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METRIC PROPERTIES OF MANIFOLDS BIMEROMORPHIC TO COMPACT KÄHLER SPACES

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Introduction

A goal of this paper is to prove that: "Every compact complex manifold M bimeromorphic to a compact Kähler manifold M' is balanced; that is, M has a hermitian metric with Kähler form ω such that $d\omega^{N-1} = 0$, $N = \dim M$" (Corollary 4.5). Of course, every Kähler manifold is balanced; the interest of the above result stems from the fact that we find out a metric property which transfers from M' to M, while it is well known that the Kähler property is not stable under bimeromorphic maps.

This introduction is mainly devoted to outline the background.

Let M and $\widetilde{M}$ be compact complex manifolds and $f\colon \widetilde{M} \to M$ be a modification. It is well known that:

(1) If f is a blow-up of M with smooth center and M is Kähler, then $\widetilde{M}$ is Kähler too [4],

however

(2) in general, if f is a modification and M is Kähler, $\widetilde{M}$ fails to be Kähler.

A counterexample is given in [12, p. 505] by a compact non-Kähler threefold X and a modification $f\colon X \to \mathbf{P}_3$. In order to illustrate Chow's lemma, Hironaka builds up also a projective threefold Y and a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & X \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle f} \\ & & \mathbf{P}_3 \end{array}$$

where g and h are obtained as a finite sequence of blow-ups with smooth centers.

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Let us consider for a moment the threefold X. Since a compact Kähler manifold cannot contain any complex curve homologous to zero, but X contains such a curve (see [17, Chapter VIII,3.3.] or [9, p. 444]), it is not Kähler. On the other hand, a compact balanced manifold contains no hypersurfaces homologous to zero and neither does X by construction: so X could be balanced. This property of compact balanced manifolds has a weak converse, if you look at hypersurfaces as positive currents of degree $(1,1)$. Indeed Michelson proved that a compact complex manifold is balanced if and only if it carries no positive currents of degree $(1, 1)$ which are components of a boundary [16, Proposition 4.5]. This result suggested that X is balanced, as we proved in [1]; yet it is only a particular case of the following general statement:

(3) If M is Kähler, then $\widetilde{M}$ is balanced [2].

Balanced manifolds have been studied from a differential point of view in [6]; other results and examples can be found of course in [16]. (3) shows how balanced manifolds can be "produced" in a very natural way by using modifications. Besides, we can also "pull back" the property of having a balanced metric (not the balanced metric itself, in general!) as is shown in [3]:

(3′) If M is balanced, $\widetilde{M}$ is balanced too.

(3′) proves that going from M to $\widetilde{M}$ via f no new hypersurface (and also positive current) which is the component of a boundary can appear (on the contrary, new curves may appear as $f\colon X \to \mathbf{P}_3$ shows).

Thus the following question arises naturally: Is the class of compact balanced manifolds invariant by modifications? In other words, can statement (3′) be reversed: If $\widetilde{M}$ is balanced, is M balanced too? The problem looks interesting even in the simplest case because, as the modification $g\colon Y \to X$ shows,

(4) even if f is a blow-up of M with smooth center and $\widetilde{M}$ is Kähler, in general M fails to be Kähler.

In this paper we prove that for a generic modification $f\colon \widetilde{M} \to M$

(5) if $\widetilde{M}$ is Kähler, then M is balanced.

The proof of (5) depends heavily on the quoted result of [16] and on our Theorem 3.9: "Suppose M and $\widetilde{M}$ are complex manifolds (not necessarily compact) and $f\colon \widetilde{M} \to M$ is a proper modification. If T is a positive

$\partial\bar\partial$-closed current on M of degree $(1, 1)$, then there exists a positive $\partial\bar\partial$-closed current $\tilde{T}$ on $\widetilde{M}$ of degree $(1, 1)$ such that $f_*\tilde{T} = T$. Moreover if $\widetilde{M}$ is compact, such a current is unique."

Since locally $\partial\bar\partial$-closed currents are components of a boundary, it is convenient to translate the condition in [16] in terms of Aeppli cohomology groups (see (1.3)). To generalize (5), we shall introduce a cohomological condition, called (B), which holds in particular for compact Kähler manifolds. In Theorem 3.9, if $\widetilde{M}$ satisfies (B), the cohomology class $[\tilde{T}]$ of $\tilde{T}$ is exactly $f^*[T]$ (see Proposition 3.10). So we get the following converse of $(3')$:

(5′) (Main Theorem 4.2) If $\widetilde{M}$is balanced and satisfies (B), then M is balanced and satisfies (B) too.

Now Corollary 4.5 announced at the beginning is simply a consequence of (5) and of a theorem of Varouchas [22].

As one can see in the literature, the most interesting case is that of compact complex manifolds which are bimeromorphic to projective varieties, that is, Moishezon manifolds. Namely, let M be a Moishezon manifold: If $\widetilde{M}$ is projective and $f: \widetilde{M} \to M$ is a modification, it is difficult to find smooth objects on M coming from $\widetilde{M}$. For instance, if $\tilde{\omega}$ is a Kähler form on $\widetilde{M}$, then $f_*\tilde{\omega}$ is not smooth: its coefficients are in L^1_{loc}. Moreover, if L is a positive line bundle on $\widetilde{M}$, although f_*L is a holomorphic line bundle on the whole of M [18], it is not, in general, positive (for a survey see [23]).

Therefore our techniques based on positive, $\partial\bar\partial$-closed currents seem to be more appropriate and allow us to assert that every Moishezon manifold carries a balanced metric.

Finally notice that Michelson's characterization theorem is not constructive, therefore if $\widetilde{M}$ and M are balanced, the results $(3')$ and $(5')$ do not give any information about the link between balanced metrics on $\widetilde{M}$ and on M. Nevertheless, we shall prove that: "For every balanced metric h on M with Kähler form ω there exists a balanced metric $\tilde{h}$ on $\widetilde{M}$ with Kähler form $\tilde{\omega}$ such that $\omega^{N-1} - f_*\tilde{\omega}^{N-1}$ is a $\partial\bar\partial$-exact current." This is a corollary of Theorem 4.8: "Let M and $\widetilde{M}$ be p-Kähler manifolds, let $f: \widetilde{M} \to M$ be a proper modification and call Y the degeneracy set, with $p > \dim Y$. For every p-Kähler form Ω on M, there exists a p-Kähler form $\tilde{\Omega}$ on $\widetilde{M}$ such that $\Omega - f_*\tilde{\Omega}$ is a $\partial\bar\partial$-exact current."

1. Preliminaries and notation

(1.1) Throughout the paper, whether explicitly stated or not, $\widetilde{M}$ and M are assumed to be complex N-dimensional manifolds. A *proper modification* $f\colon \widetilde{M} \to M$ is a proper holomorphic map such that, for a suitable analytic set Y in M, $E := f^{-1}(Y)$ (the *exceptional set of the modification*) is a hypersurface and $\widetilde{M} - E \xrightarrow{f} M - Y$ is a biholomorphism. Moreover, Y has codimension ≥ 2 or f is a biholomorphism.

In particular, if Y is smooth and f is the blow-up of M along Y, suitable coordinates can be chosen in M as follows. (As usual, $B_k(z^\circ, r)$ denotes the euclidean open ball in $\mathbf{C}^k$ with center z° and radius r. $B_k(0, r)$ is simply denoted by $B_k(r)$ and $B_k(1)$ is denoted by B_k. Take $B_0 := \{0\}$.) For every $y \in Y$ take an open neighborhood $U = B_m \times B_n$ ($m := \dim Y$ and $N := m + n$) such that $U \cap Y = B_m \times \{0\}$. Call $\widetilde{U} := f^{-1}(U)$: We can identify

$$f|_{\widetilde{U}}\colon \widetilde{U} \to U \quad \text{with the natural projection } \pi\colon B_m \times \widetilde{B}_n \to B_m \times B_n,$$

where $\widetilde{B}_n$ denotes the blow-up of B_n at $\{0\}$. In the following we shall simply say: "identify locally f with π."

Recall that if $(t_1, \cdots, t_m) \in \mathbf{C}^m$ and $(z_1, \cdots, z_n) \in \mathbf{C}^n$,

$$\widetilde{B}_n = \{(z, \xi) \in B_n \times \mathbf{P}^{n-1} \mid z_j\xi_k = z_k\xi_j,\ 1 \leq j, k \leq n\}$$

so that the natural Kähler form on $\widetilde{U}$ is given by

$$\tilde{\omega} := \frac{i}{2}\partial\bar{\partial}\|t\|^2 + \frac{i}{2}\partial\bar{\partial}\|z\|^2 + \frac{i}{2\pi}\partial\bar{\partial}\log\|\xi\|^2;$$

therefore (see [23, p. 37])

$$\pi_*\tilde{\omega} = \frac{i}{2}\partial\bar{\partial}\|t\|^2 + \frac{i}{2}\partial\bar{\partial}\|z\|^2 + \frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2.$$

(1.2) As usual, $\mathscr{E}^{k,k}(M)_{\mathbf{R}}$ (resp. $\mathscr{D}^{k,k}(M)_{\mathbf{R}}$) denotes the space of smooth real (k, k)-forms (resp. smooth real (k, k)-forms with compact support) on M. Their duals are spaces of real currents of bidimension (k, k) (or of degree $(N-k, N-k)$), i.e., real $(N-k, N-k)$-forms with distribution coefficients.

Let $\varphi \in \mathscr{E}^{k,k}(M)_{\mathbf{R}}$. In local coordinates, we shall often write

$$\varphi = \frac{ik^2}{2^k} \sum_{1\leq\alpha_1<\cdots<\alpha_k\leq N,\, 1\leq\beta_1<\cdots<\beta_k\leq N} \varphi_{\alpha_1,\cdots,\alpha_k\bar{\beta}_1,\cdots,\bar{\beta}_k}(z)\, dz_{\alpha_1} \wedge\cdots\wedge dz_{\alpha_k} \wedge d\bar{z}_{\beta_1} \wedge\cdots\wedge d\bar{z}_{\beta_k}$$

$$= \sigma_k \sum_{|A|=|B|=k}{}' \varphi_{A\bar{B}}\, dz_A \wedge d\bar{z}_B,$$

where $\sum'$ denotes the sum on strictly increasing multi-indices.

A real current T on M of bidimension (k, k) is called *positive* (in the sense of Lelong [15]) if, for every choice of $\varphi_1, \cdots, \varphi_k \in \mathcal{D}^{1,0}(M)_{\mathbf{R}}$, $T(\sigma_k \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_k) \geq 0$. Moreover T is said to be *strictly positive* if $\varphi_1 \wedge \cdots \wedge \varphi_k \neq 0$ implies $T(\sigma_k \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_k) > 0$. It is well known that a positive current is of order zero. We shall denote by $\|T\|$ the mass of T. A smooth form $\psi \in \mathcal{E}^{N-k,N-k}(M)_{\mathbf{R}}$ is *positive* (resp. *strictly positive*) if the associated current T_ψ, defined as

$$T_\psi(\varphi) = \int \varphi \wedge \psi \quad \forall \varphi \in \mathcal{D}^{k,k}(M)_{\mathbf{R}},$$

is a positive (resp. strictly positive) current.

If X is a p-dimensional irreducible analytic subset of M, we shall denote by $[X]$ the positive current defined as

$$[X](\varphi) := \int_X \varphi \quad \forall \varphi \in \mathcal{D}^{p,p}(M)_{\mathbf{R}}.$$

It is well known that $[X]$ is closed; moreover, if $u: X \to \mathbf{R}$ is a pluriharmonic function, $u[X]$ is a $\partial\overline{\partial}$-closed current.

(1.3) As regards the statement of the Main Theorem, we recall here the definition of balanced manifold and define condition (B).

1.1. Definition. Let M be a complex N-dimensional manifold. M is said to be *balanced* (or semi-Kähler) if there exists a hermitian metric h on M, called the *balanced metric,* such that its Kähler form ω satisfies $d\omega^{N-1} = 0$.

This class of manifolds obviously includes that of Kähler manifolds (for $N = 2$ they coincide) but also many important classes of non-Kähler manifolds, such as the complex solvmanifolds, twistor spaces of oriented riemannian 4-manifolds, 1-dimensional families of Kähler manifolds (see [16]), hermitian compact manifolds which are locally flat [6], manifolds obtained as modifications of compact Kähler manifolds [2]. As well as in the Kähler case (see [11]), there exists an intrinsic characterization of compact balanced manifolds by means of positive currents.

1.2. Theorem [16, Theorem 4.5]. *Suppose M is a compact complex manifold. The following conditions are equivalent:*

(i) *M is balanced.*

(ii) *If T is a positive current on M of degree $(1, 1)$ which is the component of a boundary (i.e., there exists a current S such that $T = \overline{\partial}S + \partial\overline{S}$), then $T = 0$.*

Let us say a few words on currents which are components of boundaries. If T is a current of bidimension $(N-1, N-1)$ which is the component of a boundary, or, more generally, which is a (weak) limit of currents which are components of boundaries, then $\partial\overline{\partial}T = 0$ and moreover $T(\varphi) = 0$ for every closed $\varphi \in \mathscr{D}^{N-1,N-1}(M)_{\mathbf{R}}$; that is, if we consider the operator

$$d: \mathscr{D}^{N-1,N-1}(M)_{\mathbf{R}} \to (\mathscr{D}^{N,N-1}(M) \oplus \mathscr{D}^{N-1,N}(M))_{\mathbf{R}}$$

and its dual

$$(\partial \oplus \overline{\partial}): (\mathscr{D}^{N,N-1}(M) \oplus \mathscr{D}^{N-1,N}(M))_{\mathbf{R}})' \to (\mathscr{D}^{N-1,N-1}(M)_{\mathbf{R}})',$$

then $(\operatorname{Ker} d)^{\perp} = \overline{\operatorname{Im}(\partial \oplus \overline{\partial})}$.

In [16, Lemma 4.8] it is proved that, if M is compact, $\operatorname{Im}(\partial \oplus \overline{\partial})$ is a closed subspace of $\operatorname{Ker} i\partial\overline{\partial}$; hence every current which is the limit of currents which are components of boundaries is the component of a boundary itself. Nevertheless, we shall work mainly in the noncompact case.

The *real* $(1, 1)$-*Aeppli groups* are defined as follows:

$$V^{1,1}(M)_{\mathbf{R}} = \frac{\operatorname{Ker}(i\partial\overline{\partial}: \mathscr{E}^{1,1}(M)_{\mathbf{R}} \to \mathscr{E}^{2,2}(M)_{\mathbf{R}})}{(\partial\mathscr{E}^{0,1}(M) + \overline{\partial}\mathscr{E}^{1,0}(M))_{\mathbf{R}}}$$

and

$$\Lambda^{1,1}(M)_{\mathbf{R}} = \frac{\operatorname{Ker}(d: \mathscr{E}^{1,1}(M)_{\mathbf{R}} \to (\mathscr{E}^{2,1}(M) + \mathscr{E}^{1,2}(M))_{\mathbf{R}})}{i\partial\overline{\partial}\mathscr{E}^{0,0}(M)_{\mathbf{R}}}.$$

As usual, we shall denote by $H^{1,1}(M, \mathbf{R})$ the set of classes in $H^2(M, \mathbf{R})$ which have a $(1, 1)$-representative. It is well known that all these groups can be defined also by means of real currents of degree $(1, 1)$. Thus a $\partial\overline{\partial}$-closed current T is the component of a boundary if and only if its class in $V^{1,1}(M)_{\mathbf{R}}$ is zero.

A class of $V^{1,1}(M)_{\mathbf{R}}$ is said to be *positive* if it can be represented by a positive current: hence Theorem 1.2 can be written as follows: "M is balanced if and only if every nonzero positive $\partial\overline{\partial}$-closed current of degree $(1, 1)$ represents a nonzero class in $V^{1,1}(M)_{\mathbf{R}}$." Finally, let us consider the natural maps:

$$\alpha: \Lambda^{1,1}(M)_{\mathbf{R}} \to H^{1,1}(M, \mathbf{R}),$$
$$\beta: H^{1,1}(M, \mathbf{R}) \to V^{1,1}(M)_{\mathbf{R}}$$

and the following condition:

(B) β is injective and $\operatorname{Im}\beta$ contains all positive elements of $V^{1,1}(M)_{\mathbf{R}}$.

1.3. Proposition. *If $\beta \circ \alpha \colon \Lambda^{1,1}(M, \mathbf{R})_{\mathbf{R}} \to V^{1,1}(M)_{\mathbf{R}}$ is an isomorphism, then α and β are isomorphisms. In particular, every compact Kähler manifold satisfies* (**B**).

Proof. It is enough to notice that α is always surjective. Moreover, if M is regular (in particular, if it is Kähler or Moishezon or in the class $\mathscr{C}$ of Fujiki), then $\beta \circ \alpha$ is an isomorphism (see [21] for the definition of regular manifold and its cohomological properties).

2. $\partial\overline{\partial}$-closed currents and pluriharmonic functions

We study in this section the behaviour of real $\partial\overline{\partial}$-closed currents of degree $(1, 1)$ and of order zero, whose support is contained in the exceptional set of a proper modification. If the support is "too small" (that is, it is contained in an analytic subset of dimension $< N-1$), the current vanishes (see Theorem 2.1; if T is also positive see [2, Theorem 1.5]). On the other hand, if the modification $f\colon \widetilde{M} \to M$ is obtained as a finite sequence of blow-ups with smooth centers, we get a characterization of the set of currents described above: "Every real $\partial\overline{\partial}$-closed current T of degree $(1, 1)$ and order zero supported in the exceptional set E of f is of the form $\sum_\alpha u_\alpha [E_\alpha]$, where $\{E_\alpha\}$ is the set of irreducible components of E and u_α is a pluriharmonic function on E_α" (Proposition 2.5). These results are well known for locally flat currents, but we are not in this case, as Remark 2.4 shows.

We get moreover that if such a current is limit of currents which are components of boundaries, then it vanishes. This holds also in a weaker form if f is a generic proper modification (Proposition 2.7).

2.1. Theorem. *Let Ω be an open set in $\mathbf{C}^N$, and suppose T is a real current of bidimension (p, p) and of order zero on Ω such that $\partial\overline{\partial}T = 0$. If the support of T is contained in an analytic subset Y of Ω of dimension $q < p$, then $T = 0$.*

Proof. Let $x \in \operatorname{Reg} Y$; in a neighborhood U of x choose coordinates $(z_1, \cdots, z_N)$ such that

$$Y \cap U = \{z \in U \mid z_j = 0 \text{ for } j = q+1, \cdots, N\}.$$

Call $z' = (z_1, \cdots, z_q)$ and $z'' = (z_{q+1}, \cdots, z_N)$. In U,

$$T = \sigma_{N-p} \sum_{|A|=|B|=N-p}{}' T_{A\overline{B}} dz_A \wedge d\overline{z}_B$$

where the measures $t_{A\overline{B}}$ can be written as

$$t_{A\overline{B}}(z) = r_{A\overline{B}}(z') \otimes \delta(z'')$$

because $\operatorname{supp} T \subset Y$ (see for instance [14], p. 47).

Call $I = \{q+1, \cdots, N\}$: Since $q < p$, $A \supsetneq I$ and $B \supsetneq I$ for all strictly increasing $(N-p)$-indices A and B. Choose $\alpha \in I \backslash A$, $\beta \in I \backslash B$ and let $A' = A \cup \{\alpha\}$, $B' = B \cup \{\beta\}$ (arrange indices in increasing order). Compute $i\partial\overline{\partial}T$, and notice that, in the coefficient of $dz_{A'} \wedge d\overline{z}_{B'}$, the only addendum containing $\partial^2_{\alpha\overline{\beta}}\delta$ is

$$r_{A\overline{B}}(z') \otimes \partial^2_{\alpha\overline{\beta}}\delta(z'').$$

As $\partial\overline{\partial}T = 0$, we get $r_{A\overline{B}} = 0$. Therefore $\operatorname{supp} T \subseteq \operatorname{Sing} Y$, and we get the result by induction on the dimension of Y. q.e.d.

The next result is a vanishing lemma based on the Kodaira-Nakano Vanishing Theorem.

2.2. Lemma. *Let $f: \widetilde{M} \to M$ be the blow-up of M along a submanifold Y. Then $H^0(E, \Omega^1_E(N_{E|\widetilde{M}})) = 0$, where $N_{E|\widetilde{M}}$ is the normal bundle of the exceptional set E.*

Proof. Identify f locally with the blow-up π as said in (1.1). Take $y \in U \cap Y = B_m$ and identify the singular fibre $\pi^{-1}(y)$ with $\mathbf{P}_{n-1}$. Call $E_U := E \cap U$. Let us recall some easy facts about normal and conormal bundles:

(i) The conormal bundle N^* is defined by the following exact sequence of vector spaces, for $x \in \mathbf{P}_{n-1}$:

$$0 \to N^*_{\mathbf{P}_{n-1}|E_{U,x}} \to T'^*_{E_{U,x}} \to T'^*_{\mathbf{P}_{n-1,x}} \to 0.$$

(ii) Since $E_U = B_m \times \mathbf{P}_{n-1}$, the conormal bundle $N^*_{\mathbf{P}_{n-1}|E_U}$ is trivial.

(iii) $N^*_{\mathbf{P}_{n-1}|\widetilde{B}_n} = [\mathbf{P}_{n-1}]|_{\mathbf{P}_{n-1}} = [-H]$ (notations are the standard ones, see, e.g., [8]).

From (i) we get

$$(2.1)\qquad \begin{aligned} 0 \to \mathscr{O}(N^*_{\mathbf{P}_{n-1}|E_U}) \otimes \mathscr{O}(N_{\mathbf{P}_{n-1}|\widetilde{B}_n}) &\to \Omega^1_{E_U}|_{\mathbf{P}_{n-1}} \otimes \mathscr{O}(N_{\mathbf{P}_{n-1}|\widetilde{B}_n}) \\ &\to \Omega^1_{\mathbf{P}_{n-1}} \otimes \mathscr{O}(N_{\mathbf{P}_{n-1}|\widetilde{B}_n}) \to 0 \end{aligned}$$

and from (ii) and (iii) we infer that

$$\mathscr{O}(N^*_{\mathbf{P}_{n-1}|E_U}) \otimes \mathscr{O}(N_{\mathbf{P}_{n-1}|\widetilde{B}_n}) = \mathscr{O}(-1).$$

Hence the long exact sequence of cohomology groups of (2.1) starts with

$$\begin{aligned} 0 \to H^0(\mathbf{P}_{n-1}, \mathscr{O}(-1)) &\to H^0(\mathbf{P}_{n-1}, \Omega^1_{E_U|\mathbf{P}_{n-1}} \otimes \mathscr{O}(-1)) \\ &\to H^0(\mathbf{P}_{n-1}, \Omega^1(-1)) \to \cdots. \end{aligned}$$

From the Kodaira-Nakano Vanishing Theorem (and some easy facts about Riemann surfaces for $n = 2$), we get

$$H^0(\mathbf{P}_{n-1}, \Omega^1(-1)) = H^0(\mathbf{P}_{n-1}, \mathscr{O}(-1)) = 0;$$

thus $H^0(\mathbf{P}_{n-1}, \Omega^1_{E_U}|_{\mathbf{P}_{n-1}} \otimes \mathscr{O}(-1)) = 0$.

Let $h \in H^0(E, \Omega^1_{E_U}(N_{E_U|\widetilde{U}}))$. Since

$$N_{E_U|\widetilde{U}}|_{\mathbf{P}_{n-1}} = N_{\mathbf{P}_{n-1}|\widetilde{B}_n},$$

$h|_{\mathbf{P}_{n-1}}$ is a section of $\Omega^1_{E_U}|_{\mathbf{P}_{n-1}} \otimes \mathscr{O}(N_{\mathbf{P}_{n-1}|\widetilde{B}_n}) = \Omega^1_{E_U}|_{\mathbf{P}_{n-1}} \otimes \mathscr{O}(-1)$ (by iii)). Thus $h|_{\mathbf{P}_{n-1}} = 0$; i.e. h, restricted to a generic fibre $\pi^{-1}(y)$, is zero. This achieves the proof. q.e.d.

Now we are ready to prove the following result.

2.3. Theorem. *Let $f: \widetilde{M} \to M$ be the blow-up of M along a submanifold Y. If T is a real $\partial\bar{\partial}$-closed current on $\widetilde{M}$ of order zero and degree $(1, 1)$ whose support is contained in the exceptional set E, then there exists a pluriharmonic function $h: Y \to \mathbf{R}$ such that*

$$T = (h \circ f)[E].$$

Moreover, if T is a (weak) limit of currents which are components of boundaries, then $T = 0$.

Proof. Let us fix a coordinate neighborhood $(V, v_1, \cdots, v_N) = (V, (v', v_N))$ in $\widetilde{M}$ such that $V \cap E = \{v_N = 0\}$. In V, T has the following expression:

$$T = \frac{i}{2} \sum_{\alpha,\beta=1}^{N} t_{\alpha\bar{\beta}}(v)\, dv_\alpha \wedge d\bar{v}_\beta.$$

As $\operatorname{supp} T \subseteq E$,

$$t_{\alpha\bar{\beta}}(v) = r_{\alpha\bar{\beta}}(v') \otimes \delta(v_N)$$

where $r_{\alpha\bar{\beta}}$ is a measure and $r_{\bar{\beta}\alpha} = \overline{r_{\alpha\bar{\beta}}}$.

Fix $\alpha, \beta < N$ and compute $i\partial\bar{\partial}T$. The coefficient of $dv_\alpha \wedge dv_N \wedge d\bar{v}_\beta \wedge d\bar{v}_N$, which has to vanish, is given by

$$- r_{\alpha\bar{\beta}}(v') \otimes \partial^2_{N\bar{N}}\delta(v_N) + \partial_\alpha r_{N\bar{\beta}}(v') \otimes \partial_{\bar{N}}\delta(v_N) + \partial_{\bar{\beta}} r_{\alpha\bar{N}}(v') \otimes \partial_N \delta(v_N)$$
$$- \partial^2_{\alpha\bar{\beta}} r_{N\bar{N}}(v') \otimes \delta(v_N).$$

Hence we conclude that

$$\text{(2.2)} \qquad \begin{cases} r_{\alpha\bar{\beta}} = 0 \quad \text{for } 1 \le \alpha, \beta < N, \\ r_{\alpha\bar{N}} \text{ is holomorphic for } 1 \le \alpha < N, \\ r_{N\bar{N}} \text{ is pluriharmonic.} \end{cases}$$

Let us now check what happens in another chart. Choose another coordinate neighborhood $(W, w_1, \cdots, w_N) = (W, w', w_N)$ with $W \cap V \neq \varnothing$ and $W \cap E = \{w_N = 0\}$. Assume

$$T = \frac{i}{2} \sum_{\lambda, \mu=1}^{N} s_{\lambda\overline{\mu}}(w') \otimes \delta(w_N)\, dw_\lambda \wedge d\overline{w}_\mu \quad \text{in } W,$$

by (2.2) and similar results for $\{s_{\lambda\overline{\mu}}\}$, and using the fact that $\partial v_N/\partial w_\lambda = \partial w_N/\partial v_\alpha = 0$ on E for $\alpha, \lambda < N$, we obtain the following relations:

$$r_{\alpha\overline{N}}(v') = \sum_{\lambda=1}^{N-1} s_{\lambda\overline{N}}(w'(v')) \frac{\partial v_N}{\partial w_N} \frac{\partial w_\lambda}{\partial v_\alpha} \quad \text{if } \alpha < N,$$

and

$$\begin{aligned} r_{N\overline{N}}(v') = & \sum_{\lambda=1}^{N-1} s_{\lambda\overline{N}}(w'(v')) \frac{\partial v_N}{\partial w_N} \frac{\partial w_\lambda}{\partial v_N} \\ & + \sum_{\mu=1}^{N-1} \overline{s_{\mu\overline{N}}}(w'(v')) \frac{\overline{\partial v_N}}{\partial w_N} \frac{\overline{\partial w_\mu}}{\partial v_N} + s_{N\overline{N}}(w'(v')). \end{aligned}$$

But now it is a matter of course that, if we cover $\widetilde{M}$ by charts of type (V, v', v_N), then $\{r_{1\overline{N}}, \cdots, r_{n-1\overline{N}}\}$ is nothing but a global section of $\Omega^1_E(N_E\,|_{\widetilde{M}})$ on E: Indeed, the cocycles of $N_E\,|_{\widetilde{M}}$ are given by $\partial v_N/\partial w_N$. By Lemma 2.2 we get $r_{\alpha\overline{N}} = s_{\lambda\overline{N}} = 0$, and $r_{N\overline{N}} = s_{N\overline{N}} =: h$ is a pluriharmonic map on $\widetilde{M}$. Since the fibers of f are compact, h depends only on the coordinates of Y.

To get the second part of the statement, by (1.3) we need only to prove that if $h: Y \to \mathbf{R}$ is not identically zero, then there exists a closed form $\Theta \in \mathscr{D}^{N-1, N-1}(\widetilde{M})_{\mathbf{R}}$ such that

$$T(\Theta) = \int_E (h \circ f)\Theta \neq 0.$$

Choose $y \in Y$ such that $h(y) \neq 0$ (suppose $h(y) > 0$); choose an open neighborhood U of y, biholomorphic to $B_m \times B_n$ and such that $U \cap Y \cong B_m \times \{0\}$ and $h > 0$ in $U \cap Y$; then identify $f|_{f^{-1}(U)}$ with the blow-up π. Take real functions u and v as follows:

$$u \in \mathscr{C}_0^\infty(B_m), \quad u \neq 0, \quad u(t) \geq 0,$$
$$v \in \mathscr{C}_0^\infty(B_n), \quad v(z) = 1 \text{ near the origin.}$$

Then assume

$$\psi(z) = \frac{i}{2\pi}\partial\overline{\partial}((1 - v(z))\log\|z\|^2),$$

$$\Theta = \left(\frac{i}{2\pi}\partial\overline{\partial}\log\|\xi\|^2 - \pi^*\psi\right)^{n-1} \wedge u(t)\left(\frac{i}{2}\partial\overline{\partial}\|t\|^2\right)^m.$$

It is straightforward to verify that Θ satisfies what is required. In particular, if $i\colon E \to f^{-1}(U)$ is the inclusion, $i^*\Theta = (\frac{i}{2\pi}\partial\overline{\partial}\log\|\xi\|^2)^{n-1} \wedge u(t)(\frac{i}{2}\partial\overline{\partial}\|t\|^2)^m$; hence

$$(h \circ f)[E](\Theta) = \int_E (h \circ f \circ i)(i^*\Theta) = c\int_{B_m} h(t)u(t)\left(\tfrac{i}{2}\partial\overline{\partial}\|t\|^2\right)^m > 0.$$

2.4. Remark. An example of a $\partial\overline{\partial}$-closed current T of order zero which is not locally flat is given here. Let us use the same notation of the previous theorem, and define

$$T = \frac{i}{2}c\delta(v_N)(dv_1 \wedge d\overline{v}_N + dv_N \wedge d\overline{v}_1), \qquad c \in \mathbf{R}.$$

T is a real $\partial\overline{\partial}$-closed current of degree $(1, 1)$ and order zero, and $\operatorname{supp} T \subseteq \{v_N = 0\}$. By the Support Theorem (see [10, Theorem 1.7]), if T were locally flat, there would exist a pluriharmonic function $h\colon E \to \mathbf{R}$ such that $T = h[E]$, but this is not the case, if $c \neq 0$.

2.5. Proposition. *Let $f\colon \widetilde{M} \to M$ be a proper modification which is obtained as a finite sequence of blow-ups with smooth centers. Call $\{E_\alpha\}$ the set of the irreducible components of the exceptional set E. Then:*

(i) *Every real $\partial\overline{\partial}$-closed current T of order zero and degree* $(1, 1)$ *on $\widetilde{M}$ supported in E is of the form $\sum_\alpha u_\alpha[E_\alpha]$, where u_α is a pluriharmonic function on E_α.*

(ii) *Moreover, T is a* (*weak*) *limit of currents which are components of boundaries if and only if every u_α vanishes.*

Proof. By our assumption there is a finite sequence

$$f_j\colon V_{j+1} \to V_j, \qquad 0 \le j < r,$$

of blow-ups with smooth centers $Y_j \subseteq V_j$ and exceptional sets $E'_{j+1} \subseteq V_{j+1}$ such that $V_0 = M$, $V_r = \widetilde{M}$, $f = f_0 \circ \cdots \circ f_{r-1}$. By Theorem 2.3 we get

$$(f_1 \circ \cdots \circ f_{r-1})_* T = (u_1 \circ f_0)[E'_1] \quad \text{with } u_1\colon Y_0 \to \mathbf{R} \text{ pluriharmonic.}$$

Let $\widetilde{E}'_1$ be the strict transform of E'_1 under f_1; $(u_1 \circ f_0 \circ f_1)[\widetilde{E}'_1]$ is $\partial\overline{\partial}$-closed. Therefore we can apply Theorem 2.3 again to obtain

$$(2.3) \qquad (f_2 \circ \cdots \circ f_{r-1})_* T - (u_1 \circ f_0 \circ f_1)[\widetilde{E}'_1] = (u_2 \circ f_1)[E'_2]$$

with $u_2\colon Y_1 \to \mathbf{R}$ pluriharmonic. Eventually we get

$$T = \sum_{j=1}^{r} (u_j \circ f_{j-1} \circ \cdots \circ f_{r-1})[\widetilde{E}'_j],$$

where $\widetilde{E}'_j$ is the strict transform of E'_j via $f_{j-1} \circ \cdots \circ f_{r-1}$, $1 \le j < r$, $\widetilde{E}'_r = E'_r$ and $u_j\colon Y_{j-1} \to \mathbf{R}$ is pluriharmonic.

(ii) Suppose T is a limit of components of boundaries. Then

$$(f_1 \circ \cdots \circ f_{r-1})_* T = (u_1 \circ f_0)[E'_1]$$

is limit of components of boundaries too so that by Theorem 2.3 $u_1 = 0$. From (2.3) and Theorem 2.3 we infer $u_2 = 0$, and so on. q.e.d.

Let us consider now a generic proper modification $f\colon \widetilde{M} \to M$. The following lemma is essentially contained in [13] to which we refer step by step.

2.6. Lemma. *Let $f\colon \widetilde{M} \to M$ be a proper modification; for every $x \in M$ there exist an open neighborhood V of x in M, a complex manifold Z and holomorphic maps $g\colon Z \to \widetilde{M}$, $h\colon Z \to V$ such that $h = f \circ g$. Moreover $g\colon Z \to f^{-1}(V)$ is a blow-up, and $h\colon Z \to V$ is obtained as a finite sequence of blow-ups with smooth centers.*

Proof. Locally, f is dominated by a blow-up; that is, [13, Lemma 8, p. 321] for every $x \in M$ there exist an open neighborhood V of x in M and a complex subspace $(D, \widetilde{\mathcal{O}}_D)$ of V such that, if $h'\colon V' \to V$ is the blow-up with center $(D, \widetilde{\mathcal{O}}_D)$, then there exists a holomorphic map $g'\colon V' \to \widetilde{M}$ with $h' = f \circ g'$. Let $\mathcal{J}$ be the coherent ideal sheaf in $\mathcal{O}_V$ which defines the complex space $(D, \widetilde{\mathcal{O}}_D)$; by applying Lemma 7 [13, p. 320] to V and $\mathcal{J}$ we get a suitable finite sequence

$$h_j\colon V_{j+1} \to V_j, \qquad 0 \le j < r,$$

of blow-ups with smooth centers such that $Z := V_r$ is smooth, and if $h := h_0 \circ \cdots \circ h_{r-1}$, $h^{-1}(\mathcal{J})$ is invertible (see the remark after Lemma 7 in [13]). Shrinking V we can also suppose $V_0 = V$. Therefore, by means of the universal property of blow-ups [13, Definition 1, p. 315], we get a holomorphic map $g''\colon Z \to V'$ such that $h = h' \circ g''$. If $g := g' \circ g''\colon Z \to \widetilde{M}$, then $h = f \circ g$ and $g\colon Z \to f^{-1}(V)$ becomes a blow-up since $h\colon Z \to V$ is obtained as a finite sequence of blow-ups and $f\colon f^{-1}(V) \to V$ is a proper modification (see Corollary 1, p. 320 and Lemma 4, p. 318 of [13]). q.e.d.

We can prove now the last result of this section.

2.7. Proposition. *Let* $f: \widetilde{M} \to M$ *be a proper modification and let* $\{E_\alpha\}$ *be the set of irreducible components of the exceptional set* E. *If* $T = \sum_\alpha c_\alpha [E_\alpha]$, $c_\alpha \in \mathbf{R}$, *and for every* $x \in M$ *there exists an open neighborhood* V *of* x *such that*

(2.4) $T\,|_{f^{-1}(V)}$ *is a (weak) limit of currents which are components of boundaries,*

then $c_\alpha = 0\ \forall\alpha$.

Proof. Fix α° and choose $x \in f(E_{\alpha^\circ})$. For a suitable open neighborhood V of x in M we get holomorphic maps $g: Z \to f^{-1}(V)$ and $h: Z \to V$ as in the previous lemma. Now $T\,|_{f^{-1}(V)} = \sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$ where $\{E'_{\alpha,j}\}$ is the set of connected components of $E_\alpha \cap f^{-1}(V)$. Let $\{F_\beta\}$ be the set of irreducible components of the exceptional set of $g: Z \to f^{-1}(V)$, and denote the strict transform of $E'_{\alpha,j}$ under g by $\widetilde{E}'_{\alpha,j}$. Thus $\{F_\beta\} \cup \{\widetilde{E}'_{\alpha,j}\}$ is the set of irreducible components of the exceptional set of $h: Z \to V$, and therefore the total transform $\widehat{T}$ of $\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$ under g is of the form $\widehat{T} = \sum_{\alpha,j} c_\alpha [\widetilde{E}'_{\alpha,j}] + \sum_\beta c'_\beta [F_\beta]$. By Proposition 2.5 we need only to prove that $\widehat{T}\,|_{h^{-1}(V)}$ satisfies (2.4).

Let $\varphi \in \mathscr{E}^{1,1}(f^{-1}(V))_{\mathbf{R}}$ be a representative of the fundamental class of $\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$ in $H^2(f^{-1}(V), \mathbf{R})$; i.e., $\varphi = \sum_{\alpha,j} c_\alpha [E'_{\alpha,j}] + dQ$ for a suitable current Q in $f^{-1}(V)$. Then $g^*\varphi$ represents the fundamental class of the total transform $\widehat{T}$; i.e.,

$$g^*\varphi = \widehat{T} + dQ' \tag{2.5}$$

for a suitable current Q' in Z.

The hypothesis (2.4) provides a sequence $\{R_\mu\}$ of $(1,0)$-currents in $f^{-1}(V)$ such that

$$\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}] = \lim_\mu (\overline{\partial} R_\mu + \partial \overline{R}_\mu) \quad \text{(weakly)}.$$

By smoothing R_μ and Q we get

$$\varphi = \lim_\mu (\overline{\partial} \rho_\mu + \partial \overline{\rho}_\mu) \quad \text{(weakly)}$$

where ρ_μ are smooth $(1,0)$-forms in $f^{-1}(V)$.

Let S be a closed current of degree $(N-1, N-1)$ with compact support in $f^{-1}(V)$ and let $\psi \in \mathscr{D}^{N-1,N-1}(f^{-1}(V))_{\mathbf{R}}$ such that $S = \psi + i\partial\overline{\partial} u$

for a suitable current u with compact support in $f^{-1}(V)$. Now,

$$S(\varphi) = \int \varphi \wedge \psi + i\partial\overline{\partial} u(\varphi) = \lim_{\mu} \int (\overline{\partial}\rho_\mu + \partial\overline{\rho}_\mu) \wedge \psi + i\overline{\partial} u(\partial\varphi) = 0.$$

Finally, consider the operator

$$(\partial \oplus \overline{\partial})\colon (\mathscr{E}^{0,1}(f^{-1}(V)) \oplus \mathscr{E}^{1,0}(f^{-1}(V)))_{\mathbf{R}}) \to \mathscr{E}^{1,1}(f^{-1}(V))_{\mathbf{R}}$$

and its dual

$$d\colon (\mathscr{E}^{1,1}(f^{-1}(V))_{\mathbf{R}})' \to (\mathscr{E}^{0,1}(f^{-1}(V)) \oplus \mathscr{E}^{1,0}(f^{-1}(V)))_{\mathbf{R}})';$$

we have just proved that $\varphi \in (\operatorname{Ker} d)^{\perp} = \overline{\operatorname{Im}(\partial \oplus \overline{\partial})}$. Thus φ is limit of components of boundaries in the strong sense, and the same holds for $g^*\varphi$; so, by (2.5), the same holds for $\widehat{T}$ in a weak sense.

3. Positive $\partial\overline{\partial}$-closed currents have a pullback to $\widetilde{M}$

In this section we start with the following data: $f\colon \widetilde{M} \to M$ is a proper modification and T is a positive $\partial\overline{\partial}$-closed current on M of degree $(1, 1)$, and we try to find a "nice pullback," say $\widetilde{T}$, of T to $\widetilde{M}$. If E is the exceptional set of f and $Y := f(E)$, then $\widetilde{M}-E \xrightarrow{f} M-Y$ is a biholomorphism. Therefore such a pullback must extend $((f\,|_{\widetilde{M}-E})^{-1})_*(T\,|_{M-Y})$ from $\widetilde{M}-E$ to the whole of $\widetilde{M}$. What we are looking for is a positive $\partial\overline{\partial}$-closed extension $\widetilde{T}$ on $\widetilde{M}$, which also satisfies

(3.1) $\forall x \in M$, there exists an open neighborhood W of x such that $\widetilde{T}\,|_{f^{-1}(W)}$ is a (weak) limit of currents which are components of boundaries.

But $((f\,|_{\widetilde{M}-E})^{-1})_*(T\,|_{M-Y})$ has an extension of order zero to $\widetilde{M}$ if and only if it has locally finite mass across E (see [15, p. 10]); i.e., $\forall x \in E$, there is a neighborhood V of x in $\widetilde{M}$ such that

$$\int_{V-E} ((f\,|_{\widetilde{M}-E})^{-1})_*(T\,|_{M-Y}) \wedge \theta^{N-1} < \infty, \tag{3.2}$$

where θ is a smooth strictly positive $(1, 1)$-form on $\widetilde{M}$.

Since (3.2) is a local statement, we shall carry on the computations in coordinates, starting with the case of a blow-up with smooth center. After having proved that $(f\,|_{\widetilde{M}-E})^{-1})_*(T\,|_{M-Y})$ admits extensions of order zero, we construct an extension $\widetilde{T}$ which satisfies the previous demands (see Theorem 3.9) and such that, if condition (B) holds, its class $[\widetilde{T}]$ in the Aeppli group $V^{1,1}(\widetilde{M})_{\mathbf{R}}$ coincides with $f^*([T])$ (Proposition 3.10). These properties are not enjoyed, in general, by the simple extension $\widetilde{T}^\circ$, as an example shows.

To prove the following result, we shall follow [19] (page 129 and ff. for the case $k = 1$); nevertheless, since Siu works with d-closed currents, we shall give here a sufficiently detailed proof in order to check that his arguments also work in the $\partial\overline{\partial}$-closed case and for the sake of completeness.

3.1. Proposition. *Let π be the blow-up of $U := B_m \times B_n$ with center $Y = B_m \times \{0\}$, and let $\tilde{\omega}$ be the Kähler form for $B_m \times \widetilde{B}_n$ defined in (1.1). Suppose $\{T_\varepsilon\}$ is a family of $\partial\overline{\partial}$-closed smooth positive $(1, 1)$-forms on U, such that there is a current T on U, $T = \lim_\varepsilon T_\varepsilon$ (weakly). Then $\forall t^\circ \in B_m$, there exists a neighborhood V of $(t^\circ, 0)$ in U such that*

$$\sup_\varepsilon \int_{\pi^{-1}(V)} \pi^* T_\varepsilon \wedge \tilde{\omega}^{N-1} < \infty. \tag{3.3}$$

Proof. Choose a unitary linear coordinates system $w = w(t, z) = (w_1, \cdots, w_N)$ of $\mathbf{C}^N$ such that $(w_I, z) := (w_{i_1}, \cdots, w_{i_m}, z_1, \cdots, z_n)$ form a coordinates system of $\mathbf{C}^N$ for every $I = (i_1, \cdots, i_m)$ with $1 \le i_1 < \cdots < i_m \le N$. Look at the $(1, 1)$-form $(\frac{i}{2\pi}\partial\overline{\partial}\log\|z\|^2)$. Its matrix is positive semidefinite; more precisely, at $z \neq 0$, it has 0 as simple eigenvalue, with eigendirection z, and $1/\pi\|z\|^2$ as eigenvalue of multiplicity $(n-1)$, with eigenspace $(z)^\perp$. Hence $(\frac{i}{2\pi}\partial\overline{\partial}\log\|z\|^2)^h = 0$ if $h \ge n$; this implies that there exists a constant $c > 0$ such that

$$(\pi_*\tilde{\omega})^{m+n-1} \le c \sum_{k=m}^{m+n-1} \binom{m+n-1}{k} \sum_I{}' \left(\frac{i}{2\pi}\partial\overline{\partial}\log\|z\|^2\right)^{m+n-1-k} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{k-m} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2\right)^m.$$

Let $t^\circ \in B_m$ and let $\rho_I\colon \mathbf{C}^N \to \mathbf{C}^n$ be defined by $\rho_I(t, z) = w_I(t, z)$. There exist an open ball A_I with center $w_I(t^\circ, 0)$ in $\mathbf{C}^m$ and $r_I > 0$ such that

$$X_I := \rho_I^{-1}(A_I) \cap (\mathbf{C}^m \times B_n(r_I)) \Subset U.$$

Thus, if we take $V := \bigcap_I X_I$, to check (3.3) we have only to prove that $\forall I$, $\forall k$, $m \le k \le m+n-1$,

$$\sup_\varepsilon \int_{X_I} T_\varepsilon \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log\|z\|^2\right)^{m+n-1-k} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{k-m} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2\right)^m < \infty. \tag{3.4}$$

The form which is integrated in (3.4) is smaller than or equal to

$$\frac{1}{(\pi\|z\|^2)^{N-1-k}} T_\varepsilon \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2\right)^m,$$

which is in $L^1_{\mathrm{loc}}(U)$; this implies that in (3.4) we can ignore the singularity of $\partial\overline{\partial}\log\|z\|^2$. Since $i\partial\overline{\partial}T_\varepsilon = 0$, there exist $(1, 0)$-forms S_ε on $U = B_m \times B_n$ such that $T_\varepsilon = \overline{\partial}S_\varepsilon + \partial\overline{S}_\varepsilon$. Thus denoting the (topological) boundary of X_I by bX_I

$$\begin{aligned}
\int_{X_I} & T_\varepsilon \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log\|z\|^2\right)^{m+n-1-k} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{k-m} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2\right)^m \\
&= \int_{bX_I} (S_\varepsilon + \overline{S}_\varepsilon) \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log\|z\|^2\right)^{m+n-1-k} \\
&\qquad \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{k-m} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2\right)^m \\
&= \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_{bX_I} (S_\varepsilon + \overline{S}_\varepsilon) \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2\right)^m \\
&= \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_{X_I} T_\varepsilon \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2\right)^m,
\end{aligned}$$

where the reason for the second equality is the following:

$$bX_I := [\rho_I^{-1}(bA_I) \cap (\mathbf{C}^m \times B_n(r_I))] \cup [\rho_I^{-1}(A_I) \cap (\mathbf{C}^m \times bB_n(r_I))] = Y_1 \cup Y_2;$$

integration on Y_1 gives no contribution because $(\frac{i}{2}\partial\overline{\partial}\|w_I\|^2)^m$ is a $2m$-form on the manifold bA_I of real dimension $2m-1$. On the other hand, on Y_2 we have

$$\frac{i}{2\pi}\partial\overline{\partial}\log\|z\|^2 = \frac{i}{2\pi r_I^2}\partial\overline{\partial}\|z\|^2.$$

Now we need the following result [19, p. 66].

3.2. Lemma. *Suppose $G_1 \Subset G_2 \Subset U$ are relatively compact open subsets of $\mathbf{C}^N$, φ is a product of $(N-k)$ smooth positive $(1, 1)$-forms and $\{T_\varepsilon\}$ is a sequence of positive currents on U of degree (k, k) converging (weakly) to a current T on U. Then*

$$\limsup_\varepsilon \int_{G_1} T_\varepsilon \wedge \varphi \le \int_{G_2} T \wedge \varphi \quad \textit{and} \quad \int_{G_1} T \wedge \varphi \le \liminf_\varepsilon \int_{G_2} T_\varepsilon \wedge \varphi.$$

If we choose G such that $X_I \Subset G \Subset U$, by virtue of (3.5) and Lemma 3.2 we get

$$\limsup_{\varepsilon} \int_{X_I} T_\varepsilon \wedge \left(\frac{i}{2\pi}\partial\bar\partial \log \|z\|^2\right)^{m+n-1-k} \wedge \left(\frac{i}{2}\partial\bar\partial\|z\|^2\right)^{k-m} \wedge \left(\frac{i}{2}\partial\bar\partial\|w_I\|^2\right)^{m}$$
$$\leq \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_G T \wedge \left(\frac{i}{2}\partial\bar\partial\|z\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\bar\partial\|w_I\|^2\right)^{m} < \infty.$$

Thus also

$$\sup_{\varepsilon} \int_{X_I} T_\varepsilon \wedge \left(\frac{i}{2\pi}\partial\bar\partial \log \|z\|^2\right)^{m+n-1-k} \wedge \left(\frac{i}{2}\partial\bar\partial\|z\|^2\right)^{k-m} \wedge \left(\frac{i}{2}\partial\bar\partial\|w_I\|^2\right)^{m} < \infty. \quad \text{q.e.d.}$$

We would like to mention the following easy consequence of the above lemma, which is used in what follows:

(3.6) If $\{T_\varepsilon\}$ and φ are as in Lemma 3.2, and L is a Borel set, $L \Subset U$, then

$$\lim_{\varepsilon} \int_L T_\varepsilon \wedge \varphi = \int_L T \wedge \varphi \text{ if } \|T\|(bL) = 0.$$

3.3. Proposition. *Let $f\colon \widetilde{M} \to M$ be the blow-up of M along a submanifold Y. If T is a positive $\partial\bar\partial$-closed current on M of degree $(1, 1)$, then the current $((f|_{\widetilde{M}-E})^{-1})_*(T|_{M-Y})$ has locally finite mass across E, and hence extends to a current of order zero.*

Proof. Identify f locally with the blow-up π and let $y = (t^\circ, 0) \in Y$. By smoothing T by convolutions in a suitable open neighborhood U of y in M, we get a family $\{T_\varepsilon\}$ as in Proposition 3.1. Choose $r > 0$ and a sequence $\{r_j\}$ of positive real numbers such that $r_j \downarrow 0$, $V_0 := B_m(t^\circ, r) \times B_n(r_0) \Subset U$ and, for every $j \geq 0$,

$$\|T\|(bV_j) = 0, \quad \text{where } V_j := B_m(t^\circ, r) \times B_n(r_j). \tag{3.7}$$

Since

$$\lim_j \chi_{\pi^{-1}(V_0 - V_j)} = \chi_{\pi^{-1}(V_0 - Y)},$$

where, as usual, χ_L is the characteristic function of L, we infer that

$$\lim_j \int_{\pi^{-1}(V_0-V_j)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1}$$
$$= \int_{\pi^{-1}(V_0-Y)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1},$$

where $\widetilde{U} := f^{-1}(U)$; but

$$\int_{\pi^{-1}(V_0-V_j)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1} = \int_{V_0-V_j} T \wedge \pi_*\tilde{\omega}^{N-1}.$$

By (3.7) and (3.6), we have

$$\int_{V_0-V_j} T \wedge \pi_*\tilde{\omega}^{N-1} = \lim_\varepsilon \int_{V_0-V_j} T_\varepsilon \wedge \pi_*\tilde{\omega}^{N-1}$$
$$\leq \sup_\varepsilon \int_{V_0} T_\varepsilon \wedge \pi_*\tilde{\omega}^{N-1} < \infty.$$

Thus

$$\int_{\pi^{-1}(V_0-Y)} ((\pi|_{\widetilde{U}-E})^{-1})_*(T|_{U-Y}) \wedge \tilde{\omega}^{N-1} < \infty.$$

Remark. One of the possible extensions of order zero is the "simple extension" [15], which is defined as follows:

$$\widetilde{T}^\circ(\varphi) = \int_{\widetilde{M}-E} ((f|_{\widetilde{M}-E})^{-1})_*(T|_{M-Y}) \wedge \varphi$$

for every $\varphi \in \mathscr{D}^{N-1,N-1}(\widetilde{M})_{\mathbf{R}}$. $\widetilde{T}^\circ$ is called also "extension by zero" since $\|\widetilde{T}^\circ\|(E) = 0$. Nevertheless we are interested in an extension $\widetilde{T}$ which is also $\partial\bar{\partial}$-closed and satisfies (3.1), therefore we go on otherwise.

First we recall a lemma [19, p. 69].

3.4. Lemma. *Suppose Ω is an open subset of $\mathbf{C}^N$ and θ a strictly positive $(1, 1)$-form on Ω. Suppose $\{T_\lambda\}$ is a sequence of smooth positive (k, k)-forms on Ω which satisfy*

$$\sup_\lambda \int_K T_\lambda \wedge \theta^{N-k} < \infty$$

for every compact K of Ω. Then there exists a subsequence $\{T_{\lambda_\mu}\}$ of $\{T_\lambda\}$ which converges (weakly) on Ω.

Using this result, we are able to complete Proposition 3.1 as follows.

3.5. Corollary. *In the hypotheses of Proposition 3.1, there exists a sequence $\{\varepsilon_\mu\}$, $\varepsilon_\mu \to 0$, such that $\pi^* T_{\varepsilon_\mu}$ converges (weakly) on $\widetilde{U}$ to a* *current* $\widetilde{T}_U$. *This current does depend not on the sequence* ε_μ *but only on* T.

Proof. We can take coordinates open sets Ω_j such that $\widetilde{U} = \bigcup_{j=1}^n \Omega_j$. By Proposition 3.1, Ω_j, $\pi^* T_\varepsilon |_{\Omega_j}$, and $\theta = \tilde{\omega} |_{\Omega_j}$ satisfy the hypothesis of Lemma 3.4. Therefore, if we consider subsequently $\Omega_1, \cdots, \Omega_n$, we find a sequence $\{\varepsilon_\mu\}$ such that $\pi^* T_{\varepsilon_\mu}$ converges on each Ω_j, hence on $\widetilde{U}$ to a current $\widetilde{T}_U$. Let $\{T'_{\varepsilon_\mu}\}$ be another sequence, with the same properties of $\{T_{\varepsilon_\mu}\}$, such that

$$\lim_\mu \pi^* T'_{\varepsilon_\mu} = \widetilde{T}'_U.$$

Since $U \cong B_m \times B_n$, T'_{ε_μ} and T_{ε_μ} are components of boundaries; therefore we can get $\widetilde{T}'_U = \widetilde{T}_U$ by applying Theorem 2.3 to $\widetilde{T}'_U - \widetilde{T}_U$.

3.6. Remark. Another proof of the second part of the previous corollary is given here, to emphasize the link between $\|\widetilde{T}_U\|(E)$ and a kind of "Lelong number of T along Y".

Proof. Let $\{T'_{\varepsilon_\mu}\}$ be another sequence, with the same properties of $\{T_{\varepsilon_\mu}\}$, such that

$$\lim_\mu \pi^* T'_{\varepsilon_\mu} = \widetilde{T}'_U.$$

Take an open ball $A \Subset B_m$ and $B_n(r) \Subset B_n$ such that

$$\|\widetilde{T}_U\|(b\pi^{-1}(A \times B_n(r))) = \|\widetilde{T}'_U\|(b\pi^{-1}(A \times B_n(r))) = 0. \tag{3.8}$$

Since on $bB_n(r)$

$$\frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2 = \frac{i}{2\pi r_I^2}\partial\bar{\partial}\|z\|^2,$$

we get as in the proof of Proposition 3.1

$$\begin{aligned}
&\int_{\pi^{-1}(A\times B_n(r))} \pi^* T_{\varepsilon_\mu} \wedge \left(\frac{i}{2\pi}\partial\bar{\partial}\log\|\xi\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\bar{\partial}\|t\|^2\right)^m \\
&= \int_{A\times B_n(r)} T_{\varepsilon_\mu} \wedge \left(\frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\bar{\partial}\|t\|^2\right)^m \\
&= \frac{1}{(\pi r^2)^{n-1}} \int_{A\times B_n(r)} T_{\varepsilon_\mu} \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\bar{\partial}\|t\|^2\right)^m .
\end{aligned} \tag{3.9}$$

Notice that (3.8) implies $\|T\|(b(A \times B_n(r))) = 0$; by letting $\mu \to \infty$ in

(3.9) we conclude that

$$\int_{\pi^{-1}(A\times B_n(r))} \widetilde{T}_U \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log\|\xi\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|t\|^2\right)^m$$
$$= \frac{1}{(\pi r^2)^{n-1}} \int_{A\times B_n(r)} T \wedge \left(\frac{i}{2}\partial\overline{\partial}\|z\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|t\|^2\right)^m,$$

and the same holds for $\widetilde{T}'_U$.

By applying Theorem 2.3 to the current $\widetilde{T}_U - \widetilde{T}'_U$ we get

$$0 = \int_{\pi^{-1}(A\times B_n(r))} (\widetilde{T}_U - \widetilde{T}'_U) \wedge \left(\frac{i}{2\pi}\partial\overline{\partial}\log\|\xi\|^2\right)^{n-1} \wedge \left(\frac{i}{2}\partial\overline{\partial}\|t\|^2\right)^m$$
$$= \int_A h \left(\frac{i}{2}\partial\overline{\partial}\|t\|^2\right)^m,$$

and since A is arbitrary, $h = 0$; i.e., $\widetilde{T}_U = \widetilde{T}'_U$. q.e.d.

We would like also to compare the previous proof with the analogous situation occurring in [19, p. 129]. The author has a closed current $\widetilde{T}_U$; hence he gets

$$\chi_{E_U}\widetilde{T}_U = c[E_U]$$

by deep results based on Bombieri-Hörmander estimates (see [19, Chapter 5]). Next,

$$\chi_{E_U}\widetilde{T}'_U = c'[E_U],$$

for another $\widetilde{T}'_U$ implies $\widetilde{T}_U - \widetilde{T}'_U = (c - c')[E_U]$. But the class of $[E_U]$ does not vanish in the cohomology ring of $\widetilde{U}$, while the class of $\widetilde{T}_U - \widetilde{T}'_U$ is zero, because $\widetilde{T}_U$ and $\widetilde{T}'_U$ are limits of sequences of boundaries; this implies $c = c'$.

3.7. Remark. Let $f\colon \widetilde{M} \to M$ be the blow-up of M along a submanifold Y, and let T be a positive $\partial\overline{\partial}$-closed current on M of degree $(1, 1)$. For every $x \in M$, there exists an open neighborhood U of x such that, smoothing T by convolutions, we can apply Corollary 3.5 to get a positive extension $\widetilde{T}_U$ of $((f\,|_{\widetilde{U}-E})^{-1})_*(T\,|_{U-Y})$ in $f^{-1}(U)$. By construction, such an extension is the limit of currents which are components of boundaries; hence $\widetilde{T}_U$ is $\partial\overline{\partial}$-closed. But in general $\widetilde{T}^\circ \neq \widetilde{T}_U$, as the following example shows.

Example. Let us take $Y = \{0\}$ (that is, $m = 0$ and $N = n$), and let us consider a sequence $\{T_\varepsilon\}$ as in Proposition 3.1. For $0 < r_1 < r_2 < 1$,

$$\begin{aligned}\int_{r_1<\|z\|<r_2} & T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} \\ &= \sum_{h=0}^{N-1} \binom{N-1}{h} \int_{r_1<\|z\|<r_2} T_\varepsilon \wedge \left(\frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2\right)^h \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{N-1-h} \\ &= \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\|<r_2} T_\varepsilon \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{N-1} \\ &\quad - \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_1^2)^h} \int_{\|z\|<r_1} T_\varepsilon \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{N-1}.\end{aligned}$$

For the second equality, see [20, p. 364, Remark 1] or also our (3.5). For $r_1 \to 0$,

$$\int_{\|z\|<r_2} T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} = \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\|<r_2} T_\varepsilon \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{N-1}.$$

Now choose the subsequence ε_μ of Corollary 3.5 and r_2 such that $\|T\|(bB_n(r_2)) = 0$. For $\mu \to \infty$

$$\begin{aligned}\|\widetilde{T}_U\|(\pi^{-1}(B_n(r_2))) &= \int_{\pi^{-1}(\|z\|<r_2)} \widetilde{T}_U \wedge \tilde{\omega}^{N-1} \\ &= \lim_\mu \int_{\pi^{-1}(\|z\|<r_2)} \pi^* T_{\varepsilon_\mu} \wedge \tilde{\omega}^{N-1} = \lim_\mu \int_{\|z\|<r_2} T_{\varepsilon_\mu} \wedge \pi_* \tilde{\omega}^{N-1} \\ &= \lim_\mu \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\|<r_2} T_{\varepsilon_\mu} \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{N-1};\end{aligned}$$

thus

$$\|\widetilde{T}_U\|(E \cap U) = \lim_{r_2 \to 0} \|\widetilde{T}_U\|(\pi^{-1}(B_n(r_2))) = n(T, 0). \tag{3.10}$$

Hence, if the Lelong number $n(T, 0) \neq 0$, $\widetilde{T}^\circ \neq \widetilde{T}_U$.

It is perhaps interesting to check the difference between $\widetilde{T}^\circ$ and $\widetilde{T}_U$. Take $T = [H]$, where $H = \{z_n = 0\}$; we get easily that $\widetilde{T}^\circ$ is nothing else than the strict transform of H under π. Hence $\widetilde{T}^\circ$ is closed, and therefore $\widetilde{T} - \widetilde{T}^\circ$ is $\partial\bar{\partial}$-closed; obviously

$$\widetilde{T} - \widetilde{T}^\circ \geq 0 \quad \text{and} \quad \operatorname{supp}(\widetilde{T} - \widetilde{T}^\circ) \subseteq E = \mathbf{P}_{n-1}.$$

By Theorem 1.5 in [2], there exists a constant $k \geq 0$ such that $\widetilde{T} - \widetilde{T}^\circ = k[E]$. This implies

$$\|\widetilde{T}\|(E) = k \operatorname{vol}(\mathbf{P}_{n-1}) = k,$$

but from (3.10)

$$\|\widetilde{T}\|(E) = n([H], 0) = 1.$$

Thus $\widetilde{T} = \widetilde{T}^{\circ} + [E]$.

This example reflects a more general situation, which is discussed in Proposition 3.10: that is, if T is a divisor, its total transform is $\widetilde{T}$ and not, in general, $\widetilde{T}^{\circ}$.

Let us collect our results in the following theorem.

3.8. Theorem. *Let $f: \widetilde{M} \to M$ be the blow-up of M along a submanifold Y. If T is a positive $\partial\overline{\partial}$-closed current on M of degree $(1, 1)$, then $((f|_{\widetilde{M}-E})^{-1})_*(T|_{M-Y})$ can be extended to $\widetilde{M}$, and there exists an extension $\widetilde{T}$ which is positive and $\partial\overline{\partial}$-closed, and satisfies the following condition:*

(3.1) *$\forall x \in M$, there exists an open neighborhood W of x such that $\widetilde{T}|_{f^{-1}(W)}$ is a (weak) limit of currents which are components of boundaries.*

Now we consider the general case of a proper modification.

3.9. Theorem. *Let $f: \widetilde{M} \to M$ be a proper modification and let T be a positive $\partial\overline{\partial}$-closed current on M of degree $(1, 1)$. Then the following hold:*

(i) *There exists a positive $\partial\overline{\partial}$-closed current $\widetilde{T}$ on $\widetilde{M}$ of degree $(1, 1)$ such that $f_*\widetilde{T} = T$ and (3.1) holds.*

(ii) *If M is compact, such a current $\widetilde{T}$ is unique.*

Proof. (i) Let $x \in M$ and choose an open neighborhood V of x and maps $g: Z \to f^{-1}(V)$ and $h: Z \to V$ as in Lemma 2.6. Since h is obtained as a finite sequence of blow-ups with smooth centers, by Theorem 3.8 we get a positive $\partial\overline{\partial}$-closed current $\widehat{T}$ on Z such that $h_*\widehat{T} = T$. Shrinking V, we can suppose that it is contained in a coordinate chart and is biholomorphic to an open ball, and that there exists a sequence $\{T_\varepsilon\}$ of components of boundaries, $T_\varepsilon \to T$ (e.g., by smoothing T by convolutions). By construction, $\widehat{T} = \lim_\varepsilon h^* T_\varepsilon$ and it does not depend on $\{T_\varepsilon\}$. Define

$$\widetilde{T}_V := g_*\widehat{T} = \lim_\varepsilon g_* h^* T_\varepsilon = \lim_\varepsilon (f|_{f^{-1}(V)})^* T_\varepsilon.$$

Since $\widetilde{T}_V$ does not depend on $\{T_\varepsilon\}$ nor on the factorization $h = f \circ g$, $\widetilde{T}|_{f^{-1}(V)} := \widetilde{T}_V$ defines the required current.

(ii) Let $\widetilde{T}$ and $\widetilde{T}'$ be currents on $\widetilde{M}$ which satisfy (i). Let $x \in M$ and Z, g, h be as above; using (i) for the map g, we get positive $\partial\overline{\partial}$-closed currents $\widehat{T}$ and $\widehat{T}'$ on Z such that $g_*\widehat{T} = \widetilde{T}$ and $g_*\widehat{T}' = \widetilde{T}'$ on

$f^{-1}(V)$; hence $h_*\widehat{T} = h_*\widehat{T}' = T$ on V. Let $\{E'_\gamma\}$ be the set of irreducible components of $E \cap f^{-1}(V)$, and $\{F_\beta\}$ be the set of irreducible components of the exceptional set F of $g: Z \to f^{-1}(V)$, and denote by $\widetilde{E}'_\gamma$ the strict transform of E'_γ under g. Thus $\{F_\beta\} \cup \{\widetilde{E}'_\gamma\}$ is the set of irreducible components of the exceptional set of $h: Z \to V$, and by Proposition 2.5 we get

$$\widehat{T} - \widehat{T}' = \sum_\gamma u_\gamma[\widetilde{E}'_\gamma] + \sum_\beta u_\beta[F_\beta],$$

where u_γ and u_β are pluriharmonic functions. Thus $\widetilde{T} - \widetilde{T}' = g_*(\widehat{T} - \widehat{T}') = \sum_\gamma u'_\gamma[E'_\gamma]$ on $f^{-1}(V)$, where u'_γ is a well-defined pluriharmonic function on $E'_\gamma - g(\widetilde{E}'_\gamma \cap F)$, locally bounded on E'_γ. Hence u'_γ extends to E'_γ so that we get on $\widetilde{M}$

$$\widetilde{T} - \widetilde{T}' = \sum_\alpha u_\alpha[E_\alpha],$$

where $\{E_\alpha\}$ is the set of irreducible components of E, and u_α is pluriharmonic on E_α. But each E_α is compact, so each u_α is constant. The thesis follows from Proposition 2.7. q.e.d.

Let us consider now the class in $V^{1,1}(\widetilde{M})_{\mathbf{R}}$ of the current $\widetilde{T}$ given by the previous theorem.

3.10. Proposition. *Let M, $\widetilde{M}$ be complex manifolds which satisfy condition* (B), *and let $f: \widetilde{M} \to M$ be a proper modification. Let T be a positive $\partial\bar\partial$-closed current on M of degree* $(1, 1)$, *and $\widetilde{T}$ be a current on $\widetilde{M}$ satisfying Theorem* 3.9 (i). *Let $f^*: V^{1,1}(M)_{\mathbf{R}} \to V^{1,1}(\widetilde{M})_{\mathbf{R}}$ be the natural map. Then $f^*([T] = [\widetilde{T}]$.*

Proof. By condition (B), there exist d-closed smooth real $(1, 1)$-forms φ and $\tilde\varphi$ such that $\beta([\varphi]) = [T]$ and $\tilde\beta([\tilde\varphi]) = \widetilde{T}$; i.e., $T = \varphi + \bar\partial S + \partial\overline{S}$ and $\widetilde{T} = \tilde\varphi + \bar\partial R + \partial\overline{R}$ for suitable currents S and R. Therefore

$$f_*\tilde\varphi - \varphi = \bar\partial(S - f_*R) + \partial(\overline{S} - f_*\overline{R}),$$

but β is injective; hence $f_*\tilde\varphi - \varphi$ is d-exact. Using the results on the link between the cohomology rings of M, $\widetilde{M}$, E and Y (see, e.g., [7, p. 285]) we get

$$\tilde\varphi - f^*\varphi = \sum_\alpha c_\alpha[E_\alpha] + dQ$$

for a suitable current Q.

Recall that there exist open sets V such that $\widetilde{T}\,|_{f^{-1}(V)}$ is the limit of components of boundaries; hence the same holds for $\tilde\varphi$. Moreover we can choose V such that $H^2(V, \mathbf{R}) = 0$, so that $\varphi\,|_V = d\psi$. Thus $\sum_\alpha c_\alpha [E_\alpha] = \tilde{\varphi} - f^*\varphi - dQ$ is the limit of components of boundaries in $f^{-1}(V)$; by Proposition 2.7, $c_\alpha = 0\ \forall\alpha$, so that

$$f^*([T]) = f^*\beta([\varphi]) = \tilde{\beta} f^*([\varphi]) = \tilde{\beta} f^*([\varphi]) = \tilde{\beta}([\tilde{\varphi}]) = \tilde{T}.$$

4. The main theorem

In this section we use the machinery developed in the previous sections to get some metric results. We start with a lemma concerning condition (B).

4.1. Lemma. *Let $f\colon \widetilde{M} \to M$ be a proper modification. If $\widetilde{M}$ satisfies condition* (B) (*that is, $\beta\colon H^{1,1}(M, \mathbf{R}) \to V^{1,1}(M)_{\mathbf{R}}$ is injective and* $\operatorname{Im}\beta$ *contains all positive elements of $V^{1,1}(M)_{\mathbf{R}}$*)*, then M also satisfies* (B).

Proof. Let us consider the following commutative diagram (see (1.3)):

$$\begin{array}{ccc} H^{1,1}(M, \mathbf{R}) & \xrightarrow{\beta} & V^{1,1}(M)_{\mathbf{R}} \\ \downarrow f^* & & \downarrow f^* \\ H^{1,1}(\widetilde{M}, \mathbf{R}) & \xrightarrow{\tilde{\beta}} & V^{1,1}(\widetilde{M})_{\mathbf{R}} \\ \downarrow f_* & & \downarrow f_* \\ H^{1,1}(M, \mathbf{R}) & \xrightarrow{\beta} & V^{1,1}(M)_{\mathbf{R}} \end{array}$$

where $f_* \circ f^*$ is the identity. Denote by [ ] the classes in all groups that appear in the diagram. By hypothesis, $\tilde{\beta}$ is injective; hence β is injective too. Let T be a positive $\partial\bar{\partial}$-closed current on M of degree $(1, 1)$, and $\widetilde{T}$ be a positive $\partial\bar{\partial}$-closed current on $\widetilde{M}$ of degree $(1, 1)$ given by Theorem 3.9. We know that there exists a d-closed form $\psi \in \mathscr{E}^{1,1}(\widetilde{M})_{\mathbf{R}}$ such that $\tilde{\beta}([\psi]) = [\widetilde{T}]$. Therefore

$$\beta f_*([\psi]) = f_*\tilde{\beta}([\psi]) = [f_*\widetilde{T}] = [T]. \quad \text{q.e.d.}$$

Now we can state and prove the Main Theorem. Here the manifolds are supposed to be compact, since Theorem 1.2 is needed.

4.2. Main Theorem. *Let M, $\widetilde{M}$ be compact complex manifolds, and $f\colon \widetilde{M} \to M$ be a modification. If $\widetilde{M}$ is balanced and satisfies* (B) (*in particular, if it is Kähler*)*, then M is balanced and satisfies* (B) *too.*

Proof. (B) holds for M by Lemma 4.1.

Let $T = \bar{\partial}S + \partial\bar{S}$ be a positive current of degree $(1, 1)$ on M. If we prove that $T = 0$, we get the thesis by Theorem 1.2. Let $\widetilde{T}$ be given by Theorem 3.9; then $f^*([T]) = [\widetilde{T}]$ by Proposition 3.10. Hence $\widetilde{T}$ is a positive component of a boundary on a balanced manifold. This implies $\widetilde{T} = 0$. Thus $\operatorname{supp} T \subseteq Y$, but the codimension of Y is greater than one; hence by Theorem 2.1, $T = 0$.

Remark. In the proof of the previous theorem, problems arising from changing charts are avoided. As a matter of fact, one may hope to prove the Main Theroem directly. Starting from a strictly positive $(1, 1)$-form $\tilde{\omega}$ on $\widetilde{M}$ with $d\tilde{\omega}^{N-1} = 0$, try to construct an analogous form ω on M. But obviously we cannot hope that $\omega |_{M-Y} = f_* \tilde{\omega} |_{M-Y}$, because $f_*\tilde{\omega}$ "blows up" near Y. So we should modify $f_*\tilde{\omega}$ on coordinate open sets which meet Y, and then glue together these currents. This seems to be much more complicated than our procedure, which consists basically in extending the current T from $\widetilde{M} - E$ to $\widetilde{M}$.

Theorem 4.2 has some interesting corollaries; to state them let us recall the definition of the class $\mathscr{C}$ of Fujiki [5, p. 34–35].

4.3. Definition. A reduced (compact) complex analytic space X belongs to $\mathscr{C}$ if it is a meromorphic image of a compact Kähler space.

Varouchas proved that $\mathscr{C}$ is nothing but the class of spaces bimeromorphic to some compact Kähler manifold:

4.4. Theorem [22, Theorem 3]. *If $X \in \mathscr{C}$, then there exist a compact Kähler manifold K and a modification $f\colon K \to X$.*

So we get by Theorem 4.2 a result about "nice" hermitian metrics on manifolds in the class $\mathscr{C}$.

4.5. Corollary. *Every manifold in the class $\mathscr{C}$ is balanced.*

And in particular we have

4.6. Corollary. *Moishezon manifolds are balanced.*

Notice that there exist compact balanced manifolds not in the class $\mathscr{C}$, e.g., the Ivasawa manifold I_3.

In order to study metrics in connection with modifications of balanced manifolds, let us start from a more general situation and introduce the following definition.

4.7. Definition. A complex manifold M is called *p-Kähler* if it carries a strictly weakly positive smooth closed (p, p)-form, called a *p-Kähler form.*

For more details about this subject see [2], here we may only point out that 1-Kähler is equivalent to Kähler and $(N-1)$-Kähler is equivalent to balanced.

4.8. Theorem. *Let M and $\widetilde{M}$ be compact p-Kähler manifolds, let $f\colon \widetilde{M} \to M$ be a proper modification, and call Y the degeneracy set, with $p > \dim Y$. For every p-Kähler form Ω on M, there exists a*

p-Kähler form $\tilde{\Omega}$ on $\widetilde{M}$ such that $[\Omega] = [f_* \tilde{\Omega}]$ in the (p, p)-Aeppli group $\Lambda^{p,p}(M)_{\mathbf{R}}$.

Proof. Let Ω and Ω' be p-Kähler forms on M and $\widetilde{M}$ respectively. Since $p > \dim Y$, arguing as in [22, pp. 251–252], we get an open neighborhood U of Y in M and a real current R in U such that

$$f_* \Omega' = i\partial\overline{\partial} R.$$

Since $f_* \Omega'$ is smooth and $\partial\overline{\partial}$-exact in $U - Y$, there exists a smooth real $(p-1, p-1)$-form β in $U - Y$ such that

$$f_* \Omega' |_{U-Y} = i\partial\overline{\partial}\beta.$$

Moreover, $R - \beta = \gamma + \overline{\partial} C + \partial \overline{C}$ in $U - Y$, for a suitable smooth real $\partial\overline{\partial}$-closed $(p-1, p-1)$-form γ and a current C.

Now choose an open set W such that $Y \subset W \Subset U$ and a real function $g \in \mathscr{C}_0^\infty(U)$, with $g = 1$ in W; take

$$D := g(\beta + \gamma) + \overline{\partial}(gC) + \partial(g\overline{C}).$$

Since $i\partial\overline{\partial} D$ is smooth in $M - Y$, $\chi := (f |_{\widetilde{M} - E})^* i\partial\overline{\partial} D$ is a smooth (p, p)-form in $\widetilde{M} - E$ which coincides with Ω' in $f^{-1}(W) - E$; therefore χ can be extended to a real smooth (p, p)-form on the whole of $\widetilde{M}$, which is supported in $f^{-1}(U)$ and strictly weakly positive in $f^{-1}(W)$. Choose $\varepsilon > 0$ such that

$$\tilde{\Omega} := f^* \Omega + \varepsilon \chi$$

is strictly weakly positive on $\widetilde{M}$, so we get $\Omega - f_* \tilde{\Omega} = i\partial\overline{\partial}\varepsilon D$. q.e.d.

In [2] we studied a kind of modification for which the hypotheses of Theorem 4.8 hold, so that we have now some new information about the link between p-Kähler forms on M and $\widetilde{M}$. Moreover for $p = N - 1$ we can give a metric interpretation:

4.9. Corollary. *Let M and $\widetilde{M}$ be compact balanced manifolds and $f: \widetilde{M} \to M$ a modification. For every balanced metric h on M with Kähler form ω there exists a balanced metric $\tilde{h}$ on $\widetilde{M}$ with Kähler form $\tilde{\omega}$ such that $\omega^{N-1} - f_* \tilde{\omega}^{N-1}$ is a $\partial\overline{\partial}$-exact current.*

References

[1] L. Alessandrini & G. Bassanelli, *A balanced proper modification of P_3*, Comment. Math. Helv. **66** (1991) 505–511.

[2] ——, *Positive $\partial\overline{\partial}$-closed currents and non-Kähler geometry*, J. Geometric Analysis **3** (1992).

[3] ——, *Smooth proper modifications of compact Kähler manifolds*, Proc. Internat. Workshop on Complex Analysis (Wuppertal 1990), Complex Analysis, Aspects of Mathematics E, Vol. 17, Vieweg, Germany, 1991.

[4] A. Blanchard, *Les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. **73** (1958) 157–202.

[5] A. Fujiki, *Closedness of the Douady spaces of compact Kähler spaces*, Publ. Res. Inst. Math. Sci. **14** (1978) 1–52.

[6] P. Gauduchon, *Fibrés hermitiens a endomorphisme de Ricci non négatif*, Bull. Soc. Math. France **105** (1977) 113–140.

[7] H. Grauert & O. Riemenschneider, *Verschwindungssätze für analytische Kohomologie gruppen auf komplexen Räumen*, Invent. Math. **11** (1970) 263–292.

[8] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley Interscience, New York, 1978.

[9] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.

[10] R. Harvey, *Holomorphic chains and their boundaries*, Proc. Sympos. Pure Math., Vol. 30, Part 1, Amer. Math. Soc., Providence, RI, 1977, 309–382.

[11] R. Harvey & J. R. Lawson, *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983) 169–198.

[12] H. Hironaka, *Flattening theorems in complex analytic geometry*, Amer. J. Math. **97** (1975) 503–547.

[13] H. Hironaka & H. Rossi, *On the equivalence of embeddings of exceptional complex spaces*, Math. Ann. **156** (1964) 313–333.

[14] L. Hörmander, *The analysis of linear partial differential operators*, I, Grundlehren Math. Wiss., no. 256, Springer, Berlin, 1983.

[15] P. Lelong, *Plurisubharmonic functions and positive differential forms*, Gordon and Breach, New York, 1969.

[16] M. L. Michelson, *On the existence of special metrics in complex geometry*, Acta Math. **143** (1983) 261–295.

[17] I. R. Shafarevich, *Basic algebraic geometry*, Springer, Berlin, 1977.

[18] B. Shiffman, *Extension of positive line bundles and meromorphic maps*, Invent. Math. **15** (1972) 332–347.

[19] Y. T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974) 53–156.

[20] H. Skoda, *Prolongement des courants, positif, fermés de masse finie*, Invent. Math. **66** (1982) 361–376.

[21] J. Varouchas, *Propriétés cohomologiques d'une classe de variétés analitiques complexes compactes*, Sem. d'Analyse Lelong-Dolbeault-Skoda 1983-84, Lecture Notes in Math., Vol. 1198, Springer, Berlin, 1985, 233–243.

[22] ——, *Sur l'image d'une variété kählerienne compacte*, Seminaire Norguet 1983-84, Lecture Notes in Math., Vol. 1188, Springer, Berlin, 1985, 245–259.

[23] R. O. Wells, *Moishezon spaces and Kodaira embedding theorem*, Proc. Tulane Univ. Program on Value Distribution Theory in Complex Analysis and Related Topics in Differential Geometry, 1972-73, Dekker, New York, 1974, 29–42.

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