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RIGIDITY OF PROPER HOLOMORPHIC MAPS BETWEEN SYMMETRIC DOMAINS

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In 1973 Mostow [21] proved a strong rigidity theorem to the effect that for compact Riemannian locally symmetric spaces of negative Ricci curvature the fundamental groups essentially determine the geometry (with obvious exceptions). Four years later, Margulis's Superrigidity Theorem gave as a consequence that for irreducible Riemannian locally symmetric spaces X and Y of negative Ricci curvature and finite volume, any "nondegenerate" continuous map $f: X \to Y$ is homotopic to an isometric immersion (up to normalizing constants), provided that X is of rank ≥ 2. In the last decade much work has been done in connection with the above rigidity theorems of Mostow and Margulis, e.g., [3], [4], [5], [6], [10], [19], etc. In 1978 Siu studied the strong rigidity of Kähler structures of compact quotients of bounded symmetric domains and obtained

Theorem 1 [24], [25]. *Let M be a compact quotient of an irreducible bounded symmetric domain of complex dimension ≥ 2. Suppose that X is a compact Kähler manifold homotopic to M. Then X is either biholomorphic or conjugate-biholomorphic to M.*

Siu's theorem covers the Hermitian case of Mostow's strong rigidity theorem for the reasons that any two compact $K(\pi, 1)$-spaces with isomorphic 160fundamental groups are homotopic and that any biholomorphism between Hermitian (locally) symmetric spaces of noncompact type is necessarily an isometry (up to normalizing constants). In connection with this, it is natural to study the rigidity problem for holomorphic mappings between Hermitian locally symmetric spaces of noncompact type. Indeed the following theorem was established by Mok in the compact case and by Mok and To in the finite-volume case as a consequence of their metric rigidity theorems.

Theorem 2 [16], [26]. *Let (X, g) be a Hermitian locally symmetric space of finite volume uniformized by an irreducible bounded symmetric domain Ω of rank ≥ 2. Suppose (Y, h) is any Hermitian locally*

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symmetric space of noncompact type. Then any (nonconstant) holomorphic map is necessarily a totally geodesic isometric immersion (up to normalizing constants).

In contrast with the consequence of the Superrigidity Theorem of Margulis as stated previously the result of Mok and To tells us that the holomorphic map f is already an (totally geodesic) isometric representative in its homotopy class.

In this article we are interested in the rigidity problem for holomorphic maps between bounded symmetric domains. We prove the following theorem which resolves a conjecture of Mok [18].

Main Theorem. *Let* (Ω_1, g_1) *and* (Ω_2, g_2) *be bounded symmetric domains. Suppose that* Ω_1 *is irreducible and* $\operatorname{rank}(\Omega_1) \geq \operatorname{rank}(\Omega_2) \geq 2$. *Then any proper holomorphic map* $f\colon \Omega_1 \to \Omega_2$ *is necessarily a totally geodesic isometric embedding (up to normalizing constants).*

When $\Omega_1 = \Omega_2$, our Main Theorem was proved by Henkin and Novikov [12] by using results of Bell [1], Tumanov and Henkin [28]. Namely any proper holomorphic self-map $f\colon \Omega \to \Omega$ on an irreducible bounded symmetric domain of rank ≥ 2 is an automorphism. For further connection with the subject, proper holomorphic mappings in complex analysis, we refer readers to recent (survey) articles by Bell [2] and Forstnerič [8].

In the Main Theorem the condition that $\operatorname{rank}(\Omega_1) \geq \operatorname{rank}(\Omega_2)$ is indispensible as we can see from the following example. Consider a holomorphic map $f\colon D^{\mathrm{I}}(2, 2) \to D^{\mathrm{I}}(3, 3)$ induced by the map which sends 2×2 matrices $B = (z_{ij})_{1 \leq i, j \leq 2}$ to 3×3 matrices $\widetilde{B} = (\tilde{z}_{kl})_{1 \leq k, l \leq 3}$ in such a way that $\tilde{z}_{ij} = z_{ij}$ for $1 \leq i, j \leq 2$, $\tilde{z}_{i3} = \tilde{z}_{3i} = 0$ for $1 \leq i \leq 2$ and $\tilde{z}_{33} = g$ for a holomorphic function g defined on $D^{\mathrm{I}}(2, 2)$ such that $|g| < 1$. Then clearly we can choose g so that f is proper but not totally geodesic. The condition excluding rank-one situation is also needed as there exist proper polynomial maps $f\colon B^n \to B^N$ for $N \geq 2n - 1$, which are not totally geodesic, e.g., the map $f\colon B^n \to B^{2n-1}$ defined by $f(z_1, \cdots, z_n) = (z_1, \cdots, z_{n-1}, z_1 z_n, \cdots, z_n z_n)$.

We remark that in previous rigidity theorems one usually requires the condition of compactness or finite volume imposed either on domain-manifolds or on both domain-manifolds and target-manifolds. Our result throws some light on problem of extending rigidity theorems for holomorphic maps to the situation where the finite-volume condition is relaxed. In this direction Mok conjectured that the statement of Theorem 2 remains true when the finite-volume condition on X is replaced by the weaker condition that the fundamental group $\pi_1(X)$ acts ergodically on the Shilov boundary of Ω. The case where $\operatorname{rank}(X) = \operatorname{rank}(Y)$ has been observed by Mok, based on the proof of the Main Theorem of the present article. The idea is to consider the lifted map F between their respective universal coverings and to consider nontangential limits on the Shilov boundary. It is, however, not known how to extend this result to the general situation.

In the rest of this introduction we shall present an outline of our proof of the Main Theorem. Our idea is heavily based on the framework of [16] and [20]. There are usually two main difficulties for proving this type of theorem: analytical on the one hand and geometrical on the other. Lack of boundary regularity usually presents a serious analytical difficulty. In [20] this problem was overcome by passing to a moduli map $f^\sharp$ (which is defined between moduli spaces of certain Hermitian symmetric submanifolds; see below). In the proof of the Main Theorem, the (partial) regularity of f can be thus obtained as in [20]. The genuine difficulty that we encounter here comes from the geometrical side, i.e., the problem of proving that f is totally geodesic. Our proof contains two main steps after reduction to the case where $\Omega_1 \cong D_3^{\mathrm{IV}}$ ($\operatorname{rank} D_3^{\mathrm{IV}} = 2$) and $\operatorname{rank}(\Omega_2) = 2$. The first step is to prove that f cannot be infinitesimally of rank 1, i.e., to prove that there exists a point p such that $df(T_p(\Omega_1))$ is not the tangent space of a totally geodesic three-ball. This step is connected with the work of [31] and [16]. In the second step, given that f is infinitesimally of rank two (meaning that $df(T_p(\Omega_1))$ is the tangent space of a symmetric submanifold of rank two for generic p), we prove an integrability theorem for some distribution induced by f. This leads us to the notion of invariantly geodesic submanifolds. We remark that the notion of such manifolds is already implicitly used in [20]. A new ingredient in the present article is to exploit the geometrical/algebraic aspect of invariantly geodesic submanifolds, on which the integrability theorem is based. We now explain our approach as follows. Fix an irreducible bounded symmetric domain $\Omega = G_0/K$ of rank $r \geq 2$, and denote its compact dual by X. In Ω there are distinguished 1-dimensional totally geodesic submanifolds called minimal disks Δ. The compact dual $\mathbb{P}^1$ of Δ, embedded in X, generates $H_2(X, \mathbb{Z})$ as a homology class, and is called a minimal rational curve of X. Denote by $\mathscr{M}_0$ the set of all those minimal disks containing the origin o of Ω. Each minimal disk $\Delta \in \mathscr{M}_0$ defines an element $[T_0(\Delta)]$ in $\mathbb{P}T_o(\Omega)$, and the union of all such elements is in fact the characteristic variety $\mathscr{S}_o(\Omega)$ of Ω as initially defined by Mok [16]. There are also distinguished totally geodesic submanifolds M of higher dimension and $\operatorname{rank}(M) = r - 1$, called maximal characteristic symmetric subspaces by [20]. To each M containing o there corresponds a minimal disk Δ such that $\Delta \times M$ embeds in Ω totally geodesically, and the union of all such $\Delta \times M$'s exhausts Ω. Moreover, for each $b \in \partial\Delta$, $b \times M$ is contained in $\partial\Omega$ and is in fact a "boundary component" of Ω (cf. [30]). The nullity of Ω is defined to be the dimension of M.

As in [20] we consider radial limits of submanifolds $\{t\} \times M \subseteq \Delta \times M \subseteq \Omega_1$ as $t \to \partial\Delta$, and then use the boundary structure of Ω_2 to show that each boundary component of Ω_1 is mapped into a boundary component of Ω_2. As a consequence of the maximum principle, we obtain a moduli map $f^\sharp$ which is defined from the moduli space $\mathscr{M}_1$ for characteristic symmetric subspaces of Ω_1 to the moduli space $\mathscr{M}_2$ for corresponding characteristic symmetric subspaces of Ω_2. By an induction argument it suffices for the proof of the Main Theorem to consider the case where both Ω_1 and Ω_2 are irreducible and of rank two. Since any irreducible bounded symmetric domain of rank at least two can be exhausted by totally geodesic submanifolds isomorphic to a Type-IV domain D of dimension three (which is biholomorphic to the Siegel upper half-plane for symmetric two-by-two matrices), we need only consider the case $\Omega_1 = D$. Since both domains are of rank two and D is known to be of nullity one, from the moduli map $f^\sharp$ just defined ($\mathscr{M}_1 = \{\text{minimal disks}\}$, $\mathscr{M}_2 = \{\text{rank-one characteristic symmetric submanifold}\}$) we conclude that f is characteristic, namely, $df(\mathscr{S}_p(D)) \subset \mathscr{S}_{f(p)}(\Omega_2)$ for generic points p.

We remark that minimal disks as introduced previously coincide with extremal disks in the sense considered by L. Lempert [13] who, in a different context, proved some boundary regularity results of biholomorphic mappings with extremal disks [14]. In a similar vein the notion of moduli maps was considered by J. Faran [7] where the linearity of proper holomorphic maps between balls in the low codimension case was proved.

To exploit the preceding infinitesimal information we have first of all that the characteristic subvariety of D is a rational curve of degree two in $\mathbb{P}T_0(D)$. Define $\mathscr{D}$ to be the set of rational curves of degree two in $\mathscr{S}_0(\Omega_2)$. $\mathscr{D}$ has two types of curves. Every curve of the first type is contained in the projectivized tangent space of a totally geodesic rank-one submanifold B^3. If df sends $\mathscr{S}_p(D)$ onto a curve of the first type for generic points p, then $f(D)$ is tangent to such a B^3 at $f(p)$. In this situation f is said to be infinitesimally of rank one.

When Ω_2 is of nullity one, the Main Theorem can be deduced from the fact that minimal disks of D are mapped to minimal disks of Ω_2. Ω_2 is of nullity one exactly when it is a Type-IV domain (i.e., a bounded symmetric domain dual to the hyperquadric).

Our proof of the Main Theorem contains two essential steps. First we show that f is not infinitesimally of rank one, and then that f maps D into some Type-IV domain to finish the proof.

An evidence eliminating the case where f is infinitesimally of rank one is suggested by the argument of P. Yang [31] or N. Mok [16], which says that a polydisk cannot support a complete Kähler metric of curvature pinched between negative constants. For this approach to work one would try to prove the boundary regularity of f and to do some curvature estimates. The present proof uses a different approach. We first show that $f: \Omega_1 \to \Omega_2$ has a rational extension $\tilde{f}: X_1 \to X_2$ between respective compact duals of Ω_1 and Ω_2, and then prove that the condition that f is infinitesimally of rank one cannot hold at a boundary point of Ω_1 at which $\tilde{f}$ is regular. The idea of the proof involves the analysis of the boundary structure for irreducible bounded symmetric domains of rank two.

To show that f maps D into some type-IV domain Ω' we try to prove an integrability theorem for a distribution induced by f. The present proof uses the affine structure of the Harish-Chandra realization together with the fact that characteristic varieties of a bounded symmetric domain Ω are parallel with respect to the Euclidean coordinates of Ω, and is achieved via the notion of invariantly geodesic submanifolds. In order to exploit the Euclidean coordinates of Ω_2 we will actually show that $f(D)$ is contained in some distinguished subdomain Ω_2' which is not only affine-linear in Ω_2, but is such that the transform of Ω_2' by any automorphism of Ω_2 remains affine-linear. Such a domain Ω_2' corresponds to the intersection of Ω_2 and some Hermitian symmetric submanifold M of X (X denoting the compact dual of Ω_2) with the property that M is totally geodesic with respect to any canonical metric of X. These distinguished submanifolds M and Ω_2' will be called invariantly geodesic submanifolds. Another characteristic of the subdomain Ω_2' is the complex analyticity of its orbit space under the isotropy subgroup K of Ω_2. Namely a totally geodesic subdomain Ω_2' is invariantly geodesic if (and only if) the set of its K-orbits naturally inherits a complex structure. We classify all invariantly geodesic submanifolds and prove that for any irreducible rank-two Hermitian symmetric manifold there exists an invariantly geodesic submanifold isomorphic to a hyperquadric. From the fact that f is characteristic and that f is not infinitesimally of rank one we prove that $f(D)$ is tangent at generic points p to a unique invariantly geodesic hyperquadric Q_p to the second order. Then we show that the preceding results of the order of contact and of the uniqueness are sufficient for the distribution $p \to T(Q_p)$ to be integrable. Thus $f(D)$ is contained in an invariantly geodesic hyperquadric and hence in its dual domain $\Omega_2' \subset \Omega_2$. Since the special case where D and Ω_2' are Type-IV domains has been treated already, the Main Theorem follows.

1. Preliminaries

We set up notation and review some results of [20].

Let $X_0 = G_0/K$ be an irreducible Hermitian symmetric space of noncompact type of rank $r \geq 2$, and $X = G/P = G_c/K$ be its compact dual. The Lie algebra $\mathfrak{g}_0$ of G_0 has a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{m}_0$ with respect to $\mathfrak{k}$. Complexifying $\mathfrak{m}_0$ gives $\mathfrak{m} = \mathfrak{m}_0^{\mathbb{C}} \cong \mathfrak{m}^+ \oplus \mathfrak{m}^-$ so that $\mathfrak{m}^+$ is identified with $T^{1,0}(X_0)$. The Harish-Chandra embedding $X_0 \Subset \mathfrak{m}^+ \simeq \mathbb{C}^n \subset X_0$ realizes X_0 as a bounded symmetric domain Ω. Set $P^- = \exp(\mathfrak{m}^-)$ and the complex Lie group $K^{\mathbb{C}} \subset G$ to be the complexification of K.

In Ω $(= X_0)$ there are distinguished 1-dimensional totally geodesic submanifolds called minimal disks Δ. The compact dual $\mathbb{P}^1$ of Δ, which embeds in X, is homologically a generator of $H_2(X, \mathbb{Z})$, and is called a minimal rational curve of X. By the first canonical embedding of X into some complex projective space $\mathbb{P}^N$ [22], minimal rational curves are mapped to projective lines in $\mathbb{P}^N$. To each minimal disk Δ in Ω, there corresponds an irreducible symmetric submanifold $\Omega_{\Lambda_{r-1}}$ of rank $r-1$ such that $\Delta \times \Omega_{\Lambda_{r-1}}$ with canonical metric embeds in Ω as a totally geodesic submanifold. Moreover, $b \times \Omega_{\Lambda_{r-1}}$ is a boundary component of Ω for every $b \in \partial\Delta$ (cf. [30]). These $\Omega_{\Lambda_{r-1}}$'s are called maximal characteristic symmetric subspaces of Ω by [20], and all intermediate characteristic symmetric subspaces Ω_{Λ_i} of Ω, $1 \leq i \leq r-2$, can be obtained from $\Omega_{\Lambda_{r-1}}$'s inductively.

Let $f\colon \Omega_1 \to \Omega_2$ be a proper holomorphic map between irreducible bounded symmetric domains of rank ≥ 2. By regarding $f|_{\Delta\times\Omega_{\Lambda_{r-1}}}$ as a family of bounded holomorphic functions defined on $\Omega_{\Lambda_{r-1}}$ and using the existence of radial limits for this family we showed the following in Proposition 2.2 and Proposition 2.3 of [20]:

Proposition 1.1. *In the notation as above, let Ω_{Λ_i} be a characteristic symmetric subspace of Ω_1. Then $f(\Omega_{\Lambda_i})$ is contained in some characteristic symmetric subspace of Ω_2.*

We assume now that $\mathrm{rank}(\Omega_1) \geq \mathrm{rank}(\Omega_2)$. By Proposition 1.1 and induction any characteristic symmetric subspace of rank 2 must be mapped into some characteristic symmetric subspace of rank ≤ 2. It is well known that there is no proper holomorphic map from bounded symmetric domains of higher rank into balls; we have thus shown

Proposition 1.2. *Let $f\colon \Omega_1 \to \Omega_2$ be a proper holomorphic map between irreducible bounded symmetric domains with* $\mathrm{rank}(\Omega_1) \geq \mathrm{rank}(\Omega_2) \geq 2$. *Then* $\mathrm{rank}(\Omega_1) = \mathrm{rank}(\Omega_2)$.

To state the next result we need to discuss characteristic varieties of Hermitian symmetric spaces as initially defined by [16]. Let o be the origin of Ω. The characteristic variety at o is a subset of $\mathbb{P}T_o(\Omega)$ defined to be

$$\mathscr{S}_0(\Omega) = \bigcup_{o\in\Delta} [T_o(\Delta)] \qquad (\Delta\text{'s are minimal disks}).$$

Fix $[\alpha]$ of $\mathscr{S}_o(\Omega)$ and assume that $\|\alpha\| = 1$. Then one has an orthogonal decomposition of $T_o(\Omega)$ with respect to α [16].

$$T_o(\Omega) = \mathbb{C}\alpha \oplus \mathscr{H}_\alpha \oplus \mathscr{N}_\alpha$$

such that

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} = \frac{1}{2} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$$

for any unit vector $\beta \in \mathscr{H}_\alpha$ and

$$R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} = 0$$

for any $\gamma \in \mathscr{N}_\alpha$. The dimension of $\mathscr{N}_\alpha$ is defined to be the nullity of Ω. The point is that $\mathscr{H}_\alpha$ can be naturally identified with $T_{[\alpha]}(\mathscr{S}_o(\Omega))$ by [16]. Moreover, $\mathscr{S}_o(\Omega)$ is a compact Hermitian symmetric space of rank 2, and the inclusion map $\mathscr{S}_o(\Omega) \hookrightarrow \mathbb{P}T_o(\Omega)$ is identified with the first canonical embedding of $\mathscr{S}_o(\Omega)$ except for $\Omega = D^{\mathrm{III}}(n, n)$, $n \geq 2$ (in which case the inclusion $\mathscr{S}_o(\Omega) \hookrightarrow \mathbb{P}T_o(\Omega)$ is identified with the second canonical embedding). For more detail about $\mathscr{S}_o(\Omega)$ we refer readers to [16], [17] and [18].

We now restrict ourselves to the case where $\mathrm{rank}(\Omega_1) = \mathrm{rank}(\Omega_2) = 2$. In this situation the only nontrivial characteristic symmetric subspaces of Ω_1 and Ω_2 are isomorphic to balls. Fix a characteristic symmetric subspace B^n of Ω_1 and a vector α $(\neq 0) \in T_o(B^n)$. The disk $\mathbb{C}\alpha \cap B^n = \mathbb{C}\alpha \cap \Omega_1$ is minimal in B^n, hence minimal in Ω_1. We thus have $\mathbb{P}T(B^n) \subset \mathscr{S}_o(\Omega_1)$. Conversely given $[\beta] \in \mathscr{S}_o(\Omega_1)$ by the transitivity of the K-action on $\mathscr{S}_o(\Omega)$, $\Omega_1 \cap \mathbb{C}\beta$ is contained in some characteristic symmetric subspace B^n. As a consequence, by Proposition 1.2 we have

Proposition 1.3. *In the notation of Proposition* 1.2, *assume that* f *has maximal rank at* $p \in \Omega_1$. *Then the image of* $\mathscr{S}_p(\Omega_1)$ *by the induced map* $[df_p]\colon \mathbb{P}T_p(\Omega_1) \to \mathbb{P}T_{f(p)}(\Omega_2)$ *is contained in* $\mathscr{S}_{f(p)}(\Omega_2)$.

This proposition motivates the following.

Definition 1.4. Let U be an open subset of Ω_1, and $f\colon U \to \Omega_2$ a nondegenerate holomorphic map. Then f is said to be a characteristic map if, at generic point p, $df_p(\alpha)$ is a characteristic vector of Ω_2 for any characteristic vector $\alpha \in T_p(\Omega_1)$.

2. The case where Ω_1 and Ω_2 are Type-IV domains

In this section we will prove the Main Theorem in the case where Ω_1 and Ω_2 are Type-IV domains; the general case will be reduced to this case (§5).

Recall that a hyperquadric Q^n is isomorphic to a hypersurface of $\mathbb{P}^{n+1}$ defined by

$$\{(Z_0, Z_1, \cdots, Z_{n+1}) \in \mathbb{P}^{n+1} : Z_0^2 + Z_1^2 + \cdots + Z_{n+1}^2 = 0\},$$

and a Type-IV domain D_n^{IV} is the noncompact dual of Q^n. We are going to prove the main result of this section.

Proposition 2.1. *Suppose* $f\colon (D_3^{\mathrm{IV}}, g) \to (D_n^{\mathrm{IV}}, h)$, $n \geq 3$, *is a proper holomorphic map. Then* f *is a totally geodesic isometric embedding* (*up to scaling constants*).

Our proof uses the following lemma which says essentially that f is in fact "algebraic."

Lemma 2.2. *Let* $f\colon \Omega_1 \to \Omega_2$ *be a proper holomorphic map between irreducible bounded symmetric domains of* rank ≥ 2. *Suppose that each minimal disk of* Ω_1 *is mapped into a minimal disk of* Ω_2. *Then* f *extends rationally from* X_1 *to* X_2, *where* X_i*'s are compact duals of* Ω_i*'s* ($i = 1, 2$) *respectively.*

Proof of Lemma 2.2. The proof follows Proposition 2.10 of [20]. First observe that

$$(*) \qquad f(p) = \bigcap_{\Delta \ni p} f(\Delta) \qquad (\Delta\text{'s are minimal}).$$

By using the condition that minimal disks are mapped to minimal disks, we first arrive at a rational extension $f^{\sharp}$ between moduli spaces of minimal rational curves (by the pseudoconcavity of moduli spaces). Then by regarding a point as an intersection of minimal rational curves using

$(*)$ we get from $f^\sharp$ down to a rational extension $\tilde{f}\colon X_1 \to X_2$ over base manifolds. For more detail we refer readers to [20]. q.e.d.

We are ready to give a proof of Proposition 2.1.

Proof of Proposition 2.1. Since nontrivial characteristic symmetric subspaces of D_n^{IV} are minimal disks, the condition of Proposition 2.2 is satisfied in view of Proposition 1.1. Hence the previous lemma shows that $f\colon D_3^{\mathrm{IV}} \to D_n^{\mathrm{IV}}$ is algebraic. Write $\tilde{f}\colon X_1 \to X_2$ for the rational extension between compact duals of D_3^{IV} and D_n^{IV}. We claim that there exists a minimal rational curve ℓ of X_1, such that $f|_\ell$ is a biholomorphism onto its image.

Let D be the indeterminancy of $\tilde{f}$, which is of codimension ≥ 2, and R be the subvariety where df is degenerate in $X_1 \backslash D$. Choose a point $q \in D_n^{\mathrm{IV}}$ such that $\tilde{f}^{-1}(q)$ consists of finitely many points and that $\tilde{f}^{-1}(q) \cap R = \varnothing$. Since there are only finitely many minimal rational curves connecting any two points of $\tilde{f}^{-1}(q)$, one can find a minimal rational curve ℓ such that ℓ contains just one point of $\tilde{f}^{-1}(q)$ and $\ell \cap D_3^{\mathrm{IV}} \neq \varnothing$. It follows that $\tilde{f}|_\ell$ is one-to-one. Since $\tilde{f}|_\ell$ is still minimal, it is then a biholomorphism, proving the claim. As a consequence, there exists a minimal disk D of D_3^{IV} such that $f|_{D'}$ is biholomorphic onto its image for all minimal disks D's sufficiently close to D.

Now we are going to show that f is actually a totally geodesic isometric embedding. Fix a minimal disk D_α such that $f|_{D_\alpha}$ is a biholomorphism. From §1 one knows that there exists a maximal characteristic symmetric subspace Ω_1 of D_3^{IV} (which is also a minimal disk in this case) such that $D_\alpha \times \Omega_1$ embeds as a totally geodesic submanifold of D_3^{IV}. Moreover, we have $T_o(\Omega_1) \cong \mathscr{N}_\alpha$ (Proposition 1.8 of [20]). By using the Harish-Chandra embedding we can write

$$\Omega_1 = \mathbb{C}_q \cap D_3^{\mathrm{IV}}, \qquad 0 \neq q \in T_o(\Omega_1) = \mathscr{N}_\alpha.$$

Denote by g and h the canonical, normalized metrics for D_3^{IV} and D_n^{IV} respectively. We know that the induced metrics on minimal disks are Poincaré metrics. By our assumption on f it follows from the Schwarz lemma that $f|_{D_{\alpha'}}$ is an isometry for every $[D_{\alpha'}]$ sufficiently close to D_α.

Let (z_1, z_2, z_3) stand for the Euclidean coordinates with $(0, 0, 0)$ being the origin and

$$\frac{\partial}{\partial z_1}(o) = \alpha \in T_0(D_\alpha) \quad \text{and} \quad \frac{\partial}{\partial z_2}(o) = q.$$

Note that (z_1, z_2, z_3) is also a complex geodesic coordinate system of

(D_3^{IV}, g) at the origin; i.e.,

$$\frac{\partial g_{ij}}{\partial z_k}(o) = 0 \quad \text{for } 1 \le i, j, k \le 3.$$

We assume that df has maximal rank at o. At o computing the holomorphic bisectional curvature spanned by α and q with respect to the locally defined metric $f^*(h)$, denoted also by h for simplicity, yields

$$R_{\alpha\bar{\alpha}q\bar{q}}(h)(o) = -\frac{\partial^2 h_{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \sum_{1 \le u, v \le 3} h^{u\bar{v}} \frac{\partial h_{1\bar{v}}}{\partial z_2} \frac{\partial h_{u\bar{1}}}{\partial \bar{z}_2}.$$

Since $f|_{D_{\alpha'}}$ is biholomorphic for any $D_{\alpha'}$ near to D_α, the first term in the above expression vanishes as $f^*(h)|_{D_{\alpha'}}$ is the (unique) Poincaré metric on $D_{\alpha'}$. Moreover h is seminegatively curved; it follows that

$$\frac{\partial h_{1\bar{v}}}{\partial z_2}(o) = 0 \quad \text{for } 1 \le v \le 3.$$

Since we have used complex geodesic coordinates at o, thus

$$\nabla_q h_{\alpha\bar{v}}(o) = 0 \quad \text{for } 1 \le v \le 3,$$

where ∇ is the covariant differentiation with respect to g. By a polarization argument as in [16], we have $\nabla h = 0$ almost everywhere, and hence $h \equiv g$ upon normalization. This shows that f is indeed an isometric embedding.

To see that the embedding is totally geodesic we proceed as follows. Let us retain the same notation for the images of α, q and D_3^{IV} under f. From

$$\begin{aligned} 0 &\ge R_{\alpha\bar{\alpha}q\bar{q}}(D_n^{\mathrm{IV}}, h)(f(o)) \\ &= R_{\alpha\bar{\alpha}q\bar{q}}(D_3^{\mathrm{IV}}, h|_{D_3^{\mathrm{IV}}})(o) + \|\sigma(\alpha, q)\|^2 \\ &= \|\sigma(\alpha, q)\|^2, \end{aligned}$$

where $\sigma(\alpha, q)$ is the second fundamental form for the embedding f, it follows that $\sigma(\alpha, q) = 0$. Again by polarization $\sigma \equiv 0$ at $f(o)$. Since o is arbitrary, σ vanishes everywhere. Our proof is then completed.

3. f cannot be infinitesimally of rank 1

Throughout this section we denote by Ω_1 the Type-IV domain D_3^{IV} of dimension 3, by Ω_2 an irreducible bounded symmetric domain of rank 2, and by g_1 (resp. g_2) canonical metrics of Ω_1 (resp. Ω_2). By Proposition 2.1 we can furthermore assume that Ω_2 is not a Type-IV domain.

Let f be a proper holomorphic map from Ω_1 to Ω_2. We have shown in Proposition 1.3 that f is characteristic; i.e., $df(\mathscr{S}_o(\Omega_1)) \subset \mathscr{S}_o(\Omega_2)$. By [18] one knows that $\mathscr{S}_o(D_3^{\mathrm{IV}})$ is isomorphic to a rational curve of degree 2 embedded in $\mathbb{P}T_o(D_3^{\mathrm{IV}})$. To obtain more information about f, we will examine the set $\mathscr{D}$ of all rational curves of degree 2 in $\mathscr{S}_o(\Omega_2)$. In fact $\mathscr{D}$ consists of two connected components $\mathscr{D}_1$ and $\mathscr{D}_2$. For each rational curve C in $\mathscr{D}_1$ there exists a totally geodesic submanifold B^3 such that $C \subset \mathbb{P}T_o(B^3) \subset \mathbb{P}T_o(\Omega_2)$. If it occurs that $df(T_p(\Omega_1)) \in \mathscr{D}_{1,f(p)}$ for generic point p, we will say that f is *infinitesimally of rank* 1. Our task in this section is to establish the fact that f cannot be infinitesimally of rank 1.

Let X denote the characteristic variety of Ω_2, $X \hookrightarrow \mathbb{P}^N \cong \mathbb{P}T_o(\Omega_2)$ the first canonical embedding, and $\varphi: \mathbb{P}' \hookrightarrow X \hookrightarrow \mathbb{P}^N$ a rational curve of degree 2 embedded in X. It is well known (cf. [9]) that $C = \varphi(\mathbb{P}^1)$ is contained in a projective 2-plane $\mathbb{P}^2_C$ (uniquely determined by C), and $i: C \hookrightarrow \mathbb{P}^2_C$ is essentially the Veronese map. Define two sets of embedded rational curves of degree 2 in X as follows:

$$\mathscr{D}_1 = \{[C]: \mathbb{P}^2_C \subset X\}, \qquad \mathscr{D}_2 = \{[C]: \mathbb{P}^2_C \not\subset X\}.$$

Since $\mathrm{Aut}_0(X)$ extends to a subgroup of $\mathrm{Aut}(\mathbb{P}^N)$, both $\mathscr{D}_1$ and $\mathscr{D}_2$ are preserved under $\mathrm{Aut}_0(X)$-action. Moreover, we have

Proposition 3.1. $\mathrm{Aut}_0(X)$ *acts transitively on* $\mathscr{D}_1$ *and* $\mathscr{D}_2$ *respectively.*

The proof of Proposition 3.1 will be postponed until §6. Before proceeding further, we have

Lemma 3.2. *Let* X *be a compact Hermitian symmetric space seated in* $\mathbb{P}^N$ *by its first canonical embedding. Suppose that* $H \subset \mathscr{S}_o(X) \subset \mathbb{P}T_o(X)$ *is a projective* n*-plane. Then* H *is the projectivized tangent space of a projective* $(n+1)$*-plane* $\mathbb{P}^{n+1}$ *contained in* X.

Proof. Let V be the linear subspace of $T_o(X)$ such that $\mathbb{P}V = H$. Let Ω be the noncompact dual of X and set $M = V \cap \Omega$. For any $\alpha \in V$, $\Delta_\alpha = \mathbb{C}\alpha \cap \Omega$ is a minimal disk, hence is totally geodesic in Ω. Let σ denote the second fundamental form of M in Ω. Therefore we have $\sigma(\alpha, \alpha) = 0$ for any $\alpha \in V$, and this yields $\sigma \equiv 0$. M is then a totally geodesic submanifold of constant holomorphic sectional curvature; this clearly implies the lemma. q.e.d.

By the preceding lemma we can state the main result of this section:

Proposition 3.3. *In the notation as before, suppose that* $f\colon \Omega_1 \to \Omega_2$ *is a proper holomorphic map. Then there exists an open set* U *of* Ω_1 *such that*

$$df(\mathscr{S}_p(\Omega_1)) \in \mathscr{D}_{2,f(p)}$$

for every $p \in U$, *or equivalently* f *cannot be infinitesimally of rank* 1.

The present proof of Proposition 3.3 is achieved via the rational extension $\tilde{f}$ of f. Recall that in Lemma 2.2, $\tilde{f}$ was obtained in the case where Ω_1 and Ω_2 are Type-IV domains. Here we are going to generalize it to the equal rank case. By an induction argument as in §1 it suffices to do so in the case of rank two.

Recall (§1) that if Ω is an irreducible bounded symmetric domain of rank two, its maximal boundary components and maximal characteristic symmetric subspaces are isometrically isomorphic to the ball B^k with the Poincaré metric, k denoting the nullity of Ω. To prove the existence of $\tilde{f}$ we use the following lemma.

Lemma 3.4. *Let* Ω *be an irreducible bounded symmetric domain of rank two and nullity* k. *Let* B *be a maximal boundary component, and* M *a maximal characteristic symmetric subspace of* Ω. *Suppose that* b *is a boundary point of* Ω, *which is contained in both* ∂M *and* B. *Let* H_1 *and* H_2 *be complex affine linear subspaces which contain* B *and* M *as open subsets respectively. Then we have* $H_1 \cap H_2 = \{b\}$.

Proof. There exists a totally geodesic embedding

$$\Delta \times M \hookrightarrow \Omega$$

such that $\{0\} \times M$ is identified with M of Ω. Since $b \in \partial M \cap B$ and B is a boundary component, we have

$$\Delta \times \{b\} \subset B.$$

Working with the compact dual $\mathbb{P}^1 \times \overline{M}$ (resp. X) of $\Delta \times M$ (resp. Ω) via the Harish-Chandra embedding, we have a totally geodesic embedding of $\mathbb{P}^1 \times \overline{M}$ in X

$$\begin{array}{ccc} \Delta \times M & \to & \Omega \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \overline{M} & \to & X \end{array}$$

with respect to some canonical metric h of X.

We are going to prove the lemma by contradiction. Assume that H_1 and H_2 contain a common nonzero vector ξ at b. Let $\eta \neq 0$ be a vector contained in $T_b(\Delta \times \{b\}) \subset T_b(H_1)$. The compact dual $\mathbb{P}^k$ of B contains H_1 and is totally geodesic in X. Hence

$$R_{\xi\bar{\xi}\eta\bar{\eta}}(X, h) \geq \varepsilon > 0.$$

On the other hand, with respect to the product metric g of $\mathbb{P}^1 \times M$ one has

$$R_{\xi\bar{\xi}\eta\bar{\eta}}(\mathbb{P}^1 \times M, g) = 0.$$

Since $(\mathbb{P}^1 \times M, g)$ is totally geodesic in (X, h), the contradiction is reached. The proof of the lemma is now completed. q.e.d.

We proceed to prove the existence of the rational extension $\tilde{f}$.

Lemma 3.5. *Let $f: \Omega_1 \to \Omega_2$ be a proper holomorphic map from the 3-dimensional Type-IV domain Ω_1 to an irreducible bounded symmetric domain Ω_2 of rank two. Denote by X_1 and X_2 the compact Hermitian symmetric spaces dual to Ω_1 and Ω_2 respectively. Then there exists a rational map $\tilde{f}: X_1 \to X_2$ which extends f via the Harish-Chandra embeddings $\Omega_i \subset X_i$, $i = 1, 2$.*

Proof. The proof uses the moduli map defined as follows. Denote by $\mathcal{M}_1$ the set of all minimal disks of Ω_1, and by $\overline{\mathcal{M}}_1$ the set of all minimal rational curves of X_1. One knows that $\mathcal{M}_1$ embeds in $\overline{\mathcal{M}}_1$ as an open subset. For any fixed minimal disk D of Ω_1 $(= D_3^{\mathrm{IV}})$ we have, by Proposition 1.1 and the condition $\operatorname{rank}(\Omega_2) = 2$, that $f(D)$ is contained in a characteristic symmetric subspace M of rank one. By using the Harish-Chandra embedding M is an (complex) affine linear subdomain of Ω_2, and hence $f(D)$ spans (linearly) an (complex) affine subdomain M_D of M (which is still a totally geodesic submanifold of rank 1). Let k be the dimension of M_D for a generic minimal disk D of Ω_1. Let $\mathcal{M}_2$ (resp. $\overline{\mathcal{M}}_2$) denote the set of all those k-dimensional totally geodesic submanifolds of characteristic symmetric subspaces M (resp. $\overline{M}$) in Ω_2 (resp. X_2). Then clearly $\mathcal{M}_2$ is a complex homogeneous manifold and embeds in $\overline{\mathcal{M}}_2$ as an open subset. Define a (meromorphic) map $f^\sharp: \mathcal{M}_1 \to \mathcal{M}_2$ by sending $[D] \in \mathcal{M}_1$ to $[M_D] \in \mathcal{M}_2$. Then $f^\sharp$ admits a rational extension (in the same notation) $f^\sharp: \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_2$ by the proof of Proposition 2.6 [20].

For generic points $p \in \Omega_1$, the subdomains

$$S_p = \bigcap_{D \ni p} M_D$$

are of the same dimension, say d. Let $\mathcal{M}_d$ (resp. $\overline{\mathcal{M}}_d$) be the set of all d-dimensional totally geodesic submanifolds B^d (resp. $\mathbb{P}^d$) of characteristic symmetric subspaces M (resp. compact duals $\overline{M}$) in Ω_2 (resp. in X_2). Define a (meromorphic) map $\tilde{f}: \Omega_1 \to \mathcal{M}_d$ by sending p to $[S_p] \in \mathcal{M}_d$.

We claim that $\tilde{f}$ has a rational extension from X_1 to $\overline{\mathcal{M}}_d$, which will be denoted by the same notation $\tilde{f}$. Recall that the map which arises from the assignment $D \to M_D$ admits a rational extension $f^\sharp : l \to M_l$, where l denotes minimal rational curves in X_1. Therefore, we can define a rational map $\tilde{f}$ from X_1 to $\overline{\mathscr{M}}_d$ by

$$p \to \left[\bigcap_{l \ni p} M_l\right] \in \overline{\mathscr{M}}_d,$$

which clearly give the desired extension.

To obtain a rational extension of f it suffices to show that $d = 0$ since $\overline{\mathscr{M}}_d = X_2$ in this case. Let $\Delta \times \Delta$ be a bidisk in Ω_1. We have that $f(\Delta \times \{0\})$ and $f(\{t\} \times \Delta)$ are contained in maximal characteristic symmetric subspaces M and B_t respectively by Proposition 1.1. Using Lemma 3.4 we obtain

$$B_b \cap \overline{M} = \{b\},$$

for boundary points $b \in \partial\Delta$, where B_b is the boundary component containing b. Since $f^\sharp : \overline{\mathscr{M}}_1 \to \overline{\mathscr{M}}_2$ is rational, without loss of generality we can assume that $f^\sharp$ is regular at $\{b\} \times \Delta$. Thus, for some $t \in \Delta$, $B_t \cap M$ is just the point $f(\{t\} \times \{0\})$ by the continuity of $f^\sharp$. Since those points t where $B_t \cap M$ is of dimension greater than zero form a subvariety V of δ, we have proved that $V \neq \Delta$. The proof of Lemma 3.5 is now completed. q.e.d.

We are now ready to give a proof of Proposition 3.3.

Proof of Proposition 3.3. Suppose f is infinitesimally of rank one; we shall derive a contradiction. By Lemma 3.5, $f : \Omega_1 \to \Omega_2$ admits a rational extension $\tilde{f} : X_1 \to X_2$ between compact duals of Ω_1 and Ω_2. Take a maximal boundary component B and a point $b \in B$ at which $\tilde{f}$ is regular. From the assumption that f is infinitesimally of rank one it follows that $d\tilde{f}(T_b(X_1))$ is the tangent space of a totally geodesic $\mathbb{P}^3$ at $\tilde{f}(b)$. Let M be a characteristic symmetric subspace of X_2, which contains $\mathbb{P}^3$. Since $\tilde{f}$ is regular at b, one sees that $M \cap \Omega_2 \neq \varnothing$. Take a bidisk $\Delta \times \Delta$ of Ω_1 with $b = (1, 0)$. The complex curve

$$C = \tilde{f}(\{1\} \times \Delta)$$

is therefore contained in B by the definition of boundary components (cf. [30]). Now we have

$$M \cap B \supseteq d\tilde{f}(T_b(X_1)) \cap B \supseteq T_{\tilde{f}(b)}(C).$$

Since C is regular at $\tilde{f}(b)$, $M \cap B$ contains a nonzero vector. We have reached a contradiction by Lemma 3.4. The proof of Proposition 3.3 is now completed.

4. Invariantly geodesic submanifolds

An intermediate step in the proof of the Main Theorem is the study of a class of complex submanifolds, called invariantly geodesic submanifolds, of a Hermitian symmetric space. In this section we will be interested in finding all such complex submanifolds. Fix a compact Hermitian symmetric space $X = G/P$ and a canonical metric h. Let Ω be the noncompact dual of X and $\Omega \Subset \mathfrak{m}^+ \simeq \mathbb{C}^n \subset X$ be the Harish-Chandra embedding.

Definition 4.1. Let $M \subset X$ be a complex submanifold of X and S be a subset of G. M is said to be an S-invariantly geodesic submanifold of X if $\varphi(M)$ is a totally geodesic submanifold of (X, h) for every $\varphi \in S$, or equivalently M is totally geodesic with respect to canonical metrics $\varphi^*(h)$ for any $\varphi \in S$. For simplicity by invariantly geodesic submanifolds we mean G-invariantly geodesic submanifolds.

Example 4.2. (i) *Minimal rational curves.* A rational curve ℓ of X is minimal if ℓ is a projective line by the first canonical embedding $X \hookrightarrow \mathbb{P}^N$. Since all minimal rational curves are totally geodesic in X and G-action preserves the minimality, minimal rational curves are invariantly geodesic submanifolds.

(ii) *Characteristic symmetric subspaces.* These submanifolds are defined in [20] (cf. §1). By Proposition 1.12 of [20], and Lemma 4.4 below they are invariantly geodesic submanifolds.

We give first of all a criterion for a complex submanifold to be invariantly geodesic. We will use the notation introduced in §1.

Lemma 4.3. *Let M be an invariantly geodesic submanifold of X, which contains the origin o. Then Lie brackets*

$$[[\mathfrak{m}^-, T_o(M)], T_o(M)]$$

are contained in $T_o(M)$. Conversely, suppose that $V \subset \mathfrak{m}^-$ is a complex linear space such that

$$[[\mathfrak{m}^-, V], V] \subseteq V.$$

Then V is the gangent space of an invariantly geodesic submanifold M.

Proof. Part of the proof in a special case was worked out in Proposition 1.12 of [20]; for the sake of self-containedness we give a complete proof. Suppose first of all that M is only totally geodesic. By the general theory of symmetric spaces $M \cap \mathfrak{m}^+$ is complex linear, and can be identified with $T_o(M)$. Assume now that M is invariantly geodesic; in particular $\varphi(M)$ is totally geodesic for every $\varphi \in \exp(\mathfrak{m}^-) = P^-$. Since the adjoint action of P^- is, by using $[\mathfrak{m}^-, \mathfrak{m}^+] = \mathfrak{k}^{\mathbf{C}}$, trivial on $T_o(X)$, one has that $\varphi(M) = M$ by the uniqueness of the totally geodesic submanifold. We are going to prove the first half of the lemma by making use of the fact $\varphi(M) = M$ for the invariantly geodesic submanifold M and every $\varphi \in P^-$.

Fix $\eta \in T_o(M)$. The action of $\varphi = \exp(\xi)$ on the curve $\eta(t) \equiv t\eta \subset \mathfrak{m}^+$ is given by

$$\varphi(\eta(t)) = \exp(\mathrm{Ad}(\varphi) \cdot t\eta) \cdot o.$$

To prove the first half of the lemma it suffices therefore to show that the second-order term of $\varphi(\eta(t))$ is $[\eta, [\xi, \eta]]$. Now one has for $\varphi = \exp(\xi)$

$$\mathrm{Ad}(\varphi) \cdot \eta = e^{\mathrm{ad}(\xi)} \cdot \eta = \eta + [\xi, \eta] + \tfrac{1}{2}[\xi, [\xi, \eta]]$$

by using $[\mathfrak{k}^{\mathbf{C}}, \mathfrak{m}^-] = \mathfrak{m}^-$ and $[\mathfrak{m}^-, \mathfrak{m}^-] = 0$. Hence

$$(*) \qquad \varphi \cdot (\eta(t)) = \exp(t(\eta + \gamma_1 + \gamma_2)),$$

where $\gamma_1 = [\xi, \eta] \in \mathfrak{k}^{\mathbf{C}}$ and $\gamma_2 = \frac{1}{2}[\xi, [\xi, \eta]] \in \mathfrak{m}^-$. Expanding $(*)$ out by using the Hausdorff-Campbell formula [29], we have

$$(*) = \exp(t\eta) \cdot \exp\left(\frac{t^2}{2}([\eta, \gamma_1 + \gamma_2])\right) \cdot \exp(t(\gamma_1 + \gamma_2)) \cdot o + O(t^3).$$

Since $\exp(\mathfrak{m}^- + \mathfrak{k}^{\mathbf{C}}) \cdot o = o$ and $[\eta, \gamma_2] = \frac{1}{2}[\eta, [\xi, [\xi, \eta]]] \in \mathfrak{k}^{\mathbf{C}}$, the second-order term is given by

$$\exp\left(\frac{t^2}{2}[\eta, \gamma_1]\right) \cdot \exp\left(\frac{t^2}{2}[\eta, \gamma_2]\right) \cdot o = \exp\left(\frac{t^2}{2}[\eta, [\xi, \eta]]\right) \cdot o,$$

as claimed. The first half of the lemma is proved.

For the second half of the lemma we note first of all that the condition that $[[\mathfrak{m}^-, V], V] \subseteq V$ a fortiori gives a Lie triple system:

$$[[\overline{V}, V], V] \subseteq V$$

(cf. [11]). Hence V is the tangent space of some totally geodesic submanifold M (cf. [11]). Moreover by using the Hausdorff-Campbell formula and $\exp(\mathfrak{m}^- + \mathfrak{k}^{\mathbf{C}}) \cdot o = o$ as before, one sees that $\eta(t)$ is contained in M provided that $[[\mathfrak{m}^-, V], V] \subseteq V$ (since all higher-order terms are contained in either V or $\mathfrak{m}^- + \mathfrak{k}^{\mathbf{C}}$). Hence $\varphi(M) = M$.

To finish the proof it suffices to show that M is $K^{\mathbf{C}}$-invariantly geodesic. Fix $k \in K^{\mathbf{C}}$. We have

$$k(M) = \exp(\mathrm{Ad}(k) \cdot V) \cdot o.$$

Now $\mathrm{Ad}(k)\cdot V \subset \mathfrak{m}^+$; to show that $k(M)$ is totally geodesic we need to show that $\mathrm{Ad}(k)\cdot V$ is a Lie triple system (cf. [11]). We have

$$(\sharp) \qquad [[\mathrm{ad}(k)V, \overline{\mathrm{ad}(k)V}], \mathrm{ad}(k)V] = \mathrm{ad}(k)[[V, \mathrm{ad}(k^{-1})\overline{\mathrm{ad}(k)V}], V].$$

But

$$\mathrm{ad}(k^{-1})\cdot \overline{\mathrm{ad}(k)V} \subseteq [\mathfrak{k}^{\mathbb{C}}, \mathfrak{m}^-] = \mathfrak{m}^-.$$

The condition $[[\mathfrak{m}^-, V], V] \subseteq V$ then yields that $(\sharp)$ is contained in $\mathrm{ad}(k)V$. The proof of the lemma is now completed. q.e.d.

As a consequence we have

Lemma 4.4. *If M is invariant geodesic, then $\varphi(M\cap \mathbb{C}^n)\cap \mathbb{C}^n$ is a complex affine linear subspace for $\varphi \in G$. Conversely if $\varphi(M\cap\mathbb{C}^n)\cap\mathbb{C}^n$ is complex affine linear for all $\varphi\in P^-$, then M is invariantly geodesic.*

Proof. Assume that $o \in M$. If M is invariantly geodesic, then as seen in Lemma 5.3 $\varphi(M\cap\mathbb{C}^n) = M$ for every $\varphi \in P^-$ and is complex linear. Since $K^{\mathbb{C}}$ acts on $\mathbb{C}^n \cong \mathfrak{m}^+$ as linear transformations and $P^+ = \exp(\mathfrak{m}^+)$ acts as Euclidean translations, $\varphi(M\cap\mathbb{C}^n)$ is complex affine linear for $\varphi \in P^+\cdot K^{\mathbb{C}}\cdot P^-$. The general theory of symmetric spaces ensures that $P^+\cdot K^{\mathbb{C}}\cdot P^-$ is an open dense subset of G; the first assertion follows.

Conversely suppose that $\varphi(M\cap\mathbb{C}^n)$ is complex linear for every $\varphi\in P^-$; in particular $M\cap\mathbb{C}^n$ is linear. Since $d\varphi(0) = \mathrm{id}$ for every $\varphi \in P^-$, by linearity one has $\varphi(M) = M$. The proof of Lemma 4.3 then gives

$$[[\mathfrak{m}^-, V], V] \subseteq V;$$

this in turn implies that M is invariantly geodesic by the same lemma.

Using a result of Mok [18] we now prove the following proposition, which is essential for us to find invariantly geodesic submanifolds.

Proposition 4.5. *Let M be an invariantly geodesic submanifold of X. Then* (i) *any minimal rational curve of M is also a minimal rational curve of X, and consequently $\mathscr{S}_o(M)\subset\mathscr{S}_o(X)$;* (ii) *$\mathscr{S}_o(M)$ is an invariantly geodesic submanifold of $\mathscr{S}_o(X)$.*

Proof. Let ℓ be a minimal rational curve of M. From the invariant geodesy of M in X it follows that ℓ is also invariantly geodesic in X, and we conclude that ℓ is a minimal rational curve of X (cf. Propositon 3.4 of [27]). For the second assertion one knows that $T_{[\alpha]}(\mathscr{S}_o(X))$ can be identified with $\mathscr{H}_\alpha$ for a fixed $[\alpha]\in\mathscr{S}_o(X)$; see §1. Fix a canonical metric g of X; g induces a Study-Fubini metric g_0 on $\mathbb{P}T_0(X)$. A result of Mok (the remark on p. 249 of [18]) tells us that

$$(*) \qquad R_{\eta\bar\eta\eta\bar\eta}(\mathscr{S}_o(X), g_0|_{\mathscr{S}_o(X)}) = R_{\eta\bar\eta\eta\bar\eta}(X, g),$$

for $\eta \in \mathscr{H}_\alpha$ via the identification: $T_{[\alpha]}(\mathscr{S}_o(X)) \approx \mathscr{H}_\alpha$. Suppose now that M is totally geodesic with $\mathscr{S}_o(M) \subset \mathscr{S}_o(X)$. Then from

$$R_{\eta\bar{\eta}\eta\bar{\eta}}(M, g|_M) = R_{\eta\bar{\eta}\eta\bar{\eta}}(X, g)$$

and $(*)$ we have

$$R_{\eta\bar{\eta}\eta\bar{\eta}}(\mathscr{S}_o(M), g_0|_{\mathscr{S}_o(M)}) = R_{\eta\bar{\eta}\eta\bar{\eta}}(\mathscr{S}_o(X), g_0),$$

showing that $\sigma(\eta, \eta) = 0$, where σ is the second fundamental form of $\mathscr{S}_o(M)$ in $\mathscr{S}_o(X)$. Hence $\sigma \equiv 0$, and $\mathscr{S}_o(M)$ is totally geodesic in $\mathscr{S}_o(X)$. Assume now that M is invariantly geodesic. Then we conclude that $\mathscr{S}_o(M)$ is totally geodesic with respect to these canonical metric g_0's of $\mathscr{S}_o(X)$. Since $K^{\mathbb{C}}/\text{center} \cong \text{Aut}_0(\mathscr{S}_o(X))$, g_0's exhaust all canonical metrics of $\mathscr{S}_o(X)$; the second assertion follows.

Next we are going to determine all invariantly geodesic submanifolds.

Proposition 4.6. *Let S be an invariantly geodesic submanifold of $\mathscr{S}_o(X)$. Then there exists an invariantly geodesic submanifold M of X such that $\mathscr{S}_o(M) = S$.*

Proof. We shall prove the proposition case by case.

(i) $X = G(n, m)$; $\mathscr{S}_o(X) \cong \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ (assume $m \geq n$).

The invariantly geodesic submanifolds of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ are embeddings

$$i_1 \times i_2 \colon \mathbb{P}^{r-1} \times \mathbb{P}^{s-1} \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1},$$

where $i_1 \colon \mathbb{P}^{r-1} \to \mathbb{P}^{n-1}$ and $i_2 \colon \mathbb{P}^{2-1} \to \mathbb{P}^{m-1}$ are linearly embeddings (cf. Proposition 3.4 of [27]). Then a natural embedding

$$j \colon G(r, s) \hookrightarrow G(n, m)$$

has $\mathbb{P}^{r-1} \times \mathbb{P}^{s-1}$ as characteristic variety. Precisely, the embedding is defined as follows. $G(n, m)$ has Lie algebra $\mathfrak{m}^-$ and $\mathfrak{m}^+$ consisting of $(m+n) \times (m+n)$ matrices of the following type:

$$\mathfrak{m}^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{m}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\},$$

where B (resp. C) is a $n \times m$ (resp. $m \times n$) matrix. Consider the following subspaces of $\mathfrak{m}^-$ and $\mathfrak{m}^+$, respectively:

$$\mathfrak{m}'^- = \left\{ \begin{pmatrix} 0 & 0 \\ C' & 0 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{m}'^+ = \left\{ \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix} \right\},$$

where B' (resp. C') is any $n \times m$ (resp. $m \times n$) matrix with last $(n-r)$ rows and $(m-s)$ columns (resp. $(m-s)$ rows and $(n-r)$ columns) being zero. Clearly $\mathfrak{m}'^+$ gives rise to a totally geodesic submanifold isomorphic to $G(r, s)$. A simple verification shows that $[[\mathfrak{m}^-, \mathfrak{m}'^+], \mathfrak{m}'^+] \subseteq \mathfrak{m}'^+$; hence $G(r, s)$ is also invariantly geodesic in $G(n, m)$ by Lemma 4.3. All invariantly geodesic submanifolds containing o are thus K-orbits (isotropy subgroups of isometries of X) of submanifolds $j: G(r, s) \hookrightarrow G(n, m)$, $1 \le r \le n$, $1 \le s \le m$, as described above.

(ii) $X = G^{\mathrm{II}}(n, n)$; $\mathscr{S}_o(X) \cong G(2, n-2)$.

The invariantly geodesic submanifolds S of $G(2, n-2)$ are given in case (i). Hence

$$S \cong \mathbb{P}^s, s \le n-2, \quad \text{or} \quad S \cong G(2, s), 2 \le s \le n-2.$$

In case $S \cong \mathbb{P}^s$, by Lemma 3.2 there exists an invariantly geodesic submanifold $\mathbb{P}^{s+1}$ such that $T_o(\mathbb{P}^{s+1}) = \mathbb{P}^s$. In case $S \cong G(2, s)$, a standard embedding $j: G^{\mathrm{II}}(s+2, s+2) \to G^{\mathrm{II}}(n, n)$ defined analogously as in case (i) will do the job. All invariantly geodesic submanifolds containing o are then K-orbits of the representative described above.

(iii) $X = G^{\mathrm{III}}(n, n)$; $\mathscr{S}_o(X) \cong \mathscr{P}^{n-1}$.

As before representatives of invariantly geodesic submanifolds in this case are the natural embeddings: $G^{\mathrm{III}}(m, m) \hookrightarrow G^{\mathrm{III}}(n, n)$, $m \le n$.

(iv) $X = Q^n$; $\mathscr{S}_o(X) \cong Q^{n-1}$.

We start with invariantly geodesic submanifolds of $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Those are of the forms $\mathbb{P}^1 \times \{\mathrm{pt}\}$ or $\{\mathrm{pt}\} \times \mathbb{P}^1$ and are minimal rational curves of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus by using $\mathscr{S}_o(Q^n) \cong Q^{n-2}$ [18], induction and Lemma 3.2, all invariantly geodesic submanifolds M of Q^n are of rank 1 and are projective linear by the embedding $M \hookrightarrow Q^n \hookrightarrow \mathbb{P}^{n+1}$.

(v) $X \cong E_6/\operatorname{spin}(1) \times S^1$; $\mathscr{S}_o(X) \cong G^{\mathrm{II}}(5, 5)$.

By (ii) invariantly geodesic submanifolds of $G^{\mathrm{II}}(5, 5)$ are either projective linear in $\mathbb{P}T_0(X)$ or isomorphic to $G^{\mathrm{II}}(4, 4) \cong Q^6$ (note $G^{\mathrm{II}}(3, 3) \cong \mathbb{P}^3$, $G^{\mathrm{II}}(2, 2) \cong \mathbb{P}^1$). By Lemma 3.2 we are left with finding an invariantly geodesic submanifold M of X such that $\mathscr{S}_o(M) \cong Q^6$. An easy way to see this is the following.

By the classification theory, the maximal boundary components of the irreducible bounded symmetric domain Ω' of exceptional type E_7 are isomorphic to D^{IV}_{10}, cf. [30]. One then knows (cf. §1) that maximal characteristic symmetric subspaces N of Ω' are isomorphic to D^{IV}_{10}. By Example 4.2 (ii), characteristic symmetric subspaces are invariantly geodesic, and hence $\mathscr{S}_o(N) \cong Q^8$ is invariantly geodesic in $\mathscr{S}_o(\Omega')$ by Proposition 4.5. By the classification of characteristic varieties [18] $\mathscr{S}_o(\Omega') \cong X = E_6/\operatorname{spin}(1) \times S^1$. Thus taking M to be $\mathscr{S}_o(N)$ does the job.

(vi) $X = E_7/E_6 \times S^1$; $\mathscr{S}_o(X) \cong E_6/\operatorname{spin}(10) \times S^1$.

From case (v) it follows that invariantly geodesic submanifolds of X are either of rank 1 or maximal characteristic symmetric subspaces isomorphic to Q^{10} .

We have completed the list of all invariantly geodesic submanifolds; Proposition 4.6 is proved. q.e.d.

Remark. One can see by the construction above that K-orbits (=isotropy subgroup of isometries of X) are the same as P-orbits on the set of all invariantly geodesic submanifolds containing o .

As a corollary to the proof of Proposition 4.6, together with Proposition 3.1, one has

Corollary 4.7. *Let X be an irreducible compact Hermitian symmetric space of rank 2. Suppose that $X \not\cong Q^n$. Then any rational curve ℓ of degree 2 in $\mathbb{P}T_o(X)$ with $[\ell] \in \mathscr{D}_2(\mathscr{S}_o(X))$ is the characteristic variety of a submanifold N isomoprhic to Q^3 . N is not invariantly geodesic, while it is totally geodesic with respect to some canonical metric of X . Moreover N is contained in an invariantly geodesic submanifold M (uniquely determined by N) isomoprhic to a hyperquadric.*

Proof. In fact we have the following list from the proof of Proposition 4.6:

(i) $X \cong G(2, n)$, $M \cong Q^4$;
(ii) $X \cong G^{\mathrm{II}}(5, 5)$, $M \cong Q^6$;
(iii) $X \cong E^6/\operatorname{Spin}(10) \times S^1$, $M \cong Q^8$.

5. Integrability of the distribution $y \to T_y(Q_y^n)$. Proof of the Main Theorem

We will derive the Main Theorem from the following proposition.

Proposition 5.1. *Let U be an open subset of Q^3 and $X \not\cong Q^n$ be an irreducible compact Hermitian symmetric space of rank 2. Let $f: U \to X$ be a nondegenerate holomorphic map. Suppose that f is characteristic and that f is not infinitesimally of rank 1 (§3). Then $f(U)$ is contained in some invariantly geodesic submanifold M (of rank 2). In fact M is isomorphic to a hyperquadric.*

Assuming Proposition 5.1 we can prove the Main Theorem.

Proof of the Main Theorem. We first assume that Ω_2 is irreducible. By the same proof as Proposition 2.1 it suffices to show that f maps minimal disks of Ω_1 to minimal disks of Ω_2 . The isometry group G_0 of Ω_1 acts transitively on the set of minimal disks. By Corollary 4.7

Ω_1 contains a totally geodesic submanifold M isomorphic to D_3^{IV} with $\mathscr{S}_0(D_3^{\mathrm{IV}}) \subset \mathscr{S}_o(\Omega_1)$; thus any minimal disk of Ω_1 is contained in some M as such. By Proposition 5.1 and Proposition 2.1 it follows that minimal disks are mapped to minimal disks. The Main Theorem is now proved under the assumption that Ω_2 is irreducible.

To complete the proof of the Main Theorem we are going to show that Ω_2 cannot be reducible. Suppose otherwise. Without loss of generality we can assume that $\Omega_2 = \Omega_2' \times \Omega_2''$, where Ω_1', Ω_2' are irreducible. For such reducible bounded symmetric domains as Ω_2, the boundary components (resp. characteristic symmetric subspaces) are of the form: $C' \times C''$ (resp. $D' \times D''$), where C' and C'' (resp. D' and D'') are boundary components (resp. characteristic symmetric subspaces) of Ω_2' and Ω_2'' respectively. One then sees that in the reducible case the analogous results as in §1 can also be obtained. It follows that f is characteristic. But $\mathscr{S}_o(\Omega_2)$ is a disjoint union of $\mathscr{S}_o(\Omega_2')$ and $\mathscr{S}_o(\Omega_2'')$. It follows that $df(\mathscr{S}(\Omega_1))$ is contained in either $\mathscr{S}(\Omega_2')$ or $\mathscr{S}(\Omega_2'')$, say $\mathscr{S}(\Omega_2')$. This implies that $f(\Omega_1) \subset \Omega_2'$. But now $\operatorname{rank}(\Omega_2')$ is strictly less than $\operatorname{rank}(\Omega_1)$; this is impossible in view of Proposition 1.2. The proof of the Main Theorem is now completed. q.e.d.

The remainder of this section will be devoted to the proof of Proposition 5.1.

Throughout this section denote by $\langle A \rangle$ the vector space linearly spanned by a set of vectors A. Let f and X be as stated in Proposition 5.1. Since f is characteristic and not infinitesimally of rank 1 by our assumption, from Proposition 3.3 we have that

$$[df_p(\mathscr{S}_p(U))] \in \mathscr{D}_2(\mathscr{S}_{f(p)}(X))$$

for every $p \in U$ after shrinking U if necessary. By Corollary 4.7 for each $p \in U$, $df(T_p(U))$ is thus contained in the tangent space at $f(p)$, of a uniquely determined, invariantly geodesic submanifold M_p isomorphic to a hyperquadric. (M_p depends on p a priori.) Our objective in this section is to show that $f(U)$ lies in one invariantly geodesic submanifold, i.e., M_p's are identical.

To start with, by using the Harish-Chandra embedding we can think of f as a holomorphic map from a bounded domain $U \subset \mathbb{C}^3$ to $f(U) \subset \mathbb{C}^n$ with $f(o) = o$. By a $K^{\mathbb{C}}$-transformation of X we can assume without loss of generality that $df(T_o(U))$ is the tangent space of some totally geodesic submanifold isomorphic to Q^3 (Corollary 4.7), and we will henceforth identify $T_o(U)$ with $df(T_o(U))$. We shall deduce Proposition 5.1 from the following lemmas:

Lemma 5.2. *In the notation as above, we have*

$$\langle[\mathfrak{m}^-, df(T_o(U))], df(T_o(U))]\rangle = T_o(M_o).$$

Lemma 5.3. *In the notation as above, we have*

$$\partial_\mu \partial_\nu f(o) \subseteq M_o \quad \textit{for } \mu, \nu \in T_o(U).$$

Deduction of Proposition 5.1 *from Lemma* 5.2 *and Lemma* 5.3. The idea is first of all to construct a distribution associated to f, and then by invoking the Frobenius theorem we can conclude the proof of Proposition 5.1 with Lemma 5.2 and Lemma 5.3. We proceed as follows.

Let $X = G/P = G_c/K$ be an irreducible compact Hermitian symmetric space, and M an invariantly geodesic submanifold of X. We know that the G_c-orbit of M is identical with the G-orbit of M (by the remark after Proposition 4.6). Denote this orbit by $\mathcal{M}$. Then $\mathcal{M}$ is a compact complex manifold. Write $\mathrm{Gr}_p(m, T_p(X))$ for the Grassmannian of m-planes seated in $T_p(X)$ and set

$$\mathrm{Gr}(m, T(X)) = \bigcup_{p \in X} \mathrm{Gr}_p(m, T_p(X)).$$

For an m-plane $H \subset T_p(X)$ we denote by [H] the point in $\mathrm{Gr}_p(m, T_p(X))$ it defines. Define a subset $\mathcal{GM}$ of $\mathrm{Gr}(m, T(X))$ associated to M to be

$$\mathcal{GM} = \bigcup_{M \in \mathcal{M}} [T(M)].$$

$\mathcal{GM}$ is a compact complex submanifold of $\mathrm{Gr}(m, T(X))$. Define a lifting $\widetilde{M}$ for $M \in \mathcal{M}$ by sending $x \in M$ to $[T_x(M)] \in \mathcal{GM}$. Since M's are totally geodesic, $\widetilde{M}$ is disjoint from $\widetilde{M}_2$ if $M_2 \neq M_2$, and hence $\mathcal{GM}$ is foliated by $\widetilde{M}$'s. Write $\mathcal{F}$ for this holomorphic foliation. $\mathcal{GM}$ can now be regarded as a "universal family" of M's over the orbit-space $\mathcal{M}$. Define a lifted map $\tilde{f}\colon U \to \mathcal{GM}$ for f by sending x to $[T_{f(x)}(M_x)]$.

To prove Proposition 5.1 is then equivalent to proving that $\tilde{f}(U)$ lies in one single leaf. Think of the leaves of $\mathcal{F}$ as integral manifolds of a distribution $T(\mathcal{F})$; then by the Frobenius theorem it comes down to showing that $T(\tilde{f}(U))$ constitute a subdistribution of $T(\mathcal{F})$. By using the Harish-Chandra embedding M is complex linear in $\mathfrak{m}^+ \cong \mathbb{C}^n$ and we can identify $T(\widetilde{M}) \subset T(\mathcal{F})$ with $T(M)$, and therefore with M, too.

To show that $T(\tilde{f}(U))$ constitute a subdistribution, without loss of generality, it suffices to show that

$$\partial_x[T_{f(x)}M_x](o) \subseteq T_o(M_o).$$

By Lemma 5.2, $T_{f(x)}(M_x)$ is linearly spanned by

$$\{[[\mathfrak{m}^-, \partial_\mu f(x)], \partial_\nu f(x)]\}_{1\le\mu,\nu\le 3}.$$

Hence for $w \in T_o(U)$,

$$\begin{aligned}\partial_w[T_{f(x)}M_x](o) &\subseteq \langle \partial_w[[\mathfrak{m}^-, \partial_\mu f(x)], \partial_\nu f(x)]\rangle \\ &\subseteq \langle [[\mathfrak{m}^-, \partial_w\partial_\mu f(x)], \partial_\nu f(x)\rangle + \langle [[\mathfrak{m}^-, \partial_\mu f(x)], \partial_w\partial_\nu f(x)]\rangle.\end{aligned}$$

By Lemma 5.3 we have

$$\partial_\mu\partial_\nu f(o) \subseteq T_o(M_o).$$

Hence,

$$\begin{aligned}\partial_x[T_{f(x)}M_x](o) &\subseteq \langle [[\mathfrak{m}^-, T_o(M_o)], T_o(M_o)]\rangle \\ &\subseteq T_o(M_o),\end{aligned}$$

where the second inclusion is due to Lemma 4.3 since M_0 is invariantly geodesic; Proposition 5.1 follows. q.e.d.

Now we prove Lemma 5.2.

Proof of Lemma 5.2. It suffices to prove the following (stronger) statement:

(*) For Q^n, fix $[\alpha] \in \mathcal{S}_o(Q^n)$, and $\zeta \in \mathcal{N}_\alpha$ (see §1 for the definition of $\mathcal{N}_\alpha$); set $V = \mathbb{C}\alpha \oplus \mathbb{C}\zeta$. Then $\langle [[\mathfrak{m}^-, V], V]\rangle = T_o(Q^n)$.

We divide the proof into two steps:

(i) $n = 4$.

We use the identification $D_4^{\mathrm{II}} \cong D^{\mathrm{I}}(2, 2)$. $T_o(D^{\mathrm{I}}(2, 2))$ is the set of all 2×2 matrices. From the structure of Lie algebra $\mathfrak{a}_3$ one knows that there are four noncompact positive roots γ_{ij} $(1 \le i, j \le 2)$ with root vectors $e_{\gamma_{ij}}$ corresponding to E_{ij}'s, 2×2 matrices with (i, j) entry 1 and 0 elsewhere. Let

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e_{\gamma_{11}} \quad \text{and} \quad \zeta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = e_{\gamma_{22}}.$$

Then a direct verification shows

$$\mathbb{C}[[e_{-\gamma_{ij}}, e_{\gamma_{ii}}], e_{\gamma_{jj}}] = \mathbb{C}e_{\gamma_{ji}}, \qquad 1 \le i, j \le 2,$$

establishing (*) for $n = 4$.

(ii) $n \ge 5$.

The proof is based on (i). Write $Q^n = G_c/K$, and then $\mathscr{S}_o(Q^n) = K/K_\alpha$. Recall from §1 that

$$T_o(Q^n) = \mathbb{C}\alpha + \mathscr{H}_\alpha + \mathbb{C}\zeta,$$

since $\mathscr{N}_\alpha = \mathbb{C}\zeta_\alpha$ in the case of hyperquadrics, and that

$$T_{[\alpha]}(\mathscr{S}_o(Q^n)) \cong \mathscr{H}_\alpha.$$

When $n \geq 5$, $\mathscr{S}_o(Q^n) \cong Q^{n-2}$ is Hermitian symmetric and irreducible, one can consider the characteristic variety of $\mathscr{S}_o(Q^n)$ at $[\alpha]$:

$$\mathscr{S}_{[\alpha]}(\mathscr{S}_o(Q^n)) \subset \mathbb{P}T_{[\alpha]}(\mathscr{S}_o(Q^n)),$$

and the linear span

$$\langle \mathscr{S}_{[\alpha]}(\mathscr{S}_o(Q^n))\rangle = T_{[\alpha]}(\mathscr{S}_o(Q^n)) \cong \mathscr{H}_\alpha,$$

by the irreducibility. To prove the lemma it suffices therefore to show

$$\mathbb{C}\alpha + \mathbb{C}\zeta + \langle [\mathfrak{m}^-, \alpha], \zeta\rangle \supset \mathscr{S}_{[\alpha]}(\mathscr{S}_o(Q^n)).$$

Note first of all that there exists a totally geodesic embedding $Q^4 \hookrightarrow Q^n$ with $\mathscr{S}_o(Q^4) \subset \mathscr{S}_o(Q^n)$. Assume furthermore that $T_o(Q^4) \supseteq \mathbb{C}\alpha + \mathbb{C}\zeta$. Denote the transform $k_\alpha \cdot Q^4$ by M_α for some $k_\alpha \in K_\alpha$. $T_o(M_\alpha)$ contains $\mathbb{C}\alpha + \mathbb{C}\zeta$ (since $\mathbb{C}\,\mathrm{Ad}(k_\alpha)\cdot\alpha = \mathbb{C}\alpha$ and k_α is an isometry, we have $\mathbb{C}\,\mathrm{Ad}(k_\alpha)\cdot\zeta = \mathbb{C}\zeta$ as well), and therefore by (i) for $n = 4$ we are reduced to showing

$$\bigcup_{k_\alpha \in K_\alpha} \mathbb{P}T_o(M_\alpha) \supset \mathscr{S}_{[\alpha]}(\mathscr{S}_o(Q^n)).$$

Now $\mathscr{S}_o(Q^4) \hookrightarrow \mathscr{S}_o(Q^n) \hookrightarrow \mathbb{P}T_o(Q^n)$ contains a projecting line ℓ which passes through $[\alpha]$. More precisely in the notation of (i)

$$\mathbb{P}(\mathbb{C}e_{\gamma_{11}} + \mathbb{C}e_{\gamma_{12}})$$

is a projection line contained in $\mathscr{S}_o(Q^4)$ since one can easily see that

$$(\mathbb{C}e_{\gamma_{11}} + \mathbb{C}e_{\gamma_{12}}) \cap D^{\mathrm{I}}(2,2) \cong B^2.$$

Thus

$$\mathbb{P}T_{[\alpha]}(\ell) \in \mathscr{S}_{[\alpha]}(\mathscr{S}_o(Q^n)).$$

Since K_α acts transitively on $\mathscr{S}_{[\alpha]}(\mathscr{S}_o(Q^n))$ and $\mathbb{P}T_{[\alpha]}(\ell)$ is identified with $[e_{\gamma_{12}}] \in \mathbb{P}T_o(Q^4)$, our assertion follows and the lemma is proved. q.e.d.

We turn now to

Proof of Lemma 5.3. f is characteristic; thus $[\partial_\alpha f(x)] \in \mathcal{S}_{f(x)}(X)$ for $[\alpha] \in \mathcal{S}_x(Q^3)$. Note that in the following we have used Harish-Chandra embeddings so that $\{\mathcal{S}_p(Q^3)\}_{p\in\mathbb{C}^3}$ and $\{\mathcal{S}_p(X)\}_{p\in\mathbb{C}^n}$ are parallel in the sense of Euclidean geometry [16]. Hence we have

$$\partial_\beta\partial_\alpha f(o) \in T_{[\alpha]}(\mathcal{S}_o(X)) \quad \text{for any } \beta \in T_o(Q^3).$$

Suppose furthermore that $[\beta] \in \mathcal{S}_o(Q^3)$. Then,

$$\partial_\beta\partial_\alpha f(o) = \partial_\alpha\partial_\beta f(o) \in T_{[\beta]}(\mathcal{S}_o(X)).$$

By identifying $\mathcal{H}_\alpha$ with $T_{[\alpha]}(\mathcal{S}_o(X))$ (§1) we have

$$\partial_\alpha\partial_\beta f(o) \in \mathcal{H}_\alpha \cap \mathcal{H}_\beta$$

if $[\alpha], [\beta] \in \mathcal{S}_o(Q^3)$. In the case of quadrics Q^n it is known that if $[\alpha] \in \mathcal{S}_o(Q^n)$, then $\mathcal{N}_\alpha$ is of dimension 1 and $[\mathcal{N}_\alpha] \in \mathcal{S}_o(Q^n)$ [18]. We assert

Lemma 5.4. *In the notation as above, there exists some* $[\alpha] \in \mathcal{S}_o(Q^3)$ *such that*

$$\mathcal{H}_\alpha \cap \mathcal{H}_\zeta \subset M$$

for $\zeta \in \mathcal{N}_\alpha(Q^3)$.

Assuming Lemma 5.4, we continue with the proof of Lemma 5.3. Let α_0 and ζ_0 be the vectors satisfying Lemma 5.4. By using the polarization [16] it suffices to show

$$\partial_\alpha\partial_\zeta f(o) \subseteq T_o(M)$$

for any $[\alpha] \in \mathcal{S}_o(Q^3)$ and $\zeta \in \mathcal{N}_\alpha(Q^3)$. By the previous discussion we are reduced to showing that

$$\mathcal{H}_\alpha \cap \mathcal{H}_\zeta \subset T_o(M)$$

for α, ζ as above. Now

$$df_o(\mathcal{S}_o(U)) \in \mathcal{D}_2(\mathcal{S}_o(X)),$$

and therefore $f(U)$ has the same tangent space as a totally geodesic submanifold N ($\cong Q^3$) does at o (up to $K^{\mathbb{C}}$-transformations; see Corollary 4.7). Since $df_o(\mathcal{S}_o(U)) = \mathcal{S}_o(N)$, one can then find an isometry k of N fixing o such that $k \cdot \alpha_0 = \alpha$ by using the transitivity. Since $\dim \mathcal{N}_\alpha(Q^3) = \dim \mathcal{N}_{\alpha_0}(Q^3) = 1$, one has also $k \cdot \zeta_0 = \zeta$. Therefore,

$$\mathcal{H}_\alpha \cap \mathcal{H}_\zeta = \mathcal{H}_{k\cdot\alpha_0} \cap \mathcal{H}_{k\cdot\zeta_0} = k \cdot \mathcal{H}_{\alpha_0} \cap k \cdot \mathcal{H}_{\zeta_0} \subset k \cdot T_o(M).$$

Now k extends to an isometry of $M \supset N$; we have $k \cdot M = M$. The assertion follows, and hence Lemma 5.3 is proved. q.e.d.

The remainder of this section will be devoted to the proof of Lemma 5.4.

Proof of Lemma 5.4. We shall prove the lemma case by case. The problems being local, we can work with the noncompact dual Ω of X.

Case (i). $\Omega = D^{\mathrm{I}}(2, n)$; $M \cong D_4^{\mathrm{II}}$.

Let $i: D_4^{\mathrm{IV}} \cong D^{\mathrm{I}}(2, 2) \hookrightarrow D^{\mathrm{I}}(2, n)$ be the embedding such that

$$T_o(D^{\mathrm{I}}(2, 2) = \{B = (b_{ij}) \in M(2, n; \mathbb{C}) = T_o(D^{\mathrm{I}}(2, n)) \colon b_{ij} = 0 \text{ for } j \geq 3\},$$

and $j: D_3^{\mathrm{IV}} \hookrightarrow D_4^{\mathrm{IV}}$ be the embedding such that

$$T_o(D_3^{\mathrm{IV}}) = \{B \in T_o(D^{\mathrm{I}}(2, 2)) \colon {}^tB = B\}.$$

Then i, j are totally geodesic embeddings and are characteristic. Now let

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{S}_o(D_3^{\mathrm{IV}}).$$

By using the curvature formula [18]:

$$R_{X\overline{X}Y\overline{Y}} = -\|{}^tX\overline{Y}\|^2 - \|X{}^t\overline{Y}\|^2$$

for X, $Y \in T_o(D^{\mathrm{I}}(2, n))$, one sees that

$$\zeta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{N}_\alpha(D_3^{\mathrm{IV}}),$$

$$\mathscr{N}_\alpha(D^{\mathrm{I}}(2, n)) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \end{pmatrix}_{2\times n},$$

$$\mathscr{N}_\zeta(D^{\mathrm{I}}(2, n)) = \begin{pmatrix} * & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2\times n},$$

where $(*)$ denotes constants. Hence

$$\mathscr{H}_\alpha \cap \mathscr{H}_\zeta \subset N_\alpha^\perp \cap N_\zeta^\perp = \begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ * & 0 & 0 & \ldots & 0 \end{pmatrix} \subset D_4^{\mathrm{IV}} = M,$$

as claimed.

Case (ii). $D = D^{\mathrm{II}}(5, 5)$; $M \cong D_6^{\mathrm{IV}}$.

We have

$$T_o(D^{\mathrm{II}}(5, 5)) = \{B \in M(5, 5; \mathbb{C}) = T_o(D^{\mathrm{I}}(5, 5)) \colon {}^tB = -B\},$$

and

$$T_o(D^{\mathrm{II}}(4, 4)) = \{B = (b_{ij}) \in T_o(D^{\mathrm{II}}(5, 5)) \colon b_{i5} = b_{5i} = 0,\ 1 \leq i \leq 5\}.$$

Let

$$\alpha = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & 0 \\ & 0 & 0 \end{pmatrix} \in \mathscr{S}_0(D^{\mathrm{II}}(5,5));$$

then

$$\zeta = \begin{pmatrix} 0 & & 0 & \\ & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix} \in \mathscr{N}_\alpha .$$

By using the curvature formula as in case (i), one has

$$\mathscr{N}_\alpha = \begin{pmatrix} 0 & & 0 & \\ & 0 & -C_1 & -C_2 \\ 0 & C_1 & 0 & -C_3 \\ & C_2 & C_3 & 0 \end{pmatrix}$$

and

$$\mathscr{N}_\zeta = \begin{pmatrix} 0 & -C_4 & 0 & 0 & -C_5 \\ C_4 & 0 & 0 & 0 & -C_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_5 & C_6 & 0 & 0 & 0 \end{pmatrix},$$

where C_i's $(1 \le i \le 6)$ are constants. Hence,

$$\mathscr{H}_\alpha \cap \mathscr{H}_\zeta \subset (\mathscr{N}_\alpha + \mathscr{N}_\zeta)^\perp \subset T_o(D^{\mathrm{II}}(4,4)).$$

Case (iii). $\Omega = E^6/\operatorname{spin}(10) \times S^1$; $M \cong D_8^{\mathrm{IV}}$.

To prove the lemma in this case we are going to write down explicitly an invariantly geodesic submanifold isomorphic to Q^8. The existence of such a Q^8 (or D_8^{IV}) is guaranteed by the proof of Proposition 4.6. We shall make use of Lemma 5.2 to do the job.

To start with, the root system associated to E_6 is listed as follows (cf. [Z]):

simple roots:	$x_i - x_{i+1}$, $1 \le i \le 5$; $x_4 + x_5 + x_6$
positive roots:	$x_i - x_j$, $1 \le i < j \le 6$; $x_i + x_j + x_k$, $1 < i < j < k \le 6$; $\sum_{i=1}^{6} x_i$;
positive noncompact roots:	$x_1 - x_i$, $2 \le i \le 6$; $x_1 + x_i + x_j$, $2 \le i < j \le 6$;
maximal strongly orthogonal noncompact positive roots:	$x_1 - x_2$; $x_1 + x_2 + x_3$.

Without loss of generality we assume

$$\alpha = e_{x_1 - x_2}, \quad \zeta = e_{x_1+x_2+x_3}.$$

By the proof of Lemma 5.2, we have

$$\mathbb{C}\alpha + \mathbb{C}\zeta + \langle[[\mathfrak{m}^-(Q^8), \alpha], \zeta]\rangle = T_o(Q^8).$$

Since Q^8 is invariantly geodesic, using Lemma 4.1 we obtain

$$\langle[[\mathfrak{m}^-(E_6), \alpha], \zeta]\rangle \subseteq T_o(Q^8).$$

Now $\mathfrak{m}^-(Q^8) \subset \mathfrak{m}^-(E_6)$; the above then yields

$$(*) \qquad \mathbb{C}\alpha + \mathbb{C}\zeta + \langle[[\mathfrak{m}^-(E_6), \alpha], \zeta]\rangle = T_o(Q^8).$$

We are going to find $T_o(Q^8)$ by using $(*)$. To do this we have to calculate

$$[[\mathfrak{m}^-(E_6), \alpha], \zeta].$$

In the following by $V_1 \sim V_2$ we mean $\mathbb{C}V_1 = \mathbb{C}V_2$. We have

$$[[e_{-(x_1-x_3)}, e_{x_1-x_2}], e_{x_1+x_2+x_3}] \sim [e_{x_3-x_2}, e_{x_1+x_2+x_3}] = 0$$

since $-(x_1-x_3)+(x_1-x_2) = x_3-x_2$ is a root and $(x_3-x_2)+(x_1+x_2+x_3) = 2x_3 + x_1$ is not a root by the preceding list of roots. Likewise, we have:

$$\begin{aligned}
&[[e_{-(x_1-x_i)}, e_{x_1-x_2}], e_{x_1+x_2+x_3}] \sim e_{x_1+x_3+x_i}, && 4 \le i \le 6;\\
&[[e_{-(x_1+x_3+x_i)}, e_{x_1-x_2}], e_{x_1+x_2+x_3}] \sim e_{x_1-x_i}, && 4 \le i \le 6;\\
&[[\eta, e_{x_1-x_2}], e_{x_1+x_2+x_3}] = 0,
\end{aligned}$$

for η being the following vectors:

$$e_{-(x_1+x_5+x_6)}, \quad e_{-(x_1+x_2+x_j)} (2 < j \le 6) \quad \text{and} \quad e_{-(x_1+x_4+x_j)} (j = 5, 6).$$

Hence we conclude that $T_o(Q^8)$ is linearly spanned by the following vectors:

$$(*) \qquad \begin{matrix}
e_{x_1-x_2}, & e_{x_1-x_6}, & e_{x_1+x_3+x_5},\\
e_{x_1-x_4}, & e_{x_1+x_2+x_3}, & e_{x_1+x_3+x_6},\\
e_{x_1-x_5}, & e_{x_1+x_3+x_4}. &
\end{matrix}$$

Next we want to show that

$$\mathscr{H}_\alpha \cap \mathscr{H}_\zeta \subseteq T_o(Q^8)$$

or equivalently

$$T_o(Q^8)^\perp \subseteq \mathscr{N}_\alpha + \mathscr{N}_\zeta.$$

By the classification of boundary components, the maximal boundary components of $\Omega \cong E^6/\operatorname{spin}(10)\times S^1$ are isomorphic to $\mathbb{B}^5$. Hence $\dim \mathcal{N}_\alpha = \dim \mathcal{N}_\zeta = 5$ by Proposition 1.8 of [20]. Now $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = \|[\alpha, \bar{\beta}]\|^2$ so that

$$\beta \in \mathcal{N}_\alpha$$

if and only if

$$[\alpha, \bar{\beta}] = 0.$$

By a direct verification, we have

$$(**)\quad \begin{aligned} \mathcal{N}_\alpha &= \langle e_{x_1+x_2+x_3}, e_{x_1+x_2+x_4}, e_{x_1+x_2+x_5}, e_{x_1+x_2+x_6}, e_{x_1+x_2+x_3+x_4+x_5+x_6}\rangle, \\ \mathcal{N}_\zeta &= \langle e_{x_1-x_2}, e_{x_1-x_3}, e_{x_1+x_4+x_5}, e_{x_1+x_4+x_6}, e_{x_1+x_5+x_6}\rangle. \end{aligned}$$

In view of $(*)$, $(**)$ and the root system for E_6, we obtain that

$$\mathcal{N}_\alpha + \mathcal{N}_\zeta = T_o(Q^8)^\perp + \mathbb{C}e_{x_1-x_3} + \mathbb{C}e_{x_1+x_2+x_3},$$

proving the desired. The proof of Lemma 5.4 is now completed.

6. Proof of Proposition 3.1

We will prove Proposition 3.1 in this section. Our proof is based on a case-by-case examination. Let $\Omega \ncong D_n^{\mathrm{IV}}$ be an irreducible bounded symmetric domain of rank 2. Then Ω is biholomorphic to one of the following domains:

$$D^{\mathrm{I}}(2, n), \quad D^{\mathrm{II}}(5, 5), \quad \text{and} \quad D^{\mathrm{V}}$$

where D^{V} corresponds to the exceptional Lie algebra E_6. A precise description of D^{I} and D^{II} domains is given as follows.

$$\begin{aligned} D^{\mathrm{I}}(p, q) &= \{Z \in M(p, q; \mathbb{C}) \cong \mathbb{C}^{pq} : I_q - {}^t\bar{Z}Z > 0\}, \\ D^{\mathrm{II}}(n, n) &= \{Z \in D^{\mathrm{I}}(n, n) : {}^tZ = -Z\}. \end{aligned}$$

One has also their characteristic varieties [18]:

$$\begin{aligned} \mathcal{S}_o(D^{\mathrm{I}}(2, n)) &\cong \mathbb{P}^1 \times \mathbb{P}^{n-1}, \\ \mathcal{S}_o(D^{\mathrm{II}}(5, 5)) &\cong G(2, 3), \\ \mathcal{S}_o(D^{\mathrm{V}}) &\cong G^{\mathrm{II}}(5, 5), \end{aligned}$$

where $G(2, 3)$ and $G^{\mathrm{II}}(5, 5)$ are the compact duals of $D^{\mathrm{I}}(2, 3)$ and $D^{\mathrm{II}}(5, 5)$ respectively. Moreover, the inclusion map $i: \mathcal{S}_o(\Omega) \hookrightarrow \mathbb{P}T_o(\Omega)$ is identified with the first canonical embedding of $\mathcal{S}_o(\Omega)$ (this is true for all irreducible bounded symmetric domains except Type-III domains) [22], [18].

Throughout this section set $X = \mathscr{S}_o(\Omega)$, and fix a bisphere $\mathbb{P}^1 \times \mathbb{P}^1$ of X, containing the origin o. Write φ_s for the composite map

$$\mathbb{P}^1 \overset{\Delta}{\hookrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow X,$$

where Δ is the diagonal embedding, and C_s for $\varphi_s(\mathbb{P}^1)$. The proof for $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ uses the following lemma.

Lemma 6.1. *Let $\varphi = (\varphi_1, \varphi_2): \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{2n-1}$ be an embedded rational curve of degree 2 in $\mathbb{P}^{2n-1}$. Then either*

(i) *$\varphi_1 \in \mathrm{Aut}(\mathbb{P}^1)$ and φ_2 is a projective linear embedding or*

(ii) *φ_1 is a constant map and $\varphi_2(\mathbb{P}^1)$ is an embedded rational curve of degree 2 in $\mathbb{P}^{n-1}$.*

Proof. Let H denote the hyperplane line bundle on projective spaces. The map $i: \mathscr{S}_1(\Omega) \to \mathbb{P}T(\Omega)$ is in fact the Segré map given by the complete linear system of the line bundle $\pi_1^* H \otimes \pi_2^* H$ on $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ [18], [9]. By the assumption that $\varphi(\mathbb{P}^1)$ is of degree 2, one has

$$\deg(H|_{\pi_1(\varphi(\mathbb{P}^1))}) + \deg(H|_{\pi_2(\varphi(\mathbb{P}^1))}) = 2.$$

If $d_1 = \deg(H|_{\pi_1(\varphi(\mathbb{P}^1))}) = 1$ and $d_2 = \deg(H|_{\pi_2(\varphi(\mathbb{P}^1))}) = 1$, then (i) occurs. If $d_1 = 0$ and $d_2 = 2$, then (ii) occurs. The case where $d_1 = 2$ and $d_2 = 0$ cannot occur since otherwise φ would be ramified. Hence the proof is completed.

Now each $\mathbb{P}^2_C \subseteq X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ is of the form $\{\mathrm{pt}\}\times$ a projective 2-plane. Clearly $\mathrm{Aut}_0(X)$ acts transitively on the set of those $\mathbb{P}^2_C$'s contained in X; so does $\mathrm{Aut}_0(\mathbb{P}^2_C) \subset \mathrm{Aut}_0(X)$ on the set of all rational curves of degree 2 in $\mathbb{P}^2_C$. The transitivity of the $\mathrm{Aut}_0(X)$-action on $\mathscr{D}_1$ follows; so does it on $\mathscr{D}_2$ from the lemma. Hence Proposition 3.1 in the case $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ is proved.

To prove Proposition 3.1 in the remaining cases we start with

Lemma 6.2. *Let X be an irreducible compact Hermitian symmetric space and C be an embedded rational curve of degree 2 such that $[C] \in \mathscr{D}_2$. Then $\mathbb{P}(T(C)) \cap \mathscr{S}(X) = \varnothing$.*

Proof. Suppose otherwise. Fix $[v] \in \mathbb{P}T_o(C) \cap \mathscr{S}_o(X)$. Then we can find a minimal rational curve ℓ of X such that $T_o(\ell) = \mathbb{C}v$. Since X is seated in some $\mathbb{P}^N$ via its first canonical embedding and $\mathbb{P}^2_C$ is projective linear in $\mathbb{P}^N$, it follows that ℓ is a projective line in $\mathbb{P}^N$ and is therefore contained in $\mathbb{P}^2_C$. Since $C \neq l$, $\deg(\mathbb{P}^2_C \cap X) \geq \deg(C \cup \ell) \geq 3$. On the other hand considering C_s, we have $\mathbb{P}^2_C \cap X = C_s$. Hence $\deg(\mathbb{P}^2_{C_s} \cap X) = 2$. This is a contradiction unless $\mathbb{P}^2_C \subset X$. This again violates the choice of C; the lemma is proved.

Proof of Proposition 3.1 *in the case* $X \cong G(2, 3) \hookrightarrow \mathbb{P}^9$. To facilitate the proof we work with the noncompact dual

$$D^{\mathrm{I}}(2, 3) \hookrightarrow G(2, 3).$$

By the preceding lemma and the polydisk theorem (cf. [30]) we can assume without loss of generality that

$$\varphi(0) = o, \qquad d\varphi(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

for the rational curve $\varphi\colon \mathbb{P}^1 \hookrightarrow G(2, 3)$ of degree 2. Suppose

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) & \varphi_2(t) & \varphi_5(t) \\ \varphi_3(t) & \varphi_4(t) & \varphi_6(t) \end{bmatrix},$$

where the φ_i, $1 \leq i \leq 6$, are rational functions. We claim first of all that

$$\varphi_5 \equiv \varphi_6 \equiv 0.$$

One knows that the first canonical embedding of the Grassmannian $G(n, m)$ is just its Plücker embedding. Under the Plücker embedding the point

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in \mathfrak{m}^+ \hookrightarrow G(2, 3)$$

is mapped to the point in $\mathbb{P}^9$ with coordinates given by determinants of all 2×2 submatrices of the following matrix [9]:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 \end{bmatrix}.$$

Let Δ be the least common denominator of the (inhomogeneous) coordinate-functions of φ by the Plücker embedding. Since $C = \varphi(\mathbb{P}^1)$ is of degree 2 in $\mathbb{P}^9$, one has that the degree of Δ must be less than or equal to two. Since the functions φ_5 and φ_6 are still coordinate-functions under Plücker embedding, by using the condition on $\varphi(0)$ and $d\varphi(0)$ we can set

$$\varphi_5 = \frac{at^2}{\Delta} \quad \text{and} \quad \varphi_6 = \frac{bt^2}{\Delta},$$

for some $a, b \in \mathbb{C}$. We have to show that $a = b = 0$. Suppose otherwise. Write

$$\varphi_i = \frac{t}{\Delta}\varphi_i', \qquad 1 \leq i \leq 4.$$

Then from the assumption on $d\varphi(0)$ it follows that

$$\begin{vmatrix} \varphi_1' & \varphi_2' \\ \varphi_3' & \varphi_4' \end{vmatrix}$$

is not identically zero; hence neither is

$$\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{vmatrix}.$$

Thus, without loss of generality we assume

$$\begin{vmatrix} \varphi_1 & \varphi_5 \\ \varphi_3 & \varphi_6 \end{vmatrix}$$

is not identically zero. But

$$\begin{vmatrix} \varphi_1 & \varphi_5 \\ \varphi_3 & \varphi_6 \end{vmatrix} = \frac{t^3}{\Delta^2} \begin{vmatrix} \varphi_1' & a \\ \varphi_3' & b \end{vmatrix}.$$

By the Plücker embedding

$$(*) \qquad \Delta \cdot \begin{vmatrix} \varphi_1 & \varphi_5 \\ \varphi_3 & \varphi_6 \end{vmatrix} = \frac{t^3}{\Delta} \begin{vmatrix} \varphi_1' & a \\ \varphi_3' & b \end{vmatrix}$$

should be a polynomial of degree ≤ 2. From the assumption that $\varphi(0) = 0$, we have

$$t \nmid \Delta.$$

Hence

$$\frac{1}{\Delta} \cdot \begin{vmatrix} \varphi_1' & a \\ \varphi_3' & b \end{vmatrix}$$

must be a polynomial. This will contradict the condition that $(*)$ is a polynomial of degree ≤ 2 unless

$$\begin{vmatrix} \varphi_1' & a \\ \varphi_3' & b \end{vmatrix} \equiv 0,$$

again a contradiction to our choice that

$$\begin{vmatrix} \varphi_1 & \varphi_5 \\ \varphi_3 & \varphi_6 \end{vmatrix} \not\equiv 0.$$

Our claim: $\varphi_5 \equiv \varphi_6 \equiv 0$ is now proved.

Next we are going to find a $g \in \mathrm{Aut}_0(G(2,3))$ such that $g(C) = C_s$. As before

$$\varphi_i = \frac{t\varphi_i'}{\Delta}, \qquad 1 \leq i \leq 4,$$

is one of the coordinate-functions of φ by the Plücker embedding; the φ_i''s can therefore only be linear functions. We claim that

$$(\sharp) \qquad \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix}^{-1} = \frac{1}{t}\begin{pmatrix} \varphi_4' & -\varphi_2' \\ -\varphi_3' & -\varphi_1' \end{pmatrix}.$$

Assuming the claim $(\sharp)$, write

$$\begin{aligned} \varphi_4' &= C_{11}t + 1, & -\varphi_3' &= C_{21}t, \\ -\varphi_2' &= C_{12}t, & \varphi_1' &= C_{22}t + 1, \end{aligned}$$

and define $g \in \exp(\mathfrak{m}^-)$ to be the element such that

$$g \cdot Z = Z(CZ + I_{3\times 3})^{-1},$$

where

$$C \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ 0 & 0 \end{pmatrix},$$

and Z is any 2×3 matrix. Then a direct computation by using $\varphi_5 \equiv \varphi_6 \equiv 0$ shows that $g(C_s) = C$,

$$\left(\text{note that } C_s = \varphi_s(\mathbb{P}^1), \qquad \varphi_s(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \end{bmatrix}\right),$$

proving the proposition. To see $(\sharp)$, note that

$$\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{vmatrix} = \frac{t^2}{\Delta^2}\begin{vmatrix} \varphi_1' & \varphi_2' \\ \varphi_3' & \varphi_4' \end{vmatrix} = \frac{t^2}{\Delta^2}\delta \not\equiv 0.$$

As in the first half of the proof

$$\Delta \cdot \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{vmatrix} = \frac{t^2}{\Delta}\delta$$

is a polynomial of degree ≤ 2. This yields, since $t \nmid \Delta$ as before, that $\Delta = C\delta$ for some constant C. The normalization on $\varphi(0)$ and $d\varphi(0)$ gives $C = 1$. Hence,

$$\begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix}^{-1} = \frac{\Delta}{t^2}\begin{pmatrix} \varphi_4 & -\varphi_2 \\ -\varphi_3 & \varphi_1 \end{pmatrix}.$$

Substituting

$$\varphi_i = \frac{t}{\Delta}\varphi_i'$$

into the right-hand side of the above equality proves the claim $(\sharp)$. The proof of Proposition 3.1 in the case $X \cong G(2, 3)$ is now completed.

We turn now to the proof of Propositoin 3.1 in the case $X \cong G^{\mathrm{II}}(5, 5)$. The proof is analogous to the previous case. We work with the noncompact dual $D^{\mathrm{II}}(5, 5)$. As described in the beginning of this section, $D^{\mathrm{II}}(5, 5)$ is an open subset of the space of all 5×5 antisymmetric matrices over $\mathbb{C}$. Moreover a typical characteristic vector

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right] \in T_o(D^{\mathrm{II}}(5, 5))$$

is of rank 2. Conversely any rank two, 5×5 antisymmetric matrix is a characteristic vector [18]. Define $\varphi_s \colon \mathbb{P}^1 \hookrightarrow G^{\mathrm{II}}(5, 5)$ to be

$$\varphi_s(t) = \left[\begin{array}{cccc|c} 0 & -t & & & \\ & & 0 & & \\ t & 0 & & & \\ & & 0 & -t & 0 \\ & 0 & t & 0 & \\ \hline & & 0 & & 0 \end{array}\right].$$

Normalize $\varphi \colon \mathbb{P}^1 \hookrightarrow G^{\mathrm{II}}(5, 5)$ in such a way that $\varphi(0) = 0$ and $d\varphi(0) = d\varphi_s(0)$. Let φ_{ij} be coordinate-functions of φ, $1 \le i, j \le 5$. Clearly Proposition 3.1 in this case will follow from the following two lemmas.

Lemma 6.3. $\varphi_{i5} \equiv -\varphi_{5i} \equiv 0$, $1 \le i \le 5$. *Consequently* $\varphi(C) \subset G^{\mathrm{II}}(4, 4) \cong Q^6$.

Lemma 6.4. *Suppose* C_1 *and* C_2 *are rational curves of degree* 2 *in* Q^n, $n \ge 4$, *such that* $\mathbb{P}^2_{C_i} \not\subset Q^n$ $(i = 1, 2)$. *Then* $g(C_1) = C_2$ *for some* $g \in \mathrm{Aut}_0(Q^n)$.

Proof of Lemma 6.3. Our proof is to make use of the composite map j:

$$G^{\mathrm{II}}(5, 5) \hookrightarrow G(5, 5) \hookrightarrow \mathbb{P}^N, \qquad N = \binom{10}{5} - 1,$$

where the first map is the inclusion and the second map is the Plücker embedding. One sees that a minimal rational curve ℓ of $G^{\mathrm{II}}(5, 5)$ (which homologically generates $H_2(G^{\mathrm{II}}(5, 5), \mathbb{Z})$; cf. §1) is mapped by j to a rational curve of degree 2 in $\mathbb{P}^N$; j is in fact the second canonical embedding of $G^{\mathrm{II}}(5, 5)$. Thus $C = \varphi(\mathbb{P}^1)$ is a rational curve of degree 4 in $\mathbb{P}^N$. As the proof of the lemma is similar to that in the previous case, we give only a sketch.

Let Δ be the least common denominator of inhomogeneous coordinate-functions of φ by j. The normalization condition on $\varphi(0)$ and $d\varphi(0)$

gives

$$\varphi_{i5}(5) = \frac{t^2}{\Delta} \cdot \text{ a polynomial}, \qquad 1 \le i \le 4.$$

Suppose that $\varphi_{i5}(t) \not\equiv 0$ for some $i \ge 4$ ($\varphi_{55}(t) \equiv 0$ always). By using the normalization on $\varphi(0)$ and $d\varphi(0)$ we can replace some column of the matrix $(\varphi_{ij}(t))_{1\le i,j\le 4}$ by $(\varphi_{i5}(t))_{1\le i\le 4}$ so that the determinant of the resulting matrix $(\tilde{\varphi}_{ij}(t))_{1\le i,j\le 4}$ is not identically zero. Now by normalization one has

$$\varphi_{ij}(t) = \frac{t}{\Delta}\psi_{ij},$$

where ψ_{ij} are polynomials. Thus the determinant of $(\tilde{\varphi}_{ij}(t))$ is equal to

$$|(\tilde{\varphi}_{ij}(t))| = \frac{t^5}{\Delta^4} \cdot P(t),$$

for some polynomial $P(t) \not\equiv 0$. We are going to deduce a contradiction. $|(\tilde{\varphi}_{ij})|$ can be seen to be one of coordinate-functions by j; thus $\Delta \cdot |(\tilde{\varphi}_{ij})| = (t^5/\Delta^3) \cdot P(t)$ must be a polynomial of degree less than or equal to four. From $t \nmid \Delta$, it follows that

$$\frac{t^5}{\Delta^3}P(t) = t^5 \cdot \left(\frac{P(t)}{\Delta^3}\right)$$

must be a polynomial of degree greater than four if $P(t) \not\equiv 0$. Hence our assertion that $\varphi_{i5}(t) \equiv 0$, $1 \le i \le 4$, follows.

Proof of Lemma 6.4. The lemma is true for $n = 4$ as shown in the proof of the case $X \cong G(2, 3)$. For $n \ge 5$ it suffices to find some $\chi \in \mathrm{Aut}_0(Q^n)$ such that $\chi(C_i) \subset Q^{n-1}$ for some totally geodesic submanifold Q^{n-1}. Then, from $\mathrm{Aut}_0(Q^{n-1}) \subset \mathrm{Aut}_0(Q^n)$ and induction the lemma follows.

Let $\varphi\colon \mathbb{P}^1 \hookrightarrow Q^n$ be either of the rational curves stated in the lemma, and C_s be the curve as before. Q^n is isomoprhic to a hypersurface in $\mathbb{P}^{n+1}$ defined by

$$Z_0 Z_{n+1} = \sum_{i=1}^{n} Z_i^2.$$

Let $Q^{n-1} \subset Q^n$ be defined by

$$Z_0 Z_{n+1} = \sum_{i=2}^{n} Z_i^2.$$

One can see that Q^{n-1} is totally geodesic in Q^n. Let φ_i $(0 \le i \le n+1)$ be the coordinate-functions of $\varphi\colon \mathbb{P}^1 \hookrightarrow Q^n \hookrightarrow \mathbb{P}^{n+1}$. Assume that $\varphi(0) =$

$(1, 0, \cdots, 0)$. $\varphi(\mathbb{P}^1)$ being of degree 2, then φ_i's can be chosen to be polynomials of degree less than or equal to two. Write

$$\varphi_i = a_i t^2 + b_i t, \qquad 1 \le i \le n,$$

by the normalization of $\varphi(0)$. We claim that there exist constants e_j's $\in \mathbb{C}$, $1 \le j \le n$, such that

$$(*) \qquad \sum_{j=1}^{n} e_j \varphi_j(t) \equiv 0 \quad \text{and} \quad \sum_{j=1}^{n} e_j^2 = 1.$$

Granting the claim we can then find constants e_{ij}'s $\in \mathbb{C}$, $2 \le i \le n$, $1 \le j \le n$, together with $e_{1_j} \equiv e_j$ such that the map χ' defined by sending

$$Z_i \to \sum_{j=1}^{n} e_{ij} Z_j, \qquad 1 \le i \le n,$$

preserves the quadratic form

$$Z_1^2 + Z_2^2 + \cdots + Z_n^2.$$

Define $\chi \in \mathrm{Aut}(\mathbb{P}^{n+1})$ by

$$\chi(Z_0) = Z_0,$$
$$\chi(Z_{n+1}) = Z_{n+1},$$
$$\chi(Z_i) = \chi'(Z_i), \qquad 1 \le i \le n.$$

Then clearly $\chi \in \mathrm{Aut}_0(Q^n)$. By the construction of χ', we have $\chi(C) \subset Q^{n-1}$, and the proof is then finished.

We turn now to the proof of the claim $(*)$. Actually we will take e_j's from $\mathbb{R}$. Recall that $\varphi_i = a_i t^2 + b_i t$ for $1 \le i \le n$. The condition

$$\sum_{j=1}^{n} e_j \varphi_j \equiv 0$$

is equivalent to

$$\sum_{j=1}^{n} e_j a_j = 0 \text{ and } \sum_{j=1}^{n} e_j b_j = 0.$$

If e_j's $\in \mathbb{R}$, by separating the real parts and imaginary parts of the above system we get an equivalent system of four linear equations over $\mathbb{R}$. Since $n \ge 5$, the system has a nontrivial solution e_j's $\in \mathbb{R}$. By normalizing e_j's, the condition

$$\sum_{j=1}^{n} e_j^2 = 1$$

can also be satisfied. The proof of Lemma 6.4 and the proof of Proposition 3.1 are now completed.

Remark. Lemma 6.4 is also true for $n = 3$ as can be seen from the proof in which $X \cong G(2, 3)$ by using the (totally geodesic) embedding

$$i\colon D_3^{\mathrm{IV}} \hookrightarrow D^{\mathrm{I}}(2, 2) \simeq D_4^{\mathrm{IV}}$$
$$i(D_3^{\mathrm{IV}}) = \{Z \in D^{\mathrm{I}}(2, 2)\colon {}^tZ = Z\}.$$

Then the construction of g such that $g(C) = C_s$ shows that $g \in \mathrm{Aut}(Q^3)$ if $C \subset Q^3$.

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