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ANALYTIC AND TOPOLOGICAL TORSION FOR MANIFOLDS WITH BOUNDARY AND SYMMETRY

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0. Introduction

Let G be a finite group acting on a Riemannian manifold M by isometries. We introduce *analytic torsion*

$$\rho_{\mathrm{an}}^{G}(M, M_1; V) \in \mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G),$$

PL-torsion

$$\rho_{\mathrm{pl}}^{G}(M, M_1; V) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2},$$

Poincaré torsion

$$\rho_{\mathrm{pd}}^{G}(M, M_1; V) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2},$$

and *Euler characteristic*

$$\chi^{G}(M, M_1; V) \in \mathrm{Rep}_{\mathbf{R}}(G)$$

for ∂M the disjoint union of M_1 and M_2 and V an equivariant coefficient system. The analytic torsion is defined in terms of the spectrum of the Laplace operator, the PL-torsion is based on the cellular chain complex, and Poincaré torsion measures the failure of equivariant Poincaré duality in the PL-setting, which does hold in the analytic context. Denote by $\widehat{\mathrm{Rep}}_{\mathbf{R}}(G)$ the subgroup of $\mathrm{Rep}_{\mathbf{R}}(G)$ generated by the irreducible representations of real or complex type. We define an isomorphism

$$\Gamma_1 \oplus \Gamma_2 \colon K_1(\mathbf{R}G)^{\mathbf{Z}/2} \to (\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)) \oplus (\mathbf{Z}/2 \otimes_{\mathbf{Z}} \widehat{\mathrm{Rep}}_{\mathbf{R}}(G))$$

and show under mild conditions that

$$\rho_{\mathrm{an}}^{G}(M, M_1; V) = \Gamma_1(\rho_{\mathrm{pl}}^{G}(M, M_1; V)) - \frac{1}{2} \cdot \Gamma_1(\rho_{\mathrm{pd}}^{G}(M, M_1; V)) + \frac{\ln(2)}{2} \cdot \chi^{G}(\partial M; V)$$

and

$$\Gamma_2(\rho_{\mathrm{pl}}^{G}(M, M_1; V)) = \Gamma_2(\rho_{\mathrm{pd}}^{G}(M, M_1; V)) = 0.$$

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For trivial G this reduces to the following equation in $\mathbf{R}$:

$$\rho_{\mathrm{an}}(M, M_1; V) = \ln(\rho_{\mathrm{pl}}(M, M_1; V)) + \frac{\ln(2)}{2} \cdot \chi(\partial M) \cdot \dim_{\mathbf{R}} V.$$

Torsion invariants are important invariants which relate topology to algebraic K-theory and hence to number theory (see Milnor [25]). The s-Cobordism Theorem and the classification of lens spaces are important examples. Motivated by fruitful connections between topology and analysis, e.g., the Atiyah-Singer Index Formula, Ray and Singer [29] asked whether (Reidemeister) PL-torsion can be interpreted analytically, namely, by the spectral theory of the Laplace operator. They defined analytic torsion and gave some evidence for the conjecture that analytic and PL-torsion agree. This was independently proved by Cheeger [8] and Müller [26]. Analytic torsion is used and investigated in various contexts (see, e.g., Bismut and Freed [4], [5], Fried [14], Quillen [27], Schwarz [32], and Witten [35]).

The PL-torsion is powerful as it is a very fine invariant, and there are good tools like sum and product formulas for its computation. In particular, one can chop a manifold into "elementary" pieces, determine the PL-torsion of the pieces, and use a sum formula to compute the PL-torsion of M. Notice that these pieces have boundaries even if M is closed. In order also to get a sum formula for analytic torsion, it is necessary to investigate the relation between analytic and PL-torsions also for manifolds with boundary. Inspecting the proofs of Cheeger [8] and Müller [26] one recognizes that they do not extend to the case where M has a boundary. Moreover, an easy calculation for D^1 shows that their result is not true for D^1. Now the key observation due to Cheeger (see [8, p. 320]) is that the equivariant spectrum of the Laplace operator on $M \cup_{\partial M} M$ with the $\mathbf{Z}/2$-action given by the flip and the spectrum of the Laplace operator on M for both Dirichlet and Neumann boundary conditions determine one another. Hence the problem of comparing analytic torsion and PL-torsion for manifolds with boundary can be reduced to the case of a closed manifold with a $\mathbf{Z}/2$-action. Notice that the flip on $M \cup_{\partial M} M$ reverses the orientation. Inspecting the proof of Müller [26] again it turns out that his methods carry over to closed orientable Riemannian G-manifolds with orientation preserving and isometric G-action for a finite group G. This is carried out in Lott and Rothenberg [19], and we will exploit their work. However, we will use a different setting which seems to be more appropriate for our purposes here and for more general situations (mainly an L^2-version for proper actions of infinite groups on noncompact manifolds which we will treat in forthcoming papers).

Let (M, M_1, M_2) be an m-dimensional (compact) Riemannian G-manifold triad (with G acting by isometries). There is a canonical group extension

$$0 \to \pi_1(M) \xrightarrow{i} D^G(M) \xrightarrow{q} G \to 0$$

and a $D^G(M)$-action on the universal covering $\widetilde{M}$ extending the action of the fundamental group and covering the G-action. Consider an equivariant coefficient system V, i.e., an orthogonal representation of $D^G(M)$. Such a V may be thought of as an equivariant flat G-vector bundle over M or as an equivariant flat connection. We want to allow a twisting of our invariants by such equivariant coefficient systems because analytic torsion is important for the study of moduli spaces of flat connections (see, e.g., Quillen [27], Witten [35]). Put certain boundary conditions of Dirichlet type on M_1 and of Neumann type on M_2. Then the Laplace operator Δ^p is elliptic, selfadjoint, and nonnegative definite and is compatible with the G-action. The eigenspace $E_\lambda^G(M, M_1; V)^p$ of Δ^p for the eigenvalue λ is a real G-representation. We define the *equivariant zeta-function* by meromorphic extension of

$$\zeta_p^G(M, M_1; V)(s) = \sum_{\lambda>0} \lambda^{-s} \cdot [E_\lambda^G(M, M_1; V)^p] \in \mathbf{C} \otimes_{\mathbf{Z}} \operatorname{Rep}_{\mathbf{R}}(G).$$

It is analytic in zero and we define *analytic torsion* in §1 by

$$\rho_{\mathrm{an}}^G(M, M_1; V) = \sum_{p=0}^{m} (-1)^p \cdot p \cdot \frac{d}{ds} \zeta_p^G(M, M_1; V)|_{s=0} \in \mathbf{R} \otimes_{\mathbf{Z}} \operatorname{Rep}_{\mathbf{R}}(G).$$

For $G = 1$ this agrees up to a factor 2 with the definition of Ray and Singer [29].

A finite $\mathbf{R}G$-Hilbert complex C is a finite-dimensional finitely generated $\mathbf{R}G$-chain complex C together with a $\mathbf{R}$-Hilbert structure compatible with the G-action on each C_n. Given a $\mathbf{R}G$-chain equivalence $f\colon C \to D$, we define in §2 its *Hilbert torsion*

$$\operatorname{ht}(f) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}.$$

Let C be a finite $\mathbf{R}G$-Hilbert complex. Then its homology $H(C)$ has the structure of a finite $\mathbf{R}G$-Hilbert complex with respect to the trivial differential. There is an $\mathbf{R}G$-chain map $i\colon H(C) \to C$ uniquely determined up to $\mathbf{R}G$-chain homotopy by the property $H(i) = \mathrm{id}$. Define the *Hilbert-Reidemeister torsion* of C as

$$\operatorname{hr}(C) = \operatorname{ht}(i) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}.$$

If G is trivial, then $\mathrm{hr}(C)$ is the square of Milnor's torsion defined for C, and $H(C)$ equipped with any orthonormal bases. There is also a cochain version.

Using the Hodge decomposition theorem and the cellular bases we get $\mathbf{R}G$-Hilbert structures on $H_*(M, M_1; V)$ and $C_*(M, M_1; V)$. We define the *PL-torsion* as

$$\rho_{\mathrm{pl}}^G(M, M_1; V) = \mathrm{hr}(C_*(M, M_1; V)) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}.$$

Let $\bigcap[M]\colon C^{m-*}(M, M_1; {}^wV) \to C_*(M, M_1; V)$ be the Poincaré $\mathbf{R}G$-chain equivalence; its Hilbert torsion is the *Poincaré torsion*

$$\rho_{\mathrm{pd}}^G(M, M_1; V) \in \mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G).$$

This invariant is always zero for trivial G. This follows from the proof of Poincaré duality based on the dual cell decomposition of a triangulation. In the equivariant case the dual cell structure is not compatible with the G-action, and $\rho_{\mathrm{pd}}^G(M, M_1; V)$ measures the failure. The equivariant Euler characteristic is defined by

$$\chi^G(M, M_1; V) = \sum_{p=0}^{m} (-1)^p \cdot [H^p(M, M_1; V)] \in \mathrm{Rep}_{\mathbf{R}}(G).$$

We will assume the technical condition that the equivariant coefficient system V is coherent to a G-representation. This is always satisfied if G is trivial or M^H is nonempty and connected for all $H \subset G$. Now we can state the main result of this paper.

Theorem 4.5 (Torsion formula for manifolds with boundary and symmetry). *Let M be a Riemannian G-manifold whose boundary is the disjoint union $M_1 \coprod M_2$. Let V be an equivariant coefficient system which is coherent to a G-representation. Assume that the metric is a product near the boundary. Then*

$$\rho_{\mathrm{an}}^G(M, M_1; V) = \Gamma_1(\rho_{\mathrm{pl}}^G(M, M_1; V)) - \frac{1}{2} \cdot \Gamma_1(\rho_{\mathrm{pd}}^G(M, M_1; V)) + \frac{\ln(2)}{2} \cdot \chi^G(\partial M; V).$$

If G is trivial, this reduces to the following equation of real numbers:

$$\rho_{\mathrm{an}}(M, M_1; V) = \ln(\rho_{\mathrm{pl}}(M, M_1; V)) + \frac{\ln(2)}{2} \cdot \chi(\partial M) \cdot \dim_{\mathbf{R}} V.$$

Cheeger states in [8, p. 320] without proof a formula relating analytic and PL-torsions for a manifold with boundary (without group action). His formulas are not as precise as ours since Cheeger claims only that the correction term can be computed locally at the boundary, whereas we can identify it with the Euler characteristic.

The proof of the main theorem is organized as follows. In §§1, 2, and 3 we present product and double formulas and Poincaré duality. We investigate how these invariants depend on the Riemannian metric and relate PL-torsion to the *equivariant Whitehead torsion* of a G-homotopy equivalence. Then Theorem 4.5 is verified in §4 as follows. We first give the proof under the extra conditions (i) M is orientable, (ii) G is orientation preserving, and (iii) ∂M is empty. If $\dim(M)$ is even, the assertion follows from Poincaré duality. The Poincaré duality formulas for analytic and PL-torsion differ only by the Poincaré torsion. This is the reason for the appearance of Poincaré torsion in the formula relating analytic and PL-torsions. If $\dim(M)$ is odd, we reduce the claim to the case of trivial coefficients $V = \mathbf{R}$ and then apply Lott-Rothenberg [19]. We remove condition (iii) by the various product formulas and explicit calculations for S^1 with the involution given by complex conjugation. We get rid of (i) using the orientation covering. Finally we remove (iii) by the double formula which relates the invariants for the $(G \times \mathbf{Z}/2)$-manifold $M \cup_{\partial M} M$ to the invariants of the G-manifolds $(M, \varnothing)$ and $(M, \partial M)$. The double formulas for the analytic torsion and the PL-torsion differ by a certain Euler characteristic term of the boundary, since in the analytic case the boundary is a zero set and does not affect the $\mathbf{R}G$-Hilbert structure whereas in the PL-case the cells of the boundary do contribute to the $\mathbf{R}G$-Hilbert structure. This difference in the double formulas causes the Euler characteristic term in the formula of the theorem above. The appearance of a correction term in the case of a manifold with boundary is not very surprising if one thinks of the index formula for manifolds with boundary where the η-invariant comes in (see Atiyah-Patodi-Singer [1]–[3]).

In §5 we investigate some special cases. We derive from the sum formula in the PL-case a sum formula for the analytic torsion. This is remarkable because it is in general difficult to derive the spectrum of the Laplace operator on $M \cup_f N$ for an isometric diffeomorphism $f\colon \partial M \to \partial N$ from the spectra of its restrictions to M, N, and ∂M. We express the various torsion invariants for spheres and disks of G-representations in terms of their characters. We construct an injective homomorphism based on Poincaré torsion:

$$\rho_{\mathbf{R}}^G\colon \operatorname{Rep}_{\mathbf{R}}(G) \to \mathbf{Z} \oplus \left(\bigoplus_{(H)} K_1(\mathbf{R}[WH])^{\mathbf{Z}/2} \right).$$

This reproves the theorem of de Rham that two orthogonal G-representations V and W are linearly $\mathbf{R}G$-isomorphic if and only if their unit spheres SV and SW are G-diffeomorphic. We use the sum formula for Poincaré torsion to establish a local formula for Poincaré torsion. It computes the Poincaré torsion of M in terms of the Poincaré torsion of the tangent representations of points with nontrivial isotropy group and the universal equivariant Euler characteristic of M.

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1. Analytic torsion

Let G be a finite group. A Riemannian G-manifold M is a compact smooth manifold with differentiable G-action and invariant Riemannian metric. If ∂M is the disjoint union of M_1 and M_2, we want to define the *analytic torsion* of (M, M_1) with certain coefficients V. We begin with explaining the coefficients.

Let X be a G-space with universal covering $p: \widetilde{X} \to X$. The group of covering translations is denoted by $\Delta(p)$. Let $D^G(p)$ be the discrete group

(1.1) $$D^G(p) := \{(\tilde{f}, g) \mid f : \widetilde{X} \to \widetilde{X},\ g \in G,\ p \circ \tilde{f} = l(g) \circ p\},$$

where $l(g): X \to X$ is multiplication with g. There is an obvious exact sequence

(1.2) $$0 \to \Delta(p) \xrightarrow{i(p)} D^G(p) \xrightarrow{q(p)} G \to 0$$

and an operation of $D^G(p)$ on $\widetilde{X}$ (both natural in p) making the following diagram commute:

(1.3) $$\begin{array}{ccc} \Delta(p) \times \widetilde{X} & \longrightarrow & \widetilde{X} \\ \Big\downarrow{\scriptstyle i(p)\times \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ D^G(p) \times \widetilde{X} & \longrightarrow & \widetilde{X} \\ \Big\downarrow{\scriptstyle q(p)\times p} & & \Big\downarrow{\scriptstyle p} \\ G \times X & \longrightarrow & X \end{array}$$

In the sequel we identify $\Delta(p)$ with $\pi = \pi_1(X)$ and write $D^G(X)$ instead of $D^G(p)$. The coefficients will be orthogonal $D^G(X)$-representations.

Let M be a Riemannian G-manifold of dimension m. Define the *orientation homomorphism*

$$(1.4) \qquad w^G(M)\colon D^G(M) \to \{\pm 1\}$$

as follows. For some base point $x \in M$ an element $(\tilde{f}, g) \in D^G(M)$ is given by a homotopy class of paths w from x to gx. The composition of the fiber transport of the tangent bundle TM along w and the differential of $l(g^{-1})$ at gx give an isotopy class of automorphisms of TM_x. If it is orientation preserving, resp. reversing, we set $w^G(M)(\tilde{f}, g)$ equal to, resp. -1. An equivalent definition uses the observation that $H^m(\mathrm{Hom}_{\mathbf{Z}\pi}(C_*(\widetilde{M}), \mathbf{Z}\pi))$ is an infinite cyclic group by Poincaré duality and $\tilde{f}$ induces a homomorphism $\tilde{f}_*\colon \pi \to \pi$, a $\mathbf{Z}[\tilde{f}_*]$-equivariant map $C_*(\tilde{f})\colon C_*(\widetilde{M}) \to C_*(\widetilde{M})$, and hence an automorphism of this infinite cyclic group by $H^m(\mathrm{Hom}_{\mathbf{Z}\pi}(C_*(\tilde{f}), \mathbf{Z}[f_*^{-1}]])$. Then $w^G(M)(\tilde{f}, g)$ is its degree.

If V is a $D^G(M)$-representation, let ${}^wV$ be the $w = w^G(M)$-twisted $D^G(M)$-representation given by $w(e) \cdot ev$ for $e \in D^G(M)$ and $v \in V$. The vector bundle $\widetilde{M} \times_\pi V$ over M becomes a G-vector bundle by $g(\tilde{x}, v) = (\tilde{x}\tilde{g}^{-1}, \tilde{g}v)$ for any lift $\tilde{g} \in D^G(M)$ of $g \in G$. The deRham complex $\Lambda^*(M; V)$ of differential forms with coefficients in $\widetilde{M} \times_\pi V$ is an $\mathbf{R}G$-cochain complex. A choice of a local orientation on $T\widetilde{M}_{\tilde{x}}$ for some $\tilde{x} \in \widetilde{M}$ together with the Riemannian metric determines a volume form $dM \in \Lambda^n(M; {}^w\mathbf{R})$. Using the inner product on V we get the product $\wedge\colon \Lambda^p(M; V) \otimes \Lambda^q(M; {}^wV) \to \Lambda^{p+q}(M; {}^w\mathbf{R})$. The *Hodge star operator*

$$(1.5) \qquad *\colon \Lambda^p(M; V) \to \Lambda^{m-p}(M; {}^wV)$$

is defined by $\omega \wedge (*\eta) = \langle \omega, \eta \rangle \cdot dM$, where $\langle\ ,\ \rangle$ is induced from the Riemannian metric. Since dM is G-invariant, the map $*$ is $\mathbf{R}G$-linear. The Riemannian metric induces an inner product on $\Lambda^p(M; V)$ by $\langle\langle \omega, \eta \rangle\rangle := \int_M \langle \omega, \eta \rangle\, dM$. Then $*$ is an isometry satisfying $* \circ * = (-1)^{p(m-p)}\,\mathrm{id}$. The adjoint $\delta^p\colon \Lambda^p(M; V) \to \Lambda^{p-1}(M; V)$ of the differential d^p is $(-1)^{mp+p+1} * d^{m-p} *$. Define the *Laplace operator*

$$(1.6) \qquad \Delta^p\colon \Lambda^p(M; V) \to \Lambda^p(M; V)$$

by $d^{p-1}\delta^p + \delta^{p+1}d^p$.

Let M be a Riemannian G-manifold whose boundary ∂M is the disjoint union $M_1 \coprod M_2$. We allow that M_1 or M_2 or both are empty. Consider an orthogonal $D^G(M)$-representation V. Given a p-form $\omega \in \Lambda^p(M; V)$, let $\omega_{\tan}$ be the p-form on ∂M coming from restriction with

$Ti\colon T\partial M \to TM$ for the inclusion i. Let ω_{nor} be the $(p-1)$-form $*_{\partial M}(*_M\omega)_{\tan}$. We will consider the boundary conditions

$$(1.7)\qquad \begin{aligned} b(M, M_1)\colon \omega_{\tan} = 0,\ (\delta\omega)_{\tan} = 0 \quad \text{on } M_1,\\ \omega_{\text{nor}} = 0,\ (d\omega)_{\text{nor}} = 0 \quad \text{on } M_2. \end{aligned}$$

In the sequel we write

$$\begin{aligned} &\Lambda_1^p(M, M_1; V) = \{\omega \in \Lambda^p(M; V) \mid \omega_{\tan} = 0 \text{ on } M_1 \\ &\qquad\qquad\qquad\qquad \text{and } \omega_{\text{nor}} = 0 \text{ on } M_2\},\\ (1.8)\quad &\Lambda_2^p(M, M_1; V) = \{\omega \in \Lambda^p(M; V) \mid \omega \text{ satisfies } b(M, M_1)\},\\ &H^p_{\text{harm}}(M, M_1; V) = \{\omega \in \Lambda^p(M; V) \mid \Delta\omega = 0,\\ &\qquad\qquad\qquad\qquad \omega \text{ satisfies } b(M, M_1)\}. \end{aligned}$$

The space $H^p_{\text{harm}}(M, M_1; V)$ is called *space of harmonic forms*. Denote by

$$(1.9)\qquad A(M; V)\colon \Lambda^*(M; V) \to C_s^*(M; V)$$

the *V-twisted deRham map* which is the composition

$$\begin{aligned} \Lambda^*(M; V) = \Lambda^*(M; \widetilde{M} \times_\pi V) &\xrightarrow{p^*} \Lambda^*(\widetilde{M}; \widetilde{M} \times V)^\pi \xleftarrow{j^\pi} (\Lambda^*(\widetilde{M}) \otimes_{\mathbf{R}} V)^\pi \\ &\xrightarrow{(A(\widetilde{M})\otimes_{\mathbf{R}}\text{id})^\pi} (\text{Hom}_{\mathbf{R}}(C_*^s(\widetilde{M}), \mathbf{R}) \otimes_{\mathbf{R}} V)^\pi \xrightarrow{\Phi^\pi} \text{Hom}_{\mathbf{R}}(C_*^s(\widetilde{M}), V)^\pi \\ &= \text{Hom}_{\mathbf{R}_\pi}(C_*^s(\widetilde{M}), V) =: C_s^*(M; V). \end{aligned}$$

Here $C_*^s(\widetilde{M})$ is the singular chain complex. The map p^* is induced from the projection $p\colon \widetilde{M} \times V \to \widetilde{M} \times_\pi V$. The isomorphism $j\colon \Lambda^*(\widetilde{M}) \otimes_{\mathbf{R}} V \to \Lambda^*(\widetilde{M}; \widetilde{M} \times V)$ sends $s \otimes v$, given by a section s of $\Lambda^p T^*\widetilde{M}$ and $v \in V$, to the section $x \mapsto s(x) \otimes v$. We denote by $A(\widetilde{M})$ the ordinary deRham map sending a p-form ω to the singular cosimplex $\sigma \mapsto \int \sigma^*\omega$. The isomorphism Φ maps $\phi \otimes v$ to the $\mathbf{R}$-map $C_*^s(\widetilde{M}) \to V$, $\sigma \mapsto \phi(\sigma)v$.

We denote by $L^2\Lambda^p(M; V)$ the Hilbert completion of $\Lambda^p(M; V)$ under the inner product $\langle\langle \omega, \eta \rangle\rangle := \int_M \omega \wedge *\eta$. For later purposes we state the following result whose proof can be found in Müller [26, p. 239].

Theorem 1.10 (Hodge-decomposition theorem). (a) $H^p_{\text{harm}}(M, M_1; V) = \ker(d) \cap \ker(\delta) \cap \Lambda_1^p(M, M_1; V)$.

(b) *The* $\mathbf{R}$-*modules* $\ker(\Delta) \cap \Lambda_1^p(M, M_1; V)$ *and* $H^p_{\text{harm}}(M, M_1; V)$ *are finitely generated.*

(c) *We have the orthogonal decompositions*

$$\Lambda_1^p(M, M_1; V) = H_{\text{harm}}^p(M, M_1; V) \oplus d(\Lambda_1^{p-1}(M, M_1; V)) \oplus \delta(\Lambda_1^{p+1}(M, M_1; V)),$$
$$L^2\Lambda^p(M, M_1; V) = H_{\text{harm}}^p(M, M_1; V) \oplus \text{clos}(d(\Lambda_1^{p-1}(M, M_1; V))) \oplus \text{clos}(\delta(\Lambda_1^{p-1}(M, M_1; V))).$$

(d) *The inclusion* $i\colon H_{\text{harm}}^p(M, M_1; V) \to \ker(d) \cap \Lambda_1^p(M, M_1; V)$ *composed with a deRham map has image contained in the space of cocycles in* $C_s^p(M, M_1; V)$. *We obtain an isomorphism*

$$\bar{i}\colon H_{\text{harm}}^p(M, M_1; V) \to H_s^p(M, M_1; V).$$

The Laplace operator $\Delta^p\colon \Lambda_2^p(M; V) \to \Lambda_2^p(M; V)$ is an elliptic self-adjoint differential operator, whose spectrum is a pure point spectrum consisting of nonnegative real numbers. For $\lambda \geq 0$ we put

$$E_\lambda^G(M, M_1; V)^p = \{\omega \in \Lambda_2^p(M; V) \mid \Delta^p\omega = \lambda\omega\}. \tag{1.11}$$

Since G acts on (M, M_1) by isometries, Δ is compatible with the $\mathbf{R}G$-structure. Since $E_\lambda^G(M, M_1; V)^p$ is finitely generated, it defines an element in the real representation ring $\text{Rep}_{\mathbf{R}}(G)$. We define the *equivariant zeta-function*

$$\begin{aligned}\zeta_p^G(M, M_1; V)(s) &= \sum_{\lambda>0} \lambda^{-s} \cdot [E_\lambda^G(M, M_1; V)^p] \in \mathbf{C} \otimes_{\mathbf{Z}} \text{Rep}_{\mathbf{R}}(G),\\ \zeta^G(M, M_1; V) &= \sum_{p=0}^{m} (-1)^p \cdot p \cdot \zeta_p^G(M, M_1; V)\end{aligned} \tag{1.12}$$

for $s \in \mathbf{C}$ with $\text{Real}(s) > \dim(M)/2$. Because $\text{Rep}_{\mathbf{R}}(G)$ is a finitely generated free abelian group with the isomorphism classes of irreducible representations as base, we may identify $\mathbf{C} \otimes_{\mathbf{Z}} \text{Rep}_{\mathbf{R}}(G)$ with $\mathbf{C}^r$ for $r = \text{rk}_{\mathbf{Z}}(\text{Rep}_{\mathbf{R}}(G))$. Hence it makes sense to speak of convergence in $\mathbf{C} \otimes_{\mathbf{Z}} \text{Rep}_{\mathbf{R}}(G)$. Restriction to the trivial subgroup defines a homomorphism $\text{res}\colon \mathbf{C} \otimes_{\mathbf{Z}} \text{Rep}_{\mathbf{R}}(G) \to \mathbf{C} \otimes_{\mathbf{Z}} \text{Rep}_{\mathbf{R}}(\{1\}) = \mathbf{C}$. The image of $\zeta_p^G(M, M_1; V)$ under this map is just the nonequivariant zeta-function which converges absolutely for $\text{Real}(s) > \dim(M)/2$ (see Gilkey [15, p. 79]). This implies that $\zeta_p^G(M, M_1; V)$ converges absolutely for $\text{Real}(s) > \dim(M)/2$.

Lemma 1.13. *The equivariant zeta-function* $\zeta_p^G(M, M_1; V)$ *has a meromorphic extension to* $\mathbf{C}$. *It is analytic in zero, and its derivative at zero lies in* $\mathbf{R} \otimes_{\mathbf{Z}} \text{Rep}_{\mathbf{R}}(G)$.

We defer the proof of Lemma 1.13 to §4.

Definition 1.14. The analytic torsion

$$\rho_{an}^G(M, M_1; V) \in \mathbf{R} \otimes_{\mathbf{Z}} \operatorname{Rep}_{\mathbf{R}}(G)$$

is defined by $\rho_{an}^G(M, M_1; V) = \sum_{p=0}^m (-1)^p \cdot p \cdot \frac{d}{ds}\zeta_p^G(M, M_1; V)|_{s=0}$.

Example 1.15. Fix a positive real number μ. Equip $\mathbf{R}$ with the standard metric and the unit circle S^1 with the Riemannian metric for which $\mathbf{R} \to S^1$, $t \mapsto \exp(2\pi i\mu^{-1}t)$ is isometric. Then S^1 has volume μ. Let $\mathbf{Z}/2$ act on S^1 by complex conjugation. The Laplace operator $\Delta^1 : \Lambda^1\mathbf{R} \to \Lambda^1\mathbf{R}$ maps $f(t)\,dt$ to $-f''(t)\,dt$. By checking the μ-periodic solutions of $f''(t) = -\lambda f(t)$ one shows that Δ^1 on S^1 has eigenspaces $E_\lambda(S^1)^1 = \operatorname{span}_{\mathbf{R}}(f_n\,dt, g_n\,dt)$ for $\lambda = (2\pi\mu^{-1}n)^2$, $n \geq 1$, $E_\lambda(S^1)^1 = \operatorname{span}_{\mathbf{R}}(dt)$ for $\lambda = 0$, and $E_\lambda(S^1) = \{0\}$ otherwise, if $f_n(\exp(2\pi i\mu^{-1}t)) = \cos(2\pi\mu^{-1}nt)$ and $g_n(\exp(2\pi i\mu^{-1}t)) = \sin(2\pi\mu^{-1}nt)$. The $\mathbf{Z}/2$-action on S^1 induces the $\mathbf{Z}/2$-action on $E_{(2\pi\mu^{-1}n)^2}(S^1)$ sending f_n to f_n and g_n to $-g_n$. Denote the *Riemannian zeta-function* by $\zeta_{\mathrm{Rie}}(s) = \sum_{n\geq 1} n^{-s}$. Let $\mathbf{R}$ be the trivial and $\mathbf{R}^-$ be the nontrivial one-dimensional $\mathbf{Z}/2$-representation. We get

$$\zeta_1^{\mathbf{Z}/2}(S^1; \mathbf{R}) = \left(\frac{2\pi}{\mu}\right)^{-2s} \cdot \zeta_{\mathrm{Rie}}(2s) \cdot ([\mathbf{R}] + [\mathbf{R}^-]).$$

Since $\zeta_{\mathrm{Rie}}(0) = -\frac{1}{2}$ and $\zeta'_{\mathrm{Rie}}(0) = -\ln(2\pi)/2$ hold (see Titchmarsh [34]), we obtain

$$\rho_{an}^{\mathbf{Z}/2}(S^1; \mathbf{R}) = \ln(\mu) \cdot ([\mathbf{R}] + [\mathbf{R}^{-1}]). \tag{1.16}$$

By restriction to the trivial subgroup we derive

$$\rho_{an}(S^1; \mathbf{R}) = 2 \cdot \ln(\mu). \tag{1.17}$$

Example 1.18. Equip $D^1 = [0, 1]$ with the standard metric scaled by $\mu > 0$. The volume form is then $\mu\,dt$. The Laplace operator $\Delta^1 : \Lambda^1 D \to \Lambda^1 D^1$ maps $f(t)\,dt$ to $\mu^{-2} - f''(t)\,dt$. We get

$$E_\lambda(D^1; \mathbf{R}) = \operatorname{span}_{\mathbf{R}}(\sin(\pi nt)\,dt),$$
$$E_\lambda(D^1, \partial D^1; \mathbf{R}) = \operatorname{span}_{\mathbf{R}}(\cos(\pi nt)\,dt) \quad \text{for } \lambda = (\frac{\pi}{\mu}n)^2,\ n \in \mathbf{Z},\ n \geq 1,$$

and $E_\lambda(D^1; \mathbf{R}) = E_\lambda(D^1, \partial D^1; \mathbf{R}) = \{0\}$ otherwise. Hence we get

$$\zeta_1(D^1; \mathbf{R}) = \zeta_1(D^1, \partial D^1; \mathbf{R}) = \left(\frac{\pi}{\mu}\right)^{-2s} \cdot \zeta_{\mathrm{Rie}}(2s),$$

which implies

$$(1.19)\qquad \rho_{\mathrm{an}}(D^1, \mathbf{R}) = \rho_{\mathrm{an}}(D^1, \partial D^1; \mathbf{R}) = \ln(2\mu).$$

Proposition 2.10 (Poincaré duality for analytic torsion). *Let M be an m-dimensional Riemannian G-manifold with orientation homomorphism $w = w^G(M)$. If V is an orthogonal $D^G(M)$-representation and ∂M the disjoint union $M_1 \coprod M_2$, we have*

$$\rho^G_{\mathrm{an}}(M, M_1; V) + (-1)^m \cdot \rho^G_{\mathrm{an}}(M, M_2; {}^wV) = 0.$$

Proof. Recall $E^G_\lambda(M, M_1; V)^p = \{\omega \in \Lambda^p_2(M, M_1; V) \mid \Delta\omega = \lambda\omega\}$ and $\Lambda^p_2(M, M_1; V) = \{\omega \in \Lambda^p(M; V) \mid \omega \text{ satisfies } b(M, M_1)\}$, where the boundary conditions $b(M, M_1)$ were defined in (1.7). Put

$$E'_\lambda(M, M_1; V)^p := \{\omega \in \Lambda^p_2(M, M_1; V) \mid d\delta\omega = \lambda\omega\}$$

and

$$E''_\lambda(M, M_1; V)^p := \{\omega \in \Lambda^p_2(M, M_1; V) \mid \delta\, d\omega = \lambda\omega\}.$$

For $\lambda \neq 0$ we obtain an orthogonal $\mathbf{R}G$-sum decomposition

$$(1.21)\qquad \begin{aligned} &\lambda^{-1} d\delta \oplus \lambda^{-1}\delta d: \\ &\qquad E^G_\lambda(M, M_1; V)^p \to E'_\lambda(M, M_1; V)^p \oplus E''_\lambda(M, M_1; V)^p \end{aligned}$$

and inverse isometric $\mathbf{R}G$-isomorphisms

$$(1.22)\qquad \begin{aligned} &\lambda^{-1/2}\delta\colon E'_\lambda(M, M_1; V)^{p+1} \to E''_\lambda(M, M_1; V)^p, \\ &\lambda^{-1/2} d : E''_\lambda(M, M_1; V)^p \to E'_\lambda(M, M_1; V)^{p+1}. \end{aligned}$$

The Hodge star operator $*$ induces an isometric $\mathbf{R}G$-isomorphism

$$(1.23)\qquad *\colon E^G_\lambda(M, M_1; V)^p \to E^G_\lambda(M, M_2; {}^wV)^{m-p}.$$

Now the following computation finishes the proof:

$$\begin{aligned}
\zeta^G(M, M_1; V) &= \sum_{p=0}^{m}\sum_{\lambda>0}(-1)^p p\lambda^{-s}[E_\lambda^G(M, M_1; V)^p] \\
&= \sum_{p=0}^{m}\sum_{\lambda>0}(-1)^p p\lambda^{-s}[E_\lambda^G(M, M_2; {}^wV)^{m-p}] \\
&= (-1)^m\sum_{p=0}^{m}\sum_{\lambda>0}(-1)^{m-p} p\lambda^{-s}[E_\lambda^G(M, M_2; {}^wV)^{m-p}] \\
&= (-1)^{m+1}\sum_{p=0}^{m}\sum_{\lambda>0}(-1)^{m-p}(m-p)\lambda^{-s} \\
&\quad \cdot [E_\lambda^G(M, M_2; {}^wV)^{m-p}] \\
&\quad + (-1)^m m\sum_{p=0}^{m}\sum_{\lambda>0}(-1)^{m-p}\lambda^{-s} \\
&\quad \cdot [E_\lambda^G(M, M_2; {}^wV)^{m-p}] \\
&= (-1)^{m+1}\zeta^G(M, M_2; {}^wV) + (-1)^m m\sum_{p=0}^{m}\sum_{\lambda>0}(-1)^{m-p} \\
&\quad \cdot \lambda^{-s}([E'_\lambda(M, M_1; V)^{p+1}] + [E'_\lambda(M, M_1; V)^p]) \\
&= (-1)^{m+1}\zeta^G(M, M_2; {}^wV). \quad \text{q.e.d.}
\end{aligned}$$

Suppose that M is orientable and closed, its dimension m is even, and G is orientation preserving. As w is trivial, we get $\rho_{\mathrm{an}}^G(M; V) = 0$.

Remark 1.24. We often put the condition on the Riemannian metric that it is a product near the boundary, i.e., there is an equivariant collar $f\colon \partial M \times [0, 1[$ onto an invariant neighborhood of the boundary such that f is isometric if we equip $\partial M \subset M$ and $U \subset M$ with the induced metric, $[0, 1[$ with the standard metric, and $\partial M \times [0, 1[$ with the product metric. This condition ensures that for two such Riemannian G-manifolds M and N and an isometric G-diffeomorphism $f\colon M_1 \to N_1$ between open and closed submanifolds $M_1 \subset \partial M$ and $N_1 \subset \partial N$ there is the structure of a Riemannian G-manifold on $M \cup_f N$ such that the obvious inclusions $i_M\colon M \to M \cup_f N$ and $i_N\colon N \to M \cup_f N$ are isometric G-imbeddings. Let V, resp. W, be an orthogonal $D^G(M)$-, resp. $D^G(N)$-, representation. We denote by $j_M: M_1 \to M$ and $j_N: N_1 \to N$ the inclusions. Fix an orthogonal $D^G(M_1)$-isomorphism $\bar{f}: j_M^* V \to f^* j_N^* W$. Then there is an orthogonal $D^G(M \cup_f N)$-representation $V \cup_{\bar{f}} W$ such that $j_M^*(V \cup_{\bar{f}} W)$ and V, resp. $j_N^*(V \cup_{\bar{f}} W)$ and W, agree. If G is trivial and M_1 is connected, this follows from the theorem of Seifert-van Kampen. In the general case one must apply a generalized version saying that the corresponding diagram of fundamental categories in the sense of [21] is a push out of categories. Alternatively, one may think of the representations as G-vector bundles and glue them together.

In particular, we can choose $f = \mathrm{id}$ and $\bar{f} = \mathrm{id}$ and consider $M \cup_{M_1} M$ and $V \cup_{M_1} V$. There is a canonical $\mathbf{Z}/2$-structure on $M \cup_{M_1} M$ obtained by switching the two copies of M. Hence we can consider $M \cup_{M_1} M$ as a Riemannian $(G \times \mathbf{Z}/2)$-manifold. The $\mathbf{Z}/2$-structure induces a $\mathbf{Z}/2$-action on $D^G(M \cup_{M_1} M)$ and $D^{G\times\mathbf{Z}/2}(M \cup_{M_1} M)$ is the semidirect product $D^G(M \cup_{M_1} M) \times_s \mathbf{Z}/2$, provided that M_1 is not empty. The orientation homomorphism $w^{G\times\mathbf{Z}/2}(M \cup_{M_1} M)$ maps $(u, \pm 1) \in D^{G\times\mathbf{Z}/2}(M \cup_{M_1} M)$ to $\pm 1 \cdot w^G(M \cup_{M_1} M)(u)$. One can extend $V \cup_{M_1} V$ to an orthogonal $D^{G\times\mathbf{Z}/2}(M \cup_{M_1} M)$-representation by $(u, \pm 1) \cdot v = u \cdot v$ for $u \in D^G(M \cup_{M_1} M)$ and $v \in V$, since u and $(-1) \cdot u \in D^G(M \cup_{M_1} M)$ for $\pm 1 \in \mathbf{Z}/2$ operate on $V \cup_{M_1} V$ in the same way.

The following result will allow us to reduce the case of a manifold with boundary to the closed one. We have the isomorphism

$$(1.25) \qquad \begin{gathered}(\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)) \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(H) \xrightarrow{\otimes_{\mathbf{R}}} \mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G \times H), \\ (\lambda \cdot [P]) \otimes_{\mathbf{Z}} [Q] \mapsto \lambda \cdot [P \otimes_{\mathbf{R}} Q].\end{gathered}$$

For later purposes we mention the pairing obtained from (1.25) for $G = H$, also denoted by $\otimes_{\mathbf{R}}$, and restriction to the diagonal:

$$(1.26) \qquad (\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)) \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G) \xrightarrow{\otimes_{\mathbf{R}}} \mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G).$$

For the next result put $H = \mathbf{Z}/2$ in (1.25).

Proposition 1.27 (Double formula for analytic torsion). *Let M be a Riemannian G-manifold such that the Riemannian metric is a product near the boundary. Suppose that ∂M is the disjoint union of M_1 and M_2. Let V be an orthogonal $D^G(M)$-representation. Then the following equalities* *hold*:

(a) $$E_\lambda^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V)^p = (E_\lambda^G(M; V)^p \otimes_{\mathbf{R}} \mathbf{R}) \oplus (E_\lambda^G(M, M_1; V)^p \otimes_{\mathbf{R}} \mathbf{R}^-),$$

(b) $$\zeta_p^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V) = (\zeta_p^G(M; V) \otimes_{\mathbf{C}} \mathbf{C}) \oplus (\zeta_p^G(M, M_1; V) \otimes_{\mathbf{C}} \mathbf{C}^-),$$

(c) $$\rho_{\text{an}}^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V) = (\rho_{\text{an}}^G(M; V) \otimes_{\mathbf{R}} [\mathbf{R}]) + (\rho_{\text{an}}^G(M, M_1; V) \otimes_{\mathbf{R}} [\mathbf{R}^-]).$$

Proof. Obviously (b) and (c) follow from (a). Let $\tau\colon M\cup_{M_1} M \to M\cup_{M_1} M$ be the flip map. Define

(1.28)
$$E_\lambda^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V)^p_+ = \{\omega \in E_\lambda^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V)^p \mid \tau^*\omega = \omega\},$$
$$E_\lambda^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V)^p_- = \{\omega \in E_\lambda^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V)^p \mid \tau^*\omega = -\omega\}.$$

Let $i\colon M \to M\cup_{M_1} M$ be the inclusion onto the first summand. Obviously i^* is compatible with d, $*$, δ, and Δ. Since τ is an isometry and reverses the local orientation at points in M_1, the induced map τ^* maps the volume form $d(M\cup_{M_1} M)$ to $-d(M\cup_{M_1} M)$. This implies $\tau^*\circ * = -*\circ\tau^*$. As τ is the identity on M_1, we get $(i^*\tau^*\omega)_{\tan} = (i^*\omega)_{\tan}$ on M_1. Hence $i^*\omega$ satisfies the boundary conditions $b(M, \varnothing)$ (resp. $b(M, M_1)$) (see 1.7), if $\tau^*\omega = \omega$ (resp. $\tau^*\omega = -\omega$) holds. Thus we can define $\mathbf{R}G$-maps

(1.29)
$$i^+\colon E_\lambda^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V)^p_+ \to E_\lambda^G(M; V), \qquad \omega \mapsto i^*\omega,$$
$$i^-\colon E_\lambda^{G\times \mathbf{Z}/2}(M\cup_{M_1} M; V\cup_{M_1} V)^p_- \to E_\lambda^G(M, M_1; V)^p, \qquad \omega \mapsto i^*\omega.$$

Obviously i^+ is injective, as $i^*\omega$ determines ω because of $\tau^*\omega = \omega$. Next we show that i^+ is surjective. Given $\omega \in E_\lambda^G(M; V)$, there is only one candidate as preimage, namely $\omega\cup_{M_1}\omega$. The problem is that $\omega\cup_{M_1}\omega$ is smooth on $(M\cup_{M_1} M) - M_1$ and a priori only continuous on $M\cup_{M_1} M$, but we need smoothness on $M\cup_{M_1} M$. The obvious inclusion induces an

$\mathbf{R}G$-isomorphism:

(1.30)

$$j : E_\lambda^{G\times\mathbf{Z}/2}(M \cup_{M_1} M; V \cup_{M_1} V)_+^p \oplus E_\lambda^{G\times\mathbf{Z}/2}(M \cup_{M_1} M; V \cup_{M_1} V)_-^p \to E_\lambda^{G\times\mathbf{Z}/2}(M \cup_{M_1} M; V \cup_{M_1} V)^p.$$

For any $\eta \in \Lambda^p(M \cup_{M_1} M; V)$ we get

$$\langle\langle \omega \cup_{M_1} \omega, \eta\rangle\rangle_{M\cup_{M_1} M} = \langle\langle \omega, i^*\eta\rangle\rangle_M + \langle\langle \omega, i^*\tau^*\eta\rangle\rangle_M,$$

since an integral over $M \cup_{M_1} M$ is the sum of its restrictions to the two copies of M. If $\tau^*\eta = -\eta$ holds, then $\langle \omega\cup_{M_1}\omega, \eta\rangle\rangle_{M\cup_{M_1} M} = 0$. Consider $\eta \in E_\mu^G(M \cup_{M_1} M; V \cup_{M_1} V)^p$ with $\tau^*\eta = \eta$ for $\mu \neq \lambda$. Then $i^*\eta \in E_\mu^G(M; V)$ and $\mu \neq \lambda$ imply $\langle\langle \omega \cup_{M_1} \omega, \eta\rangle\rangle = 0$. Notice that the Hilbert space $L^2\Lambda^p(M \cup_{M_1} M; V \cup_{M_1} V)$ has an orthonormal basis of smooth eigenvectors of $\Delta_{M\cup_{M_1} M}$. For $E_\lambda^{G\times\mathbf{Z}/2}(M \cup_{M_1} M; V \cup_{M_1} V)_+^p$ choose an orthonormal basis $\{\eta_1, \eta_2, \cdots, \eta_r\}$. Then the Fourier development of $\omega \cup_{M_1} \omega$ is obtained by the computations above:

$$\omega \cup_{M_1} \omega = \sum_{i=1}^r \langle\langle \omega \cup_{M_1} \omega, \eta_i\rangle\rangle \cdot \eta_i,$$

which holds in $L^2\Lambda^p(M \cup M_1 M; V \cup_{M_1} V)$. Since the both sides of the equation are represented by continuous sections, they agree as functions. The right side is smooth and hence so is $\omega \cup_{M_1} \omega$. This finishes the proof of Proposition 1.27. q.e.d.

We define the *equivariant Euler characteristic* as

$$\chi^G(M, M_1; V) = \sum_{p=0}^m (-1)^p \cdot [H_p(M, M_1; V)] \in \operatorname{Rep}_{\mathbf{R}}(G). \tag{1.31}$$

Proposition 1.32 (Product formula for analytic torsion). *Assume a Riemannian G-manifold M whose boundary is the disjoint union $M_1 \coprod M_2$ and a Riemannian H-manifold N with empty boundary. Let V (resp. W) be an orthogonal $D^G(M)$- (resp. $D^H(N)$-) representation. Then $M \times N$ is a Riemannian $(G\times H)$-manifold, and $V \otimes_{\mathbf{R}} W$ an orthogonal $D^{G\times H}(M\times N)$-representation and, using the pairing* (1.25), *we obtain*

$$\rho_{\mathrm{an}}^{G\times H}(M \times N, M_1 \times N; V \otimes_{\mathbf{R}} W) = \chi^G(M, M_1; V) \otimes_{\mathbf{R}} \rho_{\mathrm{an}}^H(N; W) + \rho_{\mathrm{an}}^G(M, M_1; V) \otimes_{\mathbf{R}} \chi^H(N; W).$$

Proof. We show the analogous statement for the zeta-functions if $\mathrm{Real}(s) > \dim(M)/2$. We conclude from (1.21) and (1.22) that

$$\sum_{p\geq 0}(-1)^p \cdot [E_\lambda^G(M, M_1; V)^p] = 0 \tag{1.33}$$

for $\lambda > 0$. We derive the equality

$$\chi^G(M, M_1; V) = \sum_{p\geq 0}(-1)^p \cdot [E_0^G(M, M_1; V)] \tag{1.34}$$

from Theorem 1.10. Notice that $\Lambda^*(M; V) \otimes_{\mathbf{R}} \Lambda^*(N; W)$ is dense in $\Lambda^*(M \times N; V \otimes_{\mathbf{R}} W)$ and on this dense subspace we have $\Delta_{M\times N} = \Delta_M \otimes_{\mathbf{R}} \mathrm{id} + \mathrm{id} \otimes_{\mathbf{R}} \Delta_N$. The eigenvalues of Δ_M build a Hilbert basis for $L^2\Lambda^*(M; V)$. We conclude that

$$\begin{aligned} &E_\gamma^{G\times H}(M \times N, M_1 \times N; V \otimes_{\mathbf{R}} W)^i \\ &\quad = \bigoplus_{p+q=i}\ \bigoplus_{\lambda+\mu=\gamma} E_\lambda^G(M, M_1; V)^p \otimes_{\mathbf{R}} E_\mu^H(N; W)^q. \end{aligned} \tag{1.35}$$

Now we compute

$$\begin{aligned} &\zeta^{G\times H}(M \times N; M_1 \times N; V \otimes_{\mathbf{R}} W) \\ &\quad = \sum_{i\geq 0}(-1)^i \cdot i \sum_{\gamma>0} \gamma^{-s} \cdot [E_\gamma^{G\times H}(M \times N; M_1 \times N; V \otimes_{\mathbf{R}} W)] \\ &\quad = \sum_{p,q}\ \sum_{\lambda+\mu>0} (\lambda+\mu)^{-s} \cdot (-1)^{p+q} \cdot (p+q) \\ &\quad \cdot [E_\lambda^G(M, M_1; V)^p \otimes_{\mathbf{R}} E_\mu^H(N; W)^q] \\ &\quad = \sum_{\lambda+\mu>0} (\lambda+\mu)^{-s} \cdot \left(\sum_{p\geq 0}(-1)^p \cdot [E_\lambda^G(M, M_1; V)^p]\right) \\ &\quad \otimes_{\mathbf{R}} \left(\sum_{q\geq 0}(-1)^q \cdot q \cdot [E_\mu^H(N; W)^q]\right) + \sum_{\lambda+\mu>0} (\lambda+\mu)^{-s} \\ &\quad \cdot \left(\sum_{p\geq 0}(-1)^p \cdot p \cdot [E_\lambda^G(M, M_1; V)]\right) \otimes_{\mathbf{R}} \left(\sum_{q\geq 0}(-1)^q \cdot [E_\mu^H(N; W)^q]\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{p\geq 0}(-1)^p \cdot [E_0^G(M, M_1; V)]\right) \\
&\quad \otimes_{\mathbf{R}} \left(\sum_{q\geq 0}(-1)^q \cdot q \cdot \sum_{\mu>0} \mu^{-s} \cdot [E_\mu^H(N; W)^q]\right) \\
&\quad + \left(\sum_{p\geq 0}(-1)^p \cdot p \cdot \sum_{\lambda>0} \lambda^{-s} \cdot [E_\lambda^G(M, M_1; V)^p]\right) \\
&\quad \otimes_{\mathbf{R}} \left(\sum_{q\geq 0}(-1)^q \cdot [E_0^H(N; W)^q]\right) \\
&\quad + \sum_{\lambda,\mu>0} (\lambda+\mu)^{-s} \cdot \left(\sum_{p\geq 0}(-1)^p \cdot p \cdot [E_\lambda^G(M, M_1; V)^p]\right) \\
&\quad \otimes_{\mathbf{R}} \left(\sum_{q\geq 0}(-1)^q \cdot [E_\mu^H(N; W)^q]\right) \\
&\quad + \sum_{\lambda,\mu>0} (\lambda+\mu)^{-s} \cdot \left(\sum_{p\geq 0}(-1)^p \cdot [E_\lambda^G(M, M_1; V)^p]\right) \\
&\quad \otimes_{\mathbf{R}} \left(\sum_{q\geq 0}(-1)^q \cdot q \cdot [E_\mu^H(N; W)^q]\right) \\
&= \chi^G(M, M_1; V) \otimes_{\mathbf{R}} \left(\sum_{q\geq 0}(-1)^q \cdot q \cdot \zeta_q^H(N; W))\right) \\
&\quad + \left(\sum_{p\geq 0}(-1)^p \cdot p \cdot \zeta_p^G(M, M_1; V)\right) \otimes_{\mathbf{R}} \chi^H(N; W) \\
&= \chi^G(M, M_1; V) \otimes_{\mathbf{R}} \zeta^H(N; W) + \zeta^G(M, M_1; V) \otimes_{\mathbf{R}} \chi^H(N; W).
\end{aligned}
$$

1.36. If H is a subgroup of G, then there are obvious restriction and induction homomorphisms for $\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{Z}}(G)$. Restriction and induction are also defined for $(M, M_1; V)$. The equivariant analytic torsion is compatible with restriction and induction.

2. Torsion invariants for chain complexes

In order to define PL-torsion invariants for G- CW-complexes and Riemannian G-manifolds it is convenient to do this for $\mathbf{R}G$-chain complexes

$C = (C_*, c_*)$. We say that C is *finite* if C_i is finitely generated for all i, and is zero for all but a finite number of $i \in \mathbf{Z}$. Let $f\colon C \to D$ be an $\mathbf{R}G$-chain equivalence of finite $\mathbf{R}G$-chain complexes and let $\phi : C_{\text{odd}} \oplus D_{\text{ev}} \to D_{\text{odd}} \oplus C_{\text{ev}}$ be an $\mathbf{R}G$-isomorphism. Denote by $\operatorname{Cone}(f)$ the mapping cone of f whose differential is also denoted by c and given by

$$c_n = \begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix} : C_{n-1} \oplus D_n \to C_{n-2} \oplus D_{n-1}.$$

Choose a chain contraction γ of $\operatorname{Cone}(f)$, i.e., a map of degree 1 such that $c \circ \gamma + \gamma \circ c = \mathrm{id}$. Then we obtain an isomorphism $(c+\gamma)\colon \operatorname{Cone}(f)_{\text{odd}} \to \operatorname{Cone}(f)_{\text{ev}}$ if $\operatorname{Cone}(f)_{\text{odd}}$, resp. $\operatorname{Cone}(f)_{\text{ev}}$, is the sum of all chain modules of odd, resp. even, dimension. If π denotes the obvious permutation map, we get an $\mathbf{R}G$-isomorphism of a finitely generated $\mathbf{R}G$-module

$$\operatorname{Cone}(f)_{\text{odd}} \xrightarrow{(c+\gamma)} \operatorname{Cone}(f)_{\text{ev}} \xrightarrow{\pi} C_{\text{odd}} \oplus D_{\text{ev}} \xrightarrow{\phi} D_{\text{odd}} \oplus C_{\text{ev}} \xrightarrow{\pi} \operatorname{Cone}(f)_{\text{odd}}.$$

Denote its class in $K_1(\mathbf{R}G)$ by

(2.1) $$t^G(f, \phi) = t(f, \phi) \in K_1(\mathbf{R}G).$$

We recall that $K_1(\mathbf{R}G)$ is the abelian group generated by automorphisms $f\colon P \to P$ of finitely generated $\mathbf{R}G$-modules with the relations $[f_2] = [f_1] + [f_3]$ for any exact sequence $\{0\} \to (P_1, f_1) \to (P_2, f_2) \to (P_3, f_3) \to \{0\}$ and $[g \circ f] = [g] + [f]$ for $f, g\colon P \to P$ and $[\mathrm{id}\colon P \to P] = 0$. We refer to Lück [21, Chapter 12] for more details about this invariant and the proof that it is well defined. The proofs of some of the results in this section are omitted since they are very similar to ones appearing in [21].

An $\mathbf{R}G$-*Hilbert complex* is an $\mathbf{R}G$-chain complex C together with a G-invariant $\mathbf{R}$-Hilbert space structure on each C_n. Let $f\colon C \to D$ be an $\mathbf{R}G$-chain equivalence of finite $\mathbf{R}G$-Hilbert complexes. Fix an isometric $\mathbf{R}G$-isomorphism $\phi\colon C_{\text{odd}} \oplus D_{\text{ev}} \to D_{\text{odd}} \oplus C_{\text{ev}}$. Its existence follows from the Polar Decomposition Theorem and the fact that $C_{\text{odd}} \oplus D_{\text{ev}}$ and $D_{\text{odd}} \oplus C_{\text{ev}}$ are $\mathbf{R}G$-isomorphic. Given an $\mathbf{R}G$-module P, let P^* be $\operatorname{Hom}_{\mathbf{R}}(P, \mathbf{R})$ equipped with the $\mathbf{R}G$-module structure $g \cdot f = f \circ l(g^{-1})$. The natural $\mathbf{R}G$-map $P \to P^{**}$ is bijective if and only if P is finitely generated. We obtain an involution

(2.2) $$*\colon K_1(\mathbf{R}G) \to K_1(\mathbf{R}G), \qquad [f] \mapsto [f^*].$$

Define the *Hilbert torsion*

$$(2.3)\qquad \operatorname{ht}^G(f) = \operatorname{ht}(f) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

by $\operatorname{ht}(f) = t(f, \phi) + *t(f, \phi)$. This is independent of the choice of ϕ. If ψ is another choice we have

$$\begin{aligned}&(t(f, \phi) + *t(f, \phi)) - (t(f, \psi) + *t(f, \psi))\\ &\quad= (t(f, \phi) - t(f, \psi)) + *(t(f, \phi) - t(f, \psi))\\ &\quad= [\psi^{-1} \circ \phi] + *[\psi^{-1} \circ \phi] = [\psi^{-1} \circ \phi] - [\psi^{-1} \circ \phi] = 0.\end{aligned}$$

Proposition 2.4. *If f and $g: C \to D$ and $\mathbf{R}G$-chain homotopic, then* $\operatorname{ht}(f) = \operatorname{ht}(g)$.

Let C be a finite $\mathbf{R}G$-Hilbert complex. Consider its homology $H(C)$ as an $\mathbf{R}G$-chain complex by the trivial differential. Suppose that additionally $H(C)$ has the structure of an $\mathbf{R}G$-Hilbert complex. Up to $\mathbf{R}G$-chain homotopy there is precisely one $\mathbf{R}G$-chain map $i: H(C) \to C$ satisfying $H(i) = \mathrm{id}$. Define the *Hilbert-Reidemeister torsion*

$$(2.5)\qquad \operatorname{hr}^G(C) = \operatorname{hr}(C) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

by $\operatorname{hr}(C) := \operatorname{ht}(i: H(C) \to C)$.

Example 2.6. Let G be the trivial group. Let C be a finite $\mathbf{R}$-Hilbert complex together with a $\mathbf{R}$-Hilbert structure on $H(C)$. Choose orthonormal bases for each C_i and $H(C)_i$. The torsion defined by Milnor [25] takes values in $\widetilde{K}_1(\mathbf{R}) = \mathbf{R}^*/\mathbf{Z}^*$. Its square is a positive real number which agrees with $\operatorname{hr}(C) \in K_1(\mathbf{R}) = \mathbf{R}^*$.

We collect the main properties of these invariants. Consider the following commutative diagram of finite $\mathbf{R}G$-Hilbert complexes whose rows are exact and whose vertical arrows are $\mathbf{R}G$-chain equivalences:

$$(2.7)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{i} & D & \xrightarrow{p} & E & \longrightarrow & 0\\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & C' & \xrightarrow{i'} & D' & \xrightarrow{p'} & E' & \longrightarrow & 0\end{array}$$

We get an acyclic finite $\mathbf{R}G$-Hilbert complex $0 \to C_n \xrightarrow{i_n} D_n \xrightarrow{p_n} E_n \to 0$ concentrated in dimensions 0, 1, and 2 for each $n \in \mathbf{Z}$. Let $\operatorname{hr}(C_n, D_n, E_n)$ be its Hilbert-Reidemeister torsion. Define

$$(2.8)\qquad \operatorname{hr}(C, D, E) = \sum_n (-1)^n \cdot \operatorname{hr}(C_n, D_n, E_n) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}.$$

Proposition 2.9 (Additivity).

$$\mathrm{ht}(f) - \mathrm{ht}(g) + \mathrm{ht}(h) = \mathrm{hr}(C, D, E) - \mathrm{hr}(C', D', E').$$

Let $0 \to C \xrightarrow{i} D \xrightarrow{p} E \to 0$ be an exact sequence of finite $\mathbf{R}G$-Hilbert complexes. Suppose that $H(C)$, $H(D)$, and $H(E)$ come with $\mathbf{R}G$-Hilbert structures. The long homology sequence $H(C, D, E)$ inherits the structure of an acyclic finite $\mathbf{R}G$-Hilbert complex. Analogously to Milnor [25] we get

Proposition 2.10 (Additivity).

$$\mathrm{hr}(C) - \mathrm{hr}(D) + \mathrm{hr}(E) = \mathrm{hr}(C, D, E) - \mathrm{hr}(H(C, D, E)).$$

Proposition 2.11 (Composition formula). *If $f: C \to D$ and $g: D \to E$ are chain equivalences of $\mathbf{R}G$-Hilbert complexes, we have*

$$\mathrm{ht}(g \circ f) = \mathrm{ht}(g) + \mathrm{ht}(f).$$

Proposition 2.12 (Comparison formula). *If $f: C \to D$ is an $\mathbf{R}G$-chain equivalence of finite $\mathbf{R}G$-Hilbert complexes and $H(C)$ and $H(D)$ come with finite $\mathbf{R}G$-Hilbert complex structures, then $H(f): H(C) \to H(D)$ is an $\mathbf{R}G$-chain equivalence of finite $\mathbf{R}G$-Hilbert complexes and we get*

$$\mathrm{ht}(f) = \mathrm{hr}(D) - \mathrm{hr}(C) + \mathrm{ht}(H(f)).$$

Proof. The proof follows from Proposition 2.11. q.e.d.

Given two finite groups G and H, there is a pairing

$$\begin{aligned} &\otimes_{\mathbf{R}}: K_0(\mathbf{R}G) \otimes_{\mathbf{Z}} K_1(\mathbf{R}H) \to K_1(\mathbf{R}G \times H), \\ &[P] \otimes_{\mathbf{Z}} [f: Q \to Q] \mapsto [\mathrm{id} \otimes_{\mathbf{R}} f: P \otimes_{\mathbf{R}} Q \to P \otimes_{\mathbf{R}} Q]. \end{aligned} \tag{2.13}$$

Notice that $K_0(\mathbf{R}G)$ is $\mathrm{Rep}_{\mathbf{R}}(G)$. Because P and P^* are (not naturally) $\mathbf{R}G$-isomorphic, we get the following induced pairings:

$$\begin{aligned} &\otimes_{\mathbf{R}}: K_0(\mathbf{R}G) \otimes_{\mathbf{Z}} K_1(\mathbf{R}H)^{\mathbf{Z}/2} \to K_1(\mathbf{R}G \times H)^{\mathbf{Z}/2}, \\ &\otimes_{\mathbf{R}}: K_0(\mathbf{R}G) \otimes_{\mathbf{Z}} K_1(\mathbf{R}G)^{\mathbf{Z}/2} \to K_1(\mathbf{R}G)^{\mathbf{Z}/2}. \end{aligned} \tag{2.14}$$

If C is a finite $\mathbf{R}G$-chain complex, define its *Euler characteristic* as

$$\chi(C) := \sum_{i>0} (-1)^i \cdot [C_i] = \sum_{i \geq 0} (-1)^i \cdot [H_i(C)] \in K_0(\mathbf{R}G). \tag{2.15}$$

Proposition 2.16 (Product formula). *Let $f\colon C' \to C$, resp. $g\colon D' \to D$, be an* **R***G-, resp.* **R***H-, chain equivalence of finite* **R***G-Hilbert complexes. Then*

$$\mathrm{ht}^{G\times H}(f \otimes_{\mathbf{R}} g) = \chi^G(C) \otimes_{\mathbf{R}} \mathrm{ht}^H(g) + \mathrm{ht}^G(f) \otimes_{\mathbf{R}} \chi^H(D).$$

(b) *Let C, resp. D, be a finite* **R***G-, resp.* **R***H-, chain complex. Assume that $H(C)$, resp. $H(D)$, possesses a finite* **R***G-, resp.* **R***H-, Hilbert complex structure. Equip $H(C \otimes_{\mathbf{R}} D)$ with the finite ($\mathbf{R}G \times H$)-Hilbert structure for which the Künneth isomorphism $H(C) \otimes_{\mathbf{R}} H(D) \cong H(C \otimes_{\mathbf{R}} D)$ becomes an isometry. Then*

$$\mathrm{hr}^{G\times H}(C \otimes_{\mathbf{R}} D) = \chi^G(C) \otimes_{\mathbf{R}} \mathrm{hr}^H(D) + \mathrm{hr}^G(C) \otimes_{\mathbf{R}} \chi^H(D).$$

2.17. If $H \subset G$ is a subgroup, there are obvious induction and restriction homomorphisms. Both ht and hr are compatible with induction and restriction.

If $C = (C_*, c_*)$ is a finite **R**G-chain complex, define its *dual* **R**G-chain complex C^{n-*} by $(C^{n-*})_r = (C_r)^*$ and $(c^{n-*})_r := (c_{n-r+1})^*$. If the **R**$G$-module P has a finite **R**G-Hilbert structure given by an **R**G-isomorphism $\phi\colon P \to P^*$ satisfying $\phi^* = \phi$, equip P^* with the finite **R**G-Hilbert structure $P^* \xrightarrow{\phi^{-1}} P \to P^{**}$. Hence C^{n-*} inherits a finite **R**G-Hilbert complex structure from C. Notice that the natural **R**G-map $C \to (C^{n-*})^{n-*}$ is an isometric **R**G-chain isomorphism.

Proposition 2.18. (a) *If $f\colon C \to D$ is an* **R***G-chain equivalence of* **R***G-Hilbert complexes, then*

$$\mathrm{ht}(f^{n-*}) = (-1)^n \cdot \mathrm{ht}(f).$$

(b) *Let C be a finite* **R***G-Hilbert complex. Assume that $H(C)$ has a finite* **R***G-Hilbert complex structure. Then*

$$\mathrm{hr}(C^{n-*}) = (-1)^{n+1} \cdot hr(C).$$

Let $f\colon C \to D$ be an **R**G-chain equivalence of finite **R**G-Hilbert complexes. Let $\kappa(C_i)$ and $\tilde{\kappa}(C_i)$ (resp. $\kappa(D_i)$ and $\tilde{\kappa}(D_i)$) be two different **R**G-Hilbert structures on C_i (resp. D_i). Then we obtain **R**G-automorphisms

(2.19)
$$\kappa(C_i)^{-1} \circ \tilde{\kappa}(C_i)\colon C_i \to C_i^* \to C_i,$$
$$\kappa(D_i)^{-1} \circ \tilde{\kappa}(D_i)\colon D_i \to D_i^* \to D_i.$$

Proposition 2.20. *We have the following equation:*

$$\mathrm{ht}(f\colon (C, \kappa(C)) \to (D, \kappa(D))) - \mathrm{ht}(f\colon (C, \tilde{\kappa}(C)) \to (D, \tilde{\kappa}(D)))$$
$$= \sum_{i\geq 0}(-1)^i \cdot [\kappa(C_i)^{-1} \circ \tilde{\kappa}(C_i)] - \sum_{i\geq 0}(-1)^i \cdot [\kappa(D_i)^{-1} \circ \tilde{\kappa}(D_i)].$$

Proof. Because of Lemma 2.11 it suffices to prove

$$\mathrm{ht}(\mathrm{id}\colon (C, \kappa(C)) \to (C, \tilde{\kappa}(C))) = \sum_{i\geq 0}(-1)^i \cdot [\kappa(C_i)^{-1} \circ \tilde{\kappa}(C_i)].$$

Because of Proposition 2.9 we can assume that C is concentrated in dimension zero. If $\phi\colon (C_0, \kappa(C_0)) \to (C_0, \tilde{\kappa}(C_0))$ is an isometric $\mathbf{R}G$-isomorphism, we compute

$$\begin{aligned}\mathrm{ht}(\mathrm{id}\colon (C, \kappa(C)) \to (C, \tilde{\kappa}(C))) &= [\phi] + [\phi^*]\\ &= [\phi] + [\tilde{\kappa}(C_0) \circ \phi^{-1} \circ \kappa(C_0)^{-1}]\\ &= [\phi \circ \tilde{\kappa}(C_0) \circ \phi^{-1} \circ \kappa(C_0)^{-1}]\\ &= [\kappa(C_0)^{-1} \circ \tilde{\kappa}(C_0)]. \quad \text{q.e.d.}\end{aligned}$$

Next we compare $K_1(\mathbf{R}G)^{\mathbf{Z}/2}$ and $\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)$. Let I be a complete set of representatives for the isomorphism classes of irreducible $\mathbf{R}G$-representations. For any finitely generated $\mathbf{R}G$-module W we have the natural $\mathbf{R}G$-isomorphism

$$(2.21) \qquad \phi : \bigoplus_{V\in I} \mathrm{Hom}_{\mathbf{R}G}(V, P) \otimes_{\mathrm{End}_{\mathbf{R}G}(V)} V \to P, \qquad f \otimes v \mapsto f(v).$$

By Schur's Lemma $\mathrm{End}_{\mathbf{R}G}(V)$ is a skew field. Hence the canonical homomorphism $\mathrm{End}_{\mathbf{R}G}(V)^* \to K_1(\mathrm{End}_{\mathbf{R}G}(V))$ induces an isomorphism from the abelianization of the group of units $\mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}} \to K_1(\mathrm{End}_{\mathbf{R}G}(V))$ (see Silvester [33, p. 133]). We obtain from (2.21) an isomorphism

$$(2.22) \qquad \begin{aligned}\Phi\colon K_1(\mathbf{R}G) &\to \bigoplus_{V\in I} \mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}},\\ [g\colon P \to P] &\mapsto \{[\mathrm{Hom}_{\mathbf{R}G}(\mathrm{id}_V, g)] \mid V \in I\}.\end{aligned}$$

There is an involution $*\colon \mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}} \to \mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}}$ sending $[f]$ to $[k^{-1} \circ f^* \circ k]$ for any $\mathbf{R}G$-isomorphism $k\colon V \to V^*$ satisfying $k = k^*$. Then ϕ is compatible with the involution $*$ on $K_1(\mathbf{R}G)$ and the direct sum of the involutions $*$ on $\mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}}$. Because $\mathrm{End}_{\mathbf{R}G}(V)$ is a skew field over $\mathbf{R}$, it is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ and we accordingly call V of real, complex, or quaternionic type. Under these isomorphisms the involution on $\mathrm{End}_{\mathbf{R}G}(V)$ corresponds to the trivial, complex,or quaternionic involution. The map $\mathbf{R}^+ \times S^3 \to \mathbf{H}^*$ sending (λ, z) to $\lambda \cdot z$ is an isomorphism of Lie groups if S^3 inherits the Lie group structure from $S^3 \subset \mathbf{H}$. The Lie group $S^3 = \mathrm{SU}(2)$ is its own commutator group. The inclusion $\mathbf{R}^+ \hookrightarrow \mathbf{H}^*$ and the norm map $\mathbf{H}^* \to \mathbf{R}$ sending $a \in \mathbf{H}$ to $\sqrt{a\bar{a}}$ induce to other inverse isomorphisms $\mathbf{R}^+ \to \mathbf{H}^*_{\mathrm{ab}}$ and $\mathbf{H}^*_{\mathrm{ab}} \to \mathbf{R}^+$. Now the inclusion $i\colon \mathbf{R}^* \to (\mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}})^{\mathbf{Z}/2}$ mapping $\lambda \in \mathbf{R}^*$ to $\lambda \cdot \mathrm{id}\colon V \to V$ induces an isomorphism

$$(2.23)\qquad \begin{aligned} &i\colon \mathbf{R}^* \to (\mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}})^{\mathbf{Z}/2}, &&\text{if } V \text{ is of real or complex type,}\\ &i\colon \mathbf{R}^+ \to (\mathrm{End}_{\mathbf{R}G}(V)^*_{\mathrm{ab}})^{\mathbf{Z}/2}, &&\text{if } V \text{ is of quaternionic type.}\end{aligned}$$

Let the isomorphism $\gamma_1\colon \mathbf{R}^+ \to \mathbf{R}$ (resp. $\gamma_1 \oplus \gamma_2)\colon \mathbf{R}^* \to \mathbf{R} \oplus \mathbf{Z}/2$, send r to $\ln(r)$ (resp. $(\ln(|r|), r/|r|)$). Denote by $\widehat{I}$ the subset of I consisting of V's of real or complex type, and by $\widehat{\mathrm{Rep}_{\mathbf{R}}}(G)$ the subgroup of $\mathrm{Rep}_{\mathbf{R}}(G)$ generated by $\widehat{I}$. For an abelian group A we get the identifications

$$(2.24)\qquad \bigoplus_{V\in I} A \to A \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G), \qquad \bigoplus_{V\in \widehat{I}} A \to A \otimes_{\mathbf{Z}} \widehat{\mathrm{Rep}_{\mathbf{R}}}(G)$$

from the $\mathbf{Z}$-bases I, resp. $\widehat{I}$. Define

$$(2.25)\qquad \Gamma_1 \oplus \Gamma_2\colon K_1(\mathbf{R}G)^{\mathbf{Z}/2} \to (\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)) \oplus (\mathbf{Z}/2 \otimes_{\mathbf{Z}} \widehat{\mathrm{Rep}_{\mathbf{R}}}(G))$$

to be the composition $((\bigoplus_{V\in\widehat{I}}\gamma_1 \oplus \gamma_2) \oplus (\bigoplus_{V\in I-\widehat{I}}\gamma_1)) \circ (\bigoplus_{V\in I} i^{-1}) \circ \Phi^{\mathbf{Z}/2}$.

Proposition 2.26. *The map $\Gamma_1 \oplus \Gamma_2$ is an isomorphism and is natural with respect to induction and restriction and the operations of $K_0(\mathbf{R}G) = \mathrm{Rep}_{\mathbf{R}}(G)$. If we denote the Schur index of V by $m(V) = \dim \mathbf{R}(\mathrm{End}_{\mathbf{R}G}(V))$, then*

$$\begin{aligned}\Gamma_1([f\colon P \to P]) &= \sum_{V\in I} \frac{1}{2\cdot m(V)} \cdot \ln((\det(\mathrm{Hom}_{\mathbf{R}G}(\mathrm{id}_V, f)) :\\ &\qquad\qquad \mathrm{Hom}_{\mathbf{R}G}(V, P) \to \mathrm{Hom}_{\mathbf{R}G}(V, P))^2),\end{aligned}$$

where the determinant is taken for a linear map over $\mathbf{R}$, and the inverse of $\Gamma_1 \oplus \Gamma_2$ maps $(\lambda \otimes_{\mathbf{R}} [V], \pm[W])$ to $[\exp(\lambda)\cdot \mathrm{id}\colon V \to V] + [\pm\,\mathrm{id}\colon W \to W]$.

2.27. Until now we have dealt with homology and chain complexes. There is also a cohomology and cochain version. If C^* is a finite $\mathbf{R}G$-Hilbert cochain complex, let $\mathrm{co}(C^*)$ be the finite $\mathbf{R}G$-Hilbert complex with $\mathrm{co}(C^*)_n := C^{-n}$ and c^{-n} the nth differential. An $\mathbf{R}G$-cochain equivalence $f^*\colon C^* \to D^*$ induces an $\mathbf{R}G$-chain equivalence $\mathrm{co}(f^*)$: $\mathrm{co}(C^*) \to \mathrm{co}(D^*)$. We define

$$\mathrm{ht}(f^*) = \mathrm{ht}(\mathrm{co}(f^*)), \qquad \mathrm{hr}(C^*) = \mathrm{hr}(\mathrm{co}(C^*)).$$

All the results of this section have cochain analogues.

Let C be a finite $\mathbf{R}G$-Hilbert chain complex with differential $c_n\colon C_n \to C_{n-1}$. The adjoint of c_{n+1} is denoted by $\gamma_n\colon C_n \to C_{n+1}$. Define a symmetric and nonnegative definite $\mathbf{R}G$-homomorphism $\Delta_n: C_n \to C_n$ by $c_{n+1} \circ \gamma_n + \gamma_{n-1} \circ c_n$. Let $E_\lambda(\Delta_n)$ be the eigenspace for $\lambda \geq 0$. Define the holomorphic zeta-function

$$\zeta_n: \mathbf{C} \to \mathbf{C} \otimes_{\mathbf{Z}} \operatorname{Rep}_{\mathbf{R}}(G), \qquad s \mapsto \sum_{\lambda > 0} \lambda^{-s} \cdot [E_\lambda(\Delta_n)]. \tag{2.28}$$

By the finite-dimensional analogue of the Hodge Decomposition Theorem 1.10 we have a canonical isomorphism $i: E_0(\Delta_n) \to H_n(C)$. Equip $E_0(\Delta_n) \subset C_n$ and $H_n(C)$ with the induced $\mathbf{R}G$-Hilbert structures. The following result motivates the definition of analytic torsion and Hilbert-Reidemeister torsion (cf. Ray-Singer [29, Proposition 1.7]).

Proposition 2.29. (a) $\Gamma_1(\operatorname{hr}(C)) = \sum_{n \geq 0} (-1)^n \cdot n \cdot \frac{d}{ds} \zeta_n|_{s=0}$.
(b) $\Gamma_2(\operatorname{hr}(C)) = 1$.

Proof. First we treat the case where $\mathbf{C}$ is acyclic. Then Δ_n is bijective. Let $f_n: C_n \to C_n$ be $\Delta_n^n = \Delta_n \circ \Delta_n \circ \cdots \circ \Delta_n$. Then the following diagram commutes:

$$\begin{array}{ccc} C_{\text{odd}} & \xrightarrow{c + c^* \Delta^{-1}} & C_{\text{ev}} \\ \Big\downarrow f_{\text{odd}} & & \Big\downarrow f_{\text{ev}} \\ C_{\text{odd}} & \xrightarrow{c\Delta^{-1} + c^*} & C_{\text{ev}} \end{array}$$

Since the lower horizontal arrow in the square above is the inverse of the adjoint of the upper horizontal arrow and $c^* \circ \Delta^{-1}$ is a chain contraction for $\mathbf{C}$, it follows that $\operatorname{hr}(C) = -\sum_{n \geq 0} (-1)^n \cdot n \cdot [\Delta_n]$ is true. One easily checks

$$\begin{aligned} \Gamma_1([\Delta_n]) &= \Gamma_1\left(\sum_{\lambda \geq 0} [\lambda \cdot \operatorname{id}: E_\lambda(\Delta_n) \to E_\lambda(\Delta_n)]\right) \\ &= \sum_{\lambda > 0} \ln(\lambda) \cdot [E_\lambda(\Delta_n)] = -\left.\frac{d}{ds}\zeta_n\right|_{s=0}. \end{aligned}$$

This proves the claim for acyclic C. The general case follows from the obvious exact sequence of a finite $\mathbf{R}G$-Hilbert complex $0 \to H(C) \xrightarrow{i} C \to \operatorname{cok}(i) \to 0$ and Proposition 2.10. We leave (b) to the reader.

3. PL-torsion

Let G be a finite group. A finite G-CW-complex X is a CW-complex X together with a G-action such that for any $g \in G$ and any open cell $e \subset X$ with $e \cap l(g)(e) \neq \emptyset$ we have $l(g)(e) = e$ and $l(g)|_e = \operatorname{id}_e$, where

$l(g)\colon X \to X$ is left-multiplication with g. We want to define *PL-torsion* for finite G-CW-complexes with an $\mathbf{R}G$-Hilbert structure on the homology, resp. cohomology, groups. We prefer the name PL-torsion instead of topological torsion, as in the equivariant setting torsion invariants are not invariants under G-homeomorphisms in general (see Cappell-Shaneson [7]). We will apply this to Riemannian G-manifolds using the equivariant triangulation theorem. Having these invariants also for G-CW-complexes gives some useful flexibility.

Consider a pair (X, Y) of finite G-CW-complexes. Suppose for simplicity that X is connected. The following definitions are easily extended to the nonconnected case. Let $p\colon \widetilde{X} \to X$ be the universal covering and $\widetilde{Y} := p^{-1}(Y)$. Then $(\widetilde{X}, \widetilde{Y})$ is a finite $D^G(X)$-CW-pair. The group $D^G(X)$ was introduced in (1.1). Its cellular $\mathbf{Z}[D^G(X)]$-chain complex $C_*(\widetilde{X}, \widetilde{Y})$ is free over $\mathbf{Z}[\pi_1(X)]$. Fix an orthogonal $D^G(X)$-representation V. As we think of $\pi = \pi_1(X)$ as a group of deck transformations, $D^G(X)$ and π act from the left on $C_*(\widetilde{X}, \widetilde{Y})$. Let $^-\colon \mathbf{Z}[D^G(X)] \to \mathbf{Z}[D^G(X)]$ be the involution sending $\sum \lambda_d \cdot d$ to $\sum \lambda_d \cdot d^{-1}$, and similarly for $\mathbf{R}\pi$. There is an induced right module structure given by $u \cdot s := \bar{s} \cdot u$ for $s \in \mathbf{R}[D^G(X)]$, $u \in C_*(\widetilde{X}, \widetilde{Y})$. Define the $\mathbf{R}G$-Hilbert complex

$$C_*(X, Y; V) := C_*(\widetilde{X}, \widetilde{Y}) \otimes_{\mathbf{R}\pi} V. \tag{3.1}$$

The $\mathbf{R}G$-module structure comes from $g \cdot (u \otimes_{\mathbf{R}\pi} v) := \tilde{g}u \otimes_{\mathbf{R}\pi} \tilde{g}v$ for any lift $\tilde{g} \in D^G(X)$ of $g \in G$ with $u \in C_*(\widetilde{X}, \widetilde{Y}$ and $v \in V$. We obtain an $\mathbf{R}G$-Hilbert structure by requiring that for one (and hence all) cellular $\mathbf{Z}\pi$-base B of $C_*(\widetilde{X}, \widetilde{Y})$ the following $\mathbf{R}$-isomorphism is an isometry with respect to the orthogonal structure on V:

$$C_*(X, Y; V) \to \bigoplus_B V, \qquad \left(\sum_{b \in B} a_b \cdot b\right) \otimes_{\mathbf{R}\pi} v \mapsto \{\bar{a}_b \cdot v \mid b \in B\}.$$

Define the $\mathbf{R}G$-Hilbert cochain complex as

$$C^*(X, Y; V) = \operatorname{Hom}_{\mathbf{R}\pi}(C_*(\widetilde{X}, \widetilde{Y}), V). \tag{3.2}$$

The $\mathbf{R}G$-structure is induced from the $D^G(X)$-action on $\operatorname{Hom}_{\mathbf{R}}(C_*(\widetilde{X}, \widetilde{Y}); V)$ given by $d \cdot f := l(d) \circ f \circ l(d^{-1})$ and the fact that the π-fixed point set is $\operatorname{Hom}_{\mathbf{R}\pi}(C_*(\widetilde{X}, \widetilde{Y}); V)$. The $\mathbf{R}G$-Hilbert structure is determined by the property that for one (and hence all) cellular $\mathbf{R}\pi$-base B of $C_*(\widetilde{X}, \widetilde{Y})$ the following $\mathbf{R}$-isomorphism is isometric:

$$C^*(X, Y; V) \to \bigoplus_B V, \qquad \phi \mapsto \{\phi(b) \mid b \in B\}.$$

We have explained before how P^* inherits an $\mathbf{R}G$-Hilbert structure from P if P is a finitely generated $\mathbf{R}G$-module with $\mathbf{R}G$-Hilbert structure. Let $H_*(X, Y; V)$ (resp. $H^*(X, Y; V)$) be the homology (resp. cohomology) of $C_*(X, Y; V)$ (resp. $C^*(X, Y; V)$). There is an isometric $\mathbf{R}G$-chain isomorphism

$$(3.3) \qquad C^*(X, Y; V) \to \mathrm{Hom}_{\mathbf{R}}(C_*(X, Y; V^*), \mathbf{R})$$

sending $\phi \in \mathrm{Hom}_{\mathbf{R}\pi}(C_*(\widetilde{X}, \widetilde{Y}), V)$ to $C_*(\widetilde{X}, \widetilde{Y}) \otimes_{\mathbf{R}\pi} V^* \to \mathbf{R}$, $u \otimes_{\mathbf{R}\pi} \psi \mapsto \psi \circ \phi(u)$. It induces an $\mathbf{R}G$-isomorphism

$$(3.4) \qquad H^*(X, Y; V) \to \mathrm{Hom}_{\mathbf{R}}(H_*(X, Y; V^*); \mathbf{R}).$$

Definition 3.5. Let (X, Y) be a finite G-CW-pair and V an orthogonal $D^G(X)$-representation. Let κ_* be an $\mathbf{R}G$-Hilbert structure on $H_*(X, Y; V)$. Define the Hilbert-PL-torsion, or briefly PL-torsion,

$$\rho^G_{\mathrm{pl}}(X, Y; V, \kappa_*) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

by $\rho^G_{\mathrm{pl}}(X, Y; V, \kappa_*) := \mathrm{hr}(C_*(X, Y; V), \kappa_*)$ (see (2.5)). Similarly, if κ^* is an $\mathbf{R}G$-Hilbert structure on $H^*(X, Y; V)$, define

$$\rho^G_{\mathrm{pl}}(X, Y; V, \kappa^*) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

by $\rho^G_{\mathrm{pl}}(X, Y; V, \kappa^*) := \mathrm{hr}(C^*(X, Y; V), \kappa^*)$ (see 3.27).

From Proposition 2.18 we derive

Proposition 3.6. *If κ_* and κ^* are compatible with* (3.4), *then*

$$\rho^G_{\mathrm{pl}}(X, Y; V, \kappa_*) = -\rho^G_{\mathrm{pl}}(X, Y; V, \kappa^*).$$

Remark 3.7. It is convenient to have both the homological and the cohomological definitions. The first one is more convenient for computations since other related torsion invariants, like Whitehead torsion, are given by chain complexes and the cellular chain complex is easier to compute than the cochain complex. The second one fits better into the context of analytic torsion and deRham cohomology.

Let $(f, f_1): (X, X_1) \to (Y, Y_1)$ be a G-homotopy equivalence of pairs of finite G-CW-complexes. In Dovermann-Rothenberg [11], Illman [17], and Lück [21] *equivariant Whitehead torsion* $\tau^G(f, f_1) \in \mathrm{Wh}^G(Y)$ is defined. Let V be an orthogonal $D^G(Y)$-representation. We consider $\mathbf{R}G$-Hilbert structures $\kappa_i(X)$ on $H_i(X, X_1; f^*V)$ and $\kappa_i(Y)$ on $H_i(Y, Y_1; f^*V)$. Let the element $u_i \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$ be given by the composition

$$H_i(X, X_1; f^*V) \overset{H_i(f, f_1)}{\to} H_i(Y, Y_1; V) \overset{\kappa_i(Y)}{\to} H_i(Y, Y_1; V)^* \overset{H_i(f, f_1)^*}{\to} H_i(X, X_1; f^*V)^* \overset{\kappa_i(X)^{-1}}{\to} H_i(X, X_1; f^*V).$$

Proposition 3.8. *There is a natural homomorphism*

$$\omega = \omega^G(Y; V): \mathrm{Wh}^G(Y) \to K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

such that

$$\rho^G_{\mathrm{pl}}(Y, Y_1; V, \kappa_*(Y)) - \rho^G_{\mathrm{pl}}(X, X_1; f^*V, \kappa_*(X)) = \omega(\tau^G(f, f_1)) - \sum_{i \geq 0}(-1)^i \cdot u_i.$$

Proof. We describe ω in the language developed in Lück [21]. A class $[k] \in \mathrm{Wh}^G(Y)$ is represented by an automorphism $k: P \to P$ of a finitely generated projective $\mathbf{Z}\Pi(G, Y)$-module P. Let $x \in \mathrm{Ob}(\Pi(G, Y))$ be represented by the G-map $x: G \to Y$ with G as domain. Let $v \in K_1(\mathbf{R}G)$ be the class of the $\mathbf{R}G$-automorphism $k(x) \otimes_{\mathbf{Z}\pi} \mathrm{id}_V$ of $P(x) \otimes_{\mathbf{Z}\pi} V$, where $k(x)$ is given by evaluating k at x. Define $\omega([g]) = v + *v$. From the definitions we get that $\omega(\tau^G(f, f_1)) = \mathrm{ht}(C_*(f, f_1; V))$. Now apply Proposition 2.12.

Remark 3.9. A G-homotopy equivalence $(f, f_1): (X, X_1) \to (Y, Y_1)$ is *simple* if $\tau^G(f, f_1)$ vanishes. Hence ρ^G_{pl} depends only on the simple G-homotopy type, provided that $H_*(Y, Y_1; V)$ vanishes.

Consider the cellular G pushout of pairs of finite G-CW-complexes, where i_1 is an inclusion of such pairs:

(3.10)
$$\begin{array}{ccc} (X_0, A_0) & \xrightarrow{i_2} & (X_2, A_2) \\ {\scriptstyle i_1}\downarrow & \searrow^{j_0} & \downarrow{\scriptstyle j_2} \\ (X_1, A_1) & \xrightarrow{j_1} & (X, A) \end{array}$$

If V is an orthogonal $D^G(X)$-representation, we get an exact sequence of $\mathbf{R}(G)$-chain complexes

$$\{0\} \to C_*(X_0, A_0, j_0^*V) \xrightarrow{i_{1*} \oplus i_{2*}} C_*(X_1, A_1; j_1^*V) \oplus C_*(X_2, A_2, j_2^*V) \xrightarrow{j_{1*} - j_{2*}} C_*(X, A; V) \to \{0\}.$$

Denote by M_* the long homology sequence of the sequence above. Suppose that we have $\mathbf{R}G$-Hilbert structures κ_i on $H_*(X_i, A_i, j_i^*V)$, for

$i = 0, 1, 2$, and κ on $H_*(X, A, V)$. Then M_* inherits the structure of an acyclic finite $\mathbf{R}G$-Hilbert complex. We derive the following proposition from Proposition 2.10.

Proposition 3.11 (Sum formula for PL-torsion).

$$\rho_{\mathrm{pl}}^G(X, A; V, \kappa) = \rho_{\mathrm{pl}}^G(X_1, A_1; j_1^* V, \kappa_1) + \rho_{\mathrm{pl}}^G(X_2, A_2; j_2^* V, \kappa_2) - \rho_{\mathrm{pl}}^G(X_0, A_0; j_1^* V, \kappa_0) + \mathrm{hr}(M_*).$$

Let (X, A) (resp. (Y, B)) be a pair of finite G- (resp. H-CW-) complexes, and let V (resp. W) be orthogonal $D^G(X)$- (resp. $D^H(Y)$-) representations. Equip $H_*(X, A, V)$ (resp. $H_*(Y, B; W)$) with $\mathbf{R}G$- (resp. $\mathbf{R}H$-) Hilbert structures $\kappa(X)$ (resp. $\kappa(Y)$). Then $(X, A) \times (Y, B)$ is a pair of finite $(G \times H)$-CW-complexes, and $V \otimes_{\mathbf{R}} W$ is an orthogonal representation of $D^{G\times H}(X \times Y) = D^G(X) \times D^H(Y)$. Put on $H_*((X, A) \times (Y, B); V \otimes_{\mathbf{R}} W)$ the $(\mathbf{R}G \times H)$-Hilbert structure $\kappa(X \times Y)$ induced by the $(\mathbf{R}G \times H)$-Künneth isomorphism from $H_*(X, A; V) \otimes_{\mathbf{R}} H_*(Y, B; W)$ to $H_*((X, A) \times (Y, B); V \otimes_{\mathbf{R}} W)$. We define the *equivariant Euler characteristic*

$$\chi^G(X, A; V) \in K_0(\mathbf{R}G) = \mathrm{Rep}_{\mathbf{R}}(G) \tag{3.12}$$

by

$$\chi^G(X, A; V) := \sum_{n \geq 0} (-1)^n \cdot [C_n(X, A; V)] = \sum_{n \geq 0} (-1)^n \cdot [H_n(X, A; V)],$$

as we have already done for manifolds in (1.31). We derive the following proposition from Proposition 2.16:

Proposition 3.13 (Product formula for PL-torsion).

$$\begin{aligned} &\rho_{\mathrm{pl}}^{G\times H}(X, A) \times (Y, B); V \otimes_{\mathbf{R}} W, \kappa(X \times Y)) \\ &\quad = \rho_{\mathrm{pl}}^G(X, A; V, \kappa(X)) \otimes_{\mathbf{R}} \chi^H(Y, B; W) \\ &\quad\quad + \chi^G(X, A; V) \otimes_{\mathbf{R}} \rho_{\mathrm{pl}}^H(Y, B; W, \kappa(Y)). \end{aligned}$$

Remark 3.14. The PL-torsion is compatible with induction and restriction by 2.17.

Next we deal with manifolds. A Riemannian G-manifold triad $(M; M_1, M_2)$ consists of a Riemannian G-manifold M together with G-invariant codimension zero submanifolds M_1 and M_2 of the boundary ∂M satisfying $\partial M = M_1 \cup M_2$ and $\partial M_1 = M_1 \cap M_2 = \partial M_2$. We do not require in this section that $M_1 \cap M_2 = \varnothing$ as we did in previous sections. Regard an equivariant triangulation $(f; f_1, f_2): (K; K_1, K_2) \to (M; M_1, M_2)$ (see Illman [18]). Roughly speaking, this is an ordinary triangulation together with a regular simplicial action of G on K for which f is G-equivariant. In particular, (K, K_1) is a pair of finite G-CW-complexes. Let V be an orthogonal $D^G(M)$-representation. The Riemannian metric on M gives an inner product $\langle\langle \omega, \eta \rangle\rangle = \int_M \langle \omega, \eta \rangle \, dM = \int_M \omega \wedge * \eta$ on $\Lambda^p(M)$. Equip the subspace of harmonic forms $H^p_{\text{harm}}(M, M_1; V)$ with the induced inner product. There is a natural $\mathbf{R}G$-isomorphism $j: H^p(K, K_1; f^*V) \to H^p_s(K, K_1, f^*V)$ between singular and cellular homology. Denote by κ^*_{harm} the $\mathbf{R}G$-Hilbert structure on $H^*(K, K_1; f^*V)$ for which the following $\mathbf{R}G$-isomorphism becomes an isometry, where $\bar{\imath}$ is the Hodge isomorphism (see Proposition 1.10):

$$H^p(K, K_1; f^*V) \xrightarrow{j} H^p_s(K, K_1; f^*V) \xrightarrow{(f^*)^{-1}} H^p_s(M, M_1; V)$$
$$\xrightarrow{\bar{\imath}^{-1}} H^p_{\text{harm}}(M, M_1; V).$$

Let κ_*^{harm} be the $\mathbf{R}G$-Hilbert structure on $H_*(K, K_1; f^*V)$ given by κ^*_{harm} and (3.4).

Definition 3.15. Define the PL-torsion

$$\rho^G_{\text{pl}}(M, M_1; V) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

by $\rho^G_{\text{pl}}(K, K_1; f^*V, \kappa_*^{\text{harm}})$.

If $(g; g_1, g_2): (L; L_1, L_2) \to (M; M_1, M_2)$ is a second triangulation, then $g \circ f^{-1}$ is simple, i.e., $\tau^G(g^{-1} \circ f) = 0$ in $\text{Wh}^G(L)$. Now Proposition 3.8 shows that the choice of equivariant triangulation does not matter. In the sequel we will identify M with a triangulation. We emphasize that our notation of PL-torsion is based on homology (cf. Proposition 3.6).

Definition 3.16. We call the $\mathbf{R}G$-Hilbert structures κ^*_{harm} and κ_*^{harm} constructed above the harmonic Hilbert structures.

In the equivariant setting there is a new invariant involved which does not occur in the nonequivariant case. For a generator

$$[M] \in H_m(C_*(\widetilde{M}, \partial \widetilde{M}) \otimes_{\mathbf{Z}\pi} {}^w\mathbf{Z}) \cong \mathbf{Z}$$

we obtain by Poincaré duality an $\mathbf{R}G$-chain equivalence unique up to homotopy:

$$(3.17) \qquad \bigcap [M]: C^{n-*}(M, M_2; {}^wV) \to C_*(M, M_1; V).$$

It induces an $\mathbf{R}G$-isomorphism

$$(3.18) \qquad H\left(\bigcap[M]\right) : H^{n-*}(M, M_2; {}^wV) \to H_*(M, M_1; V).$$

The construction of $\bigcap[M]$ as an $\mathbf{R}G$-chain map uses the existence of equivariant approximations of the diagonal G-map $M \to M \times M$. Recall that $C_*(M, M_1; V)$ and $C^{n-*}(M, M_1; V)$ have preferred $\mathbf{R}G$-Hilbert structures coming from cellular $\mathbf{R}$-bases and the inner product on V (see (3.1) and (3.2)).

Definition 3.19. Define the Hilbert-Poincaré torsion or briefly Poincaré torsion

$$\rho_{\text{pd}}^G(M, M_1; V) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

to be the Hilbert torsion of $\bigcap[M]$: $C^{n-*}(M, M_2; {}^wV) \to C_*(M, M_1; V)$.

This definition is independent of the choice of $[M]$ since the Hilbert torsion of $-\text{id}$ is always zero.

Proposition 3.20. *If G acts freely, then $\rho_{\text{pd}}^G(M, M_1; V)$ vanishes.*

Proof. If G acts freely on M, then also $D^G(M)$ acts freely on $\widetilde{M}$. The proof of Poincaré duality by the dual cell decomposition shows that the $\mathbf{Z}[D^G(M)]$-chain equivalence of finitely generated based free $\mathbf{Z}[D^G(M)]$-chain complexes $\bigcap[M]$: $C^{n-*}(M, M_2; {}^wV) \to C_*(M, M_1; V)$ is base preserving and hence has vanishing Hilbert torsion.

Remark 3.21. For nonfree actions the dual CW-complex structure obtained from an equivariant triangulation is not a G-CW-complex structure. For example, consider S^1 with the $\mathbf{Z}/2$-action given by complex conjugation. The upper and lower hemispheres give obvious equivariant triangulations. The dual cell decomposition is obtained from it by rotation about 90°. It is not an equivariant cell decomposition since $\mathbf{Z}/2$ acts nontrivially on each of the one-cells. Therefore it can happen that the Poincaré torsion is not trivial if the group acts nonfreely.

On the homology level nothing happens.

Lemma 3.22. *The $\mathbf{R}G$-map*

$$H(\bigcap[M]): H^{n-*}(M, M_2; {}^wV) \to H_*(M, M_1; V)$$

is isometric with respect to the harmonic $\mathbf{R}G$-Hilbert structures. In particular, the Hilbert torsion of it is zero.

Proof. The claim follows from the commutativity of the following diagram, where A is the deRham isomorphism and $\langle \ , \ \rangle$ denotes the Kronecker pairing, resp. is given by the Hilbert structure:

$$\begin{array}{ccc}
H^{n-p}_{\text{harm}}(M, M_2; {}^wV) & \xrightarrow{A} & H^{n-p}(M, M_2; {}^wV) \\
\downarrow * & & \downarrow H(\bigcap[M]) \\
H^p_{\text{harm}}(M, M_1; V) & & H_p(M, M_1; V) \\
\downarrow \langle\, ,\, \rangle & & \downarrow \langle\, ,\, \rangle \\
H^p_{\text{harm}}(M, M_1; V)^* & \xrightarrow{A^*} & H^p(M, M_1; V)^*
\end{array}$$

Proposition 3.23 (Poincaré duality for Poincaré and PL-torsion). *Let $(M; M_1, M_2)$ be an m-dimensional Riemannian G-manifold triad with orientation homomorphism $w\colon D^G(M) \to \{\pm 1\}$. Let V be an orthogonal $D^G(M)$-representation. Then:*

(a) $\rho^G_{\text{pl}}(M, M_1; V) + (-1)^m \cdot \rho^G_{\text{pl}}(M, M_2; {}^wV) = \rho^G_{\text{pd}}(M, M_1; V)$;

(b) $\rho^G_{\text{pd}}(M, M_1; V) = (-1)^m \cdot \rho^G_{\text{pd}}(M, M_2; {}^wV)$;

(c) $\chi^G(M, M_1; V) = (-1)^m \cdot \chi^G(M, M_2; {}^wV)$.

PROOF. (a) Because of Proposition 3.6 and Lemma 3.22 the claim follows from the comparison formula (Proposition 2.12) applied to $\bigcap[M]$: $C^{m-*}(M, M_2; V) \to C_*(M, M_1; V)$.

(b) The $\mathbf{R}G$-chain maps $\bigcap[M]$ and $\bigcap[M])^{m-*}$ from $C^{m-*}(M, M_2; V)$ to $C_*(M, M_1; V)$ are chain homotopic. Now apply Propositions 2.4 and 2.18.

(c) follows from (3.4).

Remark 3.24. If one compares the statements about Poincaré duality for analytic torsion (Proposition 1.20) and for PL-torsion (Proposition 3.23), it becomes obvious that the Poincaré torsion has to enter in a formula relating analytic and PL-torsion.

Example 3.25. Let $\mathbf{Z}/2$ act on S^1 by complex conjugation. Equip S^1 with the standard metric scaled by a factor such that the volume of S^1 is the positive real number μ. We use the $\mathbf{Z}/2$-CW-structure with the upper and lower hemispheres S^1_+ and S^1_- as one-cells and the points $-1, +1 \in \mathbf{C}$ as zero-cells. The cellular $\mathbf{R}\mathbf{Z}/2$-chain complex $C_*(S^1; \mathbf{R})$ is

$$\mathbf{R}[\mathbf{Z}/2] \overset{(-\varepsilon, \varepsilon)}{\longrightarrow} \mathbf{R} \oplus \mathbf{R},$$

where ε is the augmentation $a + b \cdot t \mapsto a + b$ for the generator $t \in \mathbf{Z}/2$, S^1_+, resp. S^1_-, corresponds to 1, resp. $t \in \mathbf{R}[\mathbf{Z}/2]$, and the points -1, 1 to $(1, 0)$, $(0, 1) \in \mathbf{R} \oplus \mathbf{R}$. The cellular $\mathbf{R}[\mathbf{Z}/2]$-Hilbert structure is given by the orthonormal **R**-bases $\{1, t\} \subset \mathbf{R}[\mathbf{Z}/2]$ and $\{(1, 0), (0, 1)\} \subset \mathbf{R} \oplus \mathbf{R}$. The 1-form $\mu^{-1/2} \cdot dvol$ for $dvol$ the volume form has norm 1 because

$$\begin{aligned} \langle\langle \mu^{-1/2} \cdot dvol, \mu^{-1/2} \cdot dvol \rangle\rangle &= \mu^{-1} \cdot \langle\langle dvol, dvol \rangle\rangle \\ &= \mu^{-1} \cdot \int_{S^1} dvol \wedge *dvol \\ &= \mu^{-1} \cdot \int_{S^1} dvol = \mu^{-1} \cdot \mu = 1. \end{aligned}$$

Notice that the generator of $H_1(S^1; \mathbf{R})$ is represented by $\mathrm{id}\colon S^1 \to S^1$ and we have

$$\int_{S^1} \mu^{-1/2} \cdot dvol = \mu^{1/2}.$$

The harmonic 0-form $S^1 \to \mathbf{R}$, $z \mapsto \mu^{-1/2}$ has norm 1, and evaluating it at the generator $(1, 1)$ of $H_0(S^1; \mathbf{R})$ yields $2 \cdot \mu^{-1/2}$. Hence the following maps are $\mathbf{R}[\mathbf{Z}/2]$-isometries, if $\mathbf{R}^-$ and $\mathbf{R}$ (resp. $H_*(S^1; \mathbf{R})$) carry the standard (resp. harmonic) $\mathbf{R}[\mathbf{Z}/2]$-Hilbert structures:

$$\begin{aligned} \mathbf{R}^- &\to H_1(S^1; \mathbf{R}), & 1 &\mapsto \mu^{-1/2} \cdot (1 - t), \\ \mathbf{R} &\to H_0(S^1; \mathbf{R}), & 1 &\mapsto 1/2 \cdot (\mu^{1/2}, \mu^{1/2}). \end{aligned}$$

We use them as identifications of $\mathbf{R}[\mathbf{Z}/2]$-Hilbert spaces. The following $\mathbf{R}[\mathbf{Z}/2]$-chain map $i\colon H_*(C^*(S^1; \mathbf{R})) \to C^*(S^1, \mathbf{R})$ satisfies $H(i) = \mathrm{id}$:

$$\begin{array}{ccc} \mathbf{R}^- & \xrightarrow{0} & \mathbf{R} \\ \Big\downarrow{\scriptstyle \mu^{-1/2} \cdot (1-t)} & & \Big\downarrow{\scriptstyle (\mu^{1/2}/2,\, \mu^{1/2}/2)} \\ \mathbf{R}[\mathbf{Z}/2] & \xrightarrow{(-\varepsilon, \varepsilon)} & \mathbf{R} \oplus \mathbf{R} \end{array}$$

Its mapping cone is concentrated in dimensions 0, 1, and 2 and has the following differential c and chain contraction γ, where $\delta\colon \mathbf{R}[\mathbf{Z}/2] \to \mathbf{R}^-$ maps $a + bt$ to $a - b$:

$$\mathbf{R} \underset{(0\ \ \mu^{1/2}/2 \cdot \delta)}{\overset{\begin{pmatrix} 0 \\ \mu^{-1/2} \cdot (1-t) \end{pmatrix}}{\rightleftarrows}} \mathbf{R} \oplus \mathbf{R}[\mathbf{Z}/2] \underset{\begin{pmatrix} \mu^{-1/2} & \mu^{-1/2} \\ (t+1)/4 & -(t+1)/4 \end{pmatrix}}{\overset{\begin{pmatrix} \mu^{1/2}/2 & \varepsilon \\ \mu^{1/2}/2 & -\varepsilon \end{pmatrix}}{\rightleftarrows}} \mathbf{R} \oplus \mathbf{R}.$$

Hence $(c + \gamma)\colon \mathrm{Cone}(i)_{\mathrm{odd}} \to \mathrm{Cone}(i)_{\mathrm{ev}}$ is the $\mathbf{R}[\mathbf{Z}/2]$-isomorphism

$$\begin{pmatrix} \mu^{-1/2}/2 & \varepsilon \\ \mu^{-1/2}/2 & -\varepsilon \\ 0 & \mu^{1/2}/2 \cdot \delta \end{pmatrix} : \mathbf{R} \oplus \mathbf{R}[\mathbf{Z}/2] \to \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}^-.$$

Since we have the $\mathbf{R}[\mathbf{Z}/2]$-isometry

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2}\cdot(t+1) & \frac{\sqrt{2}}{2}\cdot(t-1) \end{pmatrix} : \mathbf{R}\oplus\mathbf{R}\oplus\mathbf{R}^- \to \mathbf{R}\oplus\mathbf{R}[\mathbf{Z}/2],$$

$[\phi\circ(c+\gamma)] = [(c+\gamma)\circ\phi]$ is represented by

$$\begin{pmatrix} \mu^{1/2}/2 & \sqrt{2} & 0 \\ \mu^{1/2}/2 & -\sqrt{2} & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2}\cdot\mu^{1/2} \end{pmatrix} : \mathbf{R}\oplus\mathbf{R}\oplus\mathbf{R}^- \to \mathbf{R}\oplus\mathbf{R}\oplus\mathbf{R}^-.$$

This shows

$$(3.26)\qquad \rho_{\mathrm{pl}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) = [2\mu\cdot\mathrm{id}\colon \mathbf{R}\to\mathbf{R}] + [\mu/2\cdot\mathrm{id}\colon \mathbf{R}^-\to\mathbf{R}^-].$$

We get by restriction to the trivial subgroup that

$$(3.27)\qquad \rho_{\mathrm{pl}}(S^1;\mathbf{R}) = \mu^2 \in \mathbf{R}^*.$$

We conclude from (3.26) and Lemma 3.32

$$(3.28)\qquad \rho_{\mathrm{pl}}^{\mathbf{Z}/2}(S^1;\mathbf{R}^-) = [\mu/2\cdot\mathrm{id}\colon \mathbf{R}\to\mathbf{R}] + [2\mu\cdot\mathrm{id}\colon \mathbf{R}^-\to\mathbf{R}^-].$$

We derive from (3.26), (3.28), and Proposition 3.23

$$(3.29)\qquad \rho_{\mathrm{pd}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) = [4\cdot\mathrm{id}\colon \mathbf{R}\to\mathbf{R}] + [\tfrac{1}{4}\cdot\mathrm{id}\colon \mathbf{R}^-\to\mathbf{R}^-].$$

Example 3.30. Equip $D^1 = [0,1]$ with the standard metric scaled by $\mu > 0$. Then D^1 has volume μ. The cellular $\mathbf{R}$-chain complex is

$$\mathbf{R} \xrightarrow{(1,-1)} \mathbf{R}\oplus\mathbf{R}.$$

The element $(\mu^{1/2}/2, \mu^{1/2}/2) \in H_0(D^1;\mathbf{R})$ has norm 1 with respect to the harmonic Hilbert structure. Now one easily checks that

$$(3.51)\qquad \rho_{\mathrm{pl}}(D^1;\mathbf{R}) = \mu \in \mathbf{R}^*.$$

Recall the map $q\colon D^G(M)\to G$ of (1.2) and the operation of $K_0(\mathbf{R}G) = \mathrm{Rep}_{\mathbf{R}}(G)$ on $K_1(\mathbf{R}G)^{\mathbf{Z}/2}$ and $\mathbf{R}\otimes_{\mathbf{Z}}\mathrm{Rep}_{\mathbf{R}}(G)$ defined in (2.14) and (1.25).

Lemma 3.32. *Let W be an orthogonal G-representation. Then*

$$\rho_{\mathrm{an}}^G(M, M_1; V\otimes_{\mathbf{R}} q^*W) = \rho_{\mathrm{an}}^G(M, M_1; V)\otimes_{\mathbf{R}}[W],$$
$$\rho_{\mathrm{pl}}^G(M, M_1; V\otimes_{\mathbf{R}} q^*W) = \rho_{\mathrm{pl}}^G(M, M_1; V)\otimes_{\mathbf{R}}[W],$$
$$\rho_{\mathrm{pd}}^G(M, M_1; V\otimes_{\mathbf{R}} q^*W) = \rho_{\mathrm{pd}}^G(M, M_1; V)\otimes_{\mathbf{R}}[W].$$

Proof. There are natural isometric $\mathbf{R}G$-isomorphisms

$$\Lambda^*(M;V)\otimes_{\mathbf{R}} W \to \Lambda^*(M;V\otimes_{\mathbf{R}} W)$$

and

$$C_*(M, M_1; V) \otimes_{\mathbf{R}} W \to C_*(M, M_1; V \otimes_{\mathbf{R}} q^* W). \quad \text{q.e.d.}$$

Let $(M; M_1, M_2)$ and $(N; N_1, N_2)$ be m-dimensional Riemannian G-manifold triads and let $(f; f_1, f_2): (M; M_1, M_2) \to (N; N_1, N_2)$ be a G-homotopy equivalence of such triads. Let V be an orthogonal $D^G(N)$-representation, and $\kappa_*^{\text{harm}}(M)$ and κ_*^{harm} the harmonic Hilbert structures on $H_*(M, M_1; f^*V)$ and $H_*(N, N_1; V)$. The following composition represents an element $u_i \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$:

$$\begin{aligned} H_i(M, M_1; f^*V) &\xrightarrow{H_i(f, f_1)} H_i(N, N_1; V) \xrightarrow{\kappa_i^{\text{harm}}(N)} H_i(N, N_1; V)^* \\ &\xrightarrow{H_i(f, f_1)^*} H_i(M, M_1; f^*V)^* \\ &\xrightarrow{\kappa_i^{\text{harm}}(M)^{-1}} H_i(M, M_1; f^*V). \end{aligned}$$

We introduced the map $\omega^G(N; V): \mathrm{Wh}^G(N) \to K_1(\mathbf{R}G)^{\mathbf{Z}/2}$ in Proposition 3.8. We derive the following proposition from Propositions 3.8, 2.12, and 2.18.

Proposition 3.33.

(a)
$$\begin{aligned} &\rho_{\text{pl}}^G(N, N_1; V) - \rho_{\text{pl}}^G(M, M_1; f^*V) \\ &= \omega^G(N; V)(\tau^G(f, f_1)) - \sum_{i \geq 0} (-1)^i \cdot u_i. \end{aligned}$$

(b)
$$\begin{aligned} &\rho_{\text{pd}}^G(N, N_1; V) - \rho_{\text{pd}}^G(M, M_1; f^*V) \\ &= \omega^G(N; V)(\tau^G(f, f_1)) + (-1)^m \cdot \omega^G(N; {}^wV)(\tau^G(f, f_2)). \end{aligned}$$

Proposition 3.55 tells in particular how the PL-torsion varies under change of Riemannian metric. Notice that the Poincaré torsion depends only on the simple G-homotopy type. This is also true for the PL-torsion if $H_i(N, N_1; V)$ vanishes.

The following pairings are special cases of (2.14):

$$\begin{aligned} \otimes_{\mathbf{R}}&: K_1(\mathbf{R}G)^{\mathbf{Z}/2} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(\mathbf{Z}/2) \to K_1(\mathbf{R}[G \times \mathbf{Z}/2])^{\mathbf{Z}/2}, \\ \otimes_{\mathbf{R}}&: \mathrm{Rep}_R(G) \otimes_{\mathbf{Z}} K_1(\mathbf{R}[\mathbf{Z}/2])^{\mathbf{Z}/2} \to K_1(\mathbf{R}[G \times \mathbf{Z}/2])^{\mathbf{Z}/2}. \end{aligned}$$

Proposition 3.34 (Double formula for Poincaré and PL-torsion). *Let M be a Riemannian G-manifold, and V an orthogonal $D^G(M)$-representation. Suppose that ∂M is the disjoint union $M_1 \coprod M_2$ and that the metric* *is a product near the boundary. Then*

(a) $$\rho_{\mathrm{pl}}^{G\times\mathbf{Z}/2}(M\cup_{M_1} V;\, V\cup_{M_1} V) = \rho_{\mathrm{pl}}^{G}(M;\, V)\otimes_{\mathbf{R}}[\mathbf{R}] + \rho_{\mathrm{pl}}^{G}(M,\, M_1;\, V)\otimes_{\mathbf{R}}[\mathbf{R}^-] + \chi^G(M_1;\, V)\otimes_{\mathbf{R}}[2\cdot\mathrm{id}\colon \mathbf{R}\to\mathbf{R}],$$

(b) $$\rho_{\mathrm{pl}}^{G\times\mathbf{Z}/2}(M\cup_{M_1} M,\, \partial(M\cup_{M_1} M);\, V\cup_{M_1} V) = \rho_{\mathrm{pl}}^{G}(M,\, M_2;\, V)\otimes_{\mathbf{R}}[\mathbf{R}] + \rho_{\mathrm{pl}}^{G}(M,\, \partial M;\, V)\otimes_{\mathbf{R}}[\mathbf{R}^-] + \chi^G(M_1;\, V)\otimes_{\mathbf{R}}[2\cdot\mathrm{id}\colon \mathbf{R}\to\mathbf{R}],$$

(c) $$\rho_{\mathrm{pd}}^{G\times\mathbf{Z}/2}(M\cup_{M_1} M;\, V\cup_{M_1} V) = \rho_{\mathrm{pd}}^{G}(M;\, V)\otimes_{\mathbf{R}}[\mathbf{R}] + \rho_{\mathrm{pd}}^{G}(M,\, M_1;\, V)\otimes_{\mathbf{R}}[\mathbf{R}^-] + \chi^G(M_1;\, V)\otimes_{\mathbf{R}}[2\cdot\mathrm{id}\colon \mathbf{R}\to\mathbf{R}] - \chi^G(M_1;\, V)\otimes_{\mathbf{R}}[2\cdot\mathrm{id}\colon \mathbf{R}^-\to\mathbf{R}^-].$$

Proof. Let $i\colon M\to M\cup_{M_1} M$ be the inclusion onto the first summand. We obtain from this map an $\mathbf{R}[D^G(M\cup_{M_1} M)]$-chain map $C_*(i)\colon i_*C_*(\widetilde{M}) \to C_*(M\widetilde{\cup_{M_1}} M)$ for i_* the induction with the homomorphism $D^G(i)\colon D^G(M)\to D^G(M\cup_{M_1} M)$. As $i^*(V\cup_{M_1} V)$ is V, there is an $\mathbf{R}G$-chain isomorphism j from $i_*C_*(\widetilde{M})\otimes_{\mathbf{R}[D^G(M\cup_{M_1} M)]} V\cup_{M_1} V$ to $i_*C_*(M;\, V)$. The composition of $C_*(i)\otimes_{\mathbf{R}[D^G(M\cup_{M_1} M)]} V\cup_{M_1} V$ and j^{-1} is denoted by

$$i_*\colon i_*C(M;\, V)\to C(M\cup_{M_1} M;\, V\cup_{M_1} V). \tag{3.35}$$

Let $\tau\colon M\cup_{M_1} M\to M\cup_{M_1} M$ be the flip map. As $\tau^* V\cup_{M_1} V = V\cup_{M_1} V$ holds, we also obtain an $\mathbf{R}G$-chain map

$$\tau_*\colon C_*(M\cup_{M_1} M;\, V\cup_{M_1} V)\to C_*(M\cup_{M_1} M;\, V\cup_{M_1} V). \tag{3.36}$$

Define an $\mathbf{R}[G\times\mathbf{Z}/2]$-chain map

$$f_*\colon (C_*(M;\, V)\otimes_{\mathbf{R}}\mathbf{R})\oplus(C_*(M;\, M_1;\, V)\otimes_{\mathbf{R}}\mathbf{R}^-) \to C_*(M\cup_{M_1} M;\, V\cup_{M_1} V) \tag{3.37}$$

by

$$f_*(a\otimes_{\mathbf{R}} 1\oplus b\otimes_{\mathbf{R}} 1) = \frac{\sqrt{2}}{2}\cdot(i_* + \tau_*\circ i_*)(a) + \frac{\sqrt{2}}{2}\cdot(i_* - \tau_*\circ i_*)(b).$$

Counting the cellular bases we obtain isometric $\mathbf{R}(G)$-isomorphisms for $p \in \mathbf{Z}$:

$$\phi\colon C_p(M, M_1; V) \oplus C_p(M_1; V) \to C_p(M; V),$$

$$\text{(3.38)} \qquad \Psi\colon C_p(M, M_1, V) \oplus C_p(M_1; V) \oplus C_p(M, M_1; V) \to C_p(M \cup_{M_1} M; V \cup_{M_1} V).$$

The map ψ is an $\mathbf{R}[G \times \mathbf{Z}/2]$-map, if $\mathbf{Z}/2$ acts trivially on $C_p(M_1; V)$ and switches the two summands $C_p(M, M_1; V)$. We get from ϕ the following isometric $\mathbf{R}[G \times \mathbf{Z}/2]$-isomorphism:

$$(\phi \otimes_{\mathbf{R}} \mathrm{id}) \oplus \mathrm{id}\colon (C_p(M, M_1; V)) \otimes_{\mathbf{R}} \mathbf{R}) \oplus (C_p(M_1; V) \otimes_{\mathbf{R}} \mathbf{R}) \oplus (C_p(M, M_1; V) \otimes_{\mathbf{R}} \mathbf{R}^-) \to C_*(M; V) \otimes_{\mathbf{R}} \mathbf{R} \oplus C_*(M; M_1; V) \otimes_{\mathbf{R}} \mathbf{R}^-.$$

Now the composition $\psi^{-1} \circ f_p \circ ((\phi \otimes_{\mathbf{R}} \mathrm{id}) \oplus \mathrm{id})$ is given by

$$\begin{pmatrix} \frac{\sqrt{2}}{2} \cdot \mathrm{id} & 0 & \frac{\sqrt{2}}{2} \cdot \mathrm{id} \\ 0 & \sqrt{2} \cdot \mathrm{id} & 0 \\ \frac{\sqrt{2}}{2} \cdot \mathrm{id} & 0 & -\frac{\sqrt{2}}{2} \cdot \mathrm{id} \end{pmatrix} : C_p(M, M_1; V) \oplus C_p(M_1; V) \oplus C_p(M, M_1; V) \to (C_p(M, M_1; V)) \otimes_{\mathbf{R}} \mathbf{R}) \oplus (C_p(M_1; V) \otimes_{\mathbf{R}} \mathbf{R}) \oplus (C_p(M, M_1; V) \otimes_{\mathbf{R}} \mathbf{R}^-).$$

The Hilbert torsion of this $\mathbf{R}[G\times\mathbf{Z}/2]$-isomorphism of $\mathbf{R}[G\times\mathbf{Z}/2]$-Hilbert modules is easily computed as $[C_p(M_1; V)] \otimes_{\mathbf{R}} [2 \cdot \mathrm{id}\colon \mathbf{R} \to \mathbf{R}]$. For the Hilbert torsion of the $\mathbf{R}[G \times \mathbf{Z}/2]$-chain map f_* of $\mathbf{R}[G \times \mathbf{Z}/2]$-Hilbert chain complexes we get

$$\text{(3.39)} \qquad \mathrm{ht}^{G\times\mathbf{Z}/2}(f_*) = \chi^G(M_1; V) \otimes_{\mathbf{R}} [2 \cdot \mathrm{id}\colon \mathbf{R} \to \mathbf{R}].$$

Next we compute

$$\text{(3.40)} \qquad \mathrm{ht}^{G\times\mathbf{Z}/2}(H(f_*)) = 0$$

on the homology level. The following diagram commutes, where A is given by the deRham map and the isomorphism (3.4), and l is the composition of (1.29) and (1.30):

$$\begin{array}{ccc} H_p(M \cup_{M_1} M, V \cup_{M_1} V)^* & \xrightarrow{H(f_*)^*} & (H_p(M; V) \otimes_{\mathbf{R}} \mathbf{R})^* \oplus (H_p(M, M_1; V) \otimes_{\mathbf{R}} \mathbf{R}^-)^* \\ \uparrow{\scriptstyle A} & & \uparrow{\scriptstyle (A\otimes_{\mathbf{R}}\mathrm{id})\oplus(A\otimes_{\mathbf{R}}\mathrm{id})} \\ E_0(M \cup_{M_1} M; V \cup_{M_1} V) & \xrightarrow{\sqrt{2}\cdot l} & (E_0(M; V) \otimes_{\mathbf{R}} \mathbf{R}) \oplus (E_0(M, M_1; V) \otimes_{\mathbf{R}} \mathbf{R}^-) \end{array}$$

The lower vertical map is an isometry because of the following computation for $\omega, \eta \in E_0(M \cup_{M_1} M, V \cup_{M_1} V)$:

$$\begin{aligned}\langle \sqrt{2} \cdot l(\omega), \sqrt{2} \cdot l(\eta) \rangle &= \langle \sqrt{2}/2 \cdot i^*(\omega + \tau^* \omega), \sqrt{2}/2 \cdot i^*(\eta + \tau^* \eta) \rangle_M \\ &\quad + \langle \sqrt{2}/2 \cdot i^*(\omega - \tau^* \omega), \sqrt{2}/2 \cdot i^*(\eta - \tau^* \eta) \rangle_M \\ &= \langle i^* \omega, i^* \eta \rangle_M + \langle i^* \tau^* \omega, i^* \tau^* \eta \rangle_M \\ &= \langle \omega, \eta \rangle_{M \cup_{M_1} M}.\end{aligned}$$

This implies (3.40). Now claim (a) follows from the composition formula (Proposition 2.12) applied to the $\mathbf{R}[G \times \mathbf{Z}/2]$-chain isomorphism f_* defined in (3.37) if we take (3.39) and (3.40) into account. One proves (b) analogously. Then (c) follows from (a) and (b) and Poincaré duality (Proposition 3.23) since the following $\mathbf{R}[G \times \mathbf{Z}/2]$-representations are isomorphic:

$$({}^{w^G(M)}V \cup_{M_1} {}^{w^G(M)}V) \otimes \mathbf{R}^- = {}^{w^{G \times \mathbf{Z}/2}(M \cup_{M_1} M)}(V \cup_{M_1} V).$$

This finishes the proof of the double formula.

Remark 3.41. Notice that the double formulas for analytic torsion (Proposition 1.27) and PL-torsion (Proposition 3.34) differ by a Euler characteristic term depending only on the boundary. It appears in the PL-case since the cells in the boundary do contribute to the Hilbert structures. This is not true in the analytic situation where the Hilbert structures come from certain integrals and the boundary does not contribute to them since it is a zero set. These observations indicate that the Euler characteristic is the correction term in a formula relating analytic and PL-torsion for manifolds with boundary.

Example 3.42. In this example we show how the general results above can be used to compute the torsion invariants for S^1 and D^1. We will see that we get the same answers as in Examples 1.15, 1.18, 3.25, and 3.30, where we computed these invariants directly. Equip D^1 with the standard metric scaled by $\mu > 0$. Then D^1 has volume μ. The double $D^1 \cup_{\partial D^1} D^1$ is S^1 with $\mathbf{Z}/2$-action given by complex conjugation and the S^1-invariant Riemannian metric for which the volume of S^1 is $2 \cdot \mu$. From Poincaré duality (Propositions 1.20 and 3.23) and Proposition 3.20 we get

$$\rho_{\mathrm{pl}}(D^1) = \rho_{\mathrm{pl}}(D^1, \partial D^1), \qquad \rho_{\mathrm{an}}(D^1) = \rho_{\mathrm{an}}(D^1, \partial D^1).$$

By the double formulas (Propositions 1.27 and 3.34) we derive

$$\begin{aligned}
\rho_{\mathrm{an}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) &= \rho_{\mathrm{an}}(D^1)\cdot([\mathbf{R}]+[\mathbf{R}^-]),\\
\rho_{\mathrm{pl}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) &= \rho_{\mathrm{pl}}(D^1)\cdot([\mathbf{R}]+[\mathbf{R}^-]) + 2\cdot[2\cdot\mathrm{id}:\mathbf{R}\to\mathbf{R}],\\
\rho_{\mathrm{pd}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) &= \rho_{\mathrm{pd}}(D^1)\cdot([\mathbf{R}]+[\mathbf{R}^-]) + 2\cdot[2\cdot\mathrm{id}:\mathbf{R}\to\mathbf{R}]\\
&\quad - 2\cdot[2\cdot\mathrm{id}:\mathbf{R}^-\to\mathbf{R}^-].
\end{aligned}\tag{3.43}$$

We know $\rho_{\mathrm{an}}(S^1;\mathbf{R}) = \ln(\rho_{\mathrm{pl}}(S^1;\mathbf{R}))$ from Cheeger [8] and Müller [26]. Hence the restriction of (3.43) to the trivial subgroup shows

$$\rho_{\mathrm{an}}(D^1) = \ln(\rho_{\mathrm{pl}}(D^1)) + \ln(2).\tag{3.44}$$

One easily checks that the generator of $H_0(D^1;\mathbf{Z})$ has norm $\mu^{-1/2}$ with respect to the harmonic Hilbert structure when considered in $H_0(D^1;\mathbf{R})$. The projection $D^1 \to \{\mathrm{pt.}\}$ is a simple homotopy equivalence. Hence, from Proposition 3.8 we get

$$\rho_{\mathrm{pl}}(D^1) = \mu,\tag{3.45}$$

from (3.44) we get

$$\rho_{\mathrm{an}}(D^1) = \ln(2\cdot\mu),\tag{3.46}$$

and from (3.43), if $\nu = 2\cdot\mu$ is the volume of S^1, we conclude that

$$\begin{aligned}
\rho_{\mathrm{an}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) &= \ln(\nu)\cdot[\mathbf{R}] + \ln(\nu)\cdot[\mathbf{R}^-]),\\
\rho_{\mathrm{pl}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) &= [2\nu\cdot\mathrm{id}:\mathbf{R}\to\mathbf{R}] + [\nu/2\cdot\mathrm{id}:\mathbf{R}^-\to\mathbf{R}^-],\\
\rho_{\mathrm{pd}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) &= [4\cdot\mathrm{id}:\mathbf{R}\to\mathbf{R}] - [1/4\cdot\mathrm{id}:\mathbf{R}^-\to\mathbf{R}^-].
\end{aligned}\tag{3.47}$$

Let M (resp. N) be a Riemannian G- (resp. H-) manifold with boundary. Some care is necessary to put a $(G\times H)$-manifold structure on $M\times N$; one has to straighten the angle. There seems to be no canonical Riemannian metric on $M\times N$. If $f\colon K\to M$, $g\colon L\to N$, and $h\colon X\to M\times N$ are equivariant triangulations, then $h^{-1}\circ(f\times g)\colon K\times L\to X$ is a simple $(G\times H)$-homotopy equivalence. Hence we get the following proposition from Propositions 3.8 and 3.13.

Proposition 3.48 (Product formula for Poincaré and PL-torsion). *Let $(M;M_1,M_2)$ (resp. $(N;N_1,N_2)$) be a Riemannian G- (resp. H-) manifold triad and let V (resp. W) be an orthogonal $D^G(M)$- (resp. $D^H(N)$-) representation. Put on the $(G\times H)$- manifold $(M\times N; M\times N_1\cup M_1\times N, M\times N_2\cup M_2\times N)$ an invariant Riemannian metric such that the Künneth $\mathbf{R}[G\times H]$-isomorphisms from $H_*(M,M_1;V)\otimes_{\mathbf{R}} H_*(N,N_1;W)$*

to $H_*(M \times N; M_1 \times N \cup M \times N_1; V \otimes_{\mathbf{R}} W)$ *becomes an isometry with respect to the harmonic structures. Then*

(a) $$\rho_{\mathrm{pl}}^{G\times H}(M \times N, M_1 \times N \cup M \times N_1; V \otimes_{\mathbf{R}} W) = \chi^G(M, M_1; V) \otimes_{\mathbf{R}} \rho_{\mathrm{pl}}^H(N, N_1; W) + \rho_{\mathrm{pl}}^G(M, M_1; V) \otimes_{\mathbf{R}} \chi^H(N, N_1; W),$$

(b) $$\rho_{\mathrm{pd}}^{G\times H}(M \times N, M_1 \times N \cup M \times N_1; V \otimes_{\mathbf{R}} W) = \chi^G(M, M_1; V) \otimes_{\mathbf{R}} \rho_{\mathrm{pd}}^H(N, N_1; W) + \rho_{\mathrm{pd}}^G(M, M_1; V) \otimes_{\mathbf{R}} \chi^H(N, N_1; W),$$

(c) $$\chi^{G\times H}(M \times N, M_1 \times N \cup M \times N_1; V \otimes_{\mathbf{R}} W) = \chi^G(M, M_1; V) \otimes_{\mathbf{R}} \chi^H(N, N_1; W).$$

Next we examine how the Poincaré and PL-torsion behave under glueing.

Proposition 3.49 (Sum formula for Poincaré and PL-torsion). *Let* $(M; M_1, M_2)$ *and* $(N; N_1, N_2)$ *be G-manifold triads with invariant Riemannian metric. Let* V *and* W *be orthogonal* $D^G(M)$*- and* $D^G(N)$*-representations. Let* $f: M_2 \to N_1$ *be a G-diffeomorphism, and* $\bar{f}: V|M_2 \to f^* W|N_1$ *an isometric* $\mathbf{R}[D^G(M_2)]$*-isomorphism. Put an invariant Riemannian metric on* $M \cup_f N$. *Let* M_* *be the acyclic finite* **R***G-Hilbert complex given by the long Mayer-Vietoris homology sequence and the harmonic* **R***G-Hilbert structures. Define* P_* *analogously by the long homology sequence of the pair* (M, M_1). *Then:*

(a) $\rho_{\mathrm{pl}}^G(M \cup_f N, V \cup_{\bar{f}} W) = \rho_{\mathrm{pl}}^G(M; V) + \rho_{\mathrm{pl}}^G(N; W) - \rho_{\mathrm{pl}}^G(M_2; V) - \mathrm{hr}^G(M_*)$,

(b) $\rho_{\mathrm{pd}}^G(M \cup_f N, V \cup_{\bar{f}} W) = \rho_{\mathrm{pd}}^G(M; V) + \rho_{\mathrm{pd}}^G(N; W) - \rho_{\mathrm{pd}}^G(M_2; V)$,

(c) $\chi^G(M \cup_f N, V \cup_{\bar{f}} W) = \chi^G(M; V) + \chi^G(N; W) - \chi^G(M_2; V)$,

(d) $\rho_{\mathrm{pl}}^G(M, M_1; V) = \rho_{\mathrm{pl}}^G(M; V) - \rho_{\mathrm{pl}}^G(M_1; V) - \mathrm{hr}^G(P_*)$,

(e) $\rho_{\mathrm{pd}}^G(M, M_1; V) = \rho_{\mathrm{pd}}^G(M; V) - \rho_{\mathrm{pd}}^G(M_1; V)$,

(f) $\chi^G(M, M_1; V) = \chi^G(M; V) - \chi^G(M_1; V)$.

Proof. (a) and (d) follow directly from Proposition 3.11.

(e) We have exact sequences of $\mathbf{R}G$-chain complexes

$$(3.50)\qquad 0 \to C_*(M_1; V) \to C_*(M; V) \to C_*(M, M_1; V) \to 0,$$

$$(3.51)\qquad \begin{aligned} 0 \to C_*(M_1, \partial M_1; {}^wV) &\to C_*(M, M_2; {}^wV) \\ &\xrightarrow{p} C_*(M, \partial M; {}^wV) \to 0. \end{aligned}$$

We have the following canonical exact sequences if $\mathrm{Cyl}(p)$ and $\mathrm{Cone}(p)$ are the mapping cylinder and cone of p, and P_* and Q_* are the mapping cones of $C_*(M, \partial M; {}^wV)$ and $C_*(M, M_2; {}^wV)$:

$$(3.52)\qquad \begin{aligned} &0 \to C_*(M, M_2; {}^wV) \to \mathrm{Cyl}(p) \to \mathrm{Cone}(p) \to 0, \\ &0 \to C_*(M, \partial M; {}^wV) \to \mathrm{Cyl}(p) \to Q_* \to 0, \\ &0 \to \sum C_*(M_1, \partial M_1; {}^wV) \to \mathrm{Cone}(p) \to P_* \to 0. \end{aligned}$$

Since P_* and Q_* are contractible, the last two sequences in (3.52) split as exact sequences of $\mathbf{R}G$-chain complexes (see Cohen [9]). Choose such a splitting. Then, by applying $C_* \mapsto C^{m-*}$, we get the following exact sequence from (3.52):

$$(3.53)\qquad \begin{aligned} \{0\} &\to C^{m-1-*}(M_1, \partial M_1; {}^wV) \oplus P^{m-*} \\ &\to C^{m-*}(M, \partial M; {}^wV) \oplus Q^{m-*} \\ &\to C^{m-*}(M, M_2; {}^wV) \to \{0\}. \end{aligned}$$

Now one constructs the following commutative diagram of $\mathbf{R}G$-Hilbert complexes with (3.50) and (3.53) as exact rows such that f (resp. g, resp. h) represents $\bigcap[M] \circ \mathrm{pr}$ (resp. $\bigcap[M] \circ \mathrm{pr}$, resp. $\bigcap[M]$) for pr the projection:

$$\begin{array}{ccccc} C^{m-1-*}(M_1, \partial M_1; {}^wV) \oplus P^{m-*} & \to & C^{m-*}(M, \partial M; {}^wV) \oplus Q^{m-*} & \to & C^{m-*}(M, M_2; wV) \\ \downarrow f & & \downarrow g & & \downarrow h \\ C_*(M_1; V) & \longrightarrow & C_*(M; V) & \longrightarrow & C_*(M, M_1; V) \end{array}$$

Since $\mathrm{pr}^{m-*}: P^{m-*} \to \{0\}$ and $\mathrm{pr}^{m-*}: Q^{m-*} \to \{0\}$ have vanishing Hilbert torsion, we get from Propositions 2.4 and 2.11 that

$$\rho^G_{\mathrm{pd}}(M_1; V) = \mathrm{ht}^G(f), \quad \rho^G_{\mathrm{pd}}(M; V) = \mathrm{ht}^G(g), \quad \rho^G_{\mathrm{pd}}(M, M_1; V) = \mathrm{ht}^G(h).$$

Now the claim follows from Proposition 2.9.

(b) is proven analogously starting with

$$0 \to C_*(M_1; V) \to C_*(M; V) \oplus C_*(N; V) \to C_*(M \cup_f N; V \cup_{\bar{f}} W) \to 0,$$

$$\begin{aligned} \{0\} &\to C_*(M_1; \partial M_1; V) \to C_*(M, \partial M; V) \oplus C_*(N, \partial N; V) \\ &\to C_*(M \cup_f N, \partial(M \cup_f N; V \cup_{\bar{f}} W) \to \{0\}. \end{aligned}$$

(c) and (f) are obvious. This finishes the proof of the sum formula.

4. Comparison of analytic and PL-torsion

Let G be a finite group. Consider a Riemannian G-manifold M whose boundary ∂M is $M_1 \coprod M_2$. We have introduced an extension $0 \to \pi_1(M) \xrightarrow{i} D^G(p) \xrightarrow{q} G \to 0$ in (1.2) and an operation of $D^G(M)$ on the universal covering $\widetilde{M}$ of M extending the $\pi_1(M)$-action and covering the G-action in (1.3) if $\pi_1(M)$ is identified with the group of deck transformations of the universal covering. Let V be an orthogonal $D^G(M)$-representation. We introduced *analytic torsion*

$$\rho^G_{\mathrm{an}}(M, M_1; V) \in \mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)$$

in Definition 1.14, *PL-torsion*

$$\rho^G_{\mathrm{pl}}(M, M_1; V) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

in Definition 3.15, and *Poincaré torsion*

$$\rho^G_{\mathrm{pd}}(M, M_1; V) \in K_1(\mathbf{R}G)^{\mathbf{Z}/2}$$

in Definition 3.19. In (2.25) we defined an isomorphism

$$\Gamma_1 \oplus \Gamma_2 \colon K_1(\mathbf{R}G)^{\mathbf{Z}/2} \to (\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)) \oplus (\mathbf{Z}/2 \otimes_{\mathbf{Z}} \widehat{\mathrm{Rep}}_{\mathbf{R}}(G)).$$

We want to relate these invariants by this isomorphism. One easily checks

Proposition 4.1. $\Gamma_2(\rho^G_{\mathrm{pl}}(M, M_1; V)) = \Gamma_2(\rho^G_{\mathrm{pd}}(M, M_1; V)) = 0$.

Hence only the images of PL-torsion and Poincaré torsion under Γ_1 are interesting. We need the following technical condition. We call $D^G(M)$-representations V and W *coherent* if for any $H \subset G$ and $x \in M^H$ their restrictions to H by $H = D^H(\{\mathrm{pt.}\}) \xrightarrow{D^H(j(x))} D^H(M) \xrightarrow{k} D^G(M)$ are $\mathbf{R}H$-isomorphic, where $j(x)\colon \{\mathrm{pt.}\} \to M$ has $\{x\}$ as image and k is the obvious inclusion. This is equivalent to the assumption that the G-vector bundles $M \times_{\pi_1(M)} V$ and $M \times_{\pi_1(M)} W$ are locally isometrically isomorphic (cf. the proof of Lemma 4.14).

Definition 4.2. We call a $D^G(M)$-representation V coherent to a G-representation if V and $q^* W$ are coherent for an appropriate G-representation W.

Example 4.3. If M^H is nonempty and connected for all $H \subset G$, then any G-representation is coherent to a G-representation W. Namely, for a fixed element $x \in M^G$ the homomorphism $s\colon G = D^G(\{\text{pt.}\}) \xrightarrow{D^G(j(x))} D^G(M)$ splits $q\colon D^G(M) \to G$. Put $W = s^*V$. For any $y \in M^H$ the H-maps $j(x)$ and $j(y)$ are H-homotopic. Hence the composition $H \hookrightarrow G \xrightarrow{s} D^G(M)$ agrees with the composition $H = D^H(\{\text{pt.}\}) \xrightarrow{D^H(j(y))} D^H(M) \xrightarrow{k} D^G(M)$.

Example 4.4. Let $\mathbf{Z}/2$ act on S^1 by complex conjugation. For $-1, 1 \in S^1$ we get different sections $s^+, s^-\colon \mathbf{Z}/2 \to D^{\mathbf{Z}/2}(S^1)$ of q. Identify $D^{\mathbf{Z}/2}(S^1)$ with the semidirect product $\mathbf{Z} \times_s \mathbf{Z}/2$ using s^+. Then $s^-\colon \mathbf{Z}/2 \to \mathbf{Z} \times_s \mathbf{Z}/2$ sends $\overline{m}$ to $((-1)^m, \overline{m})$. Consider the one-dimensional $D^{\mathbf{Z}/2}(S^1)$-representation given by $\mathbf{Z} \times_s \mathbf{Z}/2 \to \{\pm 1\}$, $(n, \overline{m}) \mapsto (-1)^{n+m}$; it cannot be coherent to a $\mathbf{Z}/2$-representation since its restrictions with s^+ and s^- are not the same.

Our main result is:

Theorem 4.5 (Torsion formula for manifolds with boundary and symmetry). *Let M be a Riemannian G-manifold whose boundary is the disjoint union $M_1 \coprod M_2$. Let V be an equivariant coefficient system which is coherent to a G-representation. Assume that the metric is a product near the boundary. Then*

$$\rho_{\text{an}}^G(M, M_1; V) = \Gamma_1(\rho_{\text{pl}}^G(M, M_1; V)) - \frac{1}{2} \cdot \Gamma_1(\rho_{\text{pd}}^G(M, M_1; V)) + \frac{\ln(2)}{2} \cdot \chi^G(\partial M; V).$$

Recall that ${}^wV$ is the w-twisted $D^G(M)$-representation given by V and the orientation homomorphism $w\colon D^G(M) \to \{\pm 1\}$ defined in (1.4). We will give examples, computations, and applications in the next section. The remainder of this section is devoted to the proof of Theorem 4.5.

We verify Theorem 4.5 under certain additional assumptions and remove these one after the other. We will use the work of Lott-Rothenberg [19] where Theorem 4.5 is proved following Müller [26], provided that ∂M is empty, M is orientable, and V is the trivial $D^G(M)$-representation $\mathbf{R}$. Their definitions still make sense when we remove the last condition that V is trivial. Fix an orthogonal G-representation W and an orthogonal $D^G(M)$-representation V. Let $\sigma\colon G \to \operatorname{Aut}_{\mathbf{R}}(W)$ be the G-structure on W. We extend the basic definitions of Lott-Rothenberg [19]. One gets their definitions back if one puts $V = \mathbf{R}$. Notice that V plays the role of an equivariant coefficient system for M, whereas W is a base element for the representation ring when W is irreducible.

Let $\Delta^p\colon \Lambda^p(M; V) \to \Lambda^p(M; V)$ be the Laplacian and let $\mathrm{pr}^p_{\mathrm{harm}}\colon \Lambda^p(M; V) \to \Lambda^p(M; V)$ be the harmonic projection onto $H^p_{\mathrm{harm}}(M; V)$. Let $\underline{\Delta}^p\colon \Lambda^p(M; V) \to \Lambda^p(M; V)$ be the sum $\Delta^p + \mathrm{pr}^p_{\mathrm{harm}}$. Denote by $\mathrm{pr}\colon \Lambda^p(M; V) \otimes_{\mathbf{R}} W \to (\Lambda^p(M; V) \otimes_{\mathbf{R}} W)^G$ the projection operator onto the fixed point set sending $(\omega \otimes_{\mathbf{R}} w)$ to $\frac{1}{|G|} \cdot \sum_{g\in G} l(g^{-1})^*\omega \otimes_{\mathbf{R}} \sigma(g)(w)$, where $l(g^{-1})^*$ is induced from left multiplication with g^{-1}. In Lott-Rothenberg [19] a meromorphic function $\mu^p_W(s)$, analytic in 0, is constructed which for $s \in \mathbf{C}$ with $\mathrm{Real}(s) > m/2$ is given by

$$(4.6)\qquad \begin{aligned}\mu^p_W(s) = \frac{1}{\Gamma(s)} \cdot \int_0^\infty t^{s-1} \cdot \mathrm{trace}(\mathrm{pr} \circ \exp(-t(\underline{\Delta}^p \otimes_{\mathbf{R}} \mathrm{id}_W)))\, dt \\ - \dim_{\mathbf{R}}((H^p_{\mathrm{harm}}(M; V) \otimes_{\mathbf{R}} W)^G),\end{aligned}$$

where the trace is taken for operators $\Lambda^p(M; V) \otimes_{\mathbf{R}} W \to \Lambda^p(M; V) \otimes_{\mathbf{R}} W$. The existence of the meromorphic extension is based on standard approximations of the heat kernel. We recall the definition of analytic torsion in Lott-Rothenberg [19]:

$$(4.7)\qquad T_W(M; V) = \frac{1}{2} \cdot \sum_{p\geq 0} (-1)^p \cdot p \cdot \frac{d}{ds}\mu^p_W(s)\Big|_{s=0} \in \mathbf{R}.$$

The PL-torsion $\tau_W(M; V) \in \mathbf{R}^*$ in Lott-Rothenberg [19] is defined in the following way. They use the same $\mathbf{R}G$-Hilbert structures on $C^p(M; V)$ and $H^p(M; V)$ as we do and equip $(C^p(M; V) \otimes_{\mathbf{R}} W)^G$ and $H((C^p(M; V) \otimes_{\mathbf{R}} W)^G) = (H^p(M; V) \otimes_{\mathbf{R}} W)^G)$ with the $\mathbf{R}G$-Hilbert structure induced by restriction from the product structures. They choose orthonormal bases and define $\tau_W(M; V)$ by Milnor's definition of torsion for the finitely generated based free $\mathbf{R}$-cochain complex $(C^p(M; V) \otimes_{\mathbf{R}} W)^G$ with based free cohomology. One easily checks (cf. Example 2.6)

$$(4.8)\qquad \tau_W(M; V)^2 = \mathrm{hr}((C_*(M; V) \otimes_{\mathbf{R}} W)^G) \in \mathbf{R}^*.$$

Let I be a complete set of representatives of the isomorphism classes of irreducible G-representations. Let $m(W) = \dim_{\mathbf{R}}(W \otimes_{\mathbf{R}} W)^G) = \dim_{\mathbf{R}}(\mathrm{Hom}_{\mathbf{R}G}(W, W))$ be the Schur index of $W \in I$.

Lemma 4.9. (a) *For* $s \in \mathbf{C}$ *with* $\mathrm{Real}(s) > m/2$, *in* $\mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)$ *for* $p > 0$

$$\zeta_p(M; V)(s) = 2 \cdot \sum_{W\in I} m(W)^{-1} \cdot \mu^p_W(M; V)(s) \cdot [W].$$

(b) *In* $\mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G)$

$$\rho^G_{\mathrm{an}}(M; V) = 2 \cdot \sum_{W\in I} m(W)^{-1} \cdot T_W(M; V) \cdot [W].$$

(c) *In* $K_1(\mathbf{R}G)^{\mathbf{Z}/2}$

$$\rho_{\mathrm{pl}}^G(M;V) = \sum_{W\in I} m(W)^{-1}\cdot[\tau_W(M;V)^2\cdot \mathrm{id}: W\to W].$$

Proof. For any G-representation P in $\mathrm{Rep}_R(G)$ from (2.21) we obtain the following equality:

$$(4.10)\qquad [P] = \sum_{W\in I} m(W)^{-1}\cdot \dim_{\mathbf{R}}((P\otimes_{\mathbf{R}} W)^G)\cdot[W].$$

As pr is the projection operator, we have

(4.11)

$$\begin{aligned}&\mathrm{trace}(\mathrm{pr}\circ\exp(-t(\underline{\Delta}^p\otimes_{\mathbf{R}}\mathrm{id}_W))\colon \Lambda^p(M;V)\otimes_{\mathbf{R}} W\to \Lambda^p(M;V)\otimes_{\mathbf{R}} W)\\ &\quad = \mathrm{trace}(\exp(-t(\underline{\Delta}^p\otimes_{\mathbf{R}}\mathrm{id}_W)^G)\colon\\ &\qquad\qquad (\Lambda^p(M;V)\otimes_{\mathbf{R}} W)^G\to(\Lambda^p(M;V)\otimes_{\mathbf{R}} W)^G),\end{aligned}$$

which implies

$$(4.12)\qquad \begin{aligned}\mu_W^p(s) &= \frac{1}{\Gamma(s)}\cdot\int_0^\infty t^{-s}\cdot\mathrm{trace}(\exp(-t(\underline{\Delta}^p\otimes_{\mathbf{R}}\mathrm{id}_W)^G))\,dt\\ &\quad -\dim_{\mathbf{R}}((H^p_{\mathrm{harm}}(M;V)\otimes_{\mathbf{R}} W)^G)\\ &= \sum_{\lambda>0}\lambda^{-s}\cdot\dim_{\mathbf{R}}(E_\lambda((\Delta^p\otimes_{\mathbf{R}}\mathrm{id}_W)^G))\\ &= \sum_{\lambda>0}\lambda^{-s}\cdot\dim_{\mathbf{R}}((E_\lambda(\Delta^p)\otimes_{\mathbf{R}} W)^G).\end{aligned}$$

Now claim (a) follows from (4.10) and (4.12). We derive (b) from (a), and (c) is proven similarly using (4.8). q.e.d.

The main result in Lott-Rothenberg [19] is the following theorem.

Theorem 4.13 (Lott-Rothenberg). *Let M be an odd-dimensional orientable Riemannian G-manifold. Suppose that G is orientation preserving and ∂M is empty. If W is an orthogonal G-representation, then*

$$T_W(M;\mathbf{R}) = \ln(\tau_W(M;\mathbf{R})).$$

Lott and Rothenberg's proof is modelled upon the proof of Müller [26]. Notice that Müller allows arbitrary coefficients. Lott and Rothenberg define an equivariant version of the combinatorial torsion $\tau^c_W(M;V)$ based on Whitney's map, and show that the estimate of §§1–5 in Müller [26] still hold in the equivariant setting. Then they define an equivariant parametrix and generalize the estimate of §8 in Müller [26]. Since Lott and Rothenberg work with trivial coefficients V, they can leave out the first step in Müller [26], where the difference between analytic and PL-torsions is shown to be independent of V. Lott and Rothenberg carry out the second step equivariantly, where the difference is examined under surgery, and thus get their result. We want to deal with Müller's first step.

Lemma 4.14. *Let M be a closed odd-dimensional G-manifold with trivial orientation behavior $w^G(M)$. Let U and V be coherent orthogonal $D^G(M)$-representations and W an orthogonal G-representation. Then*

$$T_W(M; U) - \ln(\tau_W(M; U)) = T_W(M; V)) - \ln(\tau_W(M; V)).$$

Proof. For any $x \in M$ with isotropy group $H = \{g \in G \mid gx = x\}$ there is an open neighborhood of the shape $G \times_H S$ for an H-representation S (see Bredon [6, VI.2.4, Corollary]). The key observation is that the restrictions of the G-vector bundles with Riemannian metrics $\widetilde{M} \times_\pi U$ and $\widetilde{M} \times_\pi V$ to $G \times_H S$ are isometrically isomorphic since then the argument in Müller [26, §9] goes through following the equivariant pattern of Lott and Rothenberg. The inclusion $G/H \to G \times_H S$ sending gH to $(g, 0)$ is a G-homotopy equivalence so that it suffices to regard the restrictions to G/H. We must show for the inclusion $j: x \to M$ that the restrictions agree with $H = D^H(x) \xrightarrow{D^H(i)} D^H(M) \xrightarrow{k} D^G(M)$ of U and V for k the obvious inclusion. As $\mathbf{R}H$-isomorphic implies isometrically $\mathbf{R}H$-isomorphic, this follows from the condition that U and V are coherent.

Lemma 4.15. *Theorem* 4.5 *is true if M is closed and $w^G(M) = 0$.*

Proof. If $\dim(M)$ is even, we get $\rho_{\text{an}}^G(M; V) = 0$ from Poincaré duality (Proposition 1.20). Analogously we obtain $2 \cdot \rho_{\text{pl}}^G(M; V) = \rho_{\text{pd}}(M; V)$ from Proposition 3.23 and the claim follows. Suppose that M is odd-dimensional. We may assume $V = q^* U$ for some G-representation U by Lemma 4.14 and the assumption that V is coherent to a G-representation. Because of Lemma 3.32 we may suppose $U = \mathbf{R}$. Finally apply Lemma 4.9 and Theorem 4.13 to get the lemma. q.e.d.

Next we want to drop the condition that G is orientation preserving.

Lemma 4.16. *Theorem* 4.5 *is true if M is closed and M is orientable.*

Proof. Since M is orientable, $w^G(M): D^G(M) \to \{\pm 1\}$ factorizes over $q: D^G(M) \to G$ into $\overline{w}: G \to \{\pm 1\}$. Let K be the kernel of $\overline{w}$. Since K operates orientation preserving, Lemma 4.15 applies to $\operatorname{res}_K^G M$. It suffices to treat the case $K \neq G$. Because the maps $\operatorname{ind}_K^G \circ \operatorname{res}_G^K$ on $K_1(\mathbf{R}G)^{\mathbf{Z}/2}$ and on $\mathbf{R} \otimes_{\mathbf{Z}} \operatorname{Rep}_{\mathbf{R}}(G)$ are given by $? \otimes_{\mathbf{R}} ([\mathbf{R}] + [^{\overline{w}}(\mathbf{R})])$, and

Γ_1 is compatible with restriction, we obtain
(4.17)
$$(\rho_{\mathrm{an}}^G(M;V) - \Gamma_1(\rho_{\mathrm{pl}}^G(M;V)) + \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^G(M;V))) \otimes_{\mathbf{R}} ([\mathbf{R}] + [{}^{\overline{w}}\mathbf{R}]) = 0.$$

Let $\mathbf{Z}/2$ act on S^1 by complex conjugation. From the product formulas (Propositions 1.32 and 3.13) we derive

$$\begin{aligned}(4.18)\qquad &\rho_{\mathrm{an}}^{G\times\mathbf{Z}/2}(M\times S^1, V\otimes_{\mathbf{R}}\mathbf{R}) - \Gamma_1(\rho_{\mathrm{pl}}^{G\times\mathbf{Z}/2}(M\times S^1; V\otimes_{\mathbf{R}}\mathbf{R}))\\ &\quad + \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G\times\mathbf{Z}/2}(M\times S^1; V\otimes_{\mathbf{R}}\mathbf{R}))\\ &= \chi^G(M;V)\otimes_{\mathbf{R}}(\rho_{\mathrm{an}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) - \Gamma_1(\rho_{\mathrm{pl}}^{\mathbf{Z}/2}(S^1;V))\\ &\qquad\qquad + \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{\mathbf{Z}/2}(S^1;V)))\\ &\quad + (\rho_{\mathrm{an}}^G(M;V) - \Gamma_1(\rho_{\mathrm{pl}}^G(M;V))\\ &\qquad\qquad + \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^G(M;V)))\otimes_{\mathbf{R}}\chi^{\mathbf{Z}/2}(S^1;\mathbf{R}).\end{aligned}$$

We conclude from (1.16), (3.26), and (3.29) that

$$(4.19)\qquad \rho_{\mathrm{an}}^{\mathbf{Z}/2}(S^1;\mathbf{R}) - \Gamma_1(\rho_{\mathrm{pl}}^{\mathbf{Z}/2}(S^1;V)) + \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{\mathbf{Z}/2}(S^1;V)) = 0.$$

Obviously

$$(4.20)\qquad \chi^{\mathbf{Z}/2}(S^1;\mathbf{R}) = [\mathbf{R}] - [\mathbf{R}^-].$$

If we restrict the $G\times\mathbf{Z}/2$-operation on $M\times S^1$ to G by $\mathrm{id}\times\overline{w}\colon G\to G\times\mathbf{Z}/2$, we obtain an orientation preserving action. Hence we can apply Lemma 4.15 to obtain in consequence of (4.18), (4.19), and (4.20),
(4.21)
$$(\rho_{\mathrm{an}}^G(M;V) - \Gamma_1(\rho_{\mathrm{pl}}^G(M;V)) + \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^G(M;V))) \otimes_{\mathbf{R}} ([\mathbf{R}] - [{}^{\overline{w}}\mathbf{R}]) = 0.$$

Hence adding (4.17) and (4.21), and dividing by 2 yield the claim.

Lemma 4.22. *Theorem* 4.5 *is true if M is closed.*

Proof. Assume that M is not orientable. Let $p\colon\widehat{M}\to M$ be the orientation covering. There is a group extension $0\to\mathbf{Z}/2\to\widehat{G}\xrightarrow{\hat{g}}G\to 0$ and a $\widehat{G}$-operation on $\widehat{M}$ extending the $\mathbf{Z}/2$-action on $\widehat{M}$ and covering the G-action on M. Since $\widehat{M}$ is orientable, the claim for the $\widehat{G}$-manifold $\widehat{M}$

follows from Lemma 4.16. Applying induction with $\hat{q}$ gives the assertion also for M. q.e.d.

Now we are ready to prove Theorem 4.5. Assume that $M_1 = \emptyset$. Let the orthogonal $D^G(M)$-representation V be coherent to the G-representation W. Then $V \cup_{\partial M} V$ is coherent to $W \cup_{M_1} W$ as a $(G \times \mathbf{Z}/2)$-representation. Because of Lemma 4.22, for the closed $(G \times \mathbf{Z}/2)$-manifold $M \cup_{\partial M} M$ we get

$$\begin{aligned} &\rho_{\mathrm{an}}^{G\times\mathbf{Z}/2}(M \cup_{\partial M} M, V \cup_{\partial M} V) \\ &\quad = \Gamma_1(\rho_{\mathrm{pl}}^{G\times\mathbf{Z}/2}(M \cup_{\partial M} M; V \cup_{\partial M} V)) \\ &\quad\quad - \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G\times\mathbf{Z}/2}(M \cup_{\partial M} M; V \cup_{\partial M} V)). \end{aligned} \tag{4.23}$$

Substituting the double formulas (Propositions 1.27 and 3.34) in (4.23) gives

(4.24)

$$\begin{aligned} &\rho_{\mathrm{an}}^{G}(M; V) \otimes_{\mathbf{R}} [\mathbf{R}] + \rho_{\mathrm{an}}^{G}(M, \partial M; V) \otimes_{\mathbf{R}} [\mathbf{R}^-] \\ &= \Gamma_1(\rho_{\mathrm{pl}}^{G}(M; V)) \otimes_{\mathbf{R}} [\mathbf{R}] + \Gamma_1(\rho_{\mathrm{pl}}^{G}(M, \partial M; V)) \otimes_{\mathbf{R}} [\mathbf{R}^-] \\ &\quad + \ln(2)\cdot\chi^G(\partial M; V) \otimes_{\mathbf{R}} [\mathbf{R}] \\ &\quad - \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M; V)) \otimes_{\mathbf{R}} [\mathbf{R}] - \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M, \partial M; V)) \otimes_{\mathbf{R}} [\mathbf{R}^-] \\ &\quad - \tfrac{1}{2}\cdot\ln(2)\cdot\chi^G(\partial M; V) \otimes_{\mathbf{R}} [\mathbf{R}] + \tfrac{1}{2}\cdot\ln(2)\cdot\chi^G(\partial M; V) \otimes_{\mathbf{R}} [\mathbf{R}^-] \\ &= (\Gamma_1(\rho_{\mathrm{pl}}^{G}(M; V)) - \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M; V)) \\ &\qquad\qquad + \frac{1}{2}\cdot\ln(2)\cdot\chi^G(\partial M; V)) \otimes_{\mathbf{R}} [\mathbf{R}] \\ &\quad + (\Gamma_1(\rho_{\mathrm{pl}}^{G}(M, \partial M; V)) - \tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M, \partial M; V)) \\ &\qquad\qquad + \tfrac{1}{2}\cdot\ln(2)\cdot\chi^G(\partial M; V)) \otimes_{\mathbf{R}} [\mathbf{R}^-]. \end{aligned}$$

Now the claim in the case $M_1 = \emptyset$ follows from comparing the coefficients of $[\mathbf{R}]$ in (4.24). In the general case we repeat this argument, but now we glue along M_1 instead of ∂M. Then the equation corresponding to (4.24) looks like

(4.25)

$$
\begin{aligned}
&\rho_{\mathrm{an}}^{G}(M;V)\otimes_{\mathbf{R}}[\mathbf{R}]+\rho_{\mathrm{an}}^{G}(M,M_1;V)\otimes_{\mathbf{R}}[\mathbf{R}^-]\\
&\quad=\Gamma_1(\rho_{\mathrm{pl}}^{G}(M;V))\otimes_{\mathbf{R}}[\mathbf{R}]+\Gamma_1(\rho_{\mathrm{pl}}^{G}(M,M_1;V))\otimes_{\mathbf{R}}[\mathbf{R}^-]\\
&\quad\quad+\ln(2)\cdot\chi^{G}(M_1;V)\otimes_{\mathbf{R}}[\mathbf{R}]\\
&\quad\quad-\tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M;V))\otimes_{\mathbf{R}}[\mathbf{R}]-\tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M,M_1;V))\otimes_{\mathbf{R}}[\mathbf{R}^-]\\
&\quad\quad-\frac{\ln(2)}{2}\cdot\chi^{G}(M_1;V)\otimes_{\mathbf{R}}[\mathbf{R}]\\
&\quad\quad+\tfrac{1}{2}\cdot\ln(2)\cdot\chi^{G}(M_1;V)\otimes_{\mathbf{R}}[\mathbf{R}^-]\\
&\quad\quad+\frac{\ln(2)}{2}\cdot\chi^{G\times\mathbf{Z}/2}(\partial(M\cup_{M_1}M),V\cup_{M_1}V)\\
&\quad=(\Gamma_1(\rho_{\mathrm{pl}}^{G}(M;V))-\tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M;V))+\tfrac{1}{2}\cdot\ln(2)\\
&\qquad\qquad\qquad\qquad\cdot\chi^{G}(\partial M;V))\otimes_{\mathbf{R}}[\mathbf{R}]\\
&\quad\quad+(\Gamma_1(\rho_{\mathrm{pl}}^{G}(M,M_1;V))-\tfrac{1}{2}\cdot\Gamma_1(\rho_{\mathrm{pd}}^{G}(M,M_1;V))\\
&\qquad\qquad\qquad\qquad+\tfrac{1}{2}\cdot\ln(2)\cdot\chi^{G}(\partial M;V))\otimes_{\mathbf{R}}[\mathbf{R}^-].
\end{aligned}
$$

Hence Theorem 4.5 follows from comparing the coefficients of $\mathbf{R}^-$. q.e.d.

We end this section by indicating the proof of Lemma 1.13 about the meromorphic extension of the equivariant zeta-function. As $\mu_W^p(s)$ defined in (4.6) has a meromorphic extension to the complex plane, analytic in 0, the same is true for $\zeta(M;V)(s)$ by Lemma 4.9, provided that M is closed and $w^G(M)=0$. Since the product formula (Proposition 1.32) and double formula (Proposition 1.27) give explicit identities of zeta-functions for $\mathrm{Real}(s)>m/2$, the arguments in the proof of Theorem 4.5 can also be used to verify Lemma 1.13.

5. Some computations

In this section we treat some special cases as an illustration. First, suppose that G is trivial. Then

$$\Gamma_1\oplus\Gamma_2\colon K_1(\mathbf{R}G)^{\mathbf{Z}/2}\to(\mathbf{R}\otimes_{\mathbf{Z}}\mathrm{Rep}_{\mathbf{R}}(G))\oplus(\mathbf{Z}/2\otimes_{\mathbf{Z}}\widehat{\mathrm{Rep}}_{\mathbf{R}}(G))$$

reduces to $\mathbf{R}^*\to\mathbf{R}$, $r\mapsto\ln(r/|r|)$, and $\rho_{\mathrm{pl}}^{G}(M,M_1;V)$ is just a positive real number. Moreover, $\chi^G(\partial M;V)$ is just $\dim_{\mathbf{R}}(V)\cdot\chi(\partial M)$, where $\chi(\partial M)$ is the ordinary Euler characteristic. By Proposition 5.20, $\rho_{\mathrm{pd}}^{G}(M,M_1;V)$ vanishes. From Theorem 4.5 we get the following corollary.

Corollary 5.1. *Let M be a Riemannian manifold whose boundary is the disjoint union of M_1 and M_2. Suppose that the metric is a product near the boundary. Let V be an orthogonal $\pi_1(M)$-representation. Then*

$$\rho_{\mathrm{an}}(M, M_1; V) = \ln(\rho_{\mathrm{pl}}(M, M_1; V)) + \frac{\ln(2)}{2} \cdot \chi(\partial M) \cdot \dim_{\mathbf{R}} V.$$

Notice that our definition of analytic torsion differs from the one in Ray-Singer [29] by a factor of 2, and our PL-torsion is the square of theirs. Corollary 5.1 for closed manifolds was independently proved by Cheeger [8] and Müller [26].

For trivial G and V the PL-torsion can be computed as follows. Fix a dimension $p \geq 0$. Choose an orthonormal basis $\{\omega_1, \omega_2, \cdots, \omega_{\beta_p}\}$ for the space of harmonic p-forms $H^p_{\mathrm{harm}}(M, M_1; V)$. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_{\beta_p}\}$ be a set of cycles in the singular (or cellular) chain complex with integral coefficients of (M, M_1) such that the set of their classes in $H_p(M, M_1; \mathbf{Z})/\operatorname{Tors}(H_p(M, M_1; \mathbf{Z}))$ is an integral basis. Let r_p be the determinant of the following matrix:

$$\begin{pmatrix} \int_{\sigma_1} \omega_1 & \int_{\sigma_1} \omega_2 & \cdots & \int_{\sigma_1} \omega_{\beta_0} \\ \int_{\sigma_2} \omega_1 & \int_{\sigma_2} \omega_2 & \cdots & \int_{\sigma_2} \omega_{\beta_p} \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\sigma_{\beta_p}} \omega_1 & \int_{\sigma_{\beta_p}} \omega_2 & \cdots & \int_{\sigma_{\beta_p}} \omega_{\beta_p} \end{pmatrix}.$$

Define

$$r(M, M_1) = \prod_{p=0}^{m} r_p^{2\cdot(-1)^{p+1}}. \tag{5.2}$$

Define the *multiplicative Euler characteristic* to be

$$m\chi(M, M_1) = \prod_{p=0}^{m} |\operatorname{Tors}(H_i(M, M_1; \mathbf{Z}))|^{(-1)^p}. \tag{5.3}$$

The following result was proved in Cheeger [8, (0.1) and (1.4)].

Proposition 5.4. *Let (M, M_1, M_2) be a triad of Riemannian manifolds. Then*

$$\rho_{\mathrm{pl}}(M, M_1; V) = m\chi(M, M_1)^2 \cdot r(M, M_1).$$

Proof. Let κ_i^{int} be the Hilbert structure on $H_i(M, M_1; \mathbf{R})$ for which the basis given by $\{\sigma_1, \sigma_2, \cdots, \sigma_{\beta_p}\}$ is orthonormal. From Proposition 3.8 we derive

$$\rho_{\mathrm{pl}}(M, M_1; \mathbf{R}) = \rho_{\mathrm{pl}}(M, M_1, \kappa_{\mathrm{harm}}) = \rho_{\mathrm{pl}}(M, M_1, \kappa_*^{\mathrm{int}}) \cdot r(M, M_1).$$

Hence it suffices to show

$$\rho_{\mathrm{pl}}(M, M_1, \kappa_{\mathrm{int}}) = m\chi(M, M_1).$$

Its elementary proof can be found, for example, in Cheeger [8, (1.4)] and Lück [21, Lemma 18.34].

Corollary 5.5. *Let M be an m-dimensional Riemannian manifold.*

(a) *If M is a rational homology sphere, i.e., $H_*(M;\mathbf{Q}) = H_*(S^m;\mathbf{Q})$, then*

$$\rho_{\mathrm{an}}(M) = \ln(\rho_{\mathrm{pl}}(M)) = (1 + (-1)^{m+1}) \cdot \ln(\mathrm{Vol}(M)) + 2 \cdot \sum_{p=1}^{m-1} (-1)^p \cdot \ln(|H_p(M, \mathbf{Z})|).$$

(b) *If M is a rational homology point, i.e., $H_*(M;\mathbf{Q}) = H_*(\{point\};\mathbf{Q})$, then*

$$\rho_{\mathrm{an}}(M; \mathbf{R}) = \ln(\mathrm{Vol}(M)) + (1 + (-1)^{m+1}) \cdot \frac{\ln(2)}{2} + 2 \cdot \sum_{p=1}^{m-1} (-1)^p \cdot \ln(|H_p(M, \mathbf{Z})|).$$

Next we treat the case where M is orientable and G is orientation preserving on M, or equivalently, where $w^G(M)$ is trivial. From Theorem 4.5 and Poincaré duality (Propositions 1.20 and 3.23) we derive the following corollary.

Corollary 5.6. *Let M be a Riemannian G-manifold with invariant Riemannian metric. Suppose M is closed and $w^G(M) = 0$. Let V be an orthogonal $D^G(M)$-representation coherent to a G-representation.*

(a) *If* $\dim(M)$ *is odd, then*

$$\rho^G_{\mathrm{an}}(M; V) = \Gamma_1(\rho_{\mathrm{pl}}(M; V)), \qquad \rho_{\mathrm{pd}}(M; V) = 0.$$

(b) *If* $\dim(M)$ *is even, then*

$$\rho_{\mathrm{an}}(M; V) = 0, \qquad \rho^G_{\mathrm{pl}}(M; V) = \tfrac{1}{2}\rho^G_{\mathrm{pd}}(M; V).$$

Remark 5.7. The assumption $w^G(M) = 0$ is necessary in Corollary 5.6. We have already shown $\rho^{\mathbf{Z}/2}_{\mathrm{pd}}(S^1; \mathbf{R}) \neq 0$ for $\mathbf{Z}/2$ acting by complex conjugation in (3.29). Let $\mathbf{Z}/2$ act on S^2 by sending (x, y, z) to $(-x, y, z)$. Then S^2 is the double of D^2 for appropriate Riemannian metrics on S^2. From the double formula (Proposition 1.27), Poincaré duality (Proposition 1.20), Proposition 3.8, and Corollary 5.1 we derive that

$$\rho_{an}^{\mathbf{Z}/2}(S^2 ; \mathbf{R}) = \rho_{an}(D^2 ; \mathbf{R}) \cdot ([\mathbf{R}] - [\mathbf{R}^-]) = \ln(\mathrm{Vol}(S^2)/2) \cdot ([\mathbf{R}] - [\mathbf{R}^-]).$$

Hence $\rho_{an}^{\mathbf{Z}/2}(S^2)$ is not necessarily zero. Let $\mathbf{RP}^n$ be the n-dimensional real projective space, which is rationally a point if n is even. From Corollary 5.5 we get

$$\rho_{an}(\mathbf{RP}^{2n} ; \mathbf{R}) = \ln(\mathrm{Vol}(\mathbf{RP}^{2n})) - 2n \cdot \ln(2).$$

Again this may be nonzero.

Next we analyze how the analytic torsion changes under variation of the metric and G-homotopy equivalence. We just have to combine Theorem 4.5 and Proposition 3.33. Suppose that $(f ; f_1 , f_2): (M ; M_1 , M_2) \to (N ; N_1 , N_2)$, and V and W satisfy the same hypothesis as in Proposition 3.33, and assume additionally that $M_1 \cap M_2$ and $N_1 \cap N_2$ are empty. Using the same notation as in Proposition 3.33, we get the following proposition.

Proposition 5.8.

$$\begin{aligned} &\rho_{an}^G(N , N_1 ; V) - \rho_{an}^G(M , M_1 ; f^* V) \\ &\quad = \tfrac{1}{2} \cdot \omega^G(N ; V)(\tau^G(f , f_1)) \\ &\quad + (-1)^{m-1} \cdot \tfrac{1}{2} \cdot \omega^G(N ; {}^w V)(\tau^G(f_1 , f_2)) \\ &\quad - \sum_{i \geq 0} (-1)^i \cdot u_i . \end{aligned}$$

Consider an isometric G-diffeomorphism $f: M_2 \to N_1$ and an isometric $\mathbf{R}D^G(M_2)$-isomorphism $\bar{f}: V|_{M_2} \to W|_{N_1}$. Denote by $D^*(M \cup_f N , M , N ; V , W)$ (resp. $D^*(M , M_1 ; V)$) the acyclic finite $\mathbf{R}G$-Hilbert chain complex given by the long Mayer-Vietoris sequence (resp. the long homology sequence) of the pair and the harmonic $\mathbf{R}G$-Hilbert structures. From the sum formula (Proposition 3.49) and Theorem 4.5 we derive the following theorem.

Theorem 5.9 (Sum formula for analytic torsion).

(a)
$$\begin{aligned} \rho_{an}^G(M \cup_f N ; V \cup_{\bar f} W) = {} & \rho_{an}^G(M ; V) + \rho_{an}^G(N ; W) - \rho^G(M_2 ; V) \\ & - \Gamma_1(\mathrm{hr}(D^*(M \cup_f N , M , N ; V , W))) \\ & - \ln(2) \cdot \chi^G(M_2 ; V). \end{aligned}$$

(b)
$$\begin{aligned} \rho_{an}^G(M , M_1 ; V) = {} & \rho_{an}^G(M ; V) - \rho_{an}^G(M_1 ; V) \\ & - \Gamma_1(\mathrm{hr}(D^*(M , M_1 ; V))). \end{aligned}$$

Remark 5.10. The sum formula is very useful for computations. One may chop a manifold into elementary pieces, compute the analytic torsion for each piece, and use the sum formula to get the analytic torsion for the manifold itself. The existence of such a formula is remarkable from the analytic point of view since it is very hard to get information about the spectrum of the Laplace operator on $M \cup_f N$ from the spectrum of the Laplace operator on M, N, and M_1. We do not have a direct analytic proof of the sum formula, but such a proof may be hidden in the paper of Müller [26, §10]. If one has a proof for the sum formula not using Theorem 4.5, then one can prove Theorem 4.5 by using induction over the number of handles.

Next we compute the various torsion invariants for G-representations (cf. Ray [28]). Define for a Riemannian G-manifold

$$\begin{aligned}\hat{\rho}^G_{\mathrm{an}}(M) &= \rho^G_{\mathrm{an}}(M) + \ln(\mathrm{Vol}(M)) \cdot (\chi^G(M) - 2 \cdot [\mathbf{R}]), \\ \rho^G_{\mathrm{pl}}(M) &= \rho^G_{\mathrm{pl}}(M) + \ln(\mathrm{Vol}(M)) \cdot (\chi^G(M) - 2 \cdot [\mathbf{R}]).\end{aligned} \tag{5.11}$$

Let V be a G-representation. Choose any orthogonal structure and any invariant Riemannian metric on DV which is a product near the boundary SV. Then $\hat{\rho}^G_{\mathrm{an}}$ and $\hat{\rho}^G_{\mathrm{pl}}$ are defined for SV and DV and depend only on the G-diffeomorphism type, but not on the other choices, by Proposition 5.8 since the equivariant Whitehead torsion of a G-diffeomorphism is zero.

Lemma 5.12. (a) $\rho^G_{\mathrm{pd}}(D(V \oplus W)) = \rho^G_{\mathrm{pd}}(DV) + \rho^G_{\mathrm{pd}}(DW)$.

(b) $\hat{\rho}^G_{\mathrm{pl}}(DV) = 0$.

(c) $\hat{\rho}^G_{\mathrm{an}}(DV) = -\frac{1}{2} \cdot \Gamma_1(\rho^G_{\mathrm{pd}}(DV)) + \frac{\ln(2)}{2} \cdot \chi^G(SV)$.

(d) $\rho^G_{\mathrm{pd}}(SV) = \chi^G(SV) \otimes_{\mathbf{R}} \rho^G_{\mathrm{pd}}(DV)$.

(e) $\hat{\rho}^G_{\mathrm{pl}}(SV) = \rho^G_{\mathrm{pd}}(DV) \otimes_{\mathbf{R}} (\chi^G(SV) - [\mathbf{R}])$.

(f) $\hat{\rho}^G_{\mathrm{an}}(SV) = \Gamma_1(\rho^G_{\mathrm{pd}}(DV)) \otimes_{\mathbf{R}} (\frac{1}{2} \cdot \chi^G(SV) - [\mathbf{R}])$.

Proof. (a) is a consequence of the product formula for Poincaré torsion (Proposition 3.48).

(b) As the projection from DV to a point is a simple G-homotopy equivalence, the claim follows from Proposition 3.8.

(c) follows from Theorem 4.5 and (b).

(d) and (e) follow from Poincaré duality (Proposition 3.23) and the sum formula (Proposition 3.49).

(f) is a consequence of Theorem 4.5 and (d) and (e).

Remark 5.13. Because of Lemma 5.12 above we obtain a homomorphism

$$(5.14) \qquad \mathrm{Rep}_{\mathbf{R}}(G) \to \mathbf{R} \otimes_{\mathbf{Z}} \mathrm{Rep}_{\mathbf{R}}(G), \qquad [V] \mapsto \rho^G_{\mathrm{pd}}(DV).$$

Hence it suffices to compute the torsion invariants for irreducible G-representations in order to know it for all G-representations. Elements in $\mathbf{R}\otimes_{\mathbf{Z}}\mathrm{Rep}_{\mathbf{R}}(G)$ and hence, by Proposition 2.26, also elements in $K_1(\mathbf{R}G)^{\mathbf{Z}/2}$ are detected by restriction to cyclic subgroups so that it is enough to compute the torsion elements of irreducible G-representations for cyclic G.

Let G be the cyclic group $\mathbf{Z}/n$. Let $d > 0$ be a divisor of n. If $d \geq 3$, let $V(j, d)$ for $1 \leq j \leq d/2$ be the (real) two-dimensional G-representation given by multiplication with primitive dth root of unity $\exp(2\pi i \frac{j}{d})$ on $\mathbf{C}$. Let $v(d/2, d)$ be the unique one-dimensional nontrivial G-representation, if d is even, and $\mathbf{R}$ the trivial G-representation. Denote by $I(d)$ the set $\{j \in \mathbf{Z} \mid 1 \leq j \leq d/2,\ (d, j) = (1)\}$. Then the set

$$\{V(j, d) \mid d \geq 2,\ d \mid n,\ 1 \leq j \leq d/2,\ (j, d) = (1)\} \cup \{\mathbf{R}\}$$

is a complete set of representatives for the irreducible G-representations. Using Lemma 5.12 in the first case and the double formula (Proposition 3.34) in the second case, one easily computes

$$\begin{aligned} \rho^G_{\mathrm{pd}}(DV(j, d)) &= -\rho^G_{\mathrm{pl}}(SV(j, d)) \\ &= [d^2 \cdot \mathrm{id} : \mathbf{R} \to \mathbf{R}] \\ (5.15) \qquad &\quad - \sum_{1 \leq k \leq d/2} \left[\left| \exp\left(2\pi i \frac{jk}{d}\right) - 1 \right|^2 \cdot \mathrm{id} : V(k) \to V(k, d) \right] \\ &\qquad\qquad \text{for } d \mid n,\ d \geq 3,\ j \in I(d), \end{aligned}$$

$$\rho^G_{\mathrm{pd}}(DV(1, 2)) = [2 \cdot \mathrm{id} : \mathbf{R} \to \mathbf{R}] - [2 \cdot \mathrm{id} : V(1, 2) \to V(1, 2)],$$

$$\rho^G_{\mathrm{pd}}(D(\mathbf{R})) = 0.$$

Now one can describe the homomorphism (5.14) above by characters.

Lemma 5.16 (Character formula). *Let G be a finite group and V a G-representation with character ch_V. Then $\rho^G_{\mathrm{pd}}(DV) \in \mathbf{R}\otimes_{\mathbf{Z}}\mathrm{Rep}_{\mathbf{R}}(G)$ can be viewed as a class function on G with values in $\mathbf{R}$. Let $g \in G$ be an element. Let n be the order of the cyclic subgroup $\langle g\rangle$ generated by g. Then the value of $\rho^G_{\mathrm{pd}}(DV)$ at g is 0 if $n = 1$, and otherwise*

$$\begin{aligned}\rho_{\mathrm{pd}}^G(DV)(g) &= \sum_{d\,|\,n,\,d\geq 2}\sum_{j\in I(d)} \langle \mathrm{ch}_{V|_{\langle g\rangle}}, \mathrm{ch}_{V(j,d)}\rangle \cdot \langle \mathrm{ch}_{V(j,d)}, \mathrm{ch}_{V(j,d)}\rangle^{-1} \\ &\qquad\qquad \cdot \rho_{\mathrm{pd}}^G(DV(j,d))(g) \\ &= \sum_{1\leq l\leq n} \frac{1}{n}\cdot \mathrm{ch}_V(g^l)\cdot(-1)^l\cdot \ln(4) \\ &\quad + \sum_{d\,|\,n,\,d\geq 3}\sum_{j\in I(d)}\sum_{1\leq l\leq n} \frac{1}{n}\mathrm{ch}_V(g^l)\cdot\cos\left(2\pi\frac{jl}{d}\right) \\ &\quad \cdot\left(\ln(d^2) - \sum_{1\leq k\leq d/2} \ln\left(\sin\left(2\pi\frac{kj}{d}\right)^2 + \left(\cos\left(2\pi\frac{kj}{d}\right)-1\right)^2\right)\right. \\ &\qquad\qquad \left.\cdot 2\cdot\cos\left(2\pi\frac{k}{d}\right)\right).\end{aligned}$$

Theorem 5.17. *Let G be a finite group. Then there is an injective ring homomorphism*

$$\rho_{\mathbf{R}}^G\colon \mathrm{Rep}_{\mathbf{R}}(G) \to \mathbf{Z}\oplus\left(\bigoplus_{(H)} K_1(\mathbf{R}[WH])^{\mathbf{Z}/2}\right),$$
$$[V]\mapsto \dim_{\mathbf{R}}(V)\oplus(\rho_{\mathrm{pd}}^{\mathrm{Wh}}(DV^H))_{(H)}.$$

Proof. This homomorphism is compatible with restriction to subgroups of G for an appropriate restriction homomorphism on the right side (for the details see Lück [21]). Hence it suffices to consider the case where G is cyclic.

Let $\mathrm{Rep}_{\mathbf{R}}^{\mathrm{free}}(G)$ be the subgroup of $\mathrm{Rep}_{\mathbf{R}}(G)$ generated by all free G-representations. Let $\mathrm{res}_{G/H}^G\colon \mathrm{Rep}_{\mathbf{R}}(G/H)\to\mathrm{Rep}_{\mathbf{R}}(G)$ be restriction with the projection $G\to G/H$ for a subgroup $H\subset G$. Then we obtain the following isomorphism, provided that G is cyclic:

$$(5.18)\qquad \bigoplus_{H\subset G}\mathrm{res}_{G/H}^G\colon \bigoplus_{H\subset G}\mathrm{Rep}_{\mathbf{R}}^{\mathrm{free}}(G/H)\to\mathrm{Rep}_{\mathbf{R}}(G).$$

With respect to this splitting of the representation ring, $\rho_{\mathbf{R}}^G$ is given by an upper triangle matrix. Hence it suffices to show injectivity for the diagonal entrees, i.e., the injectivity of

$$\mathrm{Rep}_{\mathbf{R}}^{\mathrm{free}}(G)\to K_1(\mathbf{R}G)^{\mathbf{Z}/2},\qquad [V]\mapsto\rho_{\mathrm{pd}}^G(DV).$$

Consider the ring homomorphism

$$\psi_k : \mathbf{R}[\mathbf{Z}/n] \to \mathbf{C}^*$$

sending the generator to the root of unity $\exp(2\pi i \frac{k}{n})$. The set $\{V(j) \mid j \in I(n)\}$ is a basis for $\mathrm{Rep}_{\mathbf{R}}^{\mathrm{free}}(G)$. The image of $[V(j)]$ under $\psi_k \circ \rho_{\mathbf{R}}^G$ is $|\exp(2\pi i \frac{jk}{n} - 1)|^{-2}$ for $d \geq 3$, $1 \leq k < d/2$, and $1/2$ for $d = 2$. Now an application of Franz' Lemma finishes the proof (see Franz [13] and Cohen [9]). q.e.d.

We get as an immediate conclusion of Theorem 5.17 the celebrated result of deRham [30] that two orthogonal G-representations V and W are isometrically $\mathbf{R}G$-isomorphic if and only if their unit spheres are G-diffeomorphic. Similar proofs using PL-torsion (resp. analytic torsion) can be found in Rothenberg [31] and Lott-Rothenberg [19]. The result is an extension of the classification of Lens spaces, which is carried out for example in Cohen [9] and Milnor [25].

The result of deRham does not hold in the topological category. Namely, there are nonlinearly isomorphic G-representations V and W whose unit spheres are G-homeomorphic (see Cappell-Shaneson [7]). However, if G has odd order, G-homeomorphic implies G-diffeomorphic for unit spheres in G-representations as shown by Hsiang-Pardon [16] and Madsen-Rothenberg [24].

The sum formula (Proposition 3.11) implies a local formula for Poincaré torsion. For this purpose we have to recall a different notion of equivariant Euler characteristic as defined in Lück [20]. Given a G-space X, denote by $\{G/? \to X\}$ the set of all G-maps $G/H \to X$ for all subgroups H of G. $x: G/H \to X$ and $y: G/K \to X$ are called to be equivalent if there is a G-isomorphism $\sigma: G/H \to G/K$ such that $y \circ \sigma$ and x are G-homotopic. Let $\{G/? \to X\}/\sim$ be the set of equivalence classes under this equivalence relation on $\{G/? \to X\}$. Given a G-map $x: G/H \to X$, define $X^H(x)$ to be the component of X^H containing the point $x(eH)$. There is a bijection

$$(5.19)\quad \{G/? \to X\}/\sim \to \coprod_{(H)} \pi_0(X^H)/WH, \quad [x: G/H \to X] \mapsto [X^H(x)],$$

where the coproduct runs over the set of conjugacy classes (H) of subgroups of G. Let $U^G(X)$ be the free abelian group generated by $\{G/? \to X\}/\sim$. Let $X^{>H}$ be the subset of those points in X, whose isotropy group G_x satisfies $G_x \supset H$, $G_x \neq H$, and $WH(x)$ is the isotropy group of $[X^H(x)] \in \pi_0(X^H)$ under the WH-action. Given a pair (X, Y) of finite G-CW-complexes, define its *universal equivariant Euler characteristic* in the sense of Lück [20]

$$\chi^G_{\text{univ}}(X) \in U^G(X) \tag{5.20}$$

by assigning to $x\colon G/H \to X$ the integer which is given by the ordinary Euler characteristic $\chi(X^H(x)/WH(x), (X^H(x) \cap (X^{>H} \cup Y))/WH(x))$. It is connected to the equivariant Euler characteristic defined in (3.12) by the following map:

$$\theta^G(X)\colon U^G(X) \to \mathrm{Rep}_{\mathbf{R}}(G), \qquad [x\colon G/H \to X] \mapsto [\mathbf{R}[G/H]]. \tag{5.21}$$

From the universal property of χ^G_{univ} and the sum formula (Proposition 3.49) for χ^G (see Lück [20]) we derive the following lemma.

Lemma 5.22. $\theta^G(X)(\chi^G_{\text{univ}}(X, Y)) = \chi^G(X, Y)$.

Let (M, M_1, M_2) be a G-manifold triad, and V be an orthogonal $D^G(M)$-representation. Given $x\colon G/H \to X$, we define the H-representation x^*V by restricting V to H by

$$H = E^H\{\text{point}\} \xrightarrow{D^H(x|_{eH})} D^H(\mathrm{res}^G_H(X)) \hookrightarrow D^G(X).$$

Let the homomorphism

$$\Phi^G(M; V) : U^G(M) \to K_1(\mathbf{R}G)^{\mathbf{Z}/2} \tag{5.23}$$

send $[x : G/H \to X]$ to $\mathrm{ind}^G_H(\rho^H_{\text{pd}}(D(TM_x)) \otimes_{\mathbf{R}} x^*V)$, where TM_x is the tangent space of M at the point $x(eH)$, and $\otimes_{\mathbf{R}}$ is the pairing defined in (2.14). Now we obtain a local formula for the Poincaré torsion in terms of the various tangent representations of the components of the fixed point sets.

Proposition 5.24 (Local formula for Poincaré torsion).

$$\Phi^G(M; V)(\chi^G_{\text{univ}}(M, M_1)) = \rho^G_{\text{pd}}(M, M_1; V).$$

Proof. In the sequel we do not have to worry about corners and straightening the angle because ρ^G_{pd} depends only on the simple homotopy type by Proposition 3.33. Moreover, we assume $V = \mathbf{R}$ for simplicity; the general case is done similarly. Let N_1 and N_2 be G-manifolds, and let $N_0 \subset \partial N_1$ and $N_0' \subset \partial N_2$ be submanifolds of codimension 0. Let $f\colon N_0 \to N_0'$ be a G-diffeomorphism and put $N := N_1 \cup_f N_2$. Then, from the sum formula (Proposition 3.49), we get

$$\rho^G_{\text{pd}}(N) = \rho^G_{\text{pd}}(N_1) + \rho^G_{\text{pd}}(N_2) - \rho^G_{\text{pd}}(N_0). \tag{5.25}$$

Since the equivariant Euler characteristic is additive in the sense of Lück [20], we have

$$\begin{aligned}(5.26)\qquad \Phi^G(N)(\chi^G_{\mathrm{univ}}(N)) &= \Phi^G(N_1)(\chi^G_{\mathrm{univ}}(N_1)) + \Phi^G(N_2)(\chi^G_{\mathrm{univ}}(N_2)) \\ &\quad - \Phi^G(N_2)(\chi^G_{\mathrm{univ}}(N_2)).\end{aligned}$$

We first verify the claim for empty M_1. We begin with the special case $M = D\xi$, where $\xi \downarrow B$ is a G-vector bundle over a G-manifold X having precisely one orbit type, say G/H. Then X/G is a manifold, and we use induction on the number of handles. If X/G is empty the claim is trivial. Suppose that X is obtained from Y by attaching an equivariant handle:

$$\begin{array}{ccc} G/H \times S^{k-1} \times D^{d-k} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ G/H \times D^k \times D^{d-k} & \longrightarrow & X \end{array}$$

We may suppose that X is connected, otherwise treat each component separately. Let an H-representation W be the typical fiber of $\xi \downarrow X$. Then we obtain a G-pushout:

$$\begin{array}{ccc} G \times_H (S^{k-1} \times D^{d-k} \times DW) & \longrightarrow & D\xi \downarrow Y \\ \downarrow & & \downarrow \\ G \times_H (D^k \times D^{d-k} \times DW) & \longrightarrow & D\xi \downarrow X \end{array}$$

By the induction hypothesis the claim is true for $D\xi \downarrow Y$. Because of (5.25) and (5.26) it suffices to verify the claim for G-manifolds of the shape $G \times_H (Z \times DW)$ for Z a manifold with trivial H-action. One easily computes

$$\begin{aligned}\rho^G_{\mathrm{pd}}(G \times_H (Z \times DW)) &= \mathrm{ind}^G_H(\rho^H_{\mathrm{pd}}(Z \times DW)) \\ &= \mathrm{ind}^G_H(\rho^H_{\mathrm{pd}}(Z) \otimes_{\mathbf{R}} \chi^H(DW) + \chi^H(Z) \otimes_{\mathbf{R}} \rho^H_{\mathrm{pd}}(DW)) \\ &= \chi(Z) \cdot \mathrm{ind}^G_H(\rho^H_{\mathrm{pd}}(DW)) \\ &= \Phi^G(G \times_H (Z \times DW))(\chi^G(G \times_H (Z \times DW))).\end{aligned}$$

This finishes the proof for the case $M = D\xi$, $M_1 = \varnothing$. Now the claim for $M_1 = \varnothing$ follows by induction on the orbit types of M. Namely, choose $H \in \mathrm{Iso}(M)$ such that $H \subset K$, $K \in \mathrm{Iso}(M)$ implies $H = K$. Let ν be the normal bundle of $M^{(H)} = G \cdot M^H$ in M. Define $\overline{M} := M - \mathrm{int}(D\nu)$. The induction hypothesis applies to $\overline{M}$ and $S\nu$, and the considerations above apply to $D\nu$. Since M is $\overline{M} \cup_{S\nu} D\nu$, the assertion follows from (5.25) and (5.26), provided that $M_1 = \varnothing$.

The claim in general follows from the following relations:

$$\rho^G_{\mathrm{pd}}(M, M_1) = \rho^G_{\mathrm{pd}}(M) - \rho^G_{\mathrm{pd}}(M_1),$$

$$\Phi^G(M)(\chi^G(M, M_1)) = \Phi^G(M)(\chi^G(M)) - \Phi^G(M_1)(\chi^G(M_1)).$$

Example 5.27. Suppose that the G-manifold M is modelled upon the G-representation V, i.e., there is a G-representation V such that for any $x \in M$ the G_x-representations $\operatorname{res} V$ and TM_x are linearly G_x-isomorphic. This is true for example if all fixed point sets of M are connected and nonempty. Then the local formula for Poincaré torsion (Proposition 5.24) reduces to

$$\rho^G_{\mathrm{pd}}(M, M_1; \mathbf{R}) = \chi^G(M, M_1) \otimes_{\mathbf{R}} \rho^G_{\mathrm{pd}}(DV).$$

Let $\operatorname{Lef}(l(g))$ be the Lefschetz index of the map $l(g)\colon (M, M_1) \to (M, M_1)$ given by multiplication with $g \in G$. Then, in terms of class functions on G with values in $\mathbf{R}$, we obtain

$$\Gamma_1(\rho^G_{\mathrm{pd}}(M, M_1; \mathbf{R}))(g) = \operatorname{Lef}(l(g)) \cdot \Gamma_1(\rho^G_{\mathrm{pd}}(DV))(g).$$

Remark 5.28. In Connolly-Lück [10] a duality formula is established for a G-homotopy equivalence $(f, \partial f)\colon (M, \partial M) \to (N, \partial N)$. There appears a correction term which depends on the universal Euler characteristic and the G_x-representations TM_x for all $x \in M$. It is closely related first to the local formula for Poincaré torsion (Proposition 5.24), and second to Proposition 3.33 which states that the difference of the Poincaré torsion is an obstruction for a duality formula for equivariant Whitehead torsion. The duality formula is important for the proof of the equivariant π-π-theorem in the simple category (see Dovermann-Rothenberg [12] and Lück-Madsen [22], [23]).

Remark 5.29. The Euler characteristic term in our main Theorem 4.5 may also be interpreted as the index of the deRham complex. This leads to the following question.

Let P^* be an elliptic complex of partial differential operators. Denote by $\Delta(P)^*$ the associated Laplacian, which is an elliptic nonnegative selfadjoint partial differential operator in each dimension, and therefore whose analytic torsion $\rho_{\mathrm{an}}(\Delta(P)^*)$ can be defined as for the ordinary Laplace operator. Suppose that the complex restricts on the boundary of M to an elliptic complex ∂P^* in an appropriate sense. Can one find a more or less topological invariant $\rho_{\mathrm{top}}(P^*)$ such that the following equation holds?

$$\rho_{\mathrm{an}}(P^*) = \rho_{\mathrm{top}}(P^*) + \frac{\ln(2)}{2} \cdot \operatorname{index}(\partial P^*).$$

If we take P^* to be the deRham complex and put ρ_{top} to be ρ_{pl}, the above equation just becomes Corollary 5.1.

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