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THE HEAT TRACE ON SINGULAR ALGEBRAIC THREEFOLDS

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1. Introduction

Let X be a complex projective algebraic threefold with isolated singularity set Σ. Consider the Laplacian $\overline{\Delta} = \overline{\delta d}$ with respect to the induced Fubini-Study metric on the noncompact smooth locus $X - \Sigma$ acting on square integrable functions. In [7], we showed that $\overline{\delta} = \overline{d}_0^* = \overline{d}^*$, which implied the selfadjointness of the Laplacian $\overline{\Delta}$. The main result of this paper is

1.1. **Theorem.** *The trace of the heat operator* $\overline{e}^{t\overline{\Delta}}$ *is finite and satisfies*

$$\operatorname{Tr} e^{-t\overline{\Delta}} \leq K t^{-3}$$

for $t \in (0, T]$, *suitable* $T > 0$, *and* $K > 0$.

1.2. **Remarks.** The corresponding facts for curves and surfaces are respectively due to Cheeger [2], [3] and Nagase [6].

2. Reduction to local problems

Let X, Σ be as above. Then by the main results of §2,3 of [7], we may decompose

$$(1) \qquad X - \Sigma = M \cup \left(\bigcup_{\alpha=1}^{m} W_\alpha^b \right),$$

where $M = \{x \in X - \Sigma : d(x, \Sigma) \geq b\}$ for some fixed $b \in (0, 1)$, and the W_α^b are sets of the type W_{I}^b, W_{II}^b, W_{III}^b, which were introduced in [7, §2, 3]. Similarly, the ε-truncation X_ε of X is defined as

$$(2) \qquad X_\varepsilon = \{x \in X - \Sigma : d(x, \Sigma) \geq \varepsilon\} = M \cup \left(\bigcup_{\alpha=1}^{m} W_\alpha^b(\varepsilon) \right),$$

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where $W^b_\alpha(\varepsilon) = W^b_\alpha \cap \{r \geq \varepsilon\}$, r being the local radial distance function from the singular set. Clearly for

$$Y_\varepsilon = \bigcup_{\alpha=1}^{m} W^b_\alpha(\varepsilon)$$

we have

$$\partial Y_\varepsilon = \partial M \cup \partial X_\varepsilon, \tag{3}$$

$$\partial W^b_\alpha(\varepsilon) = \partial_0 W^b_\alpha(\varepsilon) \cup \partial_1 W^b_\alpha(\varepsilon), \tag{4}$$

where ∂_0 denotes the $\{r = \varepsilon\}$ part of the boundary $\partial W^b_\alpha(\varepsilon)$ of $W^b_\alpha(\varepsilon)$, and ∂_1 denotes the rest. Clearly $\partial_0 W^b_\alpha(\varepsilon) = \partial W^b_\alpha(\varepsilon) \cap \partial X_\varepsilon$.

2.1. **Lemma.** *Let Δ_ε be the Laplacian in the induced Fubini-Study metric on the ε-truncation X_ε, with Dirichlet boundary conditions on ∂X_ε, and let*

$$0 \leq \lambda_0(\varepsilon) \leq \lambda_1(\varepsilon) \leq \cdots \leq \lambda_i(\varepsilon) \cdots$$

be the eigenvalues of this selfadjoint boundary value problem, arranged in ascending order. Also let

$$0 \leq \mu_0(\varepsilon) \leq \mu_1(\varepsilon) \leq \cdots \mu_i(\varepsilon) \leq \cdots$$

be the eigenvalues of the operator Δ_M on M (with vanishing Neumann data on ∂M), and the operator Δ_{Y_ε} on the manifold Y_ε (with vanishing Dirichlet data on the ∂X_ε part of ∂Y_ε and vanishing Neumann data on the ∂M part of ∂Y_ε), all taken together and arranged in ascending order, with multiplicity if the same eigenvalue arises from two different regions. Then

$$\lambda_i(\varepsilon) \geq \mu_i(\varepsilon) \quad \forall i.$$

Proof. This is a standard fact, following from the Weyl-Courant min-imax characterisation of eigenvalues. See, e.g., Chapter 1, §5 of Chavel's book [1], and Proposition 3.2 in [6].

2.2. **Corollary.**

$$\operatorname{Tr}(e^{-t\Delta_\varepsilon}) \leq \operatorname{Tr}(e^{-t\Delta_M}) + \operatorname{Tr}(e^{-t\Delta_{Y_\varepsilon}}).$$

It is well known (by the Weyl asymptotic formula) that for the compact 6-dimensional Riemannian manifold M with boundary and Neumann conditions, we have

$$\operatorname{Tr}(e^{-t\Delta_M}) \leq Kt^{-3}$$

for a $K > 0$ and $t \in (0, T]$. Thus, in view of the above corollary, it is sufficient to prove that for the operator Δ_{Y_ε} with the boundary conditions stated in Lemma 2.1, we have the estimate

$$\operatorname{Tr}(e^{-t\Delta_{Y_\varepsilon}}) \leq Kt^{-3}, \qquad t \in (0, T],$$

where $T > 0$, and K is a positive constant *independent of* ε.

We now have to estimate the heat trace for Y_ε, in terms of heat traces for the $W^b_\alpha(\varepsilon)$. To do this we shall need the following lemma, which is a general eigenvalue comparison result similar in spirit to the lemma in Chavel cited above, but independent of it.

2.3. **Lemma.** *Let Y be a manifold with boundary $\partial Y = \partial_0 Y \cup \partial_1 Y$, and let $\{W_\alpha\}_{\alpha=1}^m$ be a finite covering of it by m normal domains, not necessarily disjoint, with boundaries meeting ∂Y transversely. Let*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

be the eigenvalues of the Laplacian Δ_Y with the mixed boundary data of Dirichlet on $\partial_0 Y$ and Neumann on $\partial_1 Y$. Further, let

$$0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots$$

be the eigenvalues of the Laplacians Δ_α on W_α with the original Y data on $W_\alpha \cap \partial Y$ and Neumann data on $\partial W_\alpha \cap (Y - \partial Y)$, arranged in ascending order, with repetition in case the same eigenvalue occurs with multiplicity, or from different W_α. Then

$$\lambda_k \geq \frac{1}{m}\mu_k$$

for all k.

Proof. We suitably modify the proof of Corollary 1 on p. 18 of Chavel [1]. Let $\Psi_1, \cdots, \Psi_{k-1}$ be the eigenfunctions corresponding to the eigenvalues $\mu_1, \cdots, \mu_{k-1}$ of the problems on W_α, $\alpha = 1, 2, \cdots, m$, stated above. Extending these by 0 to all of Y makes them admissible functions for the eigenvalue problem on Y. (See the corollary in [1] cited above for the definition of admissible.) Now let f be an admissible function for the problem on Y, which is orthogonal to the functions $\Psi_1, \cdots, \Psi_{k-1}$ in the Hilbert space $H(Y)$ of Y-admissible functions. Then f_α, the restriction of f to W_α, is in the admissible space $H(W_\alpha)$. Clearly

$$m(Df, Df)_Y \geq \sum_{\alpha=1}^{m}(Df, Df)_{W_\alpha} = \sum_{\alpha=1}^{m}(Df_\alpha', Df_\alpha),$$

where the subscripts on L^2-norms always denote the domains of integration of the pointwise norm. Since f is orthogonal to $\Psi_1, \cdots, \Psi_{k-1}$, for each α, f_α is orthogonal to those of the Ψ_j $(1 \le j \le k-1)$, which arise from W_α for each $\alpha = 1, \cdots, m$. Thus $(Df_\alpha, Df_\alpha) \ge \nu_\alpha(f_\alpha, f_\alpha)$, where ν_α is the lowest eigenvalue for the W_α problem succeeding the eigenvalues of W_α, which appear among the $\mu_1, \cdots, \mu_{k-1}$ above, for each α. By the definition of μ_k, we have $\nu_\alpha \ge \mu_k$ for all $\alpha = 1, \cdots, m$. Combining the above inequalities gives

$$m(Df, Df)_Y \ge \mu_k \sum_{\alpha=1}^{m} (f_\alpha, f_\alpha) = \mu_k \sum_{\alpha=1}^{m} (f, f)_{W_\alpha} \ge \mu_k (f, f)_Y .$$

Now, since the $\Psi_1, \cdots, \Psi_{k-1}$ span a subspace E of $H(Y)$ of dimension at most $k-1$, we have

$$\frac{\mu_k}{m} \le \min_{f \perp E} \frac{(Df, Df)}{(f, f)} \le \max_{\dim E \le k-1} \min_{f \perp E} \frac{(Df, Df)}{(f, f)} = \lambda_k$$

by the Weyl-Courant minimax characterization of eigenvalues.

2.4. **Corollary.** *If the heat-trace estimates for the Laplacians* $\Delta_{\alpha\varepsilon}$ *on* $W_\alpha^b(\varepsilon)$, $\alpha = 1, \cdots, m$, *with the boundary conditions defined in Lemma* 2.3 *satisfy the estimate*

$$\operatorname{Tr}(e^{-t\Delta_{\alpha\varepsilon}}) \le K_\alpha t^{-3} \quad \textit{for } t \in (0, T] \textit{ and some } K_\alpha > 0,$$

then the heat trace estimate for Δ_{Y_ε} *satisfies*

$$\operatorname{Tr}(e^{-t\Delta_{Y_\varepsilon}}) \le K t^{-3} \quad \textit{for } t \in (0, T'] \textit{ and some } K > 0.$$

Proof. We take $Y = Y_\varepsilon$ and $W_\alpha = W_\alpha^b(\varepsilon)$ in Lemma 2.3. Then, letting $K' = \max_\alpha K_\alpha$ we have for $t \in (0, mT]$ that

$$\operatorname{Tr}(e^{-t\Delta_{Y_\varepsilon}}) = \sum e^{-t\lambda_i} \le \sum e^{-t\mu_i/m} \le K' \left(\frac{t}{m}\right)^{-3} = K t^{-3}$$

from Lemma 2.3, where $K = K' m^3$.

Thus the problem now boils down to analyzing the $W_\alpha^b(\varepsilon)$. We will do this for the three types of $W_\alpha^b(\varepsilon)$ regions in the next section. This would establish Theorem 1.1 in view of the fact that

$$\operatorname{Tr}(e^{-t\bar{\Delta}}) = \lim_{\varepsilon \to 0} \operatorname{Tr}(e^{-t\Delta_\varepsilon})$$

as in (1.3), (1.4) of [5].

3. The estimates for the $W_\alpha^b(\varepsilon)$

For convenience, we take $b \le 1/e$ in all that follows, as we did in [7]. In any case, this b is completely immaterial, and fixed (less than 1) right at the outset.

3.1. **Proposition.** *For $W_\alpha^b(\varepsilon) = W_{\mathrm{I}}^b(\varepsilon)$ of type-I, we have the heat trace estimate (with the mixed boundary conditions stated in Lemma 2.1)*

$$\mathrm{Tr}(e^{-t\Delta_\varepsilon}) \le Kt^{-3}, \qquad t \in (0, T], \tag{5}$$

where K is a constant independent of ε.

Proof. We recall the following from 3.1.3 of [7] to define the regions W_{I}^b. It is shown in Propositions 2.2.2, 2.3.1 of [7] that for a simple-point p on the singular divisor $E = \pi^{-1}(0)$ (for a sufficiently high resolution π of the singularity $\pi\colon \widetilde{X} \to X$ as constructed in §2 of [7]), there is a (u, v, w) polydisc neighborhood U of $p = (0, 0, 0)$ such that

(i) $U \cap E = \{u = 0\}$,

(ii) the pullback $\pi^*(g)$ of the Fubini-Study metric g on $X - \{0\}$ is quasi-isometric on $U - E$ to $\sum_{i=1}^3 d\zeta_i\, d\overline{\zeta}_i$, where $\zeta_1 = u_1^{a_1}$, $\zeta_2 = u_2^{a_2}v$, and $\zeta_3 = u^{a_3}w$; $a_3 \ge a_2 \ge a_1 \ge 1$ are positive integers.

We define

$$W_{\mathrm{I}}^b = \{0 < r < b\} \cap U,$$

where r is the pullback of the radial distance function from the origin in $\mathbb{C}^N$ (the germ of the isolated singularity being embedded with the origin as the singular point) restricted to $X - \{0\}$. The metric of (ii) above further (quasi-isometrically) simplifies to the expression in (6) below with $r_1 = |\zeta_1|$ in place of r. However, the same proof as that of Lemma 3.3 of [5] shows that the quasi-isometry type is unaltered by taking r in place of r_1, which results in (6) below. Thus

$$W_{\mathrm{I}}^b = (0 < r < b) \times S^1 \times Y_1 \times Y_2,$$

where $\theta = \arg \zeta_1$ is a local coordinate on the S^1 factor, and Y_1 and Y_2 are the unit discs ($|v| \le 1$) and ($|w| \le 1$) respectively. We also proved in the section of [7] cited above that the induced Fubini-Study metric in this region W_{I}^b is quasi-isometric to the metric

$$g = dr^2 + r^2\, d\vartheta^2 + r^{2\alpha}(dx_1^2 + dy_1^2) + r^{2\beta}(dx_2^2 + dy_2^2), \tag{6}$$

where $v = x_1 + iy_1$, $w = x_2 + iy_2$, and $1 \le \alpha = a_2/a_1 \le \beta = b_2/b_1$, the a_i and b_i being as in Proposition 2.2.2 of [7].

The Laplacian corresponding to the metric g in (6) is easily seen to be

$$\begin{aligned} \Delta &= (\sqrt{g})^{-1} \sum_i \partial_i(\sqrt{g}\, g^{ij}\partial_j) \\ &= \frac{\partial^2}{\partial r^2} + \frac{2c+1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + r^{-2\alpha}\Delta_1 + r^{-2\beta}\Delta_2, \end{aligned} \tag{7}$$

where $c = \alpha + \beta$, and Δ_i are the standard Euclidean Laplacians on the discs Y_i, $i = 1, 2$.

Since quasi-isometries preserve the trace estimate which we are seeking (called the basic property BP in [6, §1]), it is enough to show that

$$\sum_{i=1}^{m} e^{-t\lambda_i(\varepsilon)} \leq K t^{-3} \quad \text{for } t \in (0, T], \text{ and } K \text{ independent of } \varepsilon ,$$

where $\lambda_i(\varepsilon)$ are the eigenvalues of the equation

(8a) $$\Delta f + \lambda f = 0,$$

Δ being given by (7) and the mixed boundary conditions

(8b) $$f = 0 \text{ on } \partial_0 W_{\mathrm{I}}^b(\varepsilon) \quad \text{and} \quad \partial_\nu f = 0 \text{ on } \partial_1 W_{\mathrm{I}}^b(\varepsilon),$$

where ∂_ν is the normal derivative.

The way to proceed now is to separate variables. Let $\{\Phi_i\}$, $\{\Psi_j\}$ be the eigenfunctions of Δ_1 and Δ_2, with vanishing Neumann data on ∂Y_1 and ∂Y_2 respectively, corresponding to the eigenvalues $\{\nu_i\}$ and $\{\mu_j\}$ respectively. Also let $\{x_k(\vartheta)\}$ be the eigenfunctions of $\partial^2/\partial\vartheta^2$ on S^1, with corresponding eigenvalues $\{\eta_k\}$. Then expanding a function f as a product $G(r)\Phi_i\Psi_j x_k$, and requiring it to be a λ-eigenfunction of (8a, b) leads to the Sturm-Liouville boundary value problem on $[\varepsilon, b]$ given by the differential equation

(9a) $$G'' + \left(\frac{2c+1}{r}\right) G' + \left(\lambda - \frac{\nu_i}{r^{2\alpha}} - \frac{\mu_j}{r^{2\beta}} - \frac{\eta_k}{r^2}\right) G = 0$$

(primes denote r-derivatives), and the boundary conditions above dictate the boundary conditions on G to be

(9b) $$G(\varepsilon) = \frac{dG}{dr}(b) = 0.$$

This can be recast, by putting $H = r^{c+1/2}G$, yielding

(10a) $$H'' + (\lambda - q_{ijk}(r))H = 0,$$

(10b) $$H(\varepsilon) = \frac{d}{dr}(r^{-c-1/2}H)(b) = 0,$$

where

(11) $$q_{ijk}(r) = \frac{\nu_i}{r^{2\alpha}} + \frac{\mu_j}{r^{2\beta}} + \frac{\eta_k}{r^2} + \frac{(c^2 - 1/4)}{r^2}.$$

Let us arrange the eigenvalues of (10a,b) in ascending order

(12) $$\lambda_{ijk0}(\varepsilon) \leq \lambda_{ijk1}(\varepsilon) \leq \cdots \leq \lambda_{ijkl}(\varepsilon) \leq \cdots .$$

These can in turn be compared with the eigenvalues of a simpler problem as follows. In (10a), replace the $q_{ijk}(r)$ by the number $p_{ijk} = \nu_i+\mu_j+\eta_k-\zeta_0$, where $\zeta_0 = \lim_{\varepsilon\to 0}\zeta_0(\varepsilon) \le \zeta_0(\varepsilon)$, and $\{\zeta_l(\varepsilon)\}$ are the eigenvalues of the problem on $[\varepsilon, b]$ given by

$$H'' + \zeta H = 0, \tag{13a}$$

$$H(\varepsilon) = \frac{d}{dr}(r^{-c-1/2}H)(b) = 0 \tag{13b}$$

arranged in ascending order of l, and the $\zeta_l = \lim_{\varepsilon\to 0}\zeta_l(\varepsilon) \le \zeta_l(\varepsilon)$ are the eigenvalues of the limiting problem as $\varepsilon \to 0$, viz., the eigenvalues of (13a) on the interval $(0, b]$, and the boundary conditions being (13b′) which is (13b) with ε replaced by 0. We note here (cf. [6, 4.2]) that $\zeta_0 < 0 < \zeta_1$, and the $\zeta_l(\varepsilon)$ diminish monotonically, as $\varepsilon \to 0$, to ζ_l.

Now if the eigenvalues of the problem, which we get by replacing $q_{ijk}(r)$ by the p_{ijk} defined above, given by the equations called (14a), (14b) respectively, are denoted by

$$0 \le \tilde{\lambda}_{ijk0}(\varepsilon) \le \tilde{\lambda}_{ijk1}(\varepsilon) \le \cdots \le \tilde{\lambda}_{ijkl}(\varepsilon) \le \cdots, \tag{15}$$

then a comparison of the Rayleigh-Ritz quotients of (10a, b) and (14a, b) shows that

$$\lambda_{ijkl}(\varepsilon) \ge \tilde{\lambda}_{ijkl}(\varepsilon) + \zeta_0 \tag{16}$$

since $q_{ijk} \ge p_{ijk} + \zeta_0$. However, since $\tilde{\lambda}_{ijkl}(\varepsilon) = \zeta_l(\varepsilon) + \nu_i + \mu_j + \eta_k - \zeta_0$, it follows that

$$\lambda_{ijkl}(\varepsilon) \ge \nu_i + \mu_j + \eta_k + \zeta_l(\varepsilon) \ge \nu_i + \mu_j + \eta_k + \zeta_l, \tag{17}$$

so that

$$\operatorname{Tr}(e^{-t\Delta_\varepsilon}) \le \left(\sum_i e^{-i\nu_i}\right)\left(\sum_j e^{-t\mu_j}\right)\left(\sum_k e^{-t\eta_k}\right)\left(\sum_l e^{-t\zeta_l}\right). \tag{18}$$

Since it is well known that ν_i and μ_j have linear growth in i, j respectively, and η_k and ζ_l have quadratic growth in k and l respectively, the proposition is proven.

3.2. **Remark.** The analysis above is very similar to that of $W(-)$ regions in the surface case of [6, Lemma 4.3].

3.3. **Proposition.** *For a region W_α^b of the type W_{II}^b (cf.* [7, *Proposition* 3.1.4]) *which satisfies the additional condition*

$$0 < (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < 1$$

(definitions of β_i, α_i below, after (20) and (21) in the proof), we have the trace estimate

(19) $$\mathrm{Tr}(e^{-t\Delta_\varepsilon}) \le Kt^{-3}$$

for $t \in (0, T]$, $K > 0$ independent of ε.

Proof. Let us recall some facts from [7] first. In Propositions 2.2.8, 2.3.1 we showed that for a double point p which lies at the intersection of exactly two components of the singular divisor E, there is a (u, v, w)-polydisc neighborhood U of $p = (0, 0, 0)$ such that

(i) $U \cap E = \{u = 0\} \cup \{v = 0\}$,

(ii) the pullback $\pi^* g$ (g defined in proof of Proposition 3.1 above) is quasi-isometric on $U - E$ to $\sum_{i=1}^{3} d\zeta_i d\bar{\zeta}_i$, where $\zeta_1 = u^{a_1}v^{b_1}$, $\zeta_2 = u^{a_2}v^{b_2}$, $\zeta_3 = u^{a_3}v^{b_3}w$, $a_3 \ge a_2 \ge a_1 \ge 1$, and $b_3 \ge b_2 \ge b_1 \ge 1$ are positive integers. One then defines

$$W_{\mathrm{II}}^b = \{0 < r < b\} \cap U,$$

where r is the radial distance function from 0 on $X - \{0\}$ pulled back to U as described in the beginning of the proof of 3.1 above.

There is also the type-B operation (see [7, §2.1], and Proposition 2.2.8) which blows up the w-axis ($u = v = 0$), viz. the global curve of intersection of the two divisor components in question on which p lies. This results in some W_{I}^b regions of the type discussed in 3.1 above, and two new type W_{II}^b regions with the same quasi-isometry class of metric described in (ii) of the above, but with *changed exponents*: viz. (a_i, b_i) are respectively replaced by the exponents $(a_i + b_i, b_i)$ and $(a_i, a_i + b_i)$ in the two new regions. This fact will be exploited in the proof of the Lemma 3.4 below.

A further (quasi-isometric) simplification of the metric in (ii) above can be achieved by introducing the new real coordinate

$$s = \frac{a_2|\log\rho| + b_2|\log\tau|}{a_1|\log\rho| + b_1|\log\tau|},$$

where $p = |u|$ and $\tau = |v|$, whose range is given below in (22). With this new coordinate, and the other coordinates defines in (20) below, one sees (cf. [7, proof of Proposition 3.1.4]), that W_{II}^b may be written as the product

(20) $$W_{\mathrm{II}}^b = (0 < r < b) \times (\alpha_1 < s < \beta_1) \times T^2 \times D^2,$$

where $1 \le \alpha_1 = a_2/a_1 < \beta_1 = b_2/b_1$. Let $\vartheta_i = \arg\zeta_i$ ($i = 1, 2$) be local coordinates on the torus factor T^2, and $D^2 = \{w \in \mathbb{C}: |w| \le 1\}$. The induced Fubini-Study metric in this region, which is quasi-isometric to the expression in (ii) above, can be further quasi-isometrically simplified, as shown in the proof of Proposition 3.1.4 of [7] cited above, to

$$g = dr^2 + r^2\, d\vartheta_1^2 + r^{2s}((\log r)^2\, ds^2 + d\vartheta_2^2) + r^{2(\lambda_1 + s\lambda_2)}\, dw\, d\overline{w}, \tag{21}$$

where the λ_i's that occur in the r-exponent of the last term are defined by $(a_3, b_3) = \lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)$, recalling that (a_1, b_1) and (a_2, b_2) are linearly independent, and if we let

$$\alpha_2 = \min\left(\frac{a_3}{a_1}, \frac{b_3}{b_1}\right) \quad \text{and} \quad \beta_2 = \max\left(\frac{a_3}{a_1}, \frac{b_3}{b_1}\right), \tag{22}$$

we have (loc. cit.) that for $s \in [\alpha_1, \beta_1]$, $(\lambda_1 + \lambda_2 s) \in [\alpha_2, \beta_2]$. Actually, (21) should contain the variable r_1 in place of r, but again by the same argument referred to in the proof of 3.1 above (Lemma 3.3 of [5]), we can replace r_1 by r.

The idea is to compare the Rayleigh-Ritz quotient for the Laplacian Δ_g of the metric in (21) with that of a simpler operator. The energy form for Δ_g is

$$\begin{aligned} E(f, f) &= \int (df, df)_g\, dV_g \\ &= \int (p_1(f_r)^2 + p_2(f_{\vartheta_1})^2 + p_3(f_s)^2 + p_4(f_{\vartheta_2})^2 \\ &\qquad + p_5(f_x)^2 + p_6(f_y)^2)\, dr\, d\vartheta_1\, ds\, d\vartheta_2\, dx\, dy, \end{aligned} \tag{23}$$

where the subscripts of f denote partial derivatives, $w = x + iy$, and $p_i = g^{ii}\sqrt{g}$. Also $\sqrt{g} = r^{2(s+\lambda_1+s\lambda_2)+1}|\log r|$. By (22), i.e., the bounds on s and $\lambda_1 + \lambda_2 s$, and by the condition that $0 < r \le b < 1$, we have the following inequalities on W_{II}^b:

$$\begin{aligned} &\sqrt{g} \le r^{2(\alpha_1+\alpha_2)+1}|\log r| = q_2 \quad \text{(say)}, \\ &p_1 = g^{11}\sqrt{g} \ge r^{2(\beta_1+\beta_2)+1}|\log r| = q_1 \quad \text{(say)}, \\ &p_2 = g^{22}\sqrt{g} = r^{-2}\sqrt{g} \ge r^{2(\beta_1+\beta_2)-1}|\log r| \ge q_2, \end{aligned}$$

since by the hypothesis of this proposition, $2(\beta_1 + \beta_2) - 1 < 2(\alpha_1 + \alpha_2) + 1$,

$$p_3 = g^{33}\sqrt{g} = r^{-2s}|\log r|^{-2}\sqrt{g} = r^{2(\lambda_1+s\lambda_2)+1}|\log r|^{-1}.$$

But,

$$\begin{aligned} \delta &= (2(\alpha_1 + \alpha_2) + 1) - (2(\lambda_1 + s\lambda_2) + 1) \ge 2(\alpha_1 + \alpha_2) - 2\beta_2 \\ &> 2\alpha_1 + 2(\beta_1 - \alpha_1) - 2 \quad \text{(by our hypothesis)} \\ &= 2\beta_1 - 2 > 0. \end{aligned}$$

Thus $r^{\delta}|\log r|^2 < 1$ (b suitably chosen), and consequently

$$p_3 \geq q_2,$$
$$p_4 = r^{-2s}\sqrt{g} \geq p_3 \geq q_2 \quad \text{by the above,}$$
$$p_5 = g^{55}\sqrt{g} = r^{-2(\lambda_1+s\lambda_2)}\sqrt{g} = r^{2s+1}|\log r|.$$

Since $1-\beta_2 \leq 0$,

$$2s+1 \leq 2\beta_1 + 1 \leq 2(\alpha_1+\alpha_2-\beta_2+1)+1$$
$$\leq 2(\alpha_1+\alpha_2)+1$$

by our hypothesis, which means $p_5 \geq q_2$. Finally, $p_6 \geq q_2$ since $p_6 = p_5$.

Thus, the Rayleigh-Ritz quotient of Δ_g on $W^b_{\mathrm{II}}(\varepsilon)$ is

$$\text{(24)} \qquad \frac{E(f,f)}{\int f^2\sqrt{g}\,dV} \geq \frac{\int (q_1(f_r)^2 + q_2(f^2_{\vartheta_1} + f^2_s + f^2_{\vartheta_2} + f^2_x + f^2_y))\,dV}{\int f^2 q_2\,dV},$$

where $dV = dr\,d\vartheta_1\,ds\,d\vartheta_2\,dx\,dy$, and E is given by (23).

But the right-hand side of (24) is the Rayleigh-Ritz quotient of the differential equation on $W^b_{\mathrm{II}}(\varepsilon)$ given by

$$\text{(25a)} \qquad \begin{aligned} &\frac{\partial}{\partial r}\left(q_1\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \vartheta_1}\left(q_2\frac{\partial f}{\partial \vartheta_1}\right) + \frac{\partial}{\partial s}\left(q_2\frac{\partial f}{\partial s}\right) + \frac{\partial}{\partial \vartheta_2}\left(q_2\frac{\partial f}{\partial \vartheta_2}\right) \\ &\quad + \frac{\partial}{\partial x}\left(q_2\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(q_2\frac{\partial f}{\partial y}\right) + \lambda q_2 f = 0 \end{aligned}$$

with the boundary conditions unchanged; viz.,

$$\text{(25b)} \qquad f = 0 \text{ on } \partial_0 W^b_{\mathrm{II}}(\varepsilon) \quad \text{and} \quad \partial_\nu f = 0 \text{ on } \partial_1 W^b_{\mathrm{II}}(\varepsilon).$$

Since $W^b_{\mathrm{II}}(\varepsilon)$ is a product region, and q_2 only involves r, we may rewrite (25a) in the form

$$\text{(26)} \qquad \frac{1}{q_2}\frac{\partial}{\partial r}\left(q_1\frac{\partial f}{\partial r}\right) + \Delta_N f + \lambda f = 0,$$

where Δ_N is the standard Euclidean Laplacian on the "link" piece N which is the $r = \text{constant}$ slice. So

$$\text{(27)} \qquad N = T^2 \times [\alpha_1, \beta_1] \times D^2,$$

a compact manifold with boundary. If $\{\lambda_i(\varepsilon)\}$ are the eigenvalues of Δ_g on $W^b_{\mathrm{II}}(\varepsilon)$ with the boundary conditions (25b), and $\{\tilde{\lambda}_i(\varepsilon)\}$ are those of (25a, b) (all in ascending order of i), then in view of inequality (24) we see that

$$\text{(28)} \qquad \lambda_i(\varepsilon) \geq \tilde{\lambda}_i(\varepsilon) \quad \forall i.$$

By separation of variables, $\{\tilde{\lambda}_i(\varepsilon)\} = \{\zeta_j(\varepsilon) + \nu_k\}_{j,k}$, where $\{\nu_k\}$ are the eigenvalues of $\Delta_N f + \nu f = 0$ on N, with vanishing Neumann data on ∂N, and $\{\zeta_j(\varepsilon)\}$ are the eigenvalues of the one-dimensional Sturm-Liouville boundary-value problem on $[\varepsilon, b]$ given by

(29a) $$\frac{1}{q_2}\frac{d}{dr}\left(q_1\frac{df}{dr}\right) + \zeta f = 0$$

with the boundary conditions

(29b) $$f(\varepsilon) = \frac{df}{dr}(b) = 0.$$

Now, it is well known that for the compact manifold N with Neumann boundary data (N has corners, but still) the heat trace $\sum_k e^{-t\nu_k} \le Kt^{-5/2}$, for $t \in (0, T]$ and $K > 0$. So all we need to show, in view of (28), is that

(30) $$\sum_j e^{-t\zeta_j(\varepsilon)} \le Kt^{-1/2}$$

for $K > 0$ and independent of ε, and t is as above.

So we are reduced to the problem (29a, b) on the interval $[\varepsilon, b]$:

(31a) $$\frac{d}{dr}\left(p\frac{df}{dr}\right) + \zeta\rho f = 0,$$

where

$$\rho = r^{2\delta}|\log r| \quad \text{and} \quad 2\delta = 2(\alpha_1 + \alpha_2) + 1 \ge 5,$$

(32) $$p = r^{2\gamma}|\log r| \quad \text{and} \quad 2\gamma = 2(\beta_1 + \beta_2) + 1 > 2\delta \ge 5,$$

and by the hypothesis of this proposition,

(33) $$0 < \gamma - \delta < 1.$$

The boundary conditions for (31a) are

(31b) $$f(\varepsilon) = \frac{df}{dr}(b) = 0.$$

This is a standard form equation, which, by the substitutions (19a), (20a) on p. 292 of Courant-Hilbert [4], may be rewritten on a new interval $[\varepsilon', b']$ as follows:

(34a) $$\frac{d^2u}{dt^2} + (\zeta - \phi)u = 0$$

with the boundary conditions

(34b) $$u(\varepsilon') = \frac{d}{dt}(\psi^{-1}u)(b') = 0,$$

where $\psi = (p\rho)^{1/4} = r^{(\delta+\gamma)/2}|\log r|^{1/2}$; $u = \psi f$; and the new variable t is defined by

$$t = \int_0^r \sqrt{\rho/p}\,dr = \int_0^r \frac{dr}{r^{\gamma-\delta}} = Cr^{\delta-\gamma+1}, \tag{35}$$

which is valid in view of (33). Clearly $t \in [\varepsilon', b']$, where the new endpoints are $\varepsilon' = C\varepsilon^{\delta-\gamma+1}$, $b' = Cb^{\delta-\gamma+1}$.

Finally, by the footnote on p. 292 (loc. cit.),

$$\phi = \frac{\psi''}{\psi} = \frac{m(m-1)}{r^2} + \frac{1}{r^2|\log r|}\left(\frac{1}{2} - m - \frac{1}{4|\log r|}\right) \tag{36}$$

where $m = (\delta + \gamma)/2$.

Since the integrand of (35) is greater than 1, we see $r \le t$. Combining this with (36) gives

$$\phi(t) \ge \frac{C'}{t^2}, \quad \text{where } C' = m(m-1) - k,$$

and k may be made arbitrarily small by making b small enough. Since $m = (\delta+\gamma)/2 > 5/2$ by (32), we may choose b and hence k small enough so that $C' \ge \frac{15}{4}$. From this one concludes that the eigenvalues $\zeta_i(\varepsilon')$ of (34a, b) are greater than or equal to those of the following Bessel-type problem, call them $\tilde{\zeta}_i(\varepsilon')$:

$$u'' + \left(\tilde{\zeta} - \frac{2^2 - 1/4}{t^2}\right)u = 0, \tag{37a}$$

$$u(\varepsilon') = \frac{d}{dt}\left(\psi^{-1}u\right)(b') = 0 \tag{37b}$$

by a comparison of the Rayleigh-Ritz quotients. We let $\{\tilde{\zeta}_i\}$ be the eigenvalues of the limiting (singular) Bessel problem, which is (37a) together with $u(0) = 0$, and the same right-hand boundary condition as (37b). It is proved in Chapter VI, §2.4, of [4] that

(a) $\tilde{\zeta}_i = \lim_{\varepsilon\to 0} \tilde{\zeta}_i(\varepsilon) \le \tilde{\zeta}_i(\varepsilon)$.

(b) The solution to the limiting problem is $\sqrt{t}J_2(t\sqrt{\tilde{\zeta}})$.

Applying the (right-hand) boundary conditions, and the facts about zeros of Bessel's functions (loc. cit.), one has that

$$\lim_{i\to\infty} \frac{\tilde{\zeta}_i}{i^2} = C(b'), \tag{38}$$

where $C(b')$ is a constant depending only on the length of the interval, b'. As a consequence we have

$$\sum_i e^{-t\zeta_i(\varepsilon)} \leq \sum_i e^{-t\tilde{\zeta}_i(\varepsilon)} \leq \sum e^{-t\tilde{\zeta}_i} \leq Kt^{-1/2}$$

for $K > 0$, independent of ε, and $t \in (0, T]$. This proves the proposition. q.e.d.

It remains to show that the condition assumed in the hypothesis of this proposition can be realised in all the regions of the type W_{II}^b. This can be achieved by enough type-B operations (cf. [7, §2.1]), as the next lemma shows.

3.4. **Lemma.** *The condition*

$$0 < (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < 1$$

*can be realised by enough type-*B *operations, in all the* W_{II}^b *type regions.*

Proof. Recall that

$$\alpha_1 = \min\left(\frac{a_2}{a_1}, \frac{b_2}{b_1}\right), \qquad \beta_1 = \max\left(\frac{a_2}{a_1}, \frac{b_2}{b_1}\right),$$
$$\alpha_2 = \min\left(\frac{a_3}{a_1}, \frac{b_3}{b_1}\right), \qquad \beta_2 = \max\left(\frac{a_3}{a_1}, \frac{b_3}{b_1}\right),$$

so that

$$(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) = \frac{|a_1 b_2 - a_2 b_1| + |a_1 b_3 - a_3 b_1|}{a_1 b_1}. \tag{39}$$

A type-B operation creates two new charts, both of type W_{II}^b, one in which a_i are replaced by $(a_i + b_i)$, and b_i are unchanged, and the other in which a_i are unchanged and b_i are replaced by $a_i + b_i$. In either case the determinants which occur in the numerator of the expression (29) above remain unchanged, whereas the denominator strictly increases by at least one. Hence in a finite number of steps we are done.

Remark. In the above, by putting $w = 0$, we will get another proof of the heat estimate for $W(+)$ regions in the surface case of [6].

We now deal with the W_{III}^b type regions. Much of the analysis is very similar to that of the W_{II}^b type regions above, so we will make it brief.

3.5. **Proposition.** *For the regions* $W_{\mathrm{III}}^b(\varepsilon)$, *with the additional condition*

$$0 < (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < 1$$

(*see definitions below*), *we have the same trace estimate as in Proposition* 3.3.

Proof. First some introductory remarks on the regions $W^b_{\rm III}$. For a triple point p, which is a point of intersection of three components of the singular divisor E, we show in [7, Propositions 2.2.11, 2.3.1] that there is a (u, v, w) polydisc neighborhood U of $p = (0, 0, 0)$ such that

(i) $U \cap E = (u = 0) \cup (v = 0) \cup (w = 0)$,

(ii) the pullback $\pi^* g$ (g defined in Proposition 3.1) is quasi-isometric on $U - E$ to $\sum_{i=1}^3 d\zeta_i d\bar{\zeta}_i$, where $\zeta_i = u^{a_i} v^{b_i} w^{c_i}$, where $a_3 \geq a_2 \geq a_1 \geq 1$, $b_3 \geq b_2 \geq b_1 \geq 1$, $c_3 \geq c_2 \geq c_1 \geq 1$ are positive integers. We define

$$W^b_{\rm III} = \{0 < r < b\} \cap U.$$

There are again two kinds of operations on such a region. The first is a type-A operation, which is blowing up the point $p = (0, 0, 0)$. This results in

(a) regions of the type W^b_α with $\alpha = \rm I, II$ of the kind dealt with above in Propositions 3.1, 3.3, together with three new type $W^b_{\rm III}$ regions centered at three new triple points, and with changed exponents, respectively $(a_i + b_i + c_i, b_i, c_i)$ corresponding to the substitution $u \to u$; $v \to uv$; $w \to uw$, $(a_i, a_i + b_i + c_i, c_i)$ corresponding to $u \to uv$; $v \to v$; $w \to vw$ and $(a_i, b_i, a_i + b_i + c_i)$ replacing the original exponents (a_i, b_i, c_i).

Similarly there are type-B curve blow-up operations which result in regions of the type W^b_α with $\alpha = \rm I, II$, as well as

(b) two new triple-point centered $W^b_{\rm III}$ regions with changed exponents. For example if this operation is performed on the w-axis $(u = v = 0)$, the two new sets of exponents are $(a_i + b_i, b_i, c_i)$ from the substitution $u \to u$; $v \to uv$; $w \to w$, and $(a_i, a_i + b_i, c_i)$ from the substitution $u \to uv$; $v \to v$; $w \to w$. Similarly, analogous changes of exponents for type-B blow-ups of the other axes. These facts will be used in proving Lemma 3.4 below.

We refer to the proof of Proposition 3.1.8 of [7], where we showed that

$$W^b_{\rm III} = (0 < r < b) \times ((s, t) \in A) \times T^3,$$

where A is a triangle in $\mathbb{R}^2$ with vertices

$$P = \left(\frac{a_2}{a_1}, \frac{a_3}{a_1}\right); \quad Q = \left(\frac{b_2}{b_1}, \frac{b_3}{b_1}\right) \quad \text{and} \quad R = \left(\frac{c_2}{c_1}, \frac{c_3}{c_1}\right).$$

Note that all these vertex coordinates are greater than or equal to one. The T^3 factor has $\vartheta_i = \arg \zeta_i$ as coordinates. In this description, the induced Fubini metric is quasi-isometric to

$$(40) \quad g = dr^2 + r^2 d\vartheta_1^2 + r^{2s}((\log r)^2 ds^2 + d\vartheta_2^2) + r^{2t}((\log r)^2 dt^2 + d\vartheta_3^2).$$

Again, we should write $r_1 = |\zeta_1|$ instead of r in (40), but as remarked in the last two propositions, the proof of Lemma 3.3 of [5] applies, and the quasi-isometry class of the metric is unaffected by replacing r_1 by r.

We now define

$$(41) \qquad \alpha_1 = \min\left(\frac{a_2}{a_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1}\right), \qquad \beta_1 = \max\left(\frac{a_2}{a_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1}\right);$$

$$(42) \qquad \alpha_2 = \min\left(\frac{a_3}{a_1}, \frac{b_3}{b_1}, \frac{c_3}{c_1}\right), \qquad \beta_2 = \max\left(\frac{a_3}{a_1}, \frac{b_3}{b_1}, \frac{c_3}{c_1}\right).$$

Note that these α_i's and β_j's are not the same as the ones defined in Proposition 3.1.8 [7] cited above. We write the Laplacian for the metric in (40), and consider its energy form

$$E = \int [p_1(f_r)^2 + p_2(f_{\vartheta_1})^2 + p_3(f_s)^2 + p_4(f_{\vartheta_2})^2 + p_5(f_t)^2 + p_6(f_{\vartheta_3})^2] \cdot dr\, d\vartheta_1\, ds\, d\vartheta_2\, dt\, d\vartheta_3,$$

where the subscripts of f denote partial derivative, $p_i = g^{ii}\sqrt{g}$, and $\sqrt{g} = r^{2s+2t+1}|\log r|^2$.

Now, applying the hypothesis $(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < 1$, together with $\alpha_i, \beta_i \geq 1$, and using the same arguments as in Proposition 3.3, we obtain

$$p_1 = \sqrt{g}g^{11} \geq r^{2\beta_1+2\beta_2+1}|\log r|^2 = q_1 \quad \text{(say)},$$

$$p_i = \sqrt{g}g^{ii} \geq r^{2\alpha_1+2\alpha_2+1}|\log r|^2 = q_2 \quad \text{(say)}, \quad \text{for } 2 \leq i \leq 6,$$

and $\sqrt{g} \leq q_2$. From this point on, the proof proceeds in exactly the same way as that of Proposition 3.3, equation (24) onwards. δ and γ are defined in the same way in terms of the α_i's and β_i's in (41) and (42), as they were in (32), (33), and obey the same inequalities. This proves the proposition.

It remains to achieve the hypothesis of the previous proposition. We do this next.

3.8. **Lemma.** *Enough type-*B *operations* (*cf.* [7, §2.1]) *ensure the condition*

$$0 < (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < 1$$

in all the W^b_{III}*-type regions.*

Proof. Let us consider the triangle A described above in the last proposition, with the vertices

$$(43) \qquad P = \left(\frac{a_2}{a_1}, \frac{a_3}{a_1}\right), \quad Q = \left(\frac{b_2}{b_1}, \frac{b_3}{b_1}\right), \quad R = \left(\frac{c_2}{c_1}, \frac{c_3}{c_1}\right).$$

A type-B operation on, say, the "a" and "c" columns leads to two new charts, both regions of type W^b_{III}. These two regions would have the new triangles PQR' and $P'QR$ associated to them, where $P' = R'$ is the new point subdividing the edge PR in the ratio $a_1 : c_1$, so that

$$P' = R' = \left(\frac{a_2 + c_2}{a_1 + c_1}, \frac{a_3 + c_3}{a_1 + c_1}\right) = \mu_1 P + (1 - \mu_1)R,$$

where

$$\mu_1 = \max\left(\frac{a_1}{a_1 + c_1}, \frac{c_1}{a_1 + c_1}\right); \qquad (1 - \mu_1) = \min\left(\frac{a_1}{a_1 + c_1}, \frac{c_1}{a_1 + c_1}\right).$$

Clearly, the lengths of the subdivided pieces satisfy

$$\|PR'\| = \|PP'\| \le \mu_1 \|PR\|; \qquad \|P'R\| = \|R'R\| \le \mu_1 \|PR\|. \tag{44}$$

In general, if we repeat type-B operations along this edge, we will have further subdivisions, so as to subdivide PR' and $P'R$ into two segments each. We will thus get two ratios $\mu_2^{(1)}$, $\mu_2^{(2)}$ analogous to μ_1 above; namely,

$$\mu_2^{(1)} = \max\left(\frac{a_1}{2a_1 + c_1}, \frac{a_1 + c_1}{2a_1 + c_1}\right) = \frac{a_1 + c_1}{2a_1 + c_1}.$$

Similarly,

$$\mu_2^{(2)} = \frac{a_1 + c_1}{a_1 + 2c_1}.$$

Therefore we see,

$$\mu_2^{(1)} = \left(1 + \frac{a_1}{a_1 + c_1}\right)^{-1} \le \left(1 + \min\left(\frac{a_1}{a_1 + c_1}, \frac{c_1}{a_1 + c_1}\right)\right)^{-1}$$
$$= (1 + (1 - \mu_1))^{-1} = (2 - \mu_1)^{-1}.$$

Similarly, $\mu_2^{(2)} \le (2 - \mu_1)^{-1}$. Thus, after n operations, the length of the largest segment, called $l_{\max}(n)$, among the 2^n segments into which PR is subdivided, satisfies

$$l_{\max}(n) \le \mu_n \mu_{n-1} \mu_{n-2} \cdots \mu_1 \|PR\|, \tag{45}$$

where

$$\mu_i \le (2 - \mu_{i-1})^{-1}. \tag{46}$$

It is easily checked by induction that if the initial $\mu_1 = a_1/(a_1 + c_1) = \frac{p}{q}$, with $q > p$ clearly, then the inequality (46) implies that

$$\mu_n \le \frac{(n-1)q - (n-2)p}{nq - (n-1)p}, \tag{47}$$

so that from (45) and (47) we have

$$l_{\max}(n) \leq \frac{p}{nq-(n-1)p}\|PR\| \leq \frac{p}{n(q-p)}\|PR\| = \frac{1}{n}\left(\frac{\mu_1}{1-\mu_1}\right)\|PR\|,$$

which clearly $\to 0$ as $n \to \infty$.

Thus enough type-B operations along the PR edge produce segments of arbitrarily small length. Similarly for any other edge. So sufficiently many type-B operations produce triangles of arbitrarily small sides. Since the quantity

$$(\beta_1-\alpha_1)+(\beta_2-\alpha_2)$$

is the sum of the lengths of the projections of the triangle A along the x and y axes, this sum can also be made arbitrarily small.

We remark here that type-A operations (cf. [7, §2.1]) will create three new charts, the triangles corresponding to which will be the three triangles formed by joining a new vertex with coordinates

$$\left(\frac{a_2+b_2+c_2}{a_1+b_1+c_1}, \frac{a_3+b_3+c_3}{a_1+b_1+c_1}\right),$$

which is created in the interior of A, to the original vertices P, Q, R.

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