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HODGE THEORY AND THE HILBERT SCHEME

ZIV RAN

In Mori's "bend and break" method [6], [7], on which his theory of extremal rays is based, a key technical role is played by a fundamental estimate, due to Grothendieck, on the dimension of the Hilbert scheme of curves in an algebraic manifold [4]. Specifically, and more generally, if X is an algebraic (or complex) manifold and $Y \subset X$ is a submanifold with normal bundle N, then Grothendieck's estimate states that any component $\mathscr{H}$ of the Hilbert scheme (or Douady space) $\mathscr{H}ilb_X$ containing $\{Y\}$ satisfies

$$\dim \mathscr{H} \geq h^0(N) - h^1(N). \tag{1}$$

In view of the fundamental, and very general, nature of the estimate (1), one naturally wonders whether it might be possible to improve it in some interesting special cases. One such improvement of a Hodge-theoretic nature is due to S. Bloch [2], generalizing some earlier work by Kodaira-Spencer in the codimension-1 case: Bloch defines a certain map

$$\pi\colon H^1(N) \to H^{p+1}(\Omega_X^{p-1}), \qquad p = \operatorname{codim}(Y, X),$$

which he names the *semiregularity* map, and proves that if π is injective, then $\mathscr{H}ilb_X$ is in fact *smooth* at $\{Y\}$, so that the estimate (1) may be improved to

$$\dim \mathscr{H} = h^0(N) \tag{2}$$

(as is well known, $h^0(N)$ is the embedding dimension at $\{Y\}$ of $\mathscr{H}ilb_X$, hence $\dim \mathscr{H} \leq h^0(N)$ always holds, with equality iff $\mathscr{H}ilb_X$ is smooth at $\{Y\}$).

Now as Bloch's semiregularity map π can rarely be injective, his dimension estimate (2) is not very useful as it stands. However, there is a natural generalization of (2) which seems a priori quite plausible (as well as more useful): namely, the estimate

$$\dim \mathscr{H} \geq h^0(N) - h^1(N) + \dim \operatorname{im}(\pi). \tag{3}$$

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The purpose of this note is to prove that the estimate (3) (in fact, a rather more precise result) is indeed valid, and to give some applications of this. The general result is as follows.

Theorem 1. *Let $f\colon Y \hookrightarrow X$ be the inclusion of a codimension-p submanifold in a compact Kähler manifold with normal bundle N, let*

$$\pi\colon H^1(N) \to H^{p+1}(\Omega_X^{p-1})$$

be the associated semiregularity map, and put $\rho = \dim \operatorname{im}(\pi)$. Then the natural map $\operatorname{Def}(f) \to \operatorname{Def}(X)$ factors through a diagram

$$\begin{array}{ccc} \operatorname{Def}(f) & \overset{j}{\hookrightarrow} & D_0 \times B \\ & \searrow & \downarrow \\ & & D_0 \subseteq \operatorname{Def}(X) \end{array}$$

in which B is a ball of dimension $h^0(N)$, D_0 is the locus of deformations of X over which the cohomology class $[Y]$ remains of Hodge type (p, p), and the image of j is defined by at most $h^1(N) - \rho$ analytic equations.

Corollary 2. *In the above situation, the estimate (3) holds for any component $\mathscr{H}$ of $\mathscr{H}ilb_X$ through $\{Y\}$, even if X is not Kähler.*

Proof. The germ of $\mathscr{H}ilb_X$ at $\{Y\}$ is nothing but the special fiber of the map $\operatorname{Def}(f) \to \operatorname{Def}(X)$; moreover, it will be seen from the proof of Theorem 1 that the part involving $\mathscr{H}ilb_X$ does not require the Kähler hypothesis.

Corollary 3. *In the above situation, suppose we have equality in (3) for some $\mathscr{H}$. Then the map $\operatorname{Def}(f) \to D_0$ is surjective; i.e., in any small deformation of X, Y lifts iff $[Y]$ remains of type (p, p), and in particular Y lifts iff any algebraic cycle homologous to a nonzero multiple of Y lifts.*

We now apply Corollary 2 to move some cycles Y. To do so, we obviously have to find some conditions under which the right-hand side of (3) may itself be estimated. This is most easily done in the case where Y is a curve. First, because of Riemann-Roch, (3) becomes in this case

$$\dim \mathscr{H} \geq \chi(N) + \dim \operatorname{im}(\pi) = -Y \cdot K_X + (n-3)(1-g) + \rho, \tag{4}$$

where g is the genus of Y and $n = \dim X$. Second, the semiregularity map π is in this case by definition (cf. [2]) dual to the map

$${}^t\pi\colon H^0(\Omega_X^2) \to H^0(N^* \otimes \Omega_Y)$$

which is deduced from the following diagram:

$$\begin{array}{ccccccc} & & \Omega_X^2 & & & & \\ & & \downarrow & & & & \\ 0 \to \bigwedge^2 N^* \to & & \Omega_X^2 \otimes \mathscr{O}_Y & \to N^* \otimes \Omega_Y \to 0 \end{array}$$

Now a couple of easy observations are in order:

(a) If ω is a 2-form on X which is *nondegenerate* at some point $y \in Y$, then ${}^t\pi(\omega)$ cannot vanish at y, and in particular π is nontrivial.

(b) In case $n = 3$, $\bigwedge^2 N^*$ is a line bundle on Y of degree $\nu = -\deg(N) = Y \cdot K_X - (2g-2)$. If, e.g., this is negative, then ${}^t\pi(\omega)$ cannot vanish unless ω itself vanishes identically on Y.

Combining the first observation with the estimate (4) leads to the following.

Corollary 4. *Let X be a symplectic complex manifold, i.e., X carries a everywhere nondegenerate holomorphic 2-form. Then any rational* (*resp. elliptic*) *curve on X moves in a family of dimension at least* $\dim X - 2$ (*resp.* 1).

Corollary 5. *Let X be a 3-dimensional complex manifold such that Ω^2_X is globally generated outside a finite set. Then*

(i) *K_X is nef;*

(ii) *any smooth curve $Y \subset X$ with $Y \cdot K_X = 0$ moves on X, hence any pluricanonical morphism of X cannot have isolated smooth 1-dimensional fibers; and*

(iii) *if in addition X is Kähler, then any Y as in* (ii) *cannot be rational.*

Proof. (i) follows from the fact that $2K_X = \det(\Omega^2_X)$ has finite base locus; (ii) follows from (a) and (4) above; as for (iii), if it were false then, by compactness of the Douady space, X would contain a ruled surface E. But since a desingularization of E cannot carry any 2-forms, we get a contradiction to our hypothesis about X carrying many 2-forms.

Remark. Hopefully, the smoothness hypothesis in (ii) can be removed.

In order to take advantage of observation (b), let us define the following invariant of X:

(5)
$$\lambda_X = \max\{k : H^0(\Omega^2_X) \text{ contains a } k\text{-dimensional subspace } A \text{ all of whose nonzero elements have isolated zeros}\}.$$

Corollary 6. *Let $Y \subset X$ be a smooth curve of genus g on a complex 3-fold, and put $\nu = Y \cdot K_X - (2g-2)$. Then Y moves on X in a family of dimension at least $-Y \cdot K_X + \lambda_X - \varepsilon$, where:*

$\varepsilon = 0$ *if* $\nu < 0$;

$\varepsilon = 1$ *if* $\nu = 0$;

$\varepsilon = \nu$ *if* $\nu \geq 0$ *and* $g \geq 1$;

$\varepsilon = \nu/2$ *if* $0 < \nu < 2g-2$ *and Y is nonhyperelliptic;*

$\varepsilon = \nu + 1$ *if* $\nu \geq 0$ *and* $g = 0$.

Proof. Easy from (4), (5), and observation (b), noting that $\ker({}^t\pi|_A)$

injects into $H^0(\bigwedge^2 N^*)$, whose dimension may be easily estimated (the fourth case uses Clifford's theorem).

In particular, if $\lambda_X > 0$ (i.e., X carries a 2-form with isolated zeros) then any smooth curve $Y \subset X$ of genus ≥ 2 with $Y \cdot K_X \leq 0$ must move. We thus have a mechanism whereby 2-forms can move curves, even ones meeting K_X nonnegatively. (This is unfortunately restricted at present to smooth curves, a restriction which we will try to remove in the future; if this can be done, one should expect some analogue of the "bend and break" method (with or without the "break" part) when λ_X is adequately large, which might imply in some cases that K_X is ample.)

Note that an obviously necessary numerical condition for a 3-fold X to have $\lambda_X > 0$ is that $c_3(\Omega_X^2) \geq 0$, i.e. $c_3(X) \leq 0$.

It remains to pay the piper and prove Theorem 1. The proof will be a modification of Bloch's argument in [2]. First of all, it is well known (and clear) that the natural map of deformation spaces (or functors) $\mathrm{Def}(f) \to \mathrm{Def}(X)$ factors through a morphism $\psi\colon \mathrm{Def}(f) \to D_0$; moreover, as the special fiber of ψ is the germ of $\mathscr{H}ilb_X$ at $\{Y\}$, we have that the relative tangent space of ψ is $H^0(N)$. By general principles of deformation theory (cf. [1], [2], [4], [6]), it will suffice to prove that the relative obstruction space of ψ is $\ker \pi$, since this has dimension $h^1(N) - \rho$.

Let (R, M) be an Artin local $\mathbb{C}$-algebra of exponent $k \geq 2$, i.e., $M^{k+1} = 0$, $M^k \neq 0$, and put $S_2 = \mathrm{Spec}(R)$, $S_1 = \mathrm{Spec}(R/M^k)$, and $S_0 = \mathrm{Spec}(R/M^{k-1})$. Suppose

$$f_1/S_1\colon Y_1/S_1 \to X_1/S_1 \tag{6}$$

is a deformation of f, and X/S is a lifting of X_1/S_1 *within* D_0. We must investigate the obstruction to lifting (6) to

$$f/S_2\colon Y/S_2 \to X/S_2. \tag{7}$$

Let X_0/S_0 be the restriction of X_1/S_1, and similarly for Y_0/S_0, etc. Now define sheaves T'_{X_i/X_i} for $i = 0, 1$ by the exact sequence

$$0 \to T'_{X_i/S_i} \to T_{X_i/S_i} \to N_{Y_i/X_i} \to 0.$$

Note the exact sequence

$$0 \to T_X \otimes (M^{k-1}/M^k) \to T_{X_1/S_1} \to T_{X_0/S_0} \to 0$$

with analogous and compatible sequences for N and T'. Now put

$$\overline{T}_{X_0/S_0} = T_{X_0/S_0} \otimes (M/M^k), \qquad \overline{T}_{X_1/S_1} = T_{X_1/S_1} \otimes M,$$

and similarly for $\overline{N}_{Y_1/X_1}$ and $\overline{T}'_{X_1/S_1}$. We thus have the following exact diagram:

$$
(8)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T'_X \otimes M^k & \to & \overline{T}'_{X_1/S_1} & \to & \overline{T}'_{X_1/S_0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_X \otimes M^k & \to & \overline{T}_{X_1/S_1} & \to & \overline{T}_{X_0/S_0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N \otimes M^k & \to & \overline{N}_{Y_1/X_1} & \to & \overline{N}_{Y_0/X_0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Now, as in [8] and [9] (see Appendix), (6) gives rise to an element $\alpha_0 \in H^1(T'_{X_0/S_0})$, while the deformation X/S gives rise to an element $\beta_1 \in H^1(\overline{T}_{X_1/S_1})$, such that α_0 and β_1 induce the same element β_0 in $H^1(\overline{T}_{X_0/S_0})$, and the problem of constructing (7) is equivalent to that of finding an element $\alpha_1 \in H^1(\overline{T}'_{X_1/S_1})$ which induces both α_0 and β_1. This leads us to study the following cohomology diagram of (8):

$$
\begin{array}{ccccccc}
 & & H^0(N)\otimes M^k & \to & H^0(\overline{N}_{Y_1/X_1}) & \to & H^0(\overline{N}_{Y_0/X_0}) \xrightarrow{\delta} H^1(N)\otimes M^k \\
 & & \downarrow & & b_1\downarrow & & \downarrow \qquad\qquad \downarrow \\
 & & H^1(T'_X)\otimes M^k & \to & H^1(\overline{T}'_{X_1/S_1}) & \xrightarrow{r_1} & H^1(\overline{T}'_{X_0/S_0}) \to H^2(T'_X)\otimes M^k \\
 & & \downarrow & & c_1\downarrow & & \downarrow \qquad\qquad \downarrow \\
 & & H^1(T_X)\otimes M^k & \to & H^1(\overline{T}_{X_1/S_1}) & \to & H^1(\overline{T}_{X_0/S_0}) \to H^2(T_X)\otimes M^k \\
 & & \downarrow & & d_1\downarrow & & \downarrow \\
H^0(\overline{N}_{Y_0/X_0}) & \xrightarrow{\delta} & H^1(N)\otimes M^k & \to & H^1(\overline{N}_{Y_1/X_1}) & \to & H^1(\overline{N}_{Y_0/X_0})
\end{array}
$$

Our problem is to find $\alpha_1 \in H^1(\overline{T}'_{X_1/S_1})$ such that $r_1(\alpha_1) = \alpha_0$ and $c_1(\alpha_1) = \beta_1$. This may be "broken up" as follows:

(i) find $\alpha'_1 \in H^1(\overline{T}'_{X_1/S_1})$ such that $c_1(\alpha'_1) = \beta_1$;

(ii) find $\nu_1 \in H^0(\overline{N}_{Y_1/X_1})$ such that $r_1(b_1(\nu_1)) = \alpha_0 - r_1(\alpha'_1)$.

Then our desired α_1 is just $\alpha'_1 + b_1(\nu_1)$.

Now, define a sheaf $\overline{\Omega}^{p-1}_{X_1/S_1}$ analogously to $\overline{T}_{X_1/S_1}$ and note the commutative diagram:

$$
\begin{array}{ccc}
H^1(\overline{T}_{X_1/S_1}) & \xrightarrow{d} & H^1(\overline{N}_{Y_1/X_1}) \\
 & \searrow^{\kappa} & \downarrow \pi_1 \\
 & & H^{p+1}(\overline{\Omega}^{p-1}_{X_1/S_1})
\end{array}
$$

where π is deduced from the semiregularity map for $f: Y \hookrightarrow X$, and κ is the cup-product with the cohomology class $[Y] \in H^p(\Omega^p_{X/S})$. As is well known [5], $\kappa(\beta)$ is essentially the obstruction for the cohomology class of $[Y]$ to remain of type (p, p) in X/S; since X/S lies in D_0 by assumption, we have $\kappa(\beta) = 0$, and in particular $d_1(\beta_1) \in \ker \pi_1$.

Of course, $d_1(\beta_1)$ is precisely the obstruction to solving problem (i) above. Now the crucial point is the following commutative diagram:

$$\begin{array}{ccccccc}
H^0(\overline{N}_{Y_0/X_0}) & \xrightarrow{\delta} & H^1(N)\otimes M^k & \longrightarrow & H^1(\overline{N}_{Y_1/X_1}) & \longrightarrow & H^1(\overline{N}_{Y_0/X_0}) \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\
 & & H^{p+1}(\Omega^{p-1}_X)\otimes M^k & \xrightarrow{h} & H^{p+1}(\overline{\Omega}^{p-1}_{X_1/S_1}) & \longrightarrow & H^{p+1}(\overline{\Omega}^{p-1}_{X_0/S_0})
\end{array}$$

in which h is *injective* by Hodge theory [3] (provided either X is Kähler or the deformation X_1/S_1 is trivial). This implies first that $\operatorname{im}\delta \subset \ker \pi$ and second, since $d_1(\beta_1)$ dies in $H^1(\overline{N}_{Y_0/X_0})$, that we may identify $d_1(\beta_1)$ with a uniquely determined element of $\ker \pi / \operatorname{im}\delta$. We have thus identified the obstruction group for problem (i) with $\ker \pi / \operatorname{im}\delta$, and to complete the proof it will suffice to show that the obstruction group to solving problem (ii) once problem (i) has been solved is $\operatorname{im}\delta$. However, this is clear: indeed given any α_1' such that $c_1(\alpha_1') = \beta_1$, $r_1(\alpha_1')$ and α_0 both map to $\beta_0 \in H^1(T_{X_0/S_0})$, so that $\alpha_0 - r_1(\alpha_1')$ comes from some $x \in H^0(\overline{N}_{Y_0/X_0})$, and the obstruction to solving problem (ii) is precisely $\delta(x) \in \operatorname{im}\delta$.

Appendix: Canonical elements over a general base

The results of [8] and [9] were formulated for deformations parametrized by "arcs," i.e., schemes of the form $\operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^k)$. Here we indicate a variant of the method of [8], valid over general Artin local rings. We will consider the case of deformations of (abstract) manifolds, as other cases are similar.

Let X_0 be a compact complex manifold, and X_1/S_1 its universal first-order deformation, $S_1 = \operatorname{Spec}(R_1)$. Then we have the canonical element $\alpha_{X_1} \in H^1(T_{X_0} \otimes M_1) = H^1(T_{X_1/S_1} \otimes M_1)$, where $M_1 \subset R_1$ is the (square-free) maximal ideal: α_{X_1} corresponds to either one of the extensions $\overline{F}_1$

or F_1 given by:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 \otimes \mathcal{O}_{X_0} & \longrightarrow & F_1 & \longrightarrow & \Omega_{X_1/S_1} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 \otimes \mathcal{O}_{X_0} & \longrightarrow & \overline{F}_1 & \longrightarrow & \Omega_{X_0} & \longrightarrow & 0 \end{array}$$

Note that $M_1 = \Omega_{R_1} \otimes (R_1/M_1)$ and we have the following R_1-linear exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \Omega_{R_1} \otimes \mathcal{O}_{X_1} & \to & \Omega_{X_1} & \to & \Omega_{X_1/S_1} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & M_1 \otimes \mathcal{O}_{X_0} & \to & F_1 & \to & \Omega_{X_1/S_1} & \to & 0 \end{array}$$

In particular, there is a natural derivation $d_1 : \mathcal{O}_{X_1} \to F_1$.

Now suppose (R_2, M_2) is a local $\mathbb{C}$-algebra with $M_2^3 = 0$ and $R_2/M_2^2 = R_1$, and put $S_2 = \operatorname{Spec}(R_2)$. We consider the problem of lifting F_1 to a sheaf $\overline{F}_2$ fitting in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2 \otimes \mathcal{O}_{X_1} & \longrightarrow & \overline{F}_2 & \longrightarrow & \Omega_{X_1/S_1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & (M_2/M_2^2) \otimes \mathcal{O}_{X_1} & \longrightarrow & F_1 & \longrightarrow & \Omega_{X_1/S_1} & \longrightarrow & 0 \end{array}$$

or equivalently, that of lifting α_{X_1} to some element $\alpha_2 \in H^1(T_{X_1/S_1} \otimes M_2)$. Obviously, the obstruction to doing this lies in $H^2(T_X) \otimes M_2^2$. We claim that to such a lifting $\overline{F}_2$ we may canonically associate a second-order deformation X_2/S_2 lifting X_1/S_1: indeed define a sheaf $\mathcal{O}_2$ by a pullback diagram

$$\begin{array}{ccc} \mathcal{O}_2 & \to & \mathcal{O}_{X_1} \\ \bar{d}_2 \downarrow & & \downarrow d_1 \\ \overline{F}_2 & \xrightarrow{\varphi} & F_1, \end{array}$$

i.e., $\mathcal{O}_2 = \{(\omega, h) \in \overline{F}_2 \oplus \mathcal{O}_{X_1} : \varphi(\omega) = d_1(h)\}$, and define multiplication on $\mathcal{O}_2$ by the rule $(\omega_1, h_1) \cdot (\omega_2, h_2) = (\omega_1 h_2 + \omega_2 h_1, h_1 h_2)$.

Then it is easy to see that this makes $\mathcal{O}_2$ into a sheaf of flat R_2-algebras, i.e., $\mathcal{O}_2 = \mathcal{O}_{X_2}$ for a flat deformation X_2/S_2 extending X_1/S_1. Moreover, the map $\bar{d}_2$ is a derivation by construction, hence if we define a sheaf F_2 by a pullback diagram

$$\begin{array}{ccc} F_2 & \to & \Omega_{X_2/S_2} \\ \downarrow & & \downarrow \\ \overline{F}_2 & \to & \Omega_{X_1/S_1} \end{array}$$

then the derivation $(\bar{d}_2, d) : \mathcal{O}_{X_2} \to \overline{F}_2 \oplus \Omega_{X_2/S_2}$ obviously factors through a derivation $d_2 : \mathcal{O}_{X_2} \to F_2$, so the construction may be continued.

Thus, quite generally, given an Artin local $\mathbb{C}$-algebra (R, M) with $M^k \neq M^{k+1} = (0)$ and a deformation $\overline{X}/\overline{S}$, $\overline{S} = \mathrm{Spec}(R/M^k)$, we may proceed inductively and define a canonical element

$$\alpha_{\overline{X}} \in H^1(T_{\overline{X}/\overline{S}} \otimes (M/M^k))$$

whose liftings to $H^1(T_{\overline{X}/\overline{S}} \otimes M)$ yield liftings of $\overline{X}/\overline{S}$ to $X/\mathrm{Spec}(R)$.

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References

[1] M. Artin, *On the solutions of analytic equations*, Invent. Math. **5** (1968) 277–291.
[2] S. Bloch, *Semi-regularity and de Rham cohomology*, Invent. Math. **17** (1972) 51–66.
[3] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968) 197–226.
[4] A. Grothendieck, *Fondements de la géométrie algébrique*, Secrétariat Math., Paris, 1962.
[5] N. M. Katz, *Algebraic solutions of differential equations*, Invent. Math. **18** (1972) 1–118.
[6] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979) 593–606.
[7] ——, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math., (2) **116** (1982) 133–176.
[8] Z. Ran, *Deformations of manifolds with torsion or negative canonical bundle*, J. Algebraic Geometry **1** (1992) 279–291.
[9] ——, *Lifting of cohomology and deformations of certain holomorphic maps*, Bull. Amer. Math. Soc. **26** (1992) 113–117.

UNIVERSITY OF CALIFORNIA, RIVERSIDE