the Kähler cone $\mathbf{C}_X$. We now investigate the nonemptiness of $E_\zeta(c_1, c_2)$ where the wall W^ζ intersects with $\mathscr{B}$. Let H and $e_1, \cdots, e_r$ be as in the proof of Lemma 2.2.4. Since $\mathscr{B}$ is compact, we can choose a positive integer a satisfying the following properties:

(P1) $(aL \pm H) \in \mathbf{C}_X$ for any L in $\mathscr{B}$;

(P2) $(aH \pm e_i) \in \mathbf{C}_X$ for any i;

(P3) $(aH - K_X)$ is strictly effective and aH is very ample.

Assume $W^\zeta \cap \mathscr{B}$ is nonempty, and $(2F - c_1) \equiv \zeta$. We separate the case where $|K_X - (2F - c_1)|$ is nonempty and the case where $|K_X - (2F - c_1)|$ is empty.

Lemma 2.2.8. *If $|K_X - (2F - c_1)|$ is nonempty, then $E_F(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathscr{B})$ where $c(X, c_1, \mathscr{B})$ is a constant depending on X, c_1 and $\mathscr{B}$.*

Proof. First of all, we show that $|\zeta^2|$ is bounded above by some constant depending on X and $\mathscr{B}$. Put $\zeta = (H \cdot \zeta)H + \sum_i \zeta_i e_i$. Let $L \in \mathscr{B} \cap W^\zeta$. We have $0 < [K_X - (2F - c_1)] \cdot (aL \pm H) = (K_X - \zeta) \cdot (aL \pm H)$, so $|H \cdot \zeta| < a|K_X \cdot L| + |K_X \cdot H|$ for $L \cdot \zeta = 0$. Now, $0 < [K_X - (2F - c_1)] \cdot (aH \pm e_i)$ implies that $|\zeta_i| < a^2|K_X \cdot L| + 2a|K_X \cdot H| + |K_X \cdot e_i|$. Thus, $|\zeta^2| \leq |H \cdot \zeta|^2 + \sum_i |\zeta_i|^2 < b_1$ where b_1 is a constant depending on X and $\mathscr{B}$.

Next, we show that $h = h^0(X, \mathscr{O}_X(K_X - (2F - c_1)))$ is also bounded above. Let $E \in |aH|$ be a smooth curve, and let g_E be its genus. Then we have

$$0 \to \mathscr{O}_X(K_X - (2F - c_1) - E) \to \mathscr{O}_X(K_X - (2F - c_1)) \to \mathscr{O}_E(D) \to 0,$$

where $\mathscr{O}_E(D)$ is the restriction of $\mathscr{O}_X(K_X - (2F - c_1))$ on E. Since $(aH - K_X)$ is strictly effective, $[K_X - (2F - c_1) - E] \cdot L = (K_X - aH) \cdot L < 0$, so $(K_X - (2F - c_1) - E)$ can never be effective. Thus, $h \leq h^0(E, \mathscr{O}_E(D))$. Note that

$$\deg(D) \leq a|K_X \cdot H| + a|\zeta \cdot H| < a^2|K_X \cdot L| + 2a|K_X \cdot H|.$$

If $h^1(E, \mathscr{O}_E(D)) = 0$, then

$$h^0(E, \mathscr{O}_E(D)) = \deg(D) + 1 - g_E \leq (a^2|K_X \cdot L| + 2a|K_X \cdot H| + 1).$$

If $h^1(E, \mathscr{O}_E(D)) > 0$, then by the Clifford's Theorem [11], $h^0(E, \mathscr{O}_E(D)) \leq 1 + \deg(D)/2 \leq (a^2|K_X \cdot L| + 2a|K_X \cdot H| + 1)$. From either case, we conclude $h \leq b_2$ where $b_2 = (a^2|K_X \cdot L| + 2a|K_X \cdot H| + 1)$ is a constant depending on X and $\mathscr{B}$. Put $c(X, c_1, \mathscr{B}) = 2b_2 + (b_1 + c_1^2)/4$.

If $c_2 > c(X, c_1, \mathscr{B})$, then

$$c_2 + (\zeta^2 - c_1^2)/4 > 2h^0(X, \mathscr{O}_X(K_X - (2F - c_1))).$$

By Corollary 2.1.2, we conclude that $E_F(c_1, c_2)$ is nonempty.

We now move to the case where $|K_X-(2F-c_1)|$ is empty. Let $\overline{O\mathscr{B}}$ be the cone with vertex O (the trivial element in $\mathrm{Num}(X)\otimes\mathbf{R}$), and spanned by elements in $\mathscr{B}$. Any wall intersecting with $\mathscr{B}$ must contain a ray in $\overline{O\mathscr{B}}$. Using this observation and the assumption that $\mathscr{B}$ is compact, we can choose another compact subset $\mathscr{B}'$ in $\mathbf{C}_X$ large enough such that any wall intersecting with $\mathscr{B}$ cuts $\mathscr{B}'$ into two parts, and that each part contains the image of some very ample divisor.

Lemma 2.2.9. *If $|K_X-(2F-c_1)|$ is empty, then $E_F(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathscr{B})$ where $c(X, c_1, \mathscr{B})$ is a constant depending on X, c_1 and $\mathscr{B}$.*

Proof. Clearly, $E_F(c_1, c_2)$ is nonempty unless $\zeta^2 = -(4c_2 - c_1^2)$ and $H^1(X, \mathscr{O}_X(2F-c_1)) = 0$. Assume $\zeta^2 = -(4c_2 - c_1^2)$ and

$$H^1(X, \mathscr{O}_X(2F-c_1)) = 0.$$

We need to show that c_2 is bounded above by some constant depending on X, c_1 and $\mathscr{B}$.

Since $\chi(\mathscr{O}_X(2F-c_1)) = 0$, by the Riemann-Roch formula, we obtain $\zeta\cdot K_X = 2\chi(\mathscr{O}_X)+\zeta^2$, so $\zeta\cdot K_X = 2\chi(\mathscr{O}_X) - (4c_2-c_1^2)$. Since W^ζ intersects with $\mathscr{B}$, from the assumption on $\mathscr{B}'$, there exists a very ample divisor H' such that its image in $\mathrm{Num}(X)\otimes\mathbf{R}$ is contained in $\mathscr{B}'$ and that $\zeta\cdot H' < 0$. Note that $2H'$ is very ample. Choose a smooth curve $E\in|2H'|$, and let g_E be its genus. Consider

$$0\to\mathscr{O}_X(K_X-(2F-c_1))\to\mathscr{O}_X(E+K_X-(2F-c_1))\to\mathscr{O}_E(D)\to 0,$$

where $\mathscr{O}_E(D)$ is the restriction of $\mathscr{O}_X(E+K_X-(2F-c_1))$ on E. Since both $h^0(X, \mathscr{O}_X(K_X-(2F-c_1)))$ and $h^1(X, \mathscr{O}_X(K_X-(2F-c_1)))$ are 0, $h^0(X, \mathscr{O}_X(E+K_X-(2F-c_1))) = h^0(E, \mathscr{O}_E(D))$. We have $\deg(D) = (2g_E-2)-2(\zeta\cdot H')$. Thus, $h^0(E, \mathscr{E}(D)) > 0$ since $(\zeta\cdot H') < 0$, so $|(E+K_X)-(2F-c_1)|$ is nonempty. As in the proof of Lemma 2.2.8, we can show that ζ^2 is bounded above by some constant depending on X, $\mathscr{B}$ (and $\mathscr{B}'$). Therefore, $|c_2| = |(c_1^2-\zeta^2)|/4$ is bounded above by some constant depending on X, c_1 and $\mathscr{B}$.

Theorem 2.2.10. *Let $\mathscr{B}$ be a compact subset in the Kähler cone $\mathbf{C}_X$. For any numerical equivalence class ζ with $W^\zeta\cap\mathscr{B}$ to be nonempty, $E_\zeta(c_1, c_2)$ is nonempty when $c_2 > c(X, c_1, \mathscr{B})$ where $c(X, c_1, \mathscr{B})$ is a constant depending on X, c_1 and $\mathscr{B}$.*

Proof. Let F be any divisor with $(2F - c_1) \equiv \zeta$. If $|K_X - (2F - c_1)|$ is nonempty, we apply Lemma 2.2.8. If $|K_X - (2F - c_1)|$ is empty, we use Lemma 2.2.9.

Corollary 2.2.11. *Let $\mathscr{B}$ be a compact subset in the Kähler cone $\mathscr{C}_X$, let W be a wall of type (c_1, c_2), and let $\mathscr{C}$ be a chamber. Assume both $W \cap \mathscr{B}$ and $W \cap \mathscr{B}$ and $W \cap \mathrm{Closure}(\mathscr{C})$ are nonempty. Then $W \cap \mathscr{E}_{\mathscr{C}}$ is empty if $c_2 > c(X, c_1, \mathscr{B})$ where $c(X, c_1, \mathscr{B})$ is a constant depending on X, c_1 and $\mathscr{B}$.*

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