J. DIFFERENTIAL GEOMETRY
36 (1992) 163-253

THE EXISTENCE OF ANTI-SELF-DUAL CONFORMAL STRUCTURES

CLIFFORD HENRY TAUBES

1. Introduction

The following is a salient feature of 4-dimensional Riemannian geometry: The conformal class of a Riemannian metric on an oriented 4-manifold defines a splitting

$$\Lambda^2 T^* \simeq \Lambda^2_+ T^* \oplus \Lambda^2_- T^*. \tag{1.1}$$

The bundles $\Lambda^2_\pm T^*$ are real, 3-plane bundles whose sections are called self-dual $(+)$ or anti-self-dual $(-)$ 2-forms.

With the metric's help, the Riemannian curvature can be thought of as a section of the symmetric endomorphisms of $\Lambda^2 T^*$. Then, with respect to (1.1), this section, $\mathscr{R}$, has the form

$$\mathscr{R} = \left(\begin{array}{c|c} \mathscr{W}_+ + \frac{s}{12}\cdot 1 & B \\ \hline B^T & \mathscr{W}_- + \frac{2}{12}\cdot 1 \end{array}\right). \tag{1.2}$$

Here, s is the usual scalar curvature, B is the traceless Ricci tensor (in an unusual guise), and the $\mathscr{W}_\pm$ are, respectively, the self-dual and anti-self-dual Weyl tensors. (The metric is Einstein if $B = 0$, and it is conformally flat if $\mathscr{W}_+$ and $\mathscr{W}_-$ are both zero.)

(a) **Existence.** Given that this preamble is understood (and [1] is the canonical reference), it can be said that the purpose of this article is to discuss metrics with $\mathscr{W}_+ = 0$. We give the main result:

Theorem 1.1. *Let M be a smooth, compact, oriented, 4-dimensional manifold. Use $\underline{\mathbb{CP}}^2$ to denote complex projective 2-space with the opposite of its complex orientation. Use $\#$ to denote the operation of connect sum. For all sufficiently large N, $M_N \equiv M\#_N\underline{\mathbb{CP}}^2$ admits a metric with $\mathscr{W}_+ \equiv 0$.*

Remark that the connect sum of manifolds X and Y is obtained from their disjoint union by cutting out an open ball in X and one in Y and then identifying the two resulting boundary 3-spheres.

Received September 16, 1991.

One more remark: The complex orientation on $\mathbb{CP}^2$ is obtained by identifying $\mathbb{CP}^2$ as $(\mathbb{C}^3\backslash\{0\})/\mathbb{C}^*$—a complex manifold. Then $\underline{\mathbb{CP}}^2$ has the *other* orientation.

Without Theorem 1.1, there were known Riemannian manifolds with $\mathscr{W}_+ \equiv 0$. First of all, any conformally flat manifold. Second, $\underline{\mathbb{CP}}^2$ with the Fubini-Study metric. Third, the $K3$ complex surfaces with Yau's Kahler-Einstein metrics [21]. Fourth, Poon [18] constructed such a metric on $\underline{\mathbb{CP}}^2\#\underline{\mathbb{CP}}^2$ and Floer [7] subsequently proved that such metrics exist on $\#_N\underline{\mathbb{CP}}^2$ for any N. Donaldson and Friedman [5] found more metrics of this type on $\#_N\underline{\mathbb{CP}}^2$, as did LeBrun [16] and Joyce [10], [11]. Donaldson and Friedman [5] also found $\mathscr{W}_+ \equiv 0$ metrics on $K3\#_N\underline{\mathbb{CP}}^2$ for N large.

Note here that not all manifolds can have metrics with $\mathscr{W}_+ \equiv 0$. Indeed, a manifold must have nonpositive signature to have such a metric, for the signature τ_X of the intersection pairing on $H_2(M)$ is, according to Hirzebruch, one-third of the first Pontrjagin number of T^*. Said characteristic number is computed via a curvature integral:

$$p_1(X) = \frac{1}{8\pi^2}\int_M d\,\mathrm{vol}_g(|\mathscr{W}_+|_g^2 - |\mathscr{W}_-|_g^2), \tag{1.3}$$

where $d\,\mathrm{vol}_g$ is the metric's volume form and $|\cdot|_g$ is the metric norm on $\mathrm{End}(\bigwedge_2 T^*)$.

(b) **Moduli spaces.** The condition $\mathscr{W}_+ \equiv 0$ is equivariant under the action of the diffeomorphism group of X. This condition is also conformally invariant: If $\mathscr{W}_+[g] = 0$, then $\mathscr{W}_+[e^u g] = 0$ for any $u \in C^\infty(X)$. Thus, when discussing the set of metrics on X with $\mathscr{W}_+ \equiv 0$, one should be considering their equivalence classes under the action of $\mathrm{Diff}(X) \ltimes C^\infty(X)$.

The space of such equivalence classes will be called the *moduli space of half* conformally flat metrics on X and denoted by $\mathscr{M}(X)$.

Various abstract properties of $\mathscr{M}(X)$ are discussed in [12]; the authors point out that $\mathscr{M}(X)$ is a priori a real analytic variety with dimension at a smooth point given by

$$\dim\mathscr{M}(X) = -\tfrac{1}{2}(15e_X + 29\tau_X), \tag{1.4}$$

where e_X is the Euler characteristic of X and τ_X is the signature.

It follows from (1.4) that

$$\dim\mathscr{M}(M\#_N\underline{\mathbb{CP}}^2) = -\tfrac{1}{2}(15e_M + 29\tau_M) + 7N, \tag{1.5}$$

which evidently increases with N.

The structure of $\mathscr{M}(M_N)$ and its behavior as $N \to \infty$ is the subject of a sequel which is now in preparation. Suffice it to say here (without proof) that Theorem 1.1 constructs smooth points of $\mathscr{M}(M_N)$ for large N.

(c) **The Penrose correspondence.** In describing Roger Penrose's work on twistors, Atiyah, Hitchin, and Singer [1] point out that the condition that $\mathscr{W}_+ \equiv 0$ for a metric on a 4-dimensional manifold implies that a certain 2-sphere bundle over said manifold has an integrable, almost complex structure. Let X be the manifold and g the metric. The sphere bundle is the unit sphere bundle,

$$Z \subset \Lambda_+^2 T^* X. \tag{1.6}$$

The conformal class of a metric on X defines an almost complex structure on Z which is integrable if and only if $\mathscr{W}_+ \equiv 0$.

As a complex 3-fold, Z is rather special, for the fibers of the projection to X are holomorphic $\mathbb{CP}^1$'s with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, multiplication by (-1) on $\Lambda_+^2 T^* X$ induces an antiholomorphic involution of Z.

Atiyah, Hitchin, and Singer [1] provide a converse to the preceding assertion:

Theorem 1.2 [1]. *Let Z be a complex 3-fold with the following properties*:

(1) *Z has a free, antiholomorphic involution, σ.*

(2) *Z has a σ-invariant foliation by $\mathbb{CP}^1$'s with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Then Z is the unit sphere bundle in $\Lambda_+^2 T^* X$ for X a smooth, oriented 4-dimensional manifold with a metric having $\mathscr{W}_+ \equiv 0$.

Donaldson and Friedman [5] discuss complex properties of the 3-folds which appear in Theorem 1.2. Hitchin [9] has shown that the only such 3-folds which are Kähler have $X = S^4$ or $\underline{\mathbb{CP}}^2$ with their canonical metrics. Campana [3], building on work of Poon [19], has shown that for one of these to be Moishezon, the corresponding X must be homeomorphic to $\#_N \underline{\mathbb{CP}}^2$. King and Kotschick [12] explain the relationship between the space of complex deformations of Z and the moduli space $\mathscr{M}(X)$.

Separately, Jim Carlson and Dieter Kotschik have pointed out to the author that Theorem 1.1 has the corollary that every finitely presentable group is the fundamental group of a compact complex 3-manifold.

2. Strategy

This section provides a section by section outline of the strategy and steps in Theorem 1.1's proof.

(a) **§3: Decreasing $\mathscr{W}_+$.** To obtain a $\mathscr{W}_+ \equiv 0$ metric on $M_N \equiv M \#_N \underline{\mathbb{CP}}^2$, start by fixing a metric g_M on M.

There is a method for decreasing $\mathscr{W}_+$ by connect summing with $\underline{\mathbb{CP}}^2$'s. Indeed, if $\mathscr{W}_+$ is large in some ball on M (as measured with g_M), then a suitable measure of $\mathscr{W}_+$ can be decreased by a predetermined fraction if enough $\underline{\mathbb{CP}}^2$'s are connect summed into said ball.

Here is why this connect sum strategy works: The Weyl curvature transforms homogeneously under a conformal change of metric. Also, the operation of connect sum is almost conformal. (It actually would be were the summing manifolds conformally flat.)

Since $\mathscr{W}_+|_{\underline{\mathbb{CP}}^2} \equiv 0$, connect summing into a very small ball on M has an almost negligible effect on any conformally insensitive measure of the size of $\mathscr{W}_+$. The L^2-norm is, for example, conformally invariant. A somewhat different norm, $\|\cdot\|_{*,p}$ (to be specified later), will actually be used.

It turns out that if the connect sum is carefully made, and only on a very small ball B, where $\mathscr{W}_+ \neq 0$ initially, then

$$\|\mathscr{W}_+|_{B\#\underline{\mathbb{CP}}^2}\|_{*,p} \leq (1-\delta_0)\cdot\|\mathscr{W}_+|_B\|_{*,p}. \tag{2.1}$$

Here, $\delta_0 > 0$ is a universal constant. Equation (2.1) holds for the L^2 norm too.

It cannot be stressed enough that (2.1) is valid in any sufficiently small ball. The ability to make B arbitrarily small is *the key* to all that follows.

After connect summing, the metric on the ball B (ε = radius B) is unchanged near the boundary of B. The metric is distorted in some annulus with inner radius ε_1. Then, an inner ball of radius ε_1 is replaced by $\underline{\mathbb{CP}}^2$. (See Figure 2.1 which pictures the connect sum of manifolds X and Y (read M and $\underline{\mathbb{CP}}^2$) with metrics g_X and g_Y, respectively.)

Note. When $Y = \underline{\mathbb{CP}}^2$ in Figure 2.1, with the Fubini-Study metric g_{FS}, one has $\mathscr{W}_+ \equiv 0$ on the region labeled "tunnel to Y" in Figure 2.1(a), and also where the metric is labeled g_Y in Figure 2.1(b).

In the problem at hand, the radius ε for the ball on M can be taken as small as needed. The ratio $\varepsilon_1/\varepsilon$ can also be assumed small.

In any event, given (2.1) and the picture of Figure 2.1, the procedure for decreasing $\mathscr{W}_+$ is clear: Fix some small ε and judiciously choose lots of disjoint balls of radius ε. As pictured in Figure 2.1, connect sum with $Y = \mathbb{CP}^2$ in each ball. The result is

$$\|\mathscr{W}_+^{(\mathrm{after})}\|_{*,p} \leq (1-\delta)\cdot\|\mathscr{W}_+^{(\mathrm{before})}\|_{*,p}, \tag{2.2}$$

where $\delta \sim \delta_0/100$.

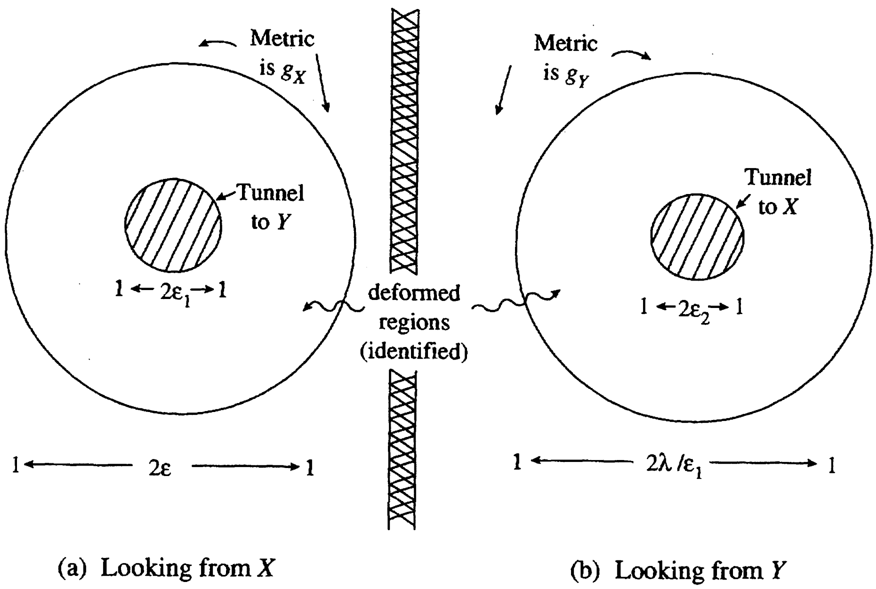

FIGURE 2.1

Now, repeat this procedure a large number, say n, times. But, be careful to make $\varepsilon \equiv \varepsilon(l)$ smaller each time (require that $\sum_{l=1}^{n} \varepsilon(l)$ be very small), and avoid connect summing on any previously added $\underline{\mathbb{CP}}^2$ (avoid the darkened tunnel region in Figure 2.1(a)). If care if taken, then n-repeats produces (2.2) with $(1-\delta)$ replaced by $(1-\delta)^n$.

After, say, $n = 3$ repeats, a ball in M can have the appearance depicted in Figure 2.2 (next page), where each blackened spot is meant to represent a tunnel to a different $\underline{\mathbb{CP}}^2$.

It is tempting to consider infinitely many repeats (producing a factor $(1-\delta)^\infty = 0$ on the right side of (2.2)), but this is not a reasonable option. Rather, one must stop after some large number of (say, $n-1$) repeats.

It proves useful to then connect sum on an additional, nth layer of $\underline{\mathbb{CP}}^2$'s according to a slightly different rule. This last step is called the "Cokernel Step" for reasons that should be evident after reading §9.

Theorem 3.15 summarizes some of the properties of the metric on $M\#_N\underline{\mathbb{CP}}^2$ which result from these n steps.

(b) **The deformation theory:** §§4–8. Let X be a compact, oriented 4-manifold with a Riemannian metric g. If $h \in \mathrm{Sym}^2(T^*X)$ has norm

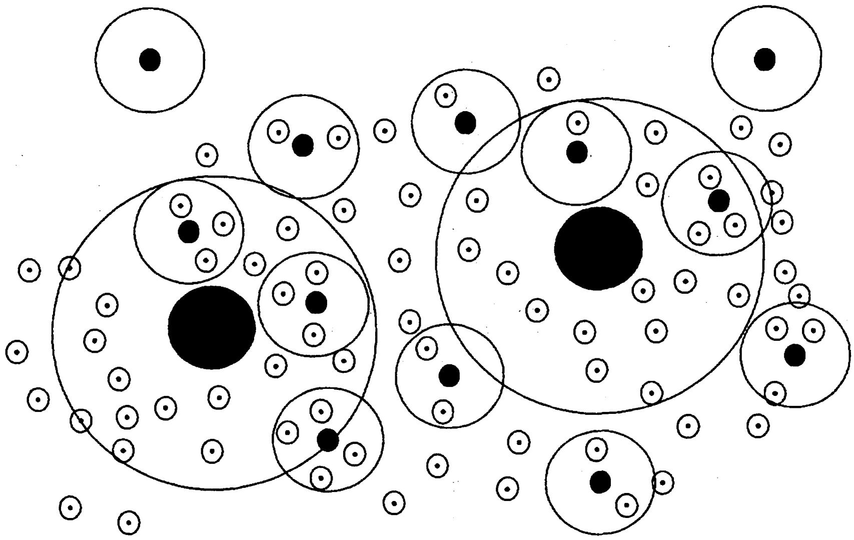

FIGURE 2.2

$|h|_g < 1$, then $g + h$ is also a Riemannian metric.

The self-dual Weyl curvature of $g+h$ has the following schematic form:

$$\mathscr{W}_+[g+h] = \mathscr{W}_+[g] + L_g \cdot h + \nabla_g \cdot (f_1(h) \cdot \nabla_g h) + f_2(h)(\nabla_g h)^{\otimes 2}. \tag{2.3}$$

Here, L_g is a second-order differential operator. Also, $f_1(h)$ and $f_2(h)$ are tensors which are analytic in h (if $|h|_g < 1$) and obey

$$|f_1(h)| \le c \cdot |h|, \quad |\nabla_g f_1(h)| \le c \cdot |\nabla_g h|, \quad |f_2(h)| \le c, \tag{2.4}$$

where c is some universal constant. (Constants which are universal, i.e., metric and manifold independent, will be denoted by the letter c.)

If $\mathscr{W}_+[g+h]$ is meant to vanish, then (2.3) defines a nonlinear differential equation for h. It is a particular example of the following inhomogeneous equation:

$$L_g \cdot h + \nabla_g(f_1(h) \cdot \nabla_g h) + f_2(h) \cdot (\nabla_g h)^{\otimes 2} = Q. \tag{2.5}$$

The relevant analytic issue is this: What properties of Q insure (2.5)'s solvability for h (with $|h|_g < 1$)?

The following line of thought leads to an answer of the preceding question: If the solution, h, is to be small ($|h|_g < 1$), then one should expect Q to be small. Therefore, (2.5) should be amenable to techniques which are essentially perturbative. (In practice, a contraction mapping theorem will be proved.)

A perturbative approach to (2.5) requires, at its start, the invertibility of L_g. This operator forms part of an elliptic complex (see, e.g., [7] or [12]) and its invertibility is insured on a finite codimension subspace of the space of sections of the traceless, symmetric endomorphisms of $\Lambda^2_+ T^*$. That is, the equation (for h_0)

$$\Pi L_g h_0 = \Pi \cdot Q \tag{2.6}$$

is solvable when Π is a projection onto a subspace of the range of L_g.

Of course, for application to (2.5), the solution to (2.6) must have small L^∞ norm. This is a nontrivial point in as much as a small L^2-bound on Q (which is conformally invariant) will not insure a small L^∞ bound for h_0. (In dimension 4, a second-order operator will not invert L^2 into L^∞.) The preceding remark gives a reason for *not* taking the L^2-norm for $\|\cdot\|_{*,p}$ in (2.1).

With the linear problem (2.6) understood, the nonlinear problem (2.5) can be analyzed as a contraction mapping question.

For this purpose, one fixes a Banach space $\mathscr{H}$, in which the solution h is to be found. Assume that

$$(\Pi L_g)^{-1} \Pi \cdot Q \in \mathscr{H}, \tag{2.7}$$

and that

$$T(h) = (\Pi L_g)^{-1} \Pi(\nabla_g \cdot (f_1(h) \nabla_g h) + f_2(h) \cdot (\nabla h)^{\otimes 2}) \tag{2.8}$$

maps $\mathscr{H}$ smoothly to itself.

Then, a small enough norm for $(\Pi L_g)^{-1} \cdot \Pi Q$ in $\mathscr{H}$ will be seen to insure the existence of a (unique) small solution h of

$$0 = \Pi L_g \cdot h + \Pi(\nabla_g (f_1(h) \nabla_g h) + f_2(h)(\nabla_g h)^{\otimes 2} + Q). \tag{2.9}$$

Of course, (2.9) is not (2.5); they are only equivalent if h solves the additional finite-dimensional system

$$(1 - \Pi) \cdot (L_g h + \nabla_g (f_1(h) \cdot \nabla_g h) + f_2(h)(\nabla_g h)^{\otimes 2} + Q) = 0. \tag{2.10}$$

(c) §§**4 and 5: Linear theory.** The nonlinear equation in (2.9) will be solved using the contraction mapping theorem. This puts all the hard work into the analysis of the linear problem, (2.6). In fact, sufficient knowledge of (2.6) will make (2.9) a formality. (Thus, four sections study essentially the linear problem (§§4, 5, 6, and 7) while the nonlinear problem, (2.9), is solved in the relatively short §8.)

With this understood, §4 is occupied with estimates for first-order operators and the Laplacian. (§5 factors the linear operator L_g into two operators, one a first-order, elliptic operator, and the second an operator with the Laplacian as its highest order piece.) These estimates in §4 are not standard (to the author's knowledge). The subtlety is in solving (2.6) for Q with an L^∞ bound on the solution h_0. One is further constrained because Q, in practice, will be controlled only in a scale-invariant norm. For example, $\mathscr{W}_+$ will be small only in the L^2-norm (which will not give an L^∞ estimate for h_0) and the norm

$$\|\mathscr{W}_+\|_{*,\rho} \equiv \sup_{x\in X}\left[\int_{B_\rho(x)} d\,\mathrm{vol}_g \frac{|\mathscr{W}_+|_g}{(\mathrm{dist}_g(x,\cdot))^2}\right], \tag{2.11}$$

where $B_\rho(x)$ is the radius ρ ball with center x and $\mathrm{dist}_g(x,\cdot)$ is the distance function from x.

In any event, §4 is occupied with basic estimates for the Laplacian and first-order operators using the norm in (2.11) and suitable generalizations. Also, the projection Π is defined in §4 as a spectral projection for the Laplacian.

§5 then takes §4's relatively abstract estimates and shows how to use them to analyze the particular equation (2.6) which comes from the $\mathscr{W}_+ \equiv 0$ problem.

The novelty (to the author's knowledge) is a decomposition of (2.6) into two coupled equations, one first-order with elliptic symbol and the other second-order with Laplacian symbol. Propositions 5.1 and 5.7 summarize §5.

By the way, §5's estimates depend intimately on properties of the metric g on X, such as its injectivity radius and curvature. But, they depend only on scale invariant properties of Q.

(d) **§§6 and 7: Linear theory on connect sums.** In the problem of interest, (2.5) is an equation on $M\#_N\overline{\mathbb{CP}}^2$, where the metric g is pictured in Figure 2.2 (and Figure 2.1), and Q is $\mathscr{W}_+[g]$.

If the problem is considered in this light, then one is forced to consider certain seemingly unpleasant issues which arise from the fact that the injectivity radius of g cannot be controlled. That is, small Q requires small injectivity radius. (This should be clear from Figure 2.2.)

This minute injectivity radius will cause headaches if one takes the classical approach to defining $(\Pi L_g)^{-1}$ (in (2.7)) using the formal adjoint L_g^*. The classical choice for L_g^{-1},

$$L_g^{-1} = L_g^*(L_gL_g^*)^{-1}, \tag{2.12}$$

breaks the conformal invariance; as a result, standard estimates for this operator involve said injectivity radius. One can imagine some sort of unpleasant feedback where the size of Q can be decreased with the addition of $\underline{\mathbb{CP}}^2$'s, but as $\underline{\mathbb{CP}}^2$'s get added, control is lost over L_g^{-1} and the projection Π. That is, $(\Pi L_g)^{-1} \cdot \Pi \cdot Q$ might *grow* and/or the cokernel of Π might increase in dimension with each additional $\underline{\mathbb{CP}}^2$. The first case would jeopardize the solvability of (2.9) and the second would threaten the solvability of (2.10).

Rather than tussel with (2.12), it proves simpler to construct $(\Pi L_g)^{-1}$ by employing a strategy of Donaldson for analyzing conformally invariant operators on connect sums. (Donaldson employs the strategy to construct anti-self-dual connections on bundles over connect sums, see [6].)

(This is an appropriate moment to remark that L_g has no unique inverse. In fact, its symbol has a 5-dimensional kernel; a manifestation of the conformal and diffeomorphism invariance of the equation $\mathscr{W}_+ \equiv 0$.)

Here (roughly) is Donaldson's strategy: Treat the equation $\Pi L_g h = \Pi \cdot Q$ on a connect sum $X \# Y$ as two equations, one on X and one on Y. However, there are matching conditions which are imposed by the tube which joins them. That is, think of $X \# Y$ as in Figure 2.1. Schematically, the equation on X is

$$\Pi_X L_{\tilde{g}_X} h_X = \Pi_X \cdot \beta_X Q, \tag{2.13a}$$

and the equation on Y is

$$\Pi_Y L_{\tilde{g}_Y} h_Y = \Pi_Y \cdot \beta_Y Q. \tag{2.13b}$$

Here, $\beta_X + \beta_Y = 1$ on $X \# Y$, with support of β_X on the complement of the shaded region in Figure 2.1(a). Then, β_Y is defined likewise on Y. Meanwhile, $\tilde{g}_X$ is a metric on X which agrees, up to a conformal factor, with the metric g on $X \# Y$ in the complement of the shaded region. Likewise, define $\tilde{g}_Y$, a metric on Y.

The matching conditions for h_X and h_Y come from the identifications of the "deformed region" in Figures 2.1(a) and 2.1(b).

If it turns out (and it will) that the metric $\tilde{g}_X$ is suitably close to the original metric on X (say, g_X), then (2.13a) will be a perturbation of an equation on X, written with the fixed metric, g_X. The same remark holds on Y also for equation (2.13b).

For the applications under consideration, $X = M$, the given manifold, and $Y = \coprod_N \underline{\mathbb{CP}}^2$. Thus, Donaldson's strategy turns the equation $\Pi L_g h = \Pi \cdot Q$ on $M \#_N \underline{\mathbb{CP}}^2$ into $N+1$ equations, one on M and one on each of the added $\underline{\mathbb{CP}}^2$'s. What is more, the metric $\tilde{g}_M$ will be seen to be suitably close to the original metric, g_M, chosen on M. Likewise, the metric $\tilde{g}_{\underline{\mathbb{CP}}^2}$ ends up being the Fubini-Study metric g_{FS} on $\underline{\mathbb{CP}}^2$.

These last points are important for they imply that the operators $L_{\tilde{g}_M}$ and $L_{\tilde{g}_{\underline{\mathbb{CP}}^2}}$ can be analyzed completely in terms of L_{g_M} and $L_{g_{\mathrm{FS}}}$, respectively. And, the analysis for the latter can be quoted verbatim from Propositions 5.1 and 5.7, respectively.

Thus, Donaldson's strategy makes it possible to sidestep completely the small injectivity radius pathologies that were predicted from §3's constructions. The cost of this sidestep is borne by the estimates in §5.

Accept these last remarks and it should be no surprise that the projection Π_M in (2.13a) is defined completely by g_M and is the same projection as in (2.6). Meanwhile, on $\underline{\mathbb{CP}}^2$, the operator $L_{g_{\mathrm{FS}}}$ is actually surjective, so the projection $\Pi_{\underline{\mathbb{CP}}^2}$ in (2.13b) can be the identity operator.

These last points are absolutely crucial, for they imply that the *number* of constraints in (2.10) is *independent* of the number N of added $\underline{\mathbb{CP}}^2$'s. (Were $\underline{\mathbb{CP}}^2$ replaced by $K3$ or T^4 as the stabilizing manifold, then the number of constraints would grow with N due to the failure of the surjectivity of L_g on these manifolds.)

§6 shows how to write (2.6) on M_N as (2.13a, b). The section ends with Theorem 6.3 which states an existence theorem for (2.13a, b).

§7 is occupied with Theorem 6.3's proof. Given (2.13a, b) the proof, though lengthy, is little more than a perturbation of the results from §5.

(e) §**8: Nonlinear theory.** §8 writes (2.9) on M_N (from §3) in the schematic form of (2.13). Think of Q in (2.13) as a sum of an h-dependent term (the nonlinear part of (2.9)) and an h-independent term (Q in (2.9)).

The linear theory for (2.13) from Theorem 6.3 was *designed* so that the nonlinear version could be simply treated with the Contraction Mapping Theorem. This is accomplished in §8 and Theorem 8.3 summarizes.

(f) §**9: Killing constraints.** If the reader will accept the plausibility of the preceding comments, then the nonlinear problem (2.9) can be solved without much trouble and one is left with (2.10). Remark that Donaldson's method as outlined interprets the finite system of (2.10) in terms of equations on M as written with the original metric g_M.

These equations assert an orthogonality to the eigenvectors with small eigenvalue of a Laplacian on M. (This Laplacian is the standard one, $\nabla^*_{g_M}\nabla_{g_M}$, on $C^\infty(\mathrm{End}\,\Lambda^2_+T^*M)$.) In fact, to a first approximation, these equations assert that the metric g on M_N has its $\mathscr{W}_+$ (suitably interpreted as a tensor on M) orthogonal to those eigenvectors of $\nabla^*_{g_M}\nabla_{g_M}$ with small eigenvalue.

With this last remark understood, recall from §2a the remark that the final step of adding $\overline{\mathbb{CP}}^2$'s was done using a set of rules which differed from those of the first $n-1$ steps. These new rules are designed so that the resulting metric on M_N is a solution to (2.10).

That is, (2.10) will be considered to be a finite set of equations on the space of parameters for the addition of $\overline{\mathbb{CP}}^2$'s in that final nth layer, the Cokernel Step.

(**Note.** The number of such equations is ultimately determined by g_M through a precise definition of the word "small" as it appears in the phrase "small eigenvalues of $\nabla^*_{g_M}\nabla_{g_M}$.")

Given the preceding remarks, the solvability of (2.10) can be analyzed by considering the manner in which the addition of a $\overline{\mathbb{CP}}^2$ affects the small eigenvalue orthogonality.

In this regard, it is important to note that the addition of a $\overline{\mathbb{CP}}^2$ is local to the ball B in Figure 2.1(a), where the $\overline{\mathbb{CP}}^2$ is summed, for the difference, $\mathscr{W}_+[g_{B\#\overline{\mathbb{CP}}^2}] - \mathscr{W}_+[g_B]$, is supported only in the deformed region in Figure 2.1(a). This localization of the affect makes the calculation straightforward. Here is the result: Adding additional $\overline{\mathbb{CP}}^2$'s in that nth, Cokernel Step can insure the vanishing of (2.10).

The details of all of this are given in §9 and Theorem 9.2 summarizes. Theorem 1.1 is a corollary to Theorem 9.2.

One last point: The ultimate number, n, of steps of adding $\overline{\mathbb{CP}}^2$'s has a lower bound which is determined in §9 from these small eigenavlue considerations. The number N of total $\overline{\mathbb{CP}}^2$'s added can be estimated from g_M, but this estimate will not likely be optimal.

(g) **Some comparisons.** The preceding strategy for the construction of metrics with $\mathscr{W}_+ \equiv 0$ is modelled heavily on a strategy that the author used in [20] to analyze the topology of moduli spaces of anti-self-dual connections on principal bundles over a fixed 4-manifold. As here, the strategy in [20] had three parts consisting of: (1) Decreasing self-dual curvature by "connect summing" a standard anti-self-dual connection from S^4. (2) Construction of a contraction mapping theorem, decomposed as in (2.9), (2.10) into infinite- and finite-dimensional parts. (3) Solving the finite-dimensional constraints by making additional "connect sums."

Remark here that the author learned of the decomposition in (2.9), (2.10) from a paper by Kuranishi [14] on deformations of complex structures.

Finally, remark that Floer's construction of metrics on $\#_N\overline{\mathbb{CP}}^2$ with $\mathscr{W}^+ \equiv 0$ is similar, in spirit, to the approach here. However, Floer was not faced with the problem of decreasing $\mathscr{W}_+$, not was he forced to deal with (2.10). For these reasons his implicit function theorem is much simpler than the one that will be presented here.

Floer's theorem is reproved here as Theorem 9.3.

3. Decreasing $\mathscr{W}_+$

Let M be a smooth, oriented, compact 4-dimensional manifold. This section will construct metrics on $M\#_N\overline{\mathbb{CP}}^2$ (N is large) which have small $\mathscr{W}_+$.

(a) Terminology. Let X be a 4-dimensional oriented manifold with Riemannian metric g. Let $\langle v, w\rangle_g$ denote the inner product on tensor bundles $(T^*X^{\otimes p} \otimes TX^{\otimes q})$ over X. Let $|v|_g$ be the norm, and ∇_g the covariant derivative as defined with the Levi-Civita connection.

The Riemann curvature, R_g, of g is defined as follows: It is a section of $\mathrm{Sym}^2(\Lambda^2 T^*)$ and evaluates on vector fields u, v, w, z according to the rule

$$R_g(u\wedge v; w\wedge z) = \langle u, [\nabla_w, \nabla_z]v - \nabla_{[w,z]}v\rangle_g. \tag{3.1}$$

As X is assumed to be oriented, the bundle $\Lambda^2 T^*$ decomposes as indicated in (1.1). With a chosen, oriented, orthonormal frame $\{e_1, e_2, e_3, e_4\}$ for T^*X at some point, $\Lambda^2_\pm T^*$ have orthonormal frames

$$\left(\frac{e_1\wedge e_4 \pm e_2\wedge e_3}{\sqrt{2}}, \frac{e_2\wedge e_4 \pm e_3\wedge e_1}{\sqrt{2}}, \frac{e_3\wedge e_4 \pm e_1\wedge e_2}{\sqrt{2}}\right). \tag{3.2}$$

The decomposition in (1.1) is due to the fact that the Lie algebra so(4) is not simple; it decomposes as $\mathrm{so}(3)\oplus\mathrm{so}(3)$.

Let FX denote the principle SO(4) bundle of oriented, orthonormal frames in TX. Let $\pi\colon FX \to X$ denote the projection.

A point $f \in FX$ defines a coordinate system on a ball B with center $x = \pi(f)$. These are the Gaussian coordinates; a diffeomorphism φ_f of a ball about the origin in $\mathbb{R}^4$ onto B. The origin is mapped to x and the geodesics through $0\in\mathbb{R}^4$ (the straight lines) are mapped to the geodesics through x. This φ_f is characterized by the preceding two qualities plus

$$d\varphi_f\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) = f. \tag{3.3}$$

The metric g pulls back by φ_f as

$$\varphi_f^* g = g_E + m_g, \tag{3.4}$$

where $g_E = \sum_{i=1}^4 dx_i \otimes dx_i$ and

$$\begin{aligned} |m|_{g_E} &\le c \cdot Z_{g,B}|x|^2, \\ |\nabla_{g_E} m|_{g_E} &\le c \cdot Z_{g,B}|x|, \\ |\nabla_{g_E}^{\otimes 2} m|_{g_E} &\le c \cdot Z_{g,B}. \end{aligned} \tag{3.5}$$

The constant $Z_{g,B}$, above, is

$$Z_{g,B} = \sup_B(|R_g|_g + \operatorname{diam}(B) \cdot |\nabla_g R_g|_g). \tag{3.6}$$

Here is one last and crucial remark about Gaussian coordinates: $\nabla_{g_E}\nabla_{g_E}\varphi_f^* g|_0$ both determines and is determined by $R_g|_x$ in that

$$\frac{\partial^2}{\partial x_i \partial x_j}\left\langle \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right\rangle \varphi_f^* g \bigg|_{x=0} \equiv K_{ijkl},$$

where

$$K_{ijkl} = \tfrac{1}{6}(R_g(e_l, e_j, e_i, e_k) + R_g(e_l, e_i, e_j, e_k)). \tag{3.7}$$

Note that (3.4)–(3.7) are all classical facts which may be found in most texts on Riemannian geometry (e.g., [13]).

One last remark about Gaussian coordinates: A ball $B \subset M$ is contained in a Gaussian coordinate chart whenever its radius is bounded by $c^{-1} \cdot r_g$, where $c \ge 1$ is a universal constant and r_g is the injectivity radius of X. The derivation of the manifold from Euclidean space will be measured in part by

$$Z_g \equiv r_g^{-2} + \sup_M(|R_g| + |\nabla R_g|^{2/3}). \tag{3.8}$$

(b) Conformal transformations. A metric g is conformally equivalent to a metric g' if $g' = e^u \cdot g$ for some $u \in C^\infty(X)$. The phrase "conformal equivalence class of a metric g" is a mouthful, so we will use the term "conformal metric."

The splitting in (1.1) only depends on the choice of a conformal metric, and the assertion that $\mathscr{W}_+ \equiv 0$ is an equation for a conformal metric. This is because $\mathscr{W}_+$ transforms homogeneously under conformal transformation. Write $e^u \equiv f$ and

$$|\mathscr{W}_+[f \cdot g]|_{fg} = f^{-1}|\mathscr{W}_+[g]|_g. \tag{3.9}$$

A diffeomorphism ψ of M will be said to be conformal for a metric g is $\psi^* g = e^u \cdot g$ for some u. Of prime interest is the following example. Think of $\mathbb{R}^4$ as the quaternions, $\mathbb{H}$. Let $\lambda > 0$ be a fixed number and define $\psi_\lambda : \mathbb{R}^4\backslash\{0\} \to \mathbb{R}^4\backslash\{0\}$ by

$$\psi_\lambda^* x = \lambda \cdot x^{-1}. \tag{3.10}$$

It is an exercise to check that

$$\psi_\lambda^* g_E = \frac{\lambda^2}{|x|^4} \cdot g_E, \tag{3.11}$$

where $g_E = \sum_{i=1}^4 dx_i \otimes dx_i$. (Note: ψ_λ is orientation preserving!)

(c) Connect sums. Let X_1 and X_2 be oriented, 4-dimensional manifolds with metrics g_1 and g_2, respectively. The connect sum $X_1\#X_2$ with a conformal metric can be constructed with the choice of points $f_1 \in FX_1$, $f_2 \in FX_2$ and positive ε_1, ε_2, and λ obeying

$$\lambda > \varepsilon_1\varepsilon_2 \quad \text{but} \quad \lambda \ll \min(\varepsilon_1 Z_{g_2}^{-1/2}, \varepsilon_2 Z_{g_1}^{-1/2}), \tag{3.12}$$

with Z_g defined in (3.8).

Use f_1 to construct the Gaussian coordinate chart on a ball $B_1 \subset X_1$ centered at $\pi(f_1)$. The chart φ_1 for B_1 identifies B_1 with a ball centered at $0 \in \mathbb{R}^4$. Let $(x_1, \cdots, x_4)$ be the Euclidean coordinates on this $\mathbb{R}^4$. Use f_2 to construct the Gaussian chart φ_2 for B_2, thus identifying B_2 with a ball about 0 in a second copy of $\mathbb{R}^4$ with coordinates $(y_1, \cdots, y_4)$.

Definition 3.1. The connect sum $X_1\#X_2$ is defined as a smooth manifold as follows: It is the union of open sets U_1, U_2:

$$U_1 \simeq X_1\backslash\varphi_1(\{x \in \mathbb{R}^4 : |x| < \varepsilon_1\}),$$
$$U_2 \simeq X_2\backslash\varphi_2(\{y \in \mathbb{R}^4 : |y| < \varepsilon_2\}).$$

These sets overlap with

$$U_1 \cap U_2 \simeq \begin{cases} \varphi_1(\{x \in \mathbb{R}^4 : \lambda\varepsilon_2^{-1} > |x| > \varepsilon_1\}) \subset X_1, \\ \varphi_2(\{y \in \mathbb{R}^4 : \lambda\varepsilon_1^{-1} > |y| > \varepsilon_2\}) \subset X_2. \end{cases}$$

The preceding identifications are

$$U_1 \cap U_2|_{X_1} = \varphi_1 \circ \psi_\lambda \circ \varphi_2^{-1}(U_1 \cap U_2|_{X_2}),$$

where $\psi_\lambda : \mathbb{R}^4\backslash\{0\} \to \mathbb{R}^4\backslash\{0\}$ is given in (3.10).

Equation (3.12) insures that $X_1\#X_2$ is a smooth, oriented manifold.

The construction of the conformal metric on $X_1 \# X_2$ requires the specification (once and for all time) of a bump function $\beta \in C^\infty([0, \infty))$ which obeys

$$\beta|_{[0,5/6]} \equiv 0 \quad \text{and} \quad \beta|_{[4/3,\infty)} \equiv 1. \tag{3.13}$$

With positive ρ chosen, introduce $\beta_\rho \in C^\infty(\mathbb{R}^4)$ as

$$\beta_\rho(x) \equiv \beta(|x|/\rho). \tag{3.14}$$

In the specification below of the conformal metric, it is implicit that the pullback by φ_1 of g_1 has the form of (3.4), and likewise for $\varphi_2^* g_2$.

A conformal class of a metric g is denoted below by $[g]$.

Definition 3.2. The conformal metric $[g]$ on $X_1 \# X_2$ is defined as follows: On $U_1 \backslash (U_1 \cap U_2)$, set $g = g_1$. On $U_2 \backslash (U_1 \cap U_2)$, set $g = g_2$. Define $[g]$ on $U_1 \cap U_2$ by $\varphi_{1,2}^*[g] = [\hat{g}_{1,2}]$, where

$$\hat{g}_1 \equiv g_E + \beta_{\varepsilon_1} \cdot m_{g_1} + \frac{|x|^4}{\lambda^2} \cdot \psi_\lambda^*(\beta_{\varepsilon_2} \cdot m_{g_2})$$

and

$$\hat{g}_2 \equiv g_E + \frac{|y|^4}{\lambda^2} \psi_\lambda^*(\beta_{\varepsilon_1} \cdot m_{g_1}) + \beta_{\varepsilon_2} \cdot m_{g_2}.$$

With reference to the definition, the astute reader will have observed that an honest metric on $X_1 \# X_2$ has not been defined, since the indicated $\hat{g}_1$ is merely conformal to the pull-back by $\varphi_1 \circ \psi_\lambda \circ \varphi_2^{-1}$ of the indicated $\hat{g}_2$.

(d) $\mathscr{W}_+$ for the conformal metric. Of sole interest here will be the case where the metric g_2 on X_2 has $\mathscr{W}_+ \equiv 0$. It will also be assumed that the constant Z_{g_2} of (3.8) is equal to 1. (When $X_2 = \mathbb{CP}^2$ with the Fubini-Study metric, the curvature is covariantly constant.)

With $\mathscr{W}_+[g_2] \equiv 0$, it is clear that $\mathscr{W}_+[g]$ has support on the open set U_1, and it differs from $\mathscr{W}_+[g_1]$ only on $U_1 \cap U_2$. Thus, it is natural to compare $\mathscr{W}_+[g]$ with $\mathscr{W}_+[g_1]$.

The comparison requires the following algebraic fact: Identify Euclidean $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ (use the inner product $\langle a, b\rangle = \text{Real}(ab^+)$). Then conjugation on $\mathbb{H}$ induces an orientation reversing, isometric involution of $\mathbb{R}^4$ and thus an isometry $\mathbb{I}\colon \Lambda_\pm^2 T^*\mathbb{R}^4 \to \Lambda_\mp^2 T^*\mathbb{R}^4$. With this understood, use $\mathbb{I}$ to denote the induced isometry

$$\mathbb{I}\colon \text{Sym}^2(\Lambda_\pm^2 T^*\mathbb{R}^4) \to \text{Sym}^2(\Lambda_\mp^2 T^*\mathbb{R}^4). \tag{3.15}$$

Proposition 3.3. *Let X_1 and X_2 be oriented, 4-dimensional Riemannian manifolds with metrics g_1 and g_2, respectively. Suppose $\mathscr{W}_+[g_2] \equiv 0$.*

Choose $f_{1,2} \in FX_{1,2}$ *and positive numbers* $\varepsilon_1, \varepsilon_2, \lambda$ *as constrained in* (3.12). *Make the connect sum* $X_1 \# X_2$ *with conformal metric* $[g]$ *as described in Definitions* 3.1 *and* 3.2. *Let* $\hat{g}_1$ *be as given in Definition* 3.2. *In the annulus* $\lambda\varepsilon_2^{-1} > |x| > \varepsilon_1$,

$$\mathscr{W}_+[\hat{g}_1]|_x = \mathscr{W}_+[g_1]|_0 + \frac{1}{4}(t^2\beta'')|_{t=(\lambda/\varepsilon_2|x|)^2} \cdot \frac{\lambda^2}{|x|^4} \cdot \mathbb{I} \cdot \mathscr{W}_-[g_2]|_{y=0} + \mathbb{R}em,$$

where $\mathbb{R}em$ *is estimated as follows:*

$$|\mathbb{R}em|_{g_1} \le c \cdot \left\{ \beta_{\varepsilon_1/2}(1-\beta_{2\varepsilon_1}) \cdot Z_1 + |x| \cdot Z_1^{3/2} + \frac{\lambda^2}{|x|^2} Z_1 + \lambda^2 Z_1^2 + \frac{\lambda^4}{|x|^4} Z_1 + \beta_{\lambda/2\varepsilon_2}(1-\beta_{2\lambda/\varepsilon_2} \cdot \frac{\lambda^3}{|x|^5} \right\}.$$

Here, $Z_1 = Z_{g_1}$ *as defined in* (3.8) *and* c *is a universal constant.*

Proof of Proposition 3.3. Of course, these formulas are obtained from Definition 3.2 by the standard formulas for calculating the curvature from a metric. With this said, the proof will be left to the reader except to remark that the estimation of the $\mathbb{R}em$ can be accomplished if one takes the following schematicized approach: Suppose that a metric g is given as

(3.16) $$g = g_E + u_1 + u_2$$

with $|u_{1,2}|_{g_E}$ small. The curvature of g can be estimated from that of $g_1 \equiv g_E + u_1$ and $g_2 \equiv g_E + u_2$ if one observes that the Levi-Civita connection (indicated by subscripted Γ) for g_1, g_2, and g is

(3.17) $$\Gamma_g = (1+a_2) \cdot \Gamma_{g_1} + (1+a_1) \cdot \Gamma_{g_2},$$

where $|a_i| \le c \cdot |u_i|$ and $|\nabla a_i| \le c \cdot |\nabla u_i|$ as long as $|u_i|_{g_E} < \frac{1}{2}$. (Under this same assumption, $|\Gamma_{g_i}| \le c \cdot |\nabla u_i|$.) q.e.d.

The application of this schematic approach obviously requires part of the following lemma.

Lemma 3.4. *Let* X_1, X_2 *and* g_1, g_2 *be as in Definitions* 3.1 *and* 3.2. *Let* $u_1 = \beta_{\varepsilon_1} m_1$ *and* $u_2 = |x|^4 \psi_\lambda^*(\beta_{\varepsilon_2} \cdot m_2)/\lambda^2$. *Then*

(1) $|u_1|_{g_E} \le Z_1 \cdot |x|^2$, $|\nabla u_1|_{g_E} \le c \cdot Z_1 |x|$.

(2) $|u_2|_{g_E} \le c \cdot Z_2 \cdot \lambda^2/|x|^2$, $|\nabla u_2|_{g_E} \le c \cdot Z_2 \cdot \lambda^2/|x|^3$, *where* $Z_i = Z_{g_i}$.

The proof of this lemma is left as an exercise.

(e) Local estimates for $\mathscr{W}_+$. The formulas in Proposition 3.3 are ultimately responsible for the assertion that connect summing with $\overline{\mathbb{CP}}^2$ can decrease $\mathscr{W}_+$.

There are two useful measures of $\mathcal{W}_+$. The first is the L^2-norm over the ball $B \subset X_1$ of radius λ/ε_2 and center $\pi(f_1)$:

$$e[g] \equiv \int_B d\,\mathrm{vol}_{\hat{g}_1} |\mathcal{W}_+[\hat{g}_1]|^2_{\hat{g}_1}. \tag{3.18}$$

(Keep in mind that $|\mathcal{W}_+[\hat{g}_1]|^2_{\hat{g}_1}$ has support in $U_1 \subset X_1$ even though g is a conformal metric on $X_1\#X_2$.)

The second measure of $\mathcal{W}_+$ is actually a function of $x \in X_1$:

$$e_*[g\,; x] \equiv \int_B d\,\mathrm{vol}_{g_1} \frac{|\mathcal{W}_+[\hat{g}_1]|_{g_1}}{(\mathrm{dist}_{g_1}(x\,,\cdot))^2}. \tag{3.19}$$

Here, $\mathrm{dist}_{g_1}(x\,,\cdot)$ is the function on X_1 which assigns to each $y \in X_1$ the distance, $\mathrm{dist}_{g_1}(x\,, y)$, from x to y as measured by the metric g_1. (Note: In (3.19), the volume form and norm are also defined with the metric g_1.)

With (3.18) and (3.19) understood, define $e[g_1]$ and $e[g_1\,; x]$ by the same equations but with g_1 replacing $\hat{g}_1$.

Proposition 3.5 below estimates $e[g]$ and $e_*[g\,; x]$. Before reading said proposition, please recall from Definition 3.1 that the connect sum $X_1\#X_2$ is defined with the choice of positive numbers $\varepsilon_1, \varepsilon_2, \lambda$ together with points $f_1 \in FX_1$ and $f_2 \in FX_2$. With $x_1 \equiv \pi(f_1) \in X_1$, Proposition 3.5 will require

$$\nu_1 \equiv |\mathcal{W}_+[g_1]|_{x_1}|_{g_1} > 0. \tag{3.20}$$

Proposition 3.5. *There exist constants $\delta > 0$ and $c_0 \geq 100$ with the following significance: Let X_1 be an oriented, compact 4-manifold with metric g_1. Let $X_2 = \underline{\mathbb{CP}}^2$ with the Fubini-Study metric. Pick $x_1 \in X_1$ where (3.20) holds. There is a constant $Z_0 \geq 1$ so that when*

(a) *$Z_1 \geq Z_0 \cdot (\nu_1 + 2)$ is first chosen;*

(b) *given Z_1, small $\varepsilon > 0$ is chosen;*

(c) *and, finally, $\varepsilon_1, \varepsilon_2$, and λ are set to be $\varepsilon_1 \equiv c_0^{-1}(\nu_1 \cdot Z_1^{-1})^{1/2} \cdot \varepsilon$, $\varepsilon_2 \equiv c_0^{-1}\nu_1^{1/2} \cdot \varepsilon$, and $\lambda \equiv c_0^{-1} \cdot \nu_1^{1/2} \cdot \varepsilon^2$,*

then there is a choice of $f_2 \in F\underline{\mathbb{CP}}^2$ and $f_1 \in FX_1|_{x_1}$ for which Definition 3.2 's conformal metric $[g]$ on $X_1\#\overline{\mathbb{CP}}^2$ obeys

$$e[g] \leq (1-\delta)e[g_1]$$

and, for all $x \in X_1$,

$$e_*[g\,; x] \leq (1-\delta)e_*[g_1\,; x].$$

(Remark that the conformal metric $[g]$ on $X_1\#X_2$ is unchanged when f_1 and f_2 are changed simultaneously by an $SO(4)$ rotation.)

With respect to Figure 2.1(a), the deformed region has outer radius $\lambda/\varepsilon_2 = \varepsilon$ and inner radius $\varepsilon_1 = c_0^{-1}(\nu_1 \cdot Z_1^{-1})^{1/2} \cdot \varepsilon$.

(f) Proof of Proposition 3.5: The L^2-norm. The value of $e[g_1]$ can be estimated by the fundamental theorem of calculus to be

$$\left| e[g_1] - \frac{\pi^2}{2} \cdot \varepsilon^4 \cdot \nu_1^2 \right| \leq c \cdot \nu_1 \cdot Z_1^{3/2} \cdot \varepsilon^5, \tag{3.21}$$

where the substitution $\lambda/\varepsilon_2 \equiv \varepsilon$ has been made.

As for $e[g]$, use Proposition 3.3's expression for $\mathscr{W}_+[\hat{g}_1]$ to compute the latter's $|\cdot|_{g_1}$-norm. Then, integrate the square of this norm (using $d\operatorname{vol} g_1$) over B. The result is

(3.22)

$$\int_B d\operatorname{vol}_{g_1} |\mathscr{W}_1[\hat{g}_1]|_{g_1}^2 = \frac{\pi^2}{2} \cdot \varepsilon^4 \nu^2 + c_1 \cdot R^2 \cdot \varepsilon^4 \cdot \langle \mathscr{W}_+[g_1]|_0, \mathbb{I}\mathscr{W}_-[g_2]|_0 \rangle_{g_1} + c_2 \cdot R^4 \varepsilon^4 + \text{Error}.$$

Here, $R \equiv \lambda/\varepsilon^2$ has been introduced. Also, $c_1, c_2 > 0$ are universal constants. The term above marked Error involves $\mathbb{R}em$ from Proposition 3.3. An estimation of the size of this term is a straightforward calculation: As long as R in (3.22) is fixed, the term "Error" in (3.22) is bounded by

$$\text{Error} \leq c_3 \cdot R^4 \cdot \varepsilon^4 \tag{3.23}$$

when ε is small and c_3 is another universal constant.

Of course, (3.22) and (3.23) measure the L^2-norm of $\mathscr{W}_+[\hat{g}_1]$ using the volume and inner product of the metric g_1. However, Lemma 3.4 insures that the replacement of g_1 by $\hat{g}_1$ can be accommodated by changing the universal constant c_3 in (3.23). Thus, (3.21)–(3.23) imply that

$$\frac{e[g]}{e[g_1]} \leq \left(1 + c_1 \frac{R^2}{\nu_1} \left(\frac{1}{\nu_1} \langle \mathscr{W}_+[g_1]|_0, \mathbb{I} \cdot \mathscr{W}_-[g_2]|_0 \rangle_{g_1} \right) + c_4 \frac{R^2}{\nu_1^2} \right). \tag{3.24}$$

Given (3.24), the proof of the first assertion of Proposition 3.5 will follow from

Lemma 3.6. *There exists a universal $c_0 > 0$ with the following significance: Let $\mathbb{A}$ be a 3×3, symmetric and traceless matrix. Let $\nu_1 = |\mathbb{A}| = (\operatorname{tr}(\mathbb{A}^2))^{1/2}$. Let g_2 denote the Fubini-Study metric on $\underline{\mathbb{CP}}^2$. Pick $y \in \underline{\mathbb{CP}}^2$ and there exists an isometry $J: \Lambda_-^2 T^*\underline{\mathbb{CP}}^2 \to \mathbb{R}^3$ with the property that*

$$\operatorname{tr}((J \cdot \mathscr{W}_-[g_2]|_y)\mathbb{A}) \leq -c_0 \cdot \nu_1. \tag{3.25}$$

Before proving Lemma 3.6, consider how it implies the first assertion of Proposition 3.5: The lemma's isometry can be realized by choosing $f_2 \in F\underline{\mathbb{CP}}^2$. With this understood, use the lemma with $\mathbb{A} = \mathbb{I} \cdot \mathscr{W}_+[g_1]|_0$. Put the result in (3.24) to find

$$\frac{e[g]}{e[g_1]} \leq \left(1 - \frac{c_1 \cdot c_0}{\nu_1} \cdot R^2 + \frac{c_4}{\nu_1^2} \cdot R^4\right). \tag{3.26}$$

With (3.26) understood, one should make $R^2 = \nu_1 \cdot c_1 \cdot c_0/(2 \cdot c_4)$, to obtain Proposition 3.5's first assertion.

Proof of Lemma 3.6. The Fubini-Study metric on $\underline{\mathbb{CP}}^2$ has covariantly constant $\mathscr{W}_-$. Furthermore, there exists an isometry J_0 between $\Lambda^2_- T^*\underline{\mathbb{CP}}^2|_y$ and $\mathbb{R}^3$ so that

$$J_0 \cdot \mathscr{W}_-[g_{FS}] = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.27}$$

From this fact, one gets (3.25) with some simple linear algebra. This last step is omitted. q.e.d.

(Remark that the lemma is also true with all signs reversed on the right side of (3.27).)

(g) Proof of Proposition 3.5's e_* assertion. The proof of the e_* assertion of Proposition 3.5 is similar in most respects to the proof of the L^2 assertion. For the purpose of the proof, it is convenient to introduce the function f on X_1 whose value at x is

$$f(x) = \int_B d\,\mathrm{vol}_{g_1} \frac{1}{(\mathrm{dist}_{g_1}(x, \cdot))^2}. \tag{3.28}$$

Here, $B \subset X_1$ is the ball of radius $\lambda/\varepsilon_2 = \varepsilon$ about x_1. This function obeys the following uniform estimate:

$$c_1^{-1} \leq \frac{(\varepsilon^2 + \mathrm{dist}(x, x_1)^2)}{\varepsilon^4} \cdot f(x) \leq c_1. \tag{3.29}$$

With $f(x)$ understood, the fundamental theorem of calculus can be used to estimate $e_*[g_1; x]$:

$$|e_*[g_1; x] - \nu_1 \cdot f(x)| \leq c \cdot Z_1^{3/2} \cdot \varepsilon \cdot f(x). \tag{3.30}$$

As for $e_*[g; x]$, if $\lambda \equiv R \cdot \varepsilon^2$ with $R = c_0^{-1}\nu_1^{1/2}$, $c_0 \geq 100$, then the second term in the expression for $\mathscr{W}_+[\hat{g}_1]$ (in Proposition 3.3) is approximately 1/100th the size of the first. Therefore, a Taylor's expansion bounds the absolute value of the first two terms by

$$\begin{aligned}(3.31)\qquad &\nu_1 + \frac{1}{\nu_1}(t^2\beta'')|_{t=(\varepsilon/|x|)^2} \cdot \frac{\lambda^2}{|x|^4}\langle \mathscr{W}_+[g_1]|_0 , \mathbb{I} \cdot \mathscr{W}_-[g_2]|_0\rangle \\ &+ \frac{c}{\nu_1} \cdot \beta_{\varepsilon/2}(1-\beta_{2\varepsilon}) \cdot \frac{\lambda^4}{|x|^8}.\end{aligned}$$

Now, integrate the preceding against the function $(\mathrm{dist}_{g_1}(x, \cdot))^{-2}$ to bound

$$\begin{aligned}(3.32)\qquad e_*[g; x] \le \nu_1 \cdot f(x) \cdot \left\{1 - \frac{c_1 c_0}{\nu_1} \cdot R^2 + \frac{c_2}{\nu_1^2} \cdot R^4\right\} \\ + \int_B d\,\mathrm{vol}_{g_1}(\mathrm{dist}_{g_1}(x, \cdot))^{-2} \cdot |\mathbb{R}em|_{g_1}.\end{aligned}$$

Here, (3.25) has been exploited and (3.29) as well.

The last term in (3.32) can be readily estimated using Proposition 3.3 and (3.29). The result is: For ε small (and R fixed),

$$(3.33)\qquad e_*[g; x] \le e_*[g_1; x]\left\{1 - \frac{c_1 c_0}{\nu_1} \cdot R^2 + \frac{c_3}{\nu_1^2} \cdot R^4 + \frac{1}{Z_0}\right\},$$

a referral to (3.30) having been made. The last term in (3.33) (the one proportional to $1/Z_0$) comes from the first term in the expression for $|\mathbb{R}em|_{g_1}$ in Proposition 3.3.

The second assertion of Proposition 3.5 follows directly from (3.33).

(h) Global decrease of $\mathscr{W}_+$. Proposition 3.5 established that connect summing with $\underline{\mathbb{CP}}^2$ can decrease $\mathscr{W}_+$. The decrease is nonnegligible near the connect sum point, but its affect globally is insignificant for small ε. To affect $\mathscr{W}_+$ globally, one must connect sum at many points at once.

When connect summing at many points, the following convention will be in effect: At each connect sum point, Definitions 3.1 and 3.2 required parameters λ, ε_1, and ε_2. The values of these parameters *will vary* from point to point and this variation will *not* be explicitly noted (usually). So be *forewarned.*

The connected sum of many $\underline{\mathbb{CP}}^2$'s into a given X_1 starts with a choice of N disjoint points in X_1. Let $\Omega_1 \subset X_1$ denote the chosen set, and introduce

$$(3.34)\qquad d = \inf_{\substack{x, y \in \Omega_1 \\ x \ne y}} (\mathrm{dist}_{g_1}(x, y)).$$

If one $\underline{\mathbb{CP}}^2$ is connect summed at each point in Ω_1 with parameters ε_1, ε_2, and λ always obeying $\lambda/\varepsilon_2 < d$, there will not be interference between the different connect sums. This requirement simplifies the accounting of the size of $\mathscr{W}_+$.

Speaking of $\mathscr{W}_+$, there are two convenient measures of its size. In fact, let Q be any section of $\mathrm{End}(\Lambda_+^2 T^*M)$. Let $U \subset X_1$ be an open set. Then, the L^2-norm of Q over U is

$$\|Q\|_{2;U} = \left[\int_U d\,\mathrm{vol}_{g_1} |Q|_{g_1}^2\right]^{1/2}. \tag{3.35}$$

It turns out that there is a better way to measure Q on U. For this purpose, one must choose $\rho > 0$. For each $x \in M$, let $B_\rho(x) \subset M$ denote the radius ρ ball with center x. Set

$$\|Q\|_{*,\rho;U} \equiv \sup_{x\in M}\left(\int_{B_\rho(x)\cap U} d\,\mathrm{vol}_{g_1} \frac{|Q|_{g_1}}{(\mathrm{dist}_{g_1}(x,\cdot))^2}\right). \tag{3.36}$$

The convention will be that $\|Q\|_2 \equiv \|Q\|_{2;X_1}$ and likewise for $\|Q\|_{*,\rho}$.

When N $\underline{\mathbb{CP}}^2$'s are connect summed to X_1 at disjoint points with $\lambda/\varepsilon_2 < d$, there results a conformal metric $[g]$ on $X_1\#_N\underline{\mathbb{CP}}^2$. The support for $\mathscr{W}_+[g]$ is restricted to

$$X_1' = X_1 \setminus \bigcup_{x\in\Omega} B_{\varepsilon_1}(x). \tag{3.37}$$

Thus, the choice of a metric $\hat{g}_1$ on X_1' in g's conformal class identifies $\mathscr{W}_+[g]$ as a section over X_1 of $\mathrm{End}(\Lambda_+^2 T^* X_1)$. As such, its size can be measured by (3.35) and (3.36).

Definition 3.2 provides a useful such $\hat{g}_1$. This metric restricts to $X_1 \setminus \bigcup_{x\in\Omega} B_{\varepsilon_1}(x)$ as g_1 and its restriction to each $B_{\lambda/\varepsilon_2}(x)$ is documented in Definition 3.2.

With this understood, one has

Proposition 3.7. *There exists a $\delta > 0$ with the following significance: Let X_1 be a compact, oriented 4-manifold with metric g_1. There is a constant Z_0 such that if one chooses,*

(a) $r > 0$,

(b) *an open* $U \subseteq X_1$, *and*

(c) $Z_1 \ge Z_0$,

then, given r, U, Z_1, and $\varepsilon > 0$, a finite set $\Omega \subset U$ can be found so that the balls $\{B_\varepsilon(x)\}_{x\in\Omega}$ are disjoint and lie in U. Furthermore, with ε_1, ε_2, and λ determined for each $x \in \Omega$ as in Proposition 3.5 (using Z_1 and ε), frames $\{f_1 \in FX_1|_x\}_{x\in\Omega}$ and $f_2 \in F\underline{\mathbb{CP}}^2$ can be found so that connect *summing* $\underline{\mathbb{CP}}^2$ *to* X_1 *at all* $x \in \Omega$ *(according to Definitions* 3.1 *and* 3.2) *produces a conformal metric* $[g]$ *on said multiple connect sum which obeys*

(1) $\|\mathscr{W}_+[\hat{g}_1]\|_{2;U} \le (1-\delta) \cdot \|\mathscr{W}_+[g_1]\|_{2;U}$.

(2) $\|\mathscr{W}_+[\hat{g}_1]\|_{*,p;U} \le (1-\delta) \cdot \|\mathscr{W}_+[g_1]\|_{*,p;U}$ *for all* $p \ge r$.

Proof of Proposition 3.7. It is first necessary to establish the following geometric fact:

Lemma 3.8. *There exist integers* n, m *with the following significance: Let* X *be a compact,* 4*-dimensional Riemannian manifold. For all sufficiently small* $\varepsilon > 0$, *there exists a finite set* $\Omega \subset X$ *of disjoint points such that:*

(1) $\bigcup_{x\in\Omega} B_{n\cdot\varepsilon}(x)$ *covers* X.

(2) *The balls* $\{B_\varepsilon(x)\}_{x\in\Omega}$ *are mutually disjoint.*

(3) $\bigcap_{x\in\Omega'} B_{n\cdot\varepsilon}(x) = \varnothing$ *if* $\Omega' \subset \Omega$ *is a subset of* m *or more disjoint points.*

Proof of Lemma 3.8. Isometrically embed X in $\mathbb{R}^{10}$, then cover a tubular neighborhood of X in $\mathbb{R}^{10}$ using balls centered on a regular, (hyper)cubic lattice of side length $4\cdot\varepsilon$. These Euclidean balls will not intersect X in balls, nor will their centers lie in X. But for ε small, this defect can be readily alleviated by small deformations. q.e.d.

Given now the preceding lemma, here is how to prove Proposition 3.7: First, pick $\mu > 0$ small (to be determined shortly). Then, choose ε and then $\Omega \subset X_1$ according to Lemma 3.8.

When ε is sufficiently small, one can require the following: If, for given $x \in \Omega$,

$$\sup_{B_{n\cdot\varepsilon}} |\mathscr{W}_+[g_1]|_{g_1} \ge \mu, \tag{3.38}$$

then

$$|\mathscr{W}_+[g_1]|_{g_1}\Big|_x \ge \tfrac{1}{2}\mu. \tag{3.39}$$

This condition insures that estimates in $B_{n\cdot\varepsilon}(x)$ for $|\mathscr{W}_+[g_1]|_{g_1}$ can be inferred from knowledge of $\mathscr{W}_+[g_1]|_x$.

With μ, ε, and Ω chosen (subject to (3.38) and (3.39)), introduce $\Omega' \subset \Omega$ as the subset of points for which $B_\varepsilon(x) \subset U$. Then, set

$$U' \equiv \bigcup_{x\in\Omega'} B_\varepsilon(x) \quad \text{and} \quad U'' \equiv U \backslash U'. \tag{3.40}$$

Now, store these definitions and use Definitions 3.1 and 3.2 and Proposition 3.5 to connect sum a $\underline{\mathbb{CP}}^2$ into each $B_\varepsilon(x)$ for $x \in \Omega'$.

With the connect summing complete, consider the proof of the second assertion of Proposition 3.7. The proof of the first goes in much the same way and is left to the reader.

To obtain the proposition's second assertion, it proves convenient to introduce one additional piece of notation: When $V \subset X_1$ is an open set and Q is a section over V of $\operatorname{End}\Lambda^2_+ T^* X_1$, introduce the function $e_*[V; Q](\cdot)$ on X_1 by

$$(3.41) \qquad e_*[V; Q](y) = \int_{B(y)\cap V} d\,\mathrm{vol}_{g_1} \frac{|Q|_{g_1}}{(\mathrm{dist}_{g_1}(y, \cdot))^2},$$

where $B(y)$ is the radius ρ ball with center y.

Introduce a function $\alpha\colon X_1 \to [0, 1]$ by

$$(3.42) \qquad e_*[U''; \mathscr{W}_+[g_1]](\cdot) \equiv (1 - \alpha(\cdot)) \cdot \|\mathscr{W}_+[g_1]\|_{*,\rho;U}.$$

Thus

$$(3.43) \qquad e_*[U'; \mathscr{W}_+[g_1]](\cdot) \le \alpha(\cdot) \cdot \|\mathscr{W}_+[g_1]\|_{*,\rho;U}.$$

A lower bound for $\alpha(\cdot)$ is needed, and here is how to get one: When (3.38) is obeyed by $x \in \Omega$, then

$$(3.44) \qquad e_*[B_{n\cdot\varepsilon}(x); \mathscr{W}_+[g_1]](\cdot) \le c \cdot e_*[B_\varepsilon(x); \mathscr{W}_+[g_1]](\cdot),$$

where c is a universal constant. This last equation implies that

$$(3.45) \quad e_*[U''; \mathscr{W}_+[g_1]](\cdot) \le c \cdot e_*[U'; \mathscr{W}_+[g_1]](\cdot) + Z \cdot \mu + Y[U; \varepsilon] \cdot Z_1.$$

Here, Z is a constant which depends on X_1 and g_1 (not x or U), while Y is a constant which depends on U and ε: Y is proportional to the volume of that subset of U which is not contained in a ball $B_{n\cdot\varepsilon}(x)$ with $x \in \Omega'$. In particular,

$$(3.46) \qquad \lim_{\varepsilon\to 0} Y[U, \varepsilon] = 0.$$

These last remarks (and (3.42), (3.43)) insure that

$$(3.47) \qquad 1 \le (1 + c) \cdot \alpha + Z \cdot (\mu + Y) \cdot \|\mathscr{W}_+[g_1]\|^{-1}_{*,\rho;U}.$$

Equation (3.7) provides a uniform lower bound for α provided

$$(3.48) \qquad \mu + Y \le \frac{1}{4} \cdot \frac{1}{(1 + Z)} \cdot \|\mathscr{W}_+[g_1]\|_{*,\rho;U},$$

a condition which will be assumed from now on. Thus,

$$(3.49) \qquad \alpha(\cdot) \ge \frac{1}{(1 + c)} > 0.$$

With $\alpha(\cdot)$ now understood, notice that $e_*[U''; \mathscr{W}_+[\hat{g}_1]](\cdot)$ is identical to $e_*[U''; \mathscr{W}_+[g_1]](\cdot)$, but

$$e_*[U', \mathscr{W}_+[\hat{g}_1]] \le (1-\delta)e_*[U'; \mathscr{W}_+[g_1]], \tag{3.50}$$

which is courtesy of Proposition 3.5. Put these last two facts together to find that

$$\|\mathscr{W}_+[\hat{g}_1]\|_{*,p;U} \le (1-\alpha\cdot\delta)\cdot\|\mathscr{W}_+[g_1]\|_{*,p;U}, \tag{3.51}$$

which, given α's lower bound, is the required estimate.

(i) Properties of $X_1\#_N\underline{\mathbb{CP}}^2$. Proposition 3.7 constructs a conformal metric $[g]$ on $X_1\#_N\underline{\mathbb{CP}}^2$ which has some useful properties. These are described in a definition and a proposition.

In both the definition and the proposition, X_1 is a compact, oriented 4-manifold with metric g_1. Also, Z is a given constant and $U \subseteq X_1$ is an open set.

When small $\varepsilon > 0$ has been fixed, it will be assumed that a finite set $\Omega \subset U$ has been specified whose points are separated by a distance $2\cdot\varepsilon$ or more. It will also be assumed that numbers $\lambda, \varepsilon_1, \varepsilon_2 > 0$ have been assigned to each $x \in \Omega$ subject to the constraints

$$\begin{aligned} &(a) \quad \lambda \le Z\cdot\varepsilon^2, \\ &(b) \quad \lambda/\varepsilon_2 = \varepsilon, \\ &(c) \quad \varepsilon_1/\varepsilon_2 \le (1+Z)^{-1}. \end{aligned} \tag{3.52}$$

Let N denote the number of points in Ω, and use $X_1\#_N\underline{\mathbb{CP}}^2$ to denote the manifold constructed in Definitions 3.1 and 3.2 by connect summing $\underline{\mathbb{CP}}^2$ to X_1 at points in Ω using the parameters λ, ε_1, and ε_2. Use $[g]$ for the resulting conformal metric.

Here is the promised definition:

Definition 3.9. *Define X' by* (3.37).

(1) *Use $\theta: X' \to X\#_N\underline{\mathbb{CP}}^2$ to denote the canonical identification as an open subset.*

(2) *Define a metric $\hat{g}$ on X' by setting $\hat{g} = g_1$ on $X_1\backslash\bigcup_{x\in\Omega}B_\varepsilon(x)$ and on $B_\varepsilon(x)\backslash B_{\varepsilon_1}(x)$, and write $\hat{g} \equiv \hat{g}_1$ as given in Definition* 3.2.

(3) *Define a metric g_1' on X_1 as follows: Set $g_1' \equiv \hat{g}$ on X_1'. When $x \in \Omega$, define g_1' on the g_1-radius ε_1 ball about x to be*

$$\varphi_1^* g_1' \equiv g_E + \beta_{\varepsilon_i} m_{g_1} + \beta_{\varepsilon_1/256}\cdot\frac{|x|^4}{\lambda^2}\cdot\psi_\lambda^*(\beta_{\varepsilon_2}\cdot m_{g_2}).$$

This notation refers back to Definition 3.2.

For the sake of an explanation, upon restriction to $X_1 \setminus \bigcup_{x\in\Omega} B_{\varepsilon_1/128}(x)$, the conformal class of g_1' is the same as that of Definition 3.2's conformal metric $[g]$ on $X_1 \#_N \overline{\mathbb{CP}}^2$. However, g_1' is a metric on X_1, not on $X_1 \#_N \overline{\mathbb{CP}}^2$. Trust for now that this g_1' has a role to play.

The role of g_1' is predicated on the fact that it is, for ε small, an insignificant perturbation of the original g_1. Here is the precise statement:

Proposition 3.10. *Let X_1 with its metric g_1 be as previously described. For $\varepsilon > 0$ small, construct the metric g_1' on X_1 as described in Definition* 3.9. *Introduce $m = g_1' - g_1$ and then*

$$\text{(1)} \quad \sup_{X_1} |m|_{g_1} < \varepsilon,$$

$$\text{(3.53)} \quad \text{(2)} \quad \sup_{y\in X_1} \int_{X_1} \frac{|\nabla_{g_1} m|^2_{g_1}}{(\operatorname{dist}_{g_1}(y,\cdot))^2} < \varepsilon,$$

$$\text{(3)} \quad \sup_{y\in X_1} \int_{X_1} \frac{|\nabla_{g_1} m|_{g_1}}{(\operatorname{dist}_{g_1}(y,\cdot))^3} < \varepsilon^{1/2}.$$

Proof of Proposition 3.10. The metrics g_1 and g_1' differ only on $\{B_\varepsilon(x)\}_{x\in\Omega}$. On one such $B_\varepsilon(x)$,

$$m = (1-\beta_{\varepsilon_1})m_{g_1} + \beta_{\varepsilon_1/256} \frac{|x|^4}{\lambda^2} \psi_\lambda^*(\beta_{\varepsilon_2} \cdot m_{g_2}). \tag{3.54}$$

Remember that the coordinate system here is that of (3.3)–(3.5). Thus,

$$|m|_{g_1} \le Z \cdot \varepsilon_1^2 + c \cdot \lambda^2/\varepsilon_1^2 \le c \cdot Z \cdot \varepsilon^2. \tag{3.55}$$

The last inequality in (3.55) used $\lambda/\varepsilon_1 = \lambda/\varepsilon_2 \cdot \varepsilon_2/\varepsilon_1$ and (3.52). Equation (3.55) proves the proposition's first assertion.

To prove the second assertion, note first that, on $B_\varepsilon(x)$,

$$|\nabla m|_{g_1} \le Z_1 \cdot \varepsilon_1 + c \cdot \frac{\lambda^2}{(\operatorname{dist}_{g_1}(\cdot, x))^3}. \tag{3.56}$$

Therefore, if $y \notin B_\varepsilon(x)$, then this $x \in \Omega$ contributes at most

$$c \cdot (\operatorname{dist}_{g_1}(y, x))^{-2} \cdot \{Z_1^2 \varepsilon^6 + \lambda^4/\varepsilon_1^2\} \tag{3.57}$$

to the integral in (2) of Proposition 3.10. If $y \in B_\varepsilon(x)$, then x contributes

$$c \cdot (Z_1^2 \varepsilon^4 + \lambda^4/\varepsilon_1^4) \le c \cdot Z_1^2 \varepsilon^2 \tag{3.58}$$

to said integral.

The integral in (2) of Proposition 3.10 can be bounded by the sum of (3.57) and (3.58) over all $x \in \Omega$. The resulting bound is

$$c \cdot Z_1^2 \cdot \varepsilon^2 \left\{ 1 + \varepsilon^4 \cdot \sum_{x \in \Omega'} (\mathrm{dist}_{g_1}(y, x))^{-2} \right\}. \tag{3.59}$$

Here, $\Omega' \subset \Omega$ is the subset of those x for which $y \notin B_\varepsilon(x)$. (Equation (3.52) had a hand in the derivation of (3.59).) Finally, (3.59) can be bounded by $c \cdot Z \cdot \varepsilon^2$, where Z depends on the metric g_1.

This last assertion follows after observing that the number of $x \in \Omega$ whose distance from y is between $d \cdot \varepsilon$ and $(d+1) \cdot \varepsilon$ must be bounded by $Z \cdot d^3$. Since the size of d is bounded by $Z \cdot \varepsilon^{-1}$, one gets

$$\varepsilon^4 \cdot \sum_{x \in \Omega'} \mathrm{dist}_{g_1}(y, x)^{-2} \le Z. \tag{3.60}$$

The bound of (3.59) by $c \cdot Z \cdot \varepsilon^2$ finishes the argument for assertion (2) of Proposition 3.10). The proof of assertion (3) is very similar and is left to the reader.

(j) Iteration. Take a compact, oriented 4-manifold M with metric g_M and apply Proposition 3.7 with $X_1 = M$ and $g_1 = g_M$. The result is $M \#_{N^{(1)}} \underline{\mathbb{CP}}^2$ with a conformal metric whose $\mathscr{W}_+$ is smaller than that of g_M. Now, repeat Proposition 3.7 with $X_1 = M \#_{N^{(1)}} \underline{\mathbb{CP}}^2$ to get a manifold and a conformal metric with even smaller $\mathscr{W}_+$. One can repeat Proposition 3.7 again and again to obtain a manifold diffeomorphic to $M \#_N \underline{\mathbb{CP}}^2$ with tiny $\mathscr{W}_+$.

The purpose of this subsection is to precisely specify the iteration just outlined. We begin with

Step 0. Select M with its metric g_M.

Also, fix $\varepsilon_0 > 0$ and small.

Suppose now that Step $n \ge 0$ has been completed with the following results:

Result 1. An integer $N^{(n)}$ with a conformal metric $[g^{(n)}]$ on $M \#_{N^{(n)}} \underline{\mathbb{CP}}^2$.

Result 2. An open subset $M^{(n)} \subseteq M$ together with an embedding $\theta^{(n)}: M^{(n)} \to M \#_{N^{(n)}} \underline{\mathbb{CP}}^2$. Identify $M^{(n)}$ with its image and require that $\mathscr{W}_+[g^{(n)}]$ have support on $M^{(n)}$.

Result 3. A metric $g_M^{(n)}$ on M which is in the same conformal class on $M^{(n)}$ as $[g^{(n)}]$. Require that $m \equiv g_M^{(n)} - g_M$ obeys (3.53) with $\varepsilon \equiv \varepsilon_0$ and $g_1 \equiv g_M$.

Given all of the above from Step n, the next step is

Step $n+1$. Apply Proposition 3.7 using $X_1 = M$, $U = M^{(n)}$, $g_1 = g_M^{(n)}$, some large Z_1, and $\varepsilon \equiv \varepsilon(n) \leq \varepsilon_0^n$.

The next proposition lists the salient results of Step $n+1$.

Proposition 3.11. *There exists a* $\delta > 0$ *with the following significance: let* M *be a compact, oriented 4-dimensional manifold. Let* g_M *be a metric on* M. *Fix* $r > 0$. *Suppose* $\varepsilon_0 > 0$ *is small, and assume that Step* $n \geq 0$ *of the iteration has been applied with results 1, 2, 3. If* $Z_1 = Z_1(n)$ *is chosen, and then* $\varepsilon \equiv \varepsilon(n) > 0$ *is chosen sufficiently small in Step* $n+1$, *there will result an integer* $N^{(n+1)} \geq N^{(n)}$, *a conformal metric* $[g^{(n+1)}]$ *on* $M\#_{N^{(n+1)}}\overline{\mathbb{CP}}^2$, *an open set* $M^{(n+1)} \subset M$ *with embedding* $\theta^{(n+1)}$ *into* $M\#_{N^{(n+1)}}\overline{\mathbb{CP}}^2$, *and a metric* $g_M^{(n+1)}$ *on* M *in* $[g^{(n+1)}]$ *on* $M^{(n+1)}$. *These obey*

(1) $\mathscr{W}_+[g^{(n+1)}]$ *is supported on* $M^{(n+1)}$.

(2) $m \equiv g_M^{(n+1)} - g_M$ *obeys* (3.53) *with* $\varepsilon = \varepsilon_0$.

(3) $\rho \geq r$, *define* $\|\cdot\|_{X,\rho;U}$ *as in* (3.36) *using* g_M. *Then*

$$\|\mathscr{W}_+[g_M^{(n+1)}]\|_{*,\rho;M^{(n+1)}} \leq (1-\delta)\cdot\|\mathscr{W}_+[g_M^{(n)}]\|_{*,\rho;M^{(n)}}.$$

(4) *For* L^2-*norms, one has*

$$\int_{M\#_{N^{(n+1)}}\overline{\mathbb{CP}}^2} d\,\mathrm{vol}_{g^{(n+1)}}\,|\mathscr{W}_+[g^{(n+1)}]|^2_{g^{(n+1)}} \leq (1-\delta)\cdot\int_{M\#_{N^{(n)}}\overline{\mathbb{CP}}^2} d\,\mathrm{vol}_{g^{(n)}}\,|\mathscr{W}_+[g^{(n)}]|^2_{g^{(n)}}.$$

Note, in particular, that this proposition affirms Results 1, 2, and 3 for Step $n+1$.

Here is a second remark: Assertion (2) of Proposition 3.11 measures $\mathscr{W}_+[g^{(n+1)}]$ and $\mathscr{W}_+[g^{(n)}]$ using the fixed metric g_M on M. It can do this because connect summing into balls in $M^{(n)}$ for the metric $g_M^{(n)}$ can be interpreted as connect summing into $M\#_{N^{(n)}}\overline{\mathbb{CP}}^2$ for a metric in the conformal class $[g^{(n)}]$.

Proof of Proposition 3.11. Take $M^{(n+1)}$ to be X_1' in (3.37). Then, take $\theta^{(n+1)}$ to be θ from Definition 3.9 and take $g_M^{(n+1)}$ to be g_1' from the same definition.

Assertion (1) of the proposition comes from Definition 3.2 and assertion (4) comes from Proposition 3.7. Assertion (3) also follows from Proposition 3.7 given

Lemma 3.12. *Let* X *be a compact Riemannian manifold. Let* g *and* $g+m$ *be metrics on* X *with* $|m|_g \leq \frac{1}{2}$. *Let* $\delta = \sup_{x\in X}|m|_g$. *Then, for* *any* $x, y \in X$,

$$\frac{\operatorname{dist}_g(x, y)}{(1+\delta)} \le \operatorname{dist}_{g+m}(x, y) \le (1+\delta)\operatorname{dist}_g(x, y).$$

Proof of Lemma 3.12. Measure the $(g+m)$-length of the shortest g-geodesic from x to y, and vice-versa. q.e.d.

Assertion (2) of Proposition 3.11 would follow automatically from Proposition 3.10 were the norms and covariant derivatives defined (in said proposition) with the metric $g_M^{(n)}$ instead of g_M. But such an inconvenience pales to insignificance using

Lemma 3.13. *Let X be a smooth manifold with metrics g and $g+m$. Suppose $|m|_g < \frac{1}{2}$. Let ω be a tensor (section of $\otimes_p TM \otimes_q T^*M$). Then*

$$(1+c|m|_g)^{-1}|\omega|_g \le |\omega|_{g+m} \le (1+c|m|_g)|\omega|_g$$

and

$$|\nabla_{g+m}\omega - \nabla_g\omega|_g \le c\cdot|\nabla_g m|\cdot|\omega|_g.$$

Proof of Lemma 3.13. This is an exercise. q.e.d.

Given this lemma, Proposition 3.11's proof becomes an exercise.

(k) The Cokernel Step. After some number, $n-1$, of iteration steps, one can stop this iteration procedure. But, it proves useful (see §9) to stifle this urge and complete one additional iteration, which will be called the Cokernel Step. This Cokernel Step will always be the final round of attaching $\underline{\mathbb{CP}}^2$'s.

This final round differs from the previous iteration steps in the manner in which the specific parameters (i.e., Ω_n, $\{f \in FM|_x\}_{x\in\Omega_n}$, and $\{(\varepsilon_1, \varepsilon_2, \lambda)|_x\}_{x\in\Omega_n}$) are chosen.

The precise details of these choices are given in §9, but a summary of some of the features is given here for use in the sections prior to §9.

First, a positive $\varepsilon(n) \equiv \varepsilon < \varepsilon_0^n$ is chosen; its upper bound is determined by the particulars of the metric $g_M^{(n-1)}$. Then, a discrete set $\Omega_n \subset M^{(n-1)}$ is chosen whose points are separated by distance 2ε or more. The details of Ω_n's specification are given in §9. Also, a frame must be chosen at each $x \in \Omega_n$, and the details here are also relegated to §9.

Numbers λ, ε, and ε_2 must also be assigned to each $x \in \Omega_n$. Their specification requires a constant $\mu_0 \in (0, 1)$ (whose value is a priori determined by g_M), and a constant $Z \ge 2$ which is determined by $g_M^{(n-1)}$.

Given μ_0 and Z, then ε_1, ε_2, and λ are defined at $x \in \Omega_n$ by

$$\lambda = \mu_x\cdot\varepsilon^2, \quad \varepsilon_1 = \mu_x\cdot Z^{-1}\cdot\varepsilon, \quad \varepsilon_2 \equiv \mu_x\cdot\varepsilon. \tag{3.61}$$

Here, $0 < \mu_x \leq 4\mu_0$. (Thus, $\lambda/\varepsilon_2 = \varepsilon$ is constant, as in all previous steps.)

The following proposition describes the Cokernel Step's result.

Proposition 3.14. *There is a constant $c > 0$ with the following significance: Let M be a compact, oriented 4-manifold with metric g_M. Fix $\mu_{0,r} > 0$ and $\varepsilon_0 > 0$ but small. Suppose that Step $(n-1) \geq 0$ of the iteration has been performed with Results 1, 2, 3. With Z fixed, when $\varepsilon \equiv \varepsilon(n) < \varepsilon_0^{(n+1)}$ is small enough, then the Cokernel Step can be performed. This step results in an integer, $N^{(n)} \geq N^{(n-1)}$, a conformal metric $[g^{(n)}]$ on $M\#_{N^{(n)}}\underline{\mathbb{CP}}^2$, an open subset $M^{(n)} \subset M$ with embedding $\theta^{(n)}$ into $M\#_{N^{(n)}}\underline{\mathbb{CP}}^2$, and a metric $g_M^{(n)}$ on M in $[g^{(n)}]$ on $M^{(n)}$. The preceding data obeys*

(1) *$\mathscr{W}_+[g^{(n)}]$ is supported in $\theta^{(n)}(M^{(n)})$.*

(2) *$m = g_M^{(n)} - g_M$ obeys (3.54) with ε replacing ε_0.*

(3) *For $\rho \geq r$, define $\|\cdot\|_{*,\rho;U}$ as in (3.36) using the metric g_M. Then*

$$\|\mathscr{W}_+[g_M^{(n)}]\|\|_{*,\rho,M^{(n)}} \leq c\cdot(\mu_0 + \|\mathscr{W}_+[g_M^{(n-1)}]\|\|_{*,\rho,M^{(n-1)}}).$$

(4) *The L^2-inequality*

$$\|\mathscr{W}_+[g^{(n)}]\|\|_{L^2(M\#_{N^{(n)}}\underline{\mathbb{CP}}^2)} \leq c\cdot(\mu_0 + \|\mathscr{W}_+[g^{(n-1)}]\|\|_{L^2(M\#_{N^{(n-1)}}\underline{\mathbb{CP}}^2)}).$$

Proof of Proposition 3.14. With $X_1 \equiv M$ with the metric $g_M^{(n-1)}$, take $M^{(n)}$ to be X_1' in (3.37). Use Definition 3.9 to take $\theta^{(n)} \equiv \theta$ and $g_M^{(n)} \equiv g_1'$.

Assertion (1) of the proposition is now automatic from Definition 3.2 and assertion (2) follows using Proposition 3.10 and Lemmas 3.12 and 3.13. Assertions (3) and (4) are left as exercises for the reader using Proposition 3.3.

(1) **Summary.** The following theorem summarizes the effects of Propositions 3.11 and 3.14. In the theorem, c is a universal constant.

Theorem 3.15. *Let M be a compact, oriented 4-manifold with metric g_M. Given $r, \mu_0, \mu_1 > 0$ and small ε_0, iterate Proposition 3.11 some large number of times using Z_1 large and ε small in each step. Apply Proposition 3.14's cokernel step using large Z and small ε and given μ_0. The result is*

(a) *an integer $N \geq 0$;*

(b) *an open set $M' \subseteq M$ and an embedding $\theta\colon M' \to M\#_N\underline{\mathbb{CP}}^2$;*

(c) *a conformal metric $[g]$ on $M\#_N\underline{\mathbb{CP}}^2$ and a metric g_M' on M.*

This data obeys the following costraints:

(1) *The support of $\mathscr{W}_+[g]$ is in M'.*

(2) *The restriction of* $[g]$ *to* M' *is* $[g'_M]$.

(3) $m \equiv g'_M - g_M$ *obeys* (1)–(3) *of Proposition* 3.10 *with* $g_1 \equiv g_M$ *and* $\varepsilon \equiv \varepsilon_0$.

(4) $\int_{M\#_N \underline{\mathbb{CP}}^2} d\,\mathrm{vol}_g\, |\mathscr{W}_+[g]|_g^2 \le \mu_0$.

(5) *For any* $p \ge r$,

$$\|\mathscr{W}_+[g'_M]\|_{*,p;M'} \le \mu_1 \cdot \|\mathscr{W}_+[g_M]\|_{*,p;M} + \mu_0,$$

where the norm $\|\cdot\|_{*,p;U}$ *is defined in* (3.36) *using the metric* g_M.

Remark. In each of Proposition 3.11's iteration steps, there is the freedom to specify the constant Z_1 ($\equiv Z_1(l)$ in the lth step) to be large, and to then specify the constant ε ($\equiv \varepsilon(l)$ in the lth step) to be small. This freedom will subsequently be exploited to add to Theorem 3.15's list of $[g]$'s properties.

The summary of the preceding subsections continues with the following alternate description of the conformal metric $[g]$ on $M_N = M\#_N\underline{\mathbb{CP}}^2$.

One can give $[g]$ by first specifying an open cover $\{U_i\}_{i=0}^N$ of M_N and honest metrics $\hat{g}_i$ on U_i which obey $[\hat{g}_i] = [g]|_{U_i}$. These $\hat{g}_i$ must obey

$$\hat{g}_i = \varphi_{ij} \cdot \hat{g}_j \quad \text{on } U_i \cap U_j \tag{3.62}$$

for a positive function φ_{ij} on $U_i \cap U_j$.

To specify $\{U_i, \hat{g}_i\}$, remember that the attachment of each $\underline{\mathbb{CP}}^2$ requires the specification of positive numbers ε_1, ε_2, and λ. These numbers will be different for each $\underline{\mathbb{CP}}^2$ (although $\varepsilon \equiv \lambda/\varepsilon_2$ is the same for all $\underline{\mathbb{CP}}^2$'s which are attached in the same stage of iteration in Proposition 3.11). To avoid clutter, the dependence of ε_1, ε_2, and λ on the particular $\underline{\mathbb{CP}}^2$ will not be indicated explicitly.

The attachment of a $\underline{\mathbb{CP}}^2$ requires the choice of a base point, $y_0 \in \underline{\mathbb{CP}}^2$.

Each open set $U_{i>0}$ corresponds to an attached $\underline{\mathbb{CP}}^2$ (these are presumed to be labeled by $\{1, 2, \cdots, N\}$). With this understood, U_i with its metric $\hat{g}_i$ is isometric to the complement in $\underline{\mathbb{CP}}^2$ (using the Fubini-Study metric) of the ball of radius $16\lambda/\varepsilon_1$ and center y_0.

Meanwhile, U_0 with its metric $\hat{g}_0$ is isometric to an open subset of M, with the metric g'_M of Theorem 3.15. This open subset (also called U_0) is the complement of N disjoint balls about N points, $\{x_1, \cdots, x_N\}$, in M. The ball about the ith point has g'_M-radius $\varepsilon_1/32$.

Remark. Taking the ball about the ith point to have radius ε_1 still gives a disjoint set of balls in M.

We will digress momentarily to discuss the set of points $\{x_1, \cdots, x_N\}$: The set is given as the union of disjoint subsets $\{\Omega_l\}_{l=1}^n$, with each Ω_l

containing some N_l points. The subset Ω_l comes from the attachment of the $\underline{\mathbb{CP}}^2$'s of the lth iteration step as described in §3j and Proposition 3.11.

As remarked, the value of $\varepsilon \equiv \varepsilon(l) = \lambda/\varepsilon_2$ for all points in Ω_l. (Note: $\varepsilon(l) \le \varepsilon(1)^l$.) Also, $\varepsilon(l) \ge \varepsilon_1$ for all points in Ω_l, and the balls of radius $\varepsilon(l)$ about the points in Ω_l are disjoint.

End of the digression to consider the metric g'_M: For this purpose, fix an orthonormal frame, f_0, for $T\underline{\mathbb{CP}}^2|_{y_0}$. This gives a Gaussian coordinate system, φ_0, a neighborhood of y_0 in $\underline{\mathbb{CP}}^2$ which sends 0 to y_0. Use g_{FS} to denote $\underline{\mathbb{CP}}^2$'s Fubini-Study metric and

$$\varphi_0^* g_{\mathrm{FS}} = g_E + m_{\mathrm{FS}}, \tag{3.63}$$

where m_{FS} obeys (3.5).

Meanwhile, Definition 3.9 implies that the radius ε_1 ball about x_i has a coordinate system φ_i, which sends 0 to x_i and which obeys

$$\varphi_i^* g'_M = g_E + \beta_{\varepsilon_1/256} \frac{|x|^4}{\lambda^4} \cdot \psi_\lambda^* m_{\mathrm{FS}}, \tag{3.64}$$

where ψ_λ is given by (3.10), β by (3.14), and $|x|$ is the function on $\mathbb{R}^4$ which measures the Euclidean distance to 0.

The reader can use (3.63) and (3.64) to verify that the data $\{U_i, \hat{g}_i\}_{i=0}^N$ indeed defines a conformal metric on M_N. For this purpose, and for others, it is important to note that

$$\text{(3.65)} \quad \begin{aligned} &\text{(a)} \quad U_i \cap U_j = \varnothing \quad \text{if } i, j > 0, \\ &\text{(b)} \quad \varphi_i^{-1}(U_i \cap U_0) = \{x \in \mathbb{R}^4 : \varepsilon_1/32 < |x| < \varepsilon_1/16\} \quad \text{if } i > 0. \end{aligned}$$

(m) Another metric on M. A metric $\overline{g}$ on M will be needed in §§6–9 which is in the conformal class $[g]$ over more of M than is g'_M. It is the purpose of this section to describe $\overline{g}$.

Fix a number $r_0 \le \frac{1}{2}$ so that the radius r_0 ball in $\underline{\mathbb{CP}}^2$ with center y_0 (measure with g_{FS}) lies in the coordinate chart φ_0. Once r_0 is chosen, one can require (without loss of generality) that

$$\lambda/\varepsilon_1 < r_0/4048 \tag{3.66}$$

for each x_i.

Let $B(i)$ denote the radius λ/r_0 ball in M with center x_i. Let $B'(i)$ denote the ball with center x_i, but with radius $2\lambda/r_0$.

Notice that the charts $\{\varphi_i\}$ on M and φ_0 on $\underline{\mathbb{CP}}^2$ naturally identify $M \setminus \bigcup_i B(i)$ as an open subset of $M \#_N \underline{\mathbb{CP}}^2$. This identification extends the map θ of Theorem 3.15 and will also be denoted by θ.

The metric $\overline{g}$ on M restricts to $M\backslash\bigcup_i B'(i)$ to be in the conformal class $[g]$ of Theorem 3.15. Also, $\overline{g}$ is flat on each $B(i)$. In fact, $\overline{g}\equiv g_1'$ on U_0, and

$$\varphi_i^*\overline{g} = g_E + \beta_{\lambda/r_0}\frac{|x|^4}{\lambda^2}\psi_\lambda^* m_{\mathrm{FS}} \tag{3.67}$$

on the radius ε_1 ball about x_i.

The following proposition describes how $\overline{g}$ and g_M differ.

Proposition 3.16. *Let M be a compact, oriented 4-manifold with metric g_M. There exists a constant Z with the following significance: Let r, μ_0, μ_1, and ε be as in Theorem 3.15. Then the conclusions of Theorem 3.15 hold, and it may be additionally assumed that the metric $\overline{g}$ of (3.67) obeys*

(1) $|\overline{g}-g_M|_{g_M}\le c\cdot r_0^2$.

Also, fix $\rho\ge r$ and use $B(x)$ to denote the g_M-radius ρ ball with center x. Then

(2) $\sup_{x\in M}\int_{B(x)} d\,\mathrm{vol}_{g_M}\cdot|\nabla_{g_M}(\overline{g}-g_M)|^2_{g_M}/(\mathrm{dist}_{g_M}(x,\cdot))^2\le Z\cdot r_0^2\rho^2+c\cdot r_0^4$,

(3) $\sup_{x\in M}\int_{B(x)} d\,\mathrm{vol}_{g_M}\cdot|\nabla_{g_M}(\overline{g}-g_M)|_{g_M}/(\mathrm{dist}_{g_M}(x,\cdot))^3\le c\cdot r_0^2$.

Proof of Proposition 3.16. Note that

$$\overline{g}-g_M=(\overline{g}-g_M')+(g_M'-g_M). \tag{3.68}$$

This fact and Lemmas 3.12 and 3.13 make the calculations simpler because it can be assumed a priori that $m\equiv g_M'-g_M$ already obeys the assertions of Proposition 3.10. With this understood, it becomes sufficient to prove the assertions of Proposition 3.16 with the metric g_M' replacing everywhere the metric g_M.

Given the preceding, the metrics $\overline{g}$ and g_M' differ only on the ball about each x_i of radius ε_1. On this ball, $\overline{m}\equiv\overline{g}-g_M'$ obeys

$$|\nabla_{g_M'}^{\otimes k}\overline{m}|_{g_M'}\le c_n\cdot(1-\beta_{\varepsilon_{1/4}})\cdot\beta_{\lambda/2r_0}\cdot\frac{\lambda^2}{|x|^{2+k}} \tag{3.69}$$

for $k=0,1,2,\cdots$. Here, c_k is a fixed constant.

Proposition 3.16's first assertion follows directly from (3.69).

It proves useful for proving the second assertion to observe first that the contribution of $x_i\in\Omega_l$ to said integral is bounded by

$$c\cdot\lambda^2\cdot r_0^2\cdot(\mathrm{dist}_{g_M'}(x,x_i)^2+\lambda/r_0^2)^{-1}. \tag{3.70}$$

The bound in the proposition's second assertion is obtained by summing (3.70) over all i. The result can be recognized most readily by making some preliminary observations. First of all, $\operatorname{dist}_{g'_M}(x, x_i) \leq \varepsilon_1/2$ for at most one i. The contribution to the sum from this i is at most

$$c \cdot r_0^4. \tag{3.71}$$

With (3.71) understood, implicitly ignore that one x_i (if it exists) in all future sums.

Next, group the points $\{x_1, \cdots, x_N\}$ into the subsets $\{\Omega_l\}_{l=1}^n$. There is at most one point $x_i \in \Omega_l$ such that

$$\varepsilon_1/2 < \operatorname{dist}_{g'_M}(x, x_i) < \varepsilon(l). \tag{3.72}$$

If such a point exists, then its contribution to the sum is bounded by

$$c \cdot \lambda^2 \cdot r_0^2/\varepsilon_1^2 \leq c \cdot Z_{n-1} \cdot \varepsilon(l)^2. \tag{3.73}$$

Here Z_{n-1} is a constant which depends on the metric $g_M^{(n-1)}$. (Propositions 3.5 and 3.14 are used at this point to identify λ and ε_1 in terms of $\varepsilon \equiv \varepsilon(l)$.) Since $\varepsilon(l)$ can be made as small as one desires without violating any of Theorem 3.15's conclusions, one may assume that (3.73) is bounded by

$$\varepsilon(l) \leq \varepsilon_0^l. \tag{3.74}$$

This is the contribution to the sum for a point in Ω_l for which (3.72) holds.

When summing (3.70) over points x_i in Ω_l, one should now implicitly delete that point (if it exists) which obeys (3.72). Let σ_l denote the said sum of (3.70) over Ω_l.

To bound σ_l for $l \leq n-1$, one should recall from Proposition 3.5 that $\lambda^2 = c_0^{-2} \cdot \nu_1 \cdot \varepsilon(l)^4$. Remember also that the points in Ω_l are a distance $2 \cdot \varepsilon(l)$ apart at least. Finally, and most importantly, remember (3.30).

With $\varepsilon(l)$ made small, one should find (after all of the above reminiscences) that

$$\sigma_l \leq c \cdot r_0^2 \cdot \|\mathscr{W}_+[g_M^{(l)}]\|_{*, p; M^{(l)}}. \tag{3.75}$$

Furthermore, use Proposition 3.11 to bound the right side of (3.75) by

$$\sigma_l \leq c \cdot r_0^2 \cdot (1-\delta)^l \cdot \|\mathscr{W}_+[g_M]\|_{*, p} \leq Z \cdot r_0^2 \cdot \rho^2 \cdot (1-\delta)^l. \tag{3.76}$$

This, of course, holds only for $l \leq n-1$.

A similar argument shows that the Cokernel Step produces σ_n with the bound $\sigma_n \leq Z \cdot r_0^2 \cdot \rho^2 \cdot \mu_0^2$.

With this last estimate understood, sum (3.76) over $l = 1, \cdots, n-1$ and use (3.73) and (3.71) to find the following as a bound for the left side of (2) in Proposition 3.16:

$$(3.77) \qquad Z \cdot (\delta^{-1} + \mu_0^2) \cdot \rho^2 \cdot r_0^2 + Z \cdot \varepsilon_0 + c \cdot r_0^4.$$

For small ε_0, this is the required estimate.

The proof of (3) in Proposition 3.16 can be made along similar lines and is left to the reader.

4. Estimates for linear operators

Suppose that M is a compact, oriented, 4-dimensional Riemannian manifold with metric g. As noted §2, a nonlinear equation for a traceless section h of $\mathrm{Sym}^2 T^*$ is defined by requiring $\mathscr{W}_+(g+h) \equiv 0$. The strategy here will be to treat nonlinear equations as perturbations of linear equations. This then provides the motivation for studying linear differential equations.

It proves simplest to treat linear equations (on 4-manifolds) first with some generality before specializing to the particular examples which arise from the linearization of $\mathscr{W}_+(g+h) \equiv 0$. The present section treats the generalization, and the specialization is postponed to §5.

This section should provide the reader with a tool kit of sorts; the tools are for application in subsequent sections. Thus, the reader can opt to study this section concurrently with the subsequent sections which require its tools.

(a) First-order operators. Let $V, W \to M$ be vector bundles which are associated to the orthonormal frame bundle of M. The Levi-Civita connection on FM induces a covariant derivative ∇_g on sections of V and of W. Assume that the metric on T^*M also induces a covariantly constant Euclidean inner product (denoted by $(\cdot, \cdot)_g$) on both V and W. Use this inner product to implicitly identify V with its dual, and W likewise.

Suppose that

$$(4.1) \qquad \sigma \in C^\infty(\mathrm{Hom}(T^*; \mathrm{Hom}(V; W)))$$

is covariantly constant ($\nabla_g \sigma \equiv 0$) and obeys

$$(4.2) \qquad \sigma(\xi)^* \sigma(\xi) = |\xi|_g^2 \cdot \mathrm{Identity}$$

for any $\xi \in T^*$.

Examples of such σ appear as the principle symbols of various naturally occurring first-order differential operators. The symbol of ∇_g is one example. For a second example, take the symbol of the composition

$$C^\infty(\Lambda^2_\pm T^*) \xrightarrow{d} C^\infty(\Lambda^3 T^*) \xrightarrow{*} C^\infty(T^*). \tag{4.3}$$

Conversely, any such σ as in (4.1), (4.2) defines the operator

$$\delta = \sigma(\nabla_g)\colon C^\infty(V) \to C^\infty(W). \tag{4.4}$$

Note that the operator δ has a formal L^2-adjoint,

$$\delta^* = -\sigma^*(\nabla_g)\colon C^\infty(W) \to C^\infty(V). \tag{4.5}$$

But the symbol σ^* of δ^* will not obey (4.2) unless σ is *elliptic*; that is, unless $\dim(V) = \dim(W)$.

(b) L^2-theory for δ. For integer $k \geq 0$, let $L^2_k(V)$ denote the completion of $C^\infty(V)$ using the norm

$$\|v\|^2_{L^2_k} \equiv \sum_{j=0}^{k} \int_M d\,\mathrm{vol}_g\, |\nabla^{(j)} v|^2_g. \tag{4.6}$$

These L^2_k spaces are the standard Sobolev spaces (see, e.g., [2]).

The following "Hörmander" lemma is completely classical (see [2]).

Lemma 4.1. *The operator δ extends to a bounded operator from $L^2_1(V)$ to $L^2(W)$ and there is a constant $z < \infty$ such that*

$$\|\delta v\|^2_{L^2} \geq \|\nabla_g u\|^2_{L^2} - Z \cdot \|v\|^2_{L^2} \tag{4.7}$$

for all $v \in L^2_1(V)$.

The preceding is parlayed in a standard way to give

Lemma 4.2. *The operator $\delta^*\delta$ extends to $L^2(V)$ as a closed, essentially selfadjoint operator with dense domain $L^2_2(V)$. The spectrum of $\delta^*\delta$ is on the nonnegative real axis. This spectrum is pure point with finite multiplicities and no accumulation points. If $m > 0$, then $\delta^*\delta + m$ is invertible and, for any $k \geq 0$, $(\delta^*\delta + m)^{-1}$ restricts the map $L^2_k(V)$ isomorphically onto $L^2_{k+2}(V)$.*

For proofs of the preceding lemmas, see, e.g., [2].

One can deduce from the preceding lemma that there exists, for each $E \geq 0$, a projector

$$\Pi_E\colon L^2(V) \to L^2(V) \tag{4.8}$$

whose kernel is the span of the eigenvectors of $\delta^*\delta$ which have eigenvalue less than E. It is a simple matter to check that $\delta^*\delta$ is invertible on $\Pi_E \cdot L^2(V)$.

This subsection ends with an estimate on the size of a solution u to the equation

$$\delta^*\delta u = Q. \tag{4.9}$$

Lemma 4.3. *There exists a constant* z *with the following significance: Let* $E > 0$ *and* $Q \in C^\infty(V)$ *be given. There exists a unique* $u \in (\Pi_E L^2(V)) \cap L_2^2(V)$ *obeying* $\delta^*\delta u = \Pi_E \cdot Q$. *Also,* $u \in C^\infty(V)$ *and*

$$\|\nabla u\|_2^2 + E\cdot\|u\|_2^2 \le \left(1+\frac{z}{E}\right)\cdot\left|\int_M d\,\mathrm{vol}_g (u, Q)_g\right|. \tag{4.10}$$

Proof of Lemma 4.3. Existence and uniqueness follow from Lemma 4.2, and (4.10) follows from (4.7).

(c) Some useful norms. Lemma 4.3's estimate for u is obviously incomplete. The typical completion replaces the right-hand side by $(1+\frac{z}{E})\cdot \|u\|_{L^2}\cdot\|Q\|_{L^2}$. Such an approach turns out to be insufficient for the applications which follow. Indeed, the foreseen applications require estimates on u and Q which "see" their point-to-point behavior.

The discussion of local estimates is facilitated with the introduction of some additional norms on $C^\infty(V)$.

Definition 4.4. Let $\rho > 0$ be given. If $x \in M$, let $B(x) \subset M$ denote the ball of radius ρ and center x.

(a) Introduce the norm $\|\cdot\|_\infty$ and $\|\cdot\|_{2*}$ on $C^\infty(V)$ by assigning to u the numbers

$$\|u\|_\infty \equiv \sup_{x\in M} |u| \tag{4.11}$$

and

$$\|u\|_{2*} \equiv \sup_{x\in M}\left[\int_{B(x)} d\,\mathrm{vol}_g \frac{|u|_g^2}{(\mathrm{dist}_g(x,\cdot))^2}\right]^{1/2}. \tag{4.12}$$

(b) Introduce the norm $\|\cdot\|_{\mathscr{S}^0}$ on $C^\infty(V)$ by assigning to u the number

$$\|u\|_{\mathscr{S}^0} \equiv \|u\|_\infty + \|\nabla_g u\|_{2*}. \tag{4.13}$$

The norm $\|\cdot\|_{\mathscr{S}^0}$ will be of particular interest after its augmentation. For, $\|\cdot\|_{\mathscr{S}^0}$ is not quite strong enough for application to the Weyl curvature problem. The norm $\mathscr{S}^0$ will be strengthened with the addition of a seminorm $\|\cdot\|_{\mathscr{S}^1}$, whose definition appears below.

This seminorm $\|\cdot\|_{\mathscr{S}^1}$ is relatively complicated and its definition requires a preliminary digression to introduce two additional notions. The first notion is notation: Use $\mathscr{S}(V) \subset C^\infty(V)$ to denote the subset of v with $\|v\|_{\mathscr{S}^0} = 1$.

For the second notion, begin the digression by fixing $f \in FM$ to define a Gaussian coordinate chart (as in (3.3)–(3.5)) with center $x = \pi(f)$. If $\rho > 0$, but small, this identifies the ball $B(x)$ (of radius ρ and center x) with the standard radius ρ ball in $\mathbb{R}^4$.

Use the Euclidean coordinates to identify sections of $T^*\mathbb{R}^4$ with maps from $\mathbb{R}^4$ to $\mathbb{R}^4$.

Let $S^3 \subset \mathbb{R}^4$ denote the standard sphere, and promote a map from S^3 to $\mathbb{R}^4$ to one from $\mathbb{R}^4\backslash\{0\}$ to $\mathbb{R}^4$ by requiring said promotion to be independent of the radial coordinate.

For $k \in \{0, 1, 2, \cdots\}$, let $L_k^2(S^3)$ denote the usual Sobolov space of functions on S^3 with square integrable derivatives of order up through k. (Use the standard metric on S^3.)

Let Γ denote the subset of smooth, $\mathbb{R}^4$-valued functions φ on S^3 which have L_2^2-norm $\|\varphi\|_{L_2^2} \equiv 1$.

With all of the above understood, and the digression, we have

Definition 4.5. Fix $\rho > 0$ as in Definition 4.4, but make ρ small. The seminorm $\|\cdot\|_{\mathscr{L}^1}$ on $C^\infty(V)$ associates to u the number

$$\|u\|_{\mathscr{L}^1} \equiv \sup_{x\in M}\left\{\sup_{\substack{v\in\mathscr{S}(V\otimes T^*)\\ \varphi\in\Gamma}}\left|\int_{B(x)} d\,\mathrm{vol}_g \frac{(v\,,\,\varphi\otimes u)_g}{(\mathrm{dist}_g(x\,,\,\cdot))^3}\right|\right\}. \tag{4.14}$$

Here is a rough explanation for the choice of these norms: First of all, the norms require the choice of $\rho > 0$. The number ρ is a measure of the locality of the norms; both $\|\cdot\|_{\mathscr{L}^0}$ and $\|\cdot\|_{\mathscr{L}^1}$ measure some sort of averaged behaviors on a ball of radius ρ. (In contrast, the L_k^2-norms of (4.6) average behavior over all of M.)

The norm $\|\cdot\|_\infty$ clearly measures u's pointwise size and $\|\cdot\|_{\mathscr{L}^0}$ therefore measures the size of u and of its derivative.

The seminorm $\|\cdot\|_{\mathscr{L}^1}$ will always be employed in one of only two contexts:

Definition 4.6. Define norms $\|\cdot\|_{\mathscr{L}}$ and $\|\cdot\|_{\mathfrak{H}}$ on $C^\infty(V)$ by assigning to u the numbers

$$\|u\|_{\mathscr{L}} \equiv \|u\|_{\mathscr{L}^0} + \|\delta u\|_{\mathscr{L}^1}, \qquad \|u\|_{\mathfrak{H}} \equiv \|u\|_{*2} + \|u\|_{\mathscr{L}^1}. \tag{4.15}$$

The reader will see in the subsequent sections that the norm $\|\cdot\|_{\mathscr{L}}$ is stronger than $\|\cdot\|_{\mathscr{L}^0}$ in that the addition of $\|\cdot\|_{\mathscr{L}^1}$ allows it to measure aspects of the second derivative. Likewise, the seminorm $\|\nabla_g u\|_{\mathfrak{H}}$ will also prove to be a useful measure of second derivatives.

The last remark in this subsection concerns the factors $\mathrm{dist}_g(x,\cdot)^{-2}$ and $\mathrm{dist}_g(x,\cdot)^{-3}$ which appear in (4.11) and (4.14). The point is that the function $\mathrm{dist}_g(x,\cdot)^{-2}$ was chosen for its resemblance to the Laplacian's Greens function with pole at x. (So $\mathrm{dist}_g(x,\cdot)^{-3}$ resembles said Greens function's derivative.)

Lemma 4.7. *There exists a constant $Z<\infty$ with the following significance: Fix $x\in M$ and let $G(x,\cdot)\in C^\infty(\mathrm{Hom}(V|_x;V|_{M\backslash x}))$ denote the Greens function with pole at x for $\nabla_g^*\nabla_g+1\colon C^\infty(V)\to C^\infty(V)$. Then*

(1) $|G(x,\cdot)-(2\pi\,\mathrm{dist}_g(x,\cdot))^{-2}|\le Z|\ln(\frac{1}{Z}\cdot\mathrm{dist}_g(x,\cdot))|$.

(2) $|\nabla_g G(x,\cdot)+\frac{1}{2\pi^2}\nabla_g\,\mathrm{dist}_g(x,\cdot)/(\mathrm{dist}_g(x,\cdot))^3|\le Z\cdot(\mathrm{dist}_g(x,\cdot))^{-1}$.

Proof of Lemma 4.7. Use [2, Proposition 4.12, Theorem 4.13, and (4.17), (4.18)]. The estimates are simple in Gaussian coordinates.

(d) Estimates for $\delta^*\delta u=Q$. Suppose that $E>0$ and $\rho>0$ have been chosen, with ρ small. Suppose that $Q\in C^\infty(V)$ obeys $(1-\Pi_E)\cdot Q=0$. Also, suppose that Q can be written as

$$Q=q+b_1\cdot\nabla_g b_2. \tag{4.16}$$

Here, b_2 is a section of some vector bundle $Y\to M$, and b_1 is a section of $C^\infty(\mathrm{Hom}(Y\otimes T^*;V))$. Meanwhile, $q\in C^\infty(V)$. Measure q with

$$\|q\|_*\equiv\sup_{x\in M}\int_{B(x)}d\,\mathrm{vol}_g\,\frac{|q|_g}{(\mathrm{dist}_g(x,\cdot))^2}, \tag{4.17}$$

with $B(x)$ denoting the ball of radius ρ around x.

Proposition 4.8. *There is a constant Z with the following significance: let $\rho>0$ be small, and let $E>0$ be given. Let q, b_1, b_2 be as described above, so that Q of* (4.22) *obeys $(1-\Pi_E)\cdot Q=0$. Let $u\in\Pi_E\cdot C^\infty(V)$ be the unique solution to $\delta^*\delta u=Q$. Then u obeys*

$$\begin{aligned}&(1)\quad \|u\|_{\mathscr{L}^0}\le Z\cdot(\rho^{-2}\|u\|_{L^2}+\|q\|_*+\|b_1\|_{\mathscr{L}^0}\|b_2\|_{\mathfrak{H}}),\\ &(2)\quad \|u\|_{\mathscr{L}^0}\le Z\cdot(1+\rho^{-6}E^{-1})\cdot(\|q\|_*+\|b_1\|_{\mathscr{L}^0}\|b_2\|_{\mathfrak{H}}).\end{aligned} \tag{4.18}$$

Before embarking on (4.18)'s proof, it proves useful to fix a smooth $\alpha\colon[0,\infty)\to[0,1]$ which equals 1 on $[0,1]$ and which vanishes on $[2,\infty)$ when $x\in M$ is specified:

$$\alpha_x(\cdot)\equiv\alpha(\rho^{-1}\mathrm{dist}_g(x,\cdot)). \tag{4.19}$$

(e) The proof of Proposition 4.8. Equation (4.2) for the symbol of δ can be used to derive the following Weitzenbach formula for $\delta^*\delta$:

$$\delta^*\delta=\nabla^*\nabla+K, \tag{4.20}$$

where K is a section of $\mathrm{End}(V)$ which is constructed from the Riemann curvature tensor.

Contract both sides of (4.9) with u and use (4.20) to find that

$$d^*d\frac{|u|_g^2}{2} + |\nabla_g u|_g^2 + (u, K\cdot u)_g = (u, Q)_g. \tag{4.21}$$

Fix $x \in M$ and multiply both sides of (4.21) by $\alpha_x \cdot G(x, \cdot)$, where $G(x, \cdot)$ is the Greens function for d^*d with pole at x. Integrate the result over M. After integrating by parts in the appropriate places, one sees (with Lemma 4.7) that

$$\begin{aligned}|u|^2(x) &+ \int_{B(x)} d\,\mathrm{vol}_g \frac{|\nabla_g u|_g^2}{(\mathrm{dist}_g(x, \cdot))^2}\\ &\leq c\cdot\left\{\rho^{-4}\int_{B'(x)} d\,\mathrm{vol}_g |u|_g^2 + Z\int_{B'(x)} d\,\mathrm{vol}_g \frac{|u|_g^2}{(\mathrm{dist}_g(x, \cdot))^2}\right.\\ &\qquad\left. + \int_{B'(x)} d\,\mathrm{vol}_g\, \alpha_x G(x, \cdot)\cdot(u, Q)_g\right\}.\end{aligned} \tag{4.22}$$

Here, $B'(x)$ is the ball of radius $2\cdot\rho$ with center x.

Evidently, (4.22) bounds $\|u\|^2_{\mathcal{L}^0}$ by the supremum as x varies in M of the right-hand side of (4.22). So, at issue is the size of the right-hand side of (4.22). With this understood, observe that this side of (4.22) is a sum of three terms; and each will be estimated in turn.

Term 1 *in* (4.22). Begin by bounding this term by

$$c\cdot\rho^{-4}\|u\|_{L^2}^2 \leq Z\cdot\rho^{-4}\cdot E^{-1}\left|\int_M d\,\mathrm{vol}_g(u, Q)_q\right|. \tag{4.23}$$

This last bound is obtained with (4.10). To estimate the integral in (4.23), first break Q as in (4.16). Then, after an integration by parts,

$$\begin{aligned}\left|\int_M d\,\mathrm{vol}_g(u, Q)_g\right| &\leq \{\|u\|_\infty(\|q\|_{L^1} + \|\nabla b_1\|_{L^2}\cdot\|b_2\|_{L^2})\\ &\qquad + \|\nabla u\|_{L^2}\|b_1\|_\infty\|b_2\|_{L^2}\}.\end{aligned} \tag{4.24}$$

Here,

$$\|q\|_{L^1} = \int_M d\,\mathrm{vol}_g |q|_g. \tag{4.25}$$

This last norm of q can be estimated in terms of $\|q\|_*$ by covering M by a finite set of balls of radius ρ. It is important to know that such a cover can be found for which the number of balls is bounded a priori by $Z \cdot \rho^{-4}$. With this understood, one has

$$\|q\|_{L^1} \le Z \cdot \rho^{-2} \cdot \|q\|_*. \tag{4.26}$$

A similar construct shows that

$$\|\nabla u\|_{L^2} \le Z \cdot \rho^{-1} \cdot \|u\|_{\mathscr{L}^0}. \tag{4.27}$$

Therefore, (4.23), (4.24), (4.26), and (4.27) estimate Term 1 in (4.22) by

$$\rho^{-4}\|u\|_{L^2}^2 \le Z \cdot \rho^{-6}E^{-1} \cdot \|u\|_{\mathscr{L}^0} \cdot (\|q\|_* + \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{2*}). \tag{4.28}$$

Term 2 *in* (4.22). This is the easy one: Term 2 can be bounded by

$$Z \cdot \rho^2 \cdot \|u\|_\infty^2 \le Z \cdot \rho^2 \cdot \|u\|_{\mathscr{L}^0}^2. \tag{4.29}$$

Term 3 *in* (4.22). This case is the most subtle, for here the $\|\cdot\|_{\mathscr{L}^1}$ norm of b_2 appears. However, the construction of the bound begins mundanely by breaking Q as in (4.16). The part with q is bounded by

$$Z \cdot \|u\|_\infty \cdot \|q\|_*. \tag{4.30}$$

This uses Lemma 4.6.

Treat the part with $b_1 \cdot \nabla_g b_2$ by first integrating by parts to find it equal to

$$\begin{aligned} -\int_{B'(x)} d\,\mathrm{vol}_g\{\alpha_x \cdot G(x\,,\cdot) \cdot ((\nabla_g u\,, b_1 \cdot b_2)_g + (u\,, \nabla_g b_1 \cdot b_2)_g) \\ + (d(\alpha_x \cdot G(x\,,\cdot)) \otimes u\,, b_1 \cdot b_2)_g\}. \end{aligned} \tag{4.31}$$

Bound the first two terms in (4.31) (using Lemma 4.7) by

$$Z \cdot \|u\|_{\mathscr{L}^0} \cdot \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{2*}. \tag{4.32}$$

The third term in (4.31) is the tricky one. Begin its analysis by replacing $d(\alpha_x \cdot (G(x\,,\cdot)))$ by $d((2\pi)^{-1} \cdot (\mathrm{dist}_g(x\,,\cdot))^{-2})$. The error can be estimated with Lemma 4.7, it only adds to the constant in (4.32). Now, go to Gaussian coordinates with center x and then replace $d((2\pi)^{-1}(\mathrm{dist}_g(x\,,\cdot))^{-2})$ by

$$(2\pi)^{-1} d(1/|\cdot|_{g_E}^2). \tag{4.33}$$

The error incurred by this replacement will only contribute to (4.32). (Use (3.3)–(3.5) to prove this.)

Notice that (4.33) can be written as

$$\varphi/|\cdot|_{g_E}^3 \tag{4.34}$$

with $\varphi \in C^\infty(S^3; \mathbb{R}^4)$. With this understood, the third term in (4.31) can be bounded by

$$Z \cdot (\|u\|_{\mathscr{L}^0} \cdot \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{2*} + \|u \otimes b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{\mathscr{L}^1}). \tag{4.35}$$

Given (4.35), the third term in (4.31) is bounded by

$$Z \cdot \|u\|_{\mathscr{L}^0} \cdot \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{\mathfrak{H}} \tag{4.36}$$

when one appeals to

Lemma 4.9. *Let V, $W \to M$ be vector bundles. Then*

$$\|v \otimes w\|_{\mathscr{L}^0} \leq \|v\|_{\mathscr{L}^0} \cdot \|w\|_{\mathscr{L}^0}.$$

Proof of Lemma 4.9. This is an exercise for the reader. q.e.d.

To finish the $\|u\|_{\mathscr{L}^0}$ estimate, put (4.28), (4.30), and (4.36) together to bound

$$\|u\|^2_{\mathscr{L}^0} \leq Z \cdot \Bigg\{ \rho^2 \|u\|^2_{\mathscr{L}^0} + \left(1 + \frac{1}{E\rho^6}\right) \cdot \|u\|_{\mathscr{L}^0} \cdot (\|q\|_* + \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{\mathfrak{H}}) \Bigg\}. \tag{4.37}$$

The final bound on $\|u\|_{\mathscr{L}^0}$ in (4.18) follows directly from (4.37) if $Z\rho^2 < 1/2$.

(f) First-order operators again. If $w \in C^\infty(W)$ is given and $u \in L^2_1(V)$ obeys

$$\delta u = w, \tag{4.38}$$

then estimates for u in terms of w can be had by applying (4.18) to the equation

$$\delta^* \delta u = \delta^* w. \tag{4.39}$$

Indeed, write $\delta^* = -\sigma^*(\nabla_g)$ and then set $b_1 \equiv -\sigma^*$ and $b_2 \equiv w$ in (4.18) to conclude

Lemma 4.10. *There is a constant Z with the following significance. Let $\rho > 0$ be small and $E > 0$. Let $w \in C^\infty(W)$ and $u \in \Pi_E \cdot C^\infty(V)$ be given to satisfy* (4.38). *Then*

$$\|u\|_{\mathscr{L}^0} \leq Z \cdot (\rho^{-2}\|u\|_{L^2} + \|w\|_{\mathfrak{H}}) \leq Z \cdot (1 + \rho^{-6}E^{-1})\|w\|_{\mathfrak{H}}. \tag{4.40}$$

If $w \equiv 0$ in (4.38), then estimates for u can still be found, but the L^2-norm of u must enter. These estimates are standard:

Lemma 4.11. *For each integer k, there exists Z_k such that if $u \in C^\infty(V)$ obeys $\delta u \equiv 0$, then*

$$\|\nabla_g^{\otimes k} u\|_\infty \le Z_k \|u\|_{L^2}.$$

(g) Estimating the $\|\cdot\|_{\mathscr{L}^1}$-norm. In this subsection, assume that the symbol σ (in (4.1)) of δ is elliptic. (Thus, for all $x \in M$ and $\xi \in T^*|_x$, $\sigma(\xi)\colon V|_x \to W|_x$ is an isomorphism.) Suppose that $w \in C^\infty(W)$ obeys

$$\delta^* w = Q, \tag{4.41}$$

with Q as described in (4.16). (Note δ^* is also elliptic.)

Proposition 4.12. *There exists a constant Z, with the following significance: Fix $\rho > 0$, but small. Suppose that Q is as described in* (4.16) *and that w obeys* (4.41) *with δ^* elliptic. Then*

$$\|w\|_{\mathscr{L}^1} \le Z \cdot (\|q\|_* + \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{\mathfrak{H}} + \|w\|_{2*}). \tag{4.42}$$

This proposition will be proved by integrating both sides of (4.41) against a suitably chosen test function. The estimate in question will result after a by-parts integration and some straightforward manipulations. The key to (4.42) is the construction of the test function.

The test function's construction requires the following short digression whose culmination is Lemma 4.13, below.

Let V_0 and W_0 be vector spaces of the same dimension, and let $\sigma \in \mathrm{Hom}(\mathbb{R}^4; \mathrm{Hom}(V_0; W_0))$ be such that $\sigma_0(\xi)$ is an isomorphism for all nonzero $\xi \in \mathbb{R}^4$. Require $\sigma_0(\xi)^*\sigma_0(\xi) = |\xi|^2 \cdot 1$ as well.

Let $\{y_\nu\}_{\nu=1}^4$ be Euclidean coordinates on $\mathbb{R}^4$ and let $\nabla_0 \equiv \sum_{\nu=1}^4 (\partial/\partial x^\nu) \otimes dx^\nu$ denote the Euclidean covariant derivative. Then

$$\delta_0 \equiv \sigma_0(\nabla_0) \tag{4.43}$$

is a first-order, elliptic constant coefficient operator on $\mathbb{R}^4$ which maps V_0-valued functions to W_0-valued functions. Likewise, it maps $(V_0 \otimes W_0)$-valued functions into $(W_0 \otimes W_0)$-valued functions.

One more remark before Lemma 4.13. Since W_0 is a Euclidean vector space, $\mathrm{End}(W_0)$ is naturally identifiable with $W_0 \otimes W_0$. Use $1 \in W_0 \otimes W_0$ to denote the identity endomorphism.

Lemma 4.13. *Let $\psi \in C^\infty(S^3)$ be given, thought of as a radially constant function on $\mathbb{R}^4\backslash\{0\}$. There exists a unique $s \in C^\infty(V_0 \otimes W_0|_{S^3})$ (thought of as a radially constant, $(V_0 \otimes W_0)$-valued function on $\mathbb{R}^4\backslash\{0\}$) which obeys*

$$\delta_0\left(\frac{s}{|\cdot|_{g_E}^2}\right) = \frac{\psi \otimes 1}{|\cdot|_{g_E}^3}. \tag{4.44}$$

Furthermore,

$$\|s\|_\infty + \|s\|_{L^2_3(S^3)} \le c \cdot \|\psi\|_{L^2_2(S^3)}, \tag{4.45}$$

where c is a universal constant and the norms $\|\psi\|_{L^2_2(S^3)}$ are the standard Sobolev norms over S^3.

Proof of Proposition 4.12. Fix $x \in M$ and a Gaussian coordinate system with center x. Make ρ small enough so that the Gaussian coordinates identify $B(x)$ with the ball $B(0) \subset \mathbb{R}^4$ of radius ρ.

The first step in the proof is to remark that a bound for $\|w\|_{\mathscr{L}^1}$ will result by bounding

$$\left| \int_{B(0)} d\,\mathrm{vol}_{g_E} \frac{(v\,,\,\varphi \otimes w)_{g_E}}{|\cdot|^3_{g_E}} \right| \tag{4.46}$$

for all $\varphi \in \Gamma$ and $v \in \mathscr{S}(W \otimes T^*)$.

Indeed, the difference between (4.46) and its twin with g_E replaced by g can be bounded by $\|w\|_{2*}$.

The Gaussian coordinate system trivializes $T^*|_{B(x)}$, for it identifies $B(x)$ with $B(0)$ and the latter has coordinates $\{y_\nu\}_{\nu=1}^4$. With respect to the basis $\{dy_\nu\}_{\nu=1}^4$ for T^*, write

$$v = \sum_{\nu=1}^4 v_\nu \otimes dy_\nu. \tag{4.47}$$

The Gaussian coordinate system also identifies $V|_{B(x)}$ and $W|_{B(x)}$ with $B(0) \times V_0$ and $B(0) \times W_0$, respectively. Here $V_0 \equiv V|_x$ and $W_0 \equiv W|_x$.

Introduce $\sigma_0 \equiv \sigma|_x$, where σ is the principal symbol of δ. Then define δ_0 as in (4.43) and note that Lemma 4.13 applies. Use said lemma to construct $\{s_\nu\}_{\nu=1}^4$ to obey (4.44) using $\{\psi = \varphi_\nu\}_{\nu=1}^4$. Here $\{\varphi_\nu\}$ are the components of φ.

Note that

$$\mathfrak{s} \equiv \sum_{\nu=1}^4 \frac{s_\nu \otimes v_\nu}{|\cdot|^2_{g_E}} \tag{4.48}$$

is a section over $B(x)$ of V. The test function for multiplication against both sides of (4.41) is going to be $\alpha_x \cdot \mathfrak{s}$, with $\alpha_x(\cdot)$ the bump function of (4.19).

Multiply $\alpha_x \cdot \mathfrak{s}$ against both sides of (4.41), then integrate by parts to find

$$\int_{B'(x)} d\,\mathrm{vol}_g(\delta(\alpha_x \cdot \mathfrak{s})\,,\,w)_g = \int_{B'(x)} d\,\mathrm{vol}_g(\alpha_x \cdot \mathfrak{s}\,,\,Q)_g. \tag{4.49}$$

This last equation will give an estimate for (4.46) after some massaging.

Analyze first the left-hand side of (4.49). For this purpose, note that

$$\delta\mathfrak{s} = \sum_{\nu=1}^{4} \frac{\varphi_\nu \cdot v_\nu}{|\cdot|_{g_E}^3} + r_0, \tag{4.50}$$

where

$$|r_0| \le Z \cdot \left\{ \frac{|s|\,|\nabla_g v|_g}{|\cdot|_{g_E}^2} + \frac{|s|\,|v_g|}{|\cdot|_{g_E}} + |\nabla_{g_E} s|\,|v_g| \right\}. \tag{4.51}$$

Thus, the left-hand side of (4.49) can be written as

$$\int_{B(x)} d\,\mathrm{vol}_{g_E} \frac{(v\,,\,\varphi\otimes w)_{g_E}}{|\cdot|_{g_E}^3} + r_1, \tag{4.52}$$

where

$$|r_1| \le z \cdot \|\varphi\|_{L_2^2(S^3)} \cdot \|v\|_{\mathscr{L}^0} \cdot \|w\|_{2*}. \tag{4.53}$$

The derivation of (4.53) is straightforward but for Lemma 4.13. It allows the replacement of $\|s\|_\infty$ and $\|s\|_{L_2^2(S^3)}$ by $z \cdot \|\varphi\|_{L_2^2(S^3)}$.

Equations (4.52) and (4.53) prove that (4.49)'s left-hand side can be used to estimate (4.46).

Turn now to the right-hand side of (4.49). It has two parts, which correspond to the breaking of Q in (4.16). Here is how to bound these terms:

Term 1. Since $|s| \le z \cdot \|\varphi\|_{L_2^2}$, the first term is bounded by

$$z \cdot \|s\|_\infty \cdot \|q\|_* \le z \cdot \|\varphi\|_{L_2^2(S^3)} \cdot \|v\|_{\mathscr{L}^0} \cdot \|q\|_*. \tag{4.54}$$

Term 2. After integration by parts, the second term on the right in (4.49) is bounded by

$$\left| \int_{B'(x)} d\,\mathrm{vol}_g \{ (\alpha_x \cdot \mathfrak{s}\,,\, \nabla_g b_1 \cdot b_2)_g + (\nabla_g(\alpha_x \cdot \mathfrak{s})\,,\, b_1 \cdot b_2)_g \} \right|. \tag{4.55}$$

The first term above is bounded by

$$z \cdot \|s\|_\infty \cdot \|v\|_\infty \cdot \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{2*}, \tag{4.56}$$

which can be replaced by

$$z \cdot \|\varphi\|_{L_2^2(S^3)} \cdot \|v\|_{\mathscr{L}^0} \cdot \|b_1\|_{\mathscr{L}^0} \cdot \|b_2\|_{2*}. \tag{4.57}$$

As for the second term in (4.55), it can be replaced by (4.57) plus

$$z \cdot \|s\|_{L_3^2(S^3)} \cdot \|v\|_\infty \cdot \|b_1\|_\infty \cdot \|b_2\|_{\mathscr{L}^1}. \tag{4.58}$$

This last term appears as the upper bound for

$$(4.59)\qquad \left|\int_{B(x)} d\,\mathrm{vol}_{g_E} \frac{(s\cdot v\otimes d|\cdot|_{g_E}, b_1\cdot b_2)_{g_E}}{|\cdot|^3_{g_E}}\right|$$

and also

$$(4.60)\qquad \left|\int_{B(x)} d\,\mathrm{vol}_{g_E} \frac{(\nabla_{g_E} s\cdot v, b_1\cdot b_2)_{g_E}}{|\cdot|^2_{g_E}}\right|.$$

Replace $\|s\|_{L^2_3(S^3)}$ in (4.58) by $Z\cdot\|\varphi\|_{L^2_2(S^3)}$ and then (4.52)–(4.54), (4.57), and (4.58) lead directly to the assertion of Proposition 4.12.

Proof of Lemma 4.13. The operator δ_0 has a Green's function, $\mathfrak{p}_p(\cdot)$, with pole at $p\in\mathbb{R}^4$, given by

$$(4.61)\qquad \mathfrak{p}_p(y) = c\cdot\frac{\sigma_0^*(y-p)}{|y-p|^4_{g_E}}.$$

Here c is a constant. Thus, (4.44) has the formal solution whose value at $p\in\mathbb{R}^4$ is

$$(4.62)\qquad \eta(p)\equiv\int_{\mathbb{R}^4} d^4y\,c\cdot\frac{\sigma_0^*(y-p)}{|y-p|^4_{g_E}}\cdot\frac{\psi(\hat{y})}{|y|^3_{g_E}}.$$

Here, $\hat{y}=y/|y|\in S^3$. As long as $p\neq 0$, the integral in (4.62) converges.

When $p\neq 0$, write $p=\hat{p}/|p|$ and (4.62) can be rewritten (by rescaling the integration variable $y\to|p|\cdot y$) as

$$(4.63)\qquad \eta(p)=\frac{\eta(\hat{p})}{|p|^2_{g_E}}.$$

Thus, with $s(\hat{p})$ defined to be $\eta(\hat{p})$ with η as in (4.62), one sees that (4.44) has a solution of the appropriate form. The estimates for s (and uniqueness) are obtained by writing (4.44) as a differential equation for s on S^3. The equation in question is readily seen to be elliptic, and standard elliptic theory gives the estimates in (4.45).

(h) The projection Π_E. This subsection will consider the projection Π_E for the operator $\nabla_g^*\nabla_g$ on $C^\infty(V)$. Of particular interest will be $\Pi_E\cdot Q$ with Q as in (4.16) and an estimate for $\|(1-\Pi_E)\cdot Q\|_*$ in terms of $\|q\|_*$ and $\|b_1\|_{\mathscr{L}^0}\cdot\|b_2\|_{2*}$.

With this goal in mind, set $\pi_E=(1-\Pi_E)$. A description of π_E requires the choice of an L^2-orthonormal basis $\{\nu_i\}_{i=1}^{N(E)}$, for the span of the eigenvectors of $\nabla_g^*\nabla_g$ with eigenvalues less than E. With this basis $\{\nu_i\}_{i=1}^{N(E)}$,

$$\pi_E \cdot Q = \sum_{i=1}^{N(E)} \nu_i \cdot \left[\int_M d\,\mathrm{vol}_g (\nu_i, Q)_g\right]. \tag{4.64}$$

Estimating $\|\pi_E \cdot Q\|_*$ is facilitated by adopting notation which distinguishes the norm in (4.17) as defined by ρ with the same norm as defined by a number $r \neq \rho$. Use $\|\cdot\|_{*,\rho}$ and $\|\cdot\|_{*,r}$ to make this distinction. Likewise, use $\|\cdot\|_{2*,\rho}$ and $\|\cdot\|_{2*,r}$ to distinguish (4.12) as defined by ρ and r. And, similarly introduce $\|\cdot\|_{\mathscr{L}^0,\rho}$ and $\|\cdot\|_{\mathscr{L}^0,r}$ for (4.13).

Here is this subsection's principle result:

Proposition 4.14. *There exists a constant Z with the following significance: Fix $E > 0$ and $\rho, r > 0$. Let Q be as in* (4.16). *Then*

$$\|\pi_E \cdot Q\|_{*\rho} \leq Z \cdot \frac{\rho^2}{r^2}(1 + r^2 E^7) \cdot (\|q\|_{*,r} + \|b_1\|_{\mathscr{L}^0,r}\|b_2\|_{2*,r}).$$

The remainder of this section is occupied with the proof of this proposition.

To prove the proposition, it is first necessary to state some basic facts about eigenfunctions of $\nabla_g^*\nabla_g$. The next two lemmas make a digression for this purpose.

Lemma 4.15. *Let M be a compact, 4-dimensional manifold with metric g. Let $Y \to M$ be a vector bundle which is associated to M's orthonormal frame bundle. Fix $n \geq 0$ and assume there is a constant Z which makes the following true: Let E be an eigenvalue of $\nabla_g^*\nabla_g$ on $C^\infty(Y)$ and let ν be an eigenfunction obeying $\|\nu\|_{L^2} = 1$. Then*

$$\|\nabla_g^{\otimes n}\nu\|_\infty \leq Z \cdot (E+1)^{1+n/2}.$$

Proof of Lemma 4.15. The $n = 0$ assertion follows from (4.18) using $\delta = \nabla_g$ and $Q = E \cdot u$. Take ρ smaller than the injectivity radius. Given the assertion for $n = k \geq 0$, one can obtain the $n = k+1$ case by differentiating both sides of $\nabla^2 u = E \cdot u$ n-times to get $\nabla^2(\nabla^{\otimes n} u) = E \cdot \nabla^{\otimes n} u + \sum_{i=0}^n \mathscr{R}_i \cdot \nabla^{\otimes i} u$, where $\mathscr{R}_i$ is constructed from the curvature of g and its derivatives to order $n - i$. Apply (4.18) to this last equation. q.e.d.

The next lemma concerns the totality of eigenfunctions of $\nabla_g^*\nabla_g$ with eigenvalue less than E.

Lemma 4.16. *Let M be a compact, 4-dimensional manifold with metric g. There are constants Z and E_0 with the following significance:*

(1) *The rank of* π_E *obeys* $N(E) \le Z \cdot (E+1)^2$.

(2) *Fix* $x \in M$ *and define* $r_2\colon C^\infty(V) \to V|_x$ *by restriction. Then* $r_x\colon \pi_{E_0} C^\infty(V) \to V|_x$ *is surjective for each* $x \in M$.

(3) *Also,* $r_x \circ \nabla_g\colon \pi_{E_0} \cdot C^\infty(V) \to (T^* \otimes V)|_x$ *is surjective for all* $x \in M$.

Proof of Lemma 4.16. The first assertion is well known (see, e.g., [4]). For the second assertion, fix $x \in M$ and choose a smooth section $\mathfrak{s}$ of V with $\mathfrak{s}(x) = 1$ and $\|\mathfrak{s}\|_{L^2} = 1$. One can assume that $\|\nabla_g \mathfrak{s}\|_{L^2} \le Z_g$. The expansion of $\mathfrak{s}$ in terms of eigenfunctions of $\nabla_g^* \nabla_g$ converges pointwise, a fact which follows from Lemma 4.2. Hence, there exists E_x such that r_x is surjective upon restriction to $\pi_{E_x} \cdot C^\infty(V)$. Lemma 4.15 implies that this surjectivity is true for $r_{x'}\colon \pi_{E_x} C^\infty(V) \to V|_{x'}$ if x' is near to x. This fact and the compactness of M completes the argument. Assertion (3) is proved with an analogous strategy.

Proof of Proposition 4.14. With r small, invoke Lemma 3.8 using $X = M$ and $\varepsilon = r$. Let $\Omega \subset M$ be a set of points for which properties (1)–(3) of Lemma 3.8 hold.

For $x \in M$, define $\alpha_x(\cdot)$ by (4.19) using $n \cdot r$ instead of ρ, with n as in Lemma 3.8. Define

$$\alpha \equiv \sum_{x \in \Omega} \alpha_x \tag{4.65}$$

so that $\{\underline{\alpha}_x = \alpha_x/\alpha\}_{x \in \Omega}$ is a partition of unity which is subordinate to the cover $\{B_{n \cdot r}(x)\}_{x \in \Omega}$ of M. Note that there is a constant c so that

$$|\nabla \underline{\alpha}_x| \le c \cdot r^{-1}. \tag{4.66}$$

Now, write

$$Q = \sum_{x \in \Omega} \underline{\alpha}_x \cdot Q. \tag{4.67}$$

Since π_E is linear, an estimate of $\pi_E \cdot Q$ follows from estimates of $\{\pi_E \cdot (\underline{\alpha}_x \cdot Q)\}_{x \in \Omega}$.

To estimate $\pi_E \cdot \underline{\alpha}_x Q$, choose an L^2-orthonormal basis $\{\nu_i\}_{i=1}^{N(E)}$ for $\pi_E \cdot C^\infty(V)$ so that the following conditions are met:

$$\begin{aligned} &(1) \quad \{\nu_i\}_{i=1}^{N(E)} \subset \pi_{E_0} C^\infty(V). \\ &(2) \quad \text{If } i > N(E_0), \text{ then } r_x \nu_i = 0 \text{ and } r_x \circ \nabla_g \nu_i = 0. \end{aligned} \tag{4.68}$$

The existence of such a basis is insured by Lemma 4.16.

If $i \le N(E_0)$, then

$$\left| \int_M d\,\mathrm{vol}_g (\nu_i, \underline{\alpha}_x Q)_g \right| \le Z \cdot r^2 (\|q\|_{*,r} + \|b_1\|_{\mathcal{L}^0, r} \cdot \|b_2\|_{2*, r}). \tag{4.69}$$

Here, Lemma 4.15 is used to bound ν_i and $\nabla\nu_i$ uniformly. (This is possible because each ν_i is a linear combination of eigenvectors with eigenvalue $N(E_0)$ or less of which there are at most $Z\cdot(E_0+1)^2$.) In deriving (4.69), integration by parts is used to remove the derivative from b_2.

Now, for $i > N(E_0)$, one has in $B_{2n\cdot r}(x)$

$$|\nu_i| \le Z\cdot r^2E^3, \qquad |\nabla_g\nu_i| \le Z\cdot rE^3. \tag{4.70}$$

This means that when $i > N(E_0)$,

$$\left|\int_M d\,\mathrm{vol}_g(\nu_i, \underline{\alpha}_i Q)_g\right| \le Z\cdot r^4E^3\cdot(\|q\|_{*,r} + \|b_1\|_{\mathscr{L}^0,r}\cdot\|b_2\|_{2*,r}). \tag{4.71}$$

Meanwhile, when $i \le N(E_0)$, one has

$$\|\nu_i\|_{*,p} \le Z\cdot\rho^2, \tag{4.72}$$

and for $i > N(E_0)$ one has

$$\|\nu_i\|_{*p} \le Z\cdot\rho^2\cdot E^2. \tag{4.73}$$

Together, (4.69) and (4.71)–(4.73), with Lemma 4.15, estimate

$$\|\pi_E\cdot\underline{\alpha}_x\cdot Q\|_{*p} \le Z\cdot\rho^2\cdot r^2(1+r^2E^7)\cdot(\|q\|_{*,r}+\|b_1\|_{\mathscr{L}^0,r}\cdot\|b_2\|_{2*,r}). \tag{4.74}$$

Since the number of points in Ω can be bounded by $Z\cdot r^{-4}$, equation (4.74) gives the required estimate.

5. Linear theory for $\mathscr{W}_+$

Suppose that M is a compact, oriented, 4-dimensional manifold with metric g. Specify a section Q of the bundle of symmetric, traceless endomorphisms of $\Lambda^2_+T^*M$. Consider the equation below for a section h of $\mathrm{Sym}^2\,T^*$:

$$\frac{d}{dt}\mathscr{W}_+(g+th)|_{t=0} \equiv L_g h = Q. \tag{5.1}$$

Equation (5.1) is a linearization about g of the equation for prescribing $\mathscr{W}^+$; (5.1) is the subject for this section.

(a) Solving for h. With g fixed, $\mathscr{W}_+(g+h)$ depends analytically on h as long as $|g|_g < 1$ everywhere. Thus, the left-hand side of (5.1) defines a second-order, linear, differential operator L_g.

The operator L_g takes a section of the bundle of symmetric endomorphisms of T^*M and gives a section of the bundle V_+ of symmetric, trace zero endomorphisms of $\Lambda^2_+ \equiv \Lambda^2_+ T^*M$. (Simplify notation here and use $\Lambda^2_\pm$ for $\Lambda^2_\pm T^*M$.) Thus,

$$L_g\colon C^\infty(\mathrm{Sym}^2(T^*)) \to C^\infty(V_+). \tag{5.2}$$

There is, of course, always a cokernel obstruction to solving $L_g h = Q$ for general Q. This obstruction will eventually be dealt with, but for now it is convenient to circumvent the whole cokernel issue by considering the equation

$$\Pi_E \cdot L_g h = \Pi_E \cdot Q. \tag{5.3}$$

Here, $E \geq 0$ and $\Pi_E\colon C^\infty(V_+) \to C^\infty(V_+)$ is the L^2-orthogonal projection onto the span of the eigenvectors of $\nabla_g^*\nabla_g$ with eigenvalue E or greater. (Note that Π_E is *not* defined by $L_g L_g^*$!)

To measure solutions to (5.3), use $\|\cdot\|_{\mathscr{L}^0,\rho}$, $\|\cdot\|_{\mathscr{L}^1,\rho}$, and $\|\cdot\|_{\mathfrak{H},\rho}$ to indicate that the norms in Definition 4.6 have been defined using the parameter $\rho > 0$. Likewise, use $\|\cdot\|_{*,\rho}$ for (4.17).

Proposition 5.1. *Let M be a compact, oriented 4-manifold with metric g. Let $Y \to M$ be a vector bundle. There exists Z which makes the following true:*

(1) *Fix $E \geq Z$ and assume that there is a continuous map $H\colon C^\infty(V_+) \to C^\infty(\mathrm{Sym}^2 T^*)$ for which $h \equiv H[Q]$ obeys (5.3).*

(2) *Suppose that $Q = q + b_1 \cdot \nabla b_2$, where $b_2 \in C^\infty(Y)$ and $b_1 \in C^\infty(\mathrm{Hom}(Y \otimes T^*; V_+))$. If ρ, r obey $E\rho^{12} > Z^{-1}$ and $r^2E^7 < Z$; then*

$$\|h\|_{\mathscr{L},\rho} \equiv \|h\|_{\mathscr{L}^0,\rho} + \|\nabla_g h\|_{\mathscr{L}^1,\rho} \leq Z \cdot \left(\langle Q\rangle_\rho + \frac{\rho^2}{r^2}\langle Q\rangle_r\right). \tag{5.4}$$

Here

$$\langle Q\rangle_\rho \equiv \|q\|_{*\rho} + \|b_1\|_{\mathscr{L}^0,\rho}\|b_2\|_{\mathfrak{H},\rho}. \tag{5.5}$$

The remainder of this section is occupied with the proof of this proposition and with a special case (Proposition 5.7). However, properties of the solution h to (5.3) are derived along the way.

Before turning to the proof, here is a word of explanation for the use of Π_E: The analysis of (5.3) using Π_E, as opposed to $L_g L_g^*$'s projection, seems easier to the author. The reason being that $\nabla_g^*\nabla_g$ is a second-order operator while $L_g L_g^*$ is a fourth-order operator. There are disadvantages to the use of Π_E, but they prove to be of little consequence.

(b) The linearization of $\mathscr{W}_+$. The metric g defines the orthgonal decomposition

$$\mathrm{Sym}^2 T^* \simeq \mathbb{R} \oplus (\Lambda^2_- \otimes \Lambda^2_+). \tag{5.6}$$

The first factor corresponds to multiples of g. With respect to (5.6), L_g sends $f \cdot g$ (with $f \in C^\infty(M)$) to $f \cdot \mathscr{W}_+$.

On a section of $\Lambda^2_- \otimes \Lambda^2_+$, the operator L_g is identical to the orthogonal projection,

$$\mathfrak{s}\colon \mathrm{End}(\Lambda^2_+) \simeq \Lambda^2_+ \otimes \Lambda^2_+ \to V_+, \tag{5.7}$$

of an operator $\widehat{L}_g\colon C^\infty(\Lambda^2_- \otimes \Lambda^2_+) \to C^\infty(\Lambda^2_+ \otimes \Lambda^2_+)$. This $\widehat{L}_g$ has the form

$$\widehat{L}_g = \partial^*_{+g}\partial_{-g} + B_g. \tag{5.8}$$

The various terms in (5.8) are as follows: First, $B_g \in C^\infty(\mathrm{Hom}(\Lambda^2_-\,;\, \Lambda^2_+))$ is the traceless Ricci tensor in (1.2). Second, $\partial_{\pm g}$ are defined as follows: Take the composition $*d\colon C^\infty(\Lambda^2_\pm) \to C^\infty(T^*)$ of (4.3) and extend it to

$$\partial_{\pm g}\colon C^\infty(\Lambda^2_\pm \otimes \Lambda^2_+) \to C^\infty(T^* \otimes \Lambda^2_+) \tag{5.9}$$

using the Levi-Civita connection on Λ^2_+. Finally, $\partial^*_{\pm g}$ is the formal, L^2-adjoint of $\partial_{\pm g}$. Thus,

$$\partial^*_{\pm g}\colon C^\infty(T^* \otimes \Lambda^2_+) \to C^\infty(\Lambda^2_\pm \otimes \Lambda^2_+) \tag{5.10}$$

is the covariant extension (using the Levi-Civita connection) of the composition

$$C^\infty(T^*) \xrightarrow{d} C^\infty(\Lambda^2 T^*) \xrightarrow{P_\pm} C^\infty(\Lambda^2_\pm), \tag{5.11}$$

where the $P_\pm$ are the orthogonal projections for (1.1).

The factorization of $\widehat{L}_g$ as in (5.8) appears in [7].

(c) The extensions of $\partial_{\pm g}$. Neither of the $\partial_{\pm g}$ in (5.9) are elliptic, but both have injective symbols which obey (4.2). (Much of the discussion in §4 will be applied here to $\partial_{\pm g}$.) Both $\partial_{\pm g}$ come as part of naturally occurring elliptic operators,

$$D_{\pm g}\colon C^\infty(\Lambda^2_\pm \otimes \Lambda^2_+) \oplus C^\infty(\Lambda^2_+) \to C^\infty(T^* \otimes \Lambda^2_+). \tag{5.12}$$

Here,

$$D_{\pm g}(u\,,\, u_0) \equiv \partial_{\pm g} u + \nabla_g u_0. \tag{5.13}$$

Use $D^*_{\pm g} = (\partial^*_{\pm g}, \nabla^*_g)$ to denote the formal L^2-adjoints of $D_{\pm g}$.

Since $D_{\pm g}$ are elliptic, there is the usual "Hodge theorem":

Lemma 5.2. *Each $a \in C^\infty(T^* \otimes \Lambda^2_+)$ has a unique decomposition $a = v_\pm + \hat{a}_\pm$, where $v_\pm \in \ker(D^*_{\pm g})$ and $\hat{a}_\pm \in \operatorname{Im}(D_{\pm g})$.*

(d) A change of view. When $u \in C^\infty(V_+)$, then $\partial_{+g} u \in C^\infty(T^* \otimes \Lambda^2_+)$ and one can use Lemma 5.2 to decompose

$$\partial_{+g} u = v_- + \partial_{-g} h + \nabla_g \varphi, \tag{5.14}$$

where $v_- \in \ker D^*_{-g}$, $h \in C^\infty(\Lambda^2_- \otimes \Lambda^2_+)$, and $\varphi \in C^\infty(\Lambda^2_+)$. If, in addition, u obeys

$$\mathfrak{s} \cdot \partial^*_{+g} \partial_{+g} u - \mathfrak{s}(\partial^*_{+g} v_- + \partial^*_{+g} \nabla_g \varphi) + \mathfrak{s} \cdot B_g \cdot h = Q, \tag{5.15}$$

then h obeys (5.1), that is,

$$L_g h = Q. \tag{5.16}$$

The point of view that will be stressed here and henceforth is that (5.15) is an equation for u with h, φ, and v_- implicit functions of u which are determined by (5.14) and Lemma 5.2.

The replacement of (5.16) (an equation for h) by (5.15) (an equation for $u \in C^\infty(V_+)$) can be justified at two levels. First, (5.16) is a second-order equation without injective symbol. It has been replaced by (5.14) and (5.15). The former, an equation for h, φ, and v_-, is an elliptic, first-order equation. The latter, (5.15), is a second-order equation whose symbol is that of the *Laplacian*, $\xi \in T^* \mapsto |\xi|^2 \cdot 1$. In particular, the machinery of §4 is designed for (5.14) and (5.15).

The second justification for (5.15)'s replacement of (5.16) is not so practical, but theologically much more satisfying: To understand this second justification, one must realize that the failure of L_g's symbol to be injective is the linear manifestation of the natural equivariance of the equation $\mathscr{W}_+(g+h) = Q$ under the action of Diff(M). Indeed, L_g is a *zeroth*-order operator upon restriction to the linear subspace of $C^\infty(\Lambda^2_- \oplus \Lambda^2_+)$ which is tangent to the orbit through g of Diff(M). (This subspace is the image of $C^\infty(T^*)$ under the map

$$C^\infty(T^*) \xrightarrow{\nabla_g} C^\infty(T^* \otimes T^*) \xrightarrow{p} C^\infty(\Lambda^2_- \oplus \Lambda^2_+), \tag{5.17}$$

where p is orthogonal projection.)

The degeneracy of L_g's symbol implies that there is an infinite-dimensional ambiguity in L_g^{-1} since L_g will have an infinite-dimensional kernel.

To contrast, an elliptic operator has at most a finite-dimensional kernel, so a finite-dimensional ambiguity to its inverse.

With the above understood, take comfort in the fact that the replacement of (5.16) (with its great indeterminacy as an equation for h) by (5.15) (with its small indeterminacy as an equation for u) can be thought of as a choice of a particular resolution of (5.16)'s indeterminacy. Indeed, given any h satisfying (5.16), one can find u, v_-, and φ to solve (5.14) and (5.15) *if* one is willing to allow u to be a section of $\Lambda^2_+ \otimes \Lambda^2_+$ rather than a section of $V_+ \subset \Lambda^2_+ \otimes \Lambda^2_+$. (Apply both $\pm$ versions of Lemma 5.2 to prove this.) Now,

$$\Lambda^2_+ \otimes \Lambda^2_+ \simeq V_+ \oplus \Lambda^2_+ \oplus \mathbb{R}, \tag{5.18}$$

where the last two terms correspond to the skew-symmetric endomorphisms of Λ^2_+ and the multiples of the identity, respectively. On the symbol level, the composition

$$C^\infty(T^*) \xrightarrow{p\circ\nabla_g} C^\infty(\Lambda^2_- \otimes \Lambda^2_+) \xrightarrow{\partial^*_{+g}\partial_{-g}} C^\infty(\Lambda^2_+ \otimes \Lambda^2_+) \tag{5.19}$$

is an isomorphism *onto* the last two summands in (5.18). This means that, up to a finite-dimensional vector subspace, any h solving (5.16) can be adjusted by adding elements in the kernel of L_g to insure that the resulting u solving (5.14) and (5.15) lies everywhere in V_+.

(e) Equations (5.14) and (5.15) for u. Having said all of the above, return to (5.14) and (5.15) as an equation for $u \in C^\infty(V_+)$. To bring (5.15) into better focus, remark that the symbol for $\partial^*_{+g}\partial_{+g}$ is a multiple of the identity. This means that

$$\partial^*_{+g}\partial_{+g}u = \nabla^*_g\nabla_g u + \mathscr{R}_0 \cdot u, \tag{5.20}$$

where $\mathscr{R}_0 \in C^\infty(\mathrm{Hom}(V_+, \Lambda^2_+ \otimes \Lambda^2_+))$ is a linear function of the Riemann curvature tensor. (In fact, it depends only on $\mathscr{W}_+$ and s of (1.2).)

Also, the symbol sequence for

$$C^\infty(\Lambda^2_+) \xrightarrow{\nabla_g} C^\infty(T^* \otimes \Lambda^2_+) \xrightarrow{\partial^*_{+g}} C^\infty(\Lambda^2_+ \otimes \Lambda^2_+) \tag{5.21}$$

is exact. This (and (5.2)) allows (5.15) to be rewritten as

$$\nabla^*_g\nabla_g u + \mathfrak{s} \cdot (\partial^*_{+g}v_- + \mathscr{R}_0 \cdot u + \mathscr{R}_1(h, \varphi)) = Q, \tag{5.22}$$

where $\mathscr{R}_1$ is a section of $\mathrm{Hom}(\Lambda^2_- \otimes \Lambda^2_+ \oplus \Lambda^2_+ ; \Lambda^2_+ \otimes \Lambda^2_+)$ which is a linear function of the entries $\mathscr{W}_+$, s, and B in (1.2).

Together, (5.22) and (5.14) define an elliptic, second-order pseudodifferential equation for u. The equation's solvability cannot generally be insured without modifications which avoid any cokernel to the adjoint equation. The simplest way to avoid these cokernel obstructions is to choose $E > 0$ and replace (5.22) by

$$\nabla_g^*\nabla_g u + \Pi_E \cdot \mathfrak{s} \cdot (\partial_{+g}^* v_- + \mathscr{R}_0 \cdot u + \mathscr{R}_1(h, \varphi)) = \Pi_E \cdot Q. \tag{5.23}$$

Note. If u satisfies (5.23) with (v_-, h, φ) determined by (5.14), then h satisfies (5.3).

(f) Existence and uniqueness for (5.23). The analysis of (5.23) is rooted in the following basic result:

Proposition 5.3. *Let M be a compact, oriented 4-manifold with metric g. There exists Z such that when $E \geq Z$, the following is true: Specify $Q \in C^\infty(V_+)$ and (5.23) has a unique solution u when subject to the additional constraints that $u \in \Pi_E \cdot C^\infty(V)$ and that (h, φ) be L^2-orthogonal to $\ker D_{-g}$. Furthermore, this association defines a continuous, linear map from $C^\infty(V_+)$ to $\Pi_E C^\infty(V_+)$.*

Proof of Proposition 5.3. A weak solution to (5.23) has u in L_1^2, $(h, \varphi) \in L_1^2$, and $v_- \in C^\infty$. Elliptic regularity (see Lemma 4.2) implies that u, h, φ are smooth.

The L_1^2 solution will be found after rewriting (5.23) as a fixed point equation on $\Pi_E \cdot L_1^2(V_+)$. Existence and uniqueness will follow using a contraction mapping argument.

To begin the rewriting of (5.23), consider (5.14) to be an equation for determining (h, φ) and v_- in terms of $u \in L_1^2(V_+)$. Hodge theory provides that there is a unique (h, φ) and v_- solving (5.14) provided that $(h, \varphi) \in \ker D_{-g}$.

Lemmas 4.1 and 4.2 estimate

$$\|v_-\|_{L_1^2} \leq Z \cdot \|u\|_{L_1^2} \tag{5.24}$$

and

$$\|(h, \varphi)\|_{L_1^2} \leq Z \cdot \|u\|_{L_1^2}. \tag{5.25}$$

These last estimates can be bootstrapped to show that the assignment of (h, φ) and v_- to u gives a bounded, linear map from $L_k^2(V_+)$ to $L_k^2(\Lambda_-^2 \otimes \Lambda_+^2 \oplus \Lambda_+^2) \times \ker D_{-g}^*$.

With (5.14) understood, turn to (5.23). With $u \in L_1^2$, one has $\Pi_E \cdot \mathfrak{s} \cdot (\partial_{+g}^* v_- + \mathscr{R}_0 \cdot u + \mathscr{R}_1(h, \varphi))$ in $L_1^2(V_+)$.

Lemma 4.3 provides a unique $w \in \Pi_E \cdot L_3^2(V_+)$ which obeys

$$\nabla_g^*\nabla_g w = -\Pi_E \cdot \mathfrak{s} \cdot (\partial_{+g}^* v_- + \mathscr{R}_0 \cdot u + \mathscr{R}_1(h, \varphi)). \tag{5.26}$$

Lemma 4.2 implies that the assignment to $u \in L_1^2(V_+)$ of $w[u] \in L_3^2(V_+)$ is linear and continuous.

Meanwhile, Lemma 4.3 also provides a unique $u_Q \in \Pi_E C^\infty(V_+)$ which obeys

$$\nabla_g^* \nabla_g u_Q = \Pi_E \cdot Q. \tag{5.27}$$

With (5.26) and (5.27), one can rewrite (5.23) as the following fixed point equation on $\Pi_E L_1^2(V_+)$:

$$u = w[u] + u_Q. \tag{5.28}$$

This equation will have a unique solution if $w[\cdot]$ is a contraction mapping from $\Pi_E L_1^2(V_+)$ to itself. The L_1^2-norm of w is estimated using (5.26), Lemma 4.3, and Holder's inequality:

$$\|w\|_{L_1^2} \le Z \cdot E^{-1/2} \|u\|_{L_1^2}. \tag{5.29}$$

If $E > Z^2$, then (5.29) implies that (5.28) is a contraction mapping. Therefore, Proposition 5.3 is a consequence of (5.29).

(g) Local estimates for u in (5.23). Fix E so that Proposition 5.3 holds. With $\rho > 0$ fixed, the purpose of this subsection is to estimate the size of u as measured by

$$\|u\|_{\mathscr{L},\rho} \equiv \|u\|_{\mathscr{L}^0,\rho} + \|\nabla u\|_{\mathscr{L}^1,\rho}. \tag{5.30}$$

(Refer here to Definitions 4.4–4.6.)

Proposition 5.4. *There exists a constant Z which makes the following true: Fix $E > 0$ so that Proposition 5.3 holds, and let $u \in \Pi_E C^\infty(V_+)$ be the solution to (5.23) given by said proposition. Suppose $\rho, r > 0$ but that $\rho^6 E > Z^{-1}$, $\rho < Z^{-1}$, and $r^2 E^7 < Z$. Then*

$$\|u\|_{\mathscr{L},\rho} \le Z \cdot (\langle Q\rangle_\rho + \rho^2/r^2 \cdot \langle Q\rangle_r), \tag{5.31}$$

where $\langle Q\rangle_\rho$ is defined in (5.5).

Here is how the proposition will be proved: First, Propositions 4.8 and 4.14 be used to bound the $\|\cdot\|_{\mathscr{L}^0,\rho}$ norm of u in terms of $\langle Q\rangle_\rho$, $\langle Q\rangle_r$, and $\|(h, \varphi)\|_\infty$. Second, Lemma 4.10 will be used to bound $\|(h, \varphi)\|_\infty$ in terms of $\|u\|_{\mathscr{L},\rho}$. Third, $\|\nabla_g u\|_{\mathscr{L}^1}$ will be bounded in terms of $\langle Q\rangle_\rho$, $\langle Q\rangle_r$, and $\|u\|_{\mathscr{L},\rho}$ using Proposition 4.12. Steps 1–3 will give a bound on $\|u\|_{\mathscr{L},\rho}$ in terms of $\langle Q\rangle_\rho$, $\langle Q\rangle_r$, and $\|u\|_{\mathscr{L},\rho}$ again. Finally, when the bounds on ρ, E, and r are obeyed, this last bound will be parlayed into (5.31).

To set up the first step, one must write (5.23) as

$$\nabla_g^* \nabla_g u = \Pi_E \cdot (Q + q'), \tag{5.32}$$

where

$$q' \equiv -\mathfrak{s}\cdot(\partial^*_{+g}v_- + \mathscr{R}_0\cdot u + \mathscr{R}_1\cdot(h\,,\,\varphi)). \tag{5.33}$$

An appeal to Proposition 4.8 requires an estimate for $\|\Pi_E\cdot q'\|_{\mathscr{L}^0,\rho}$. This number can be estimated using Proposition 4.14 from a bound on $\|q'\|_{*,\rho}$ and $\|q'\|_{*,r}$. These, finally, are estimated by

Lemma 5.5. *There exists Z such that if u obeys* (5.23) *and $\rho\,,\,r>0$, then*

$$\|q'\|_{*,r} \le Z\cdot r^2(1+\rho^{-3}E^{-1/4})\cdot\|u\|_{\mathscr{L},\rho}. \tag{5.34}$$

This lemma is proved below. With Lemma 5.5, the proof of Proposition 5.4 requires one additional result:

Lemma 5.6. *There exists Z such that if u obeys* (5.23), *then for $\rho\,,\,r>0$,*

$$\begin{aligned}\|\nabla_g u\|_{\mathscr{L}^1} \le Z\cdot(1+\rho^{-6}E^{-1})\cdot(\langle Q\rangle_\rho + \tfrac{\rho^2}{r^2}(1+r^2E^7)\cdot\langle Q\rangle_r + (1+r^2E^7)\\ \cdot(\rho^2+\rho^{-1}E^{-1/4})\cdot\|u\|_{\mathscr{L},\rho}) + Z\cdot\|u\|_{\mathscr{L}^0,\rho}.\end{aligned} \tag{5.35}$$

This last lemma is also proved below.

Proof of Proposition 5.4. Plug (5.34) into Propositions 4.8 and 4.14 to find that

$$\begin{aligned}\|u\|_{\mathscr{L}^0} \le Z\cdot(1+\rho^{-6}E^{-1})\cdot(\langle Q\rangle_\rho + \tfrac{\rho^2}{r^2}(1+r^2E^7)\cdot\langle Q\rangle_r + (1+r^2E^7)\\ \cdot(\rho^2+\rho^{-1}E^{-1/4})\cdot\|u\|_{\mathscr{L},\rho}).\end{aligned} \tag{5.36}$$

Meanwhile, Lemma 5.6 bounds $\|\nabla_g u\|_{\mathscr{L}^1}$ by the same expression as on the right in (5.36) but for the addition of $Z\cdot\|u\|_{\mathscr{L}^0,\rho}$. Multiply both sides of (5.36) by $2\cdot Z$ and add the result to (5.35). This yields Proposition 5.4's asserted bound when $\rho^6E \ge>^{-1}$, $\rho\le Z^{-1}$, and $r^2E^7\le Z$.

Proof of Lemma 5.5. As

$$\|q'\|_{*,r} \le Z\cdot r^2\cdot\|q'\|_\infty, \tag{5.37}$$

it is sufficient to estimate the L^∞ norm of each of the three terms in (5.35).

Term 1 *in* (5.33). Use Lemma 4.11 to conclude that

$$\|\partial^*_{+g}v_-\|_\infty \le Z\cdot\|u\|_{L^2} \le Z\cdot\|u\|_\infty. \tag{5.38}$$

Term 2 *in* (5.33). This term is bounded by

$$Z\cdot\|u\|_\infty. \tag{5.39}$$

Term 3 *in* (5.33). This term is bounded by $Z \cdot \|(h, \varphi)\|_\infty$ which must now be estimated. For this purpose, write $\psi = (h, \varphi)$. Since $D_{-g}\psi = \partial_{+g}u - v_-$, Lemma 4.10 can be applied if E in said lemma is taken to be the first nonzero eigenvalue of $D^*_{-g}D_{-g}$. Use the first estimate to bound

$$(5.40) \qquad \|(h, \varphi)\|_{\mathscr{L}^0} \le Z \cdot (\rho^{-2}\|(h, \varphi)\|_{L^2} + \|\nabla_g u\|_{\mathfrak{h}}).$$

To estimate the L^2 norm of $\psi = (h, \varphi)$, pick $\lambda > 0$ and write $\psi = \psi_\lambda + \overline{\psi}_\lambda$, where $\overline{\psi}_\lambda = \Pi_\lambda \cdot \psi$, with Π_λ the projection onto the span of the eigenvectors of $D^*_{-g}D_{-g}$ which have eigenvalue λ or greater. One has

$$(5.41) \qquad \|\overline{\psi}_\lambda\|_{L^2} \le \frac{1}{\sqrt{\lambda}}\|D_{-g}\overline{\psi}_\lambda\|_{L^2} \le \frac{z}{\sqrt{\lambda}}\|u\|_{L^2_1}.$$

As for $\|\psi_\lambda\|_{L^2}$, one has

$$(5.42) \qquad \|\psi_\lambda\|^2_{L^2} \le Z \cdot \|D_{-g}\psi_\lambda\|^2_{L^2} \le Z \cdot \int_M d\,\mathrm{vol}_g(D_{-g}\psi_\lambda, \partial_{+g}u)_g.$$

The last term on the right of (5.42) can be bounded by

$$(5.43) \qquad \|\psi_\lambda\|_{L^2_2} \cdot \|u\|_{L^2} \le Z \cdot \lambda \cdot \|\psi_\lambda\|_{L^2} \cdot \|u\|_{L^2}.$$

Together, (5.41)–(5.43) imply that

$$(5.44) \qquad \|\psi\|_{L^2} \le Z \cdot \left(\frac{1}{\sqrt{\lambda}} + \frac{\lambda}{\sqrt{E}}\right) \cdot \|u\|_{L^2_1}.$$

Now, set $\lambda = E^{1/4}$. This gives

$$(5.45) \qquad \|(h, \varphi)\|_{\mathscr{L}^0, \rho} \le Z_g \cdot (\rho^{-3}E^{-1/4} \cdot \|u\|_{\mathscr{L}^0, \rho} + \|\nabla_g u\|_{\mathfrak{h}, \rho}).$$

Equations (5.38), (5.39), and (5.44) give Lemma 5.5.

Proof of Lemma 5.6. The estimate for $\|\nabla_g u\|_{\mathscr{L}^1, \rho}$ will come from Proposition 4.12. To invoke said proposition, set $w = \nabla_g \cdot u$. Note that (5.32) describes $\nabla^*_g w$. Meanwhile,

$$(5.46) \qquad \partial^*_{+g}w = \mathscr{R}_2 \cdot u,$$

where $\mathscr{R}_2 \in C^\infty(\mathrm{End}\, V_+)$ is a linear function of the components of $\mathscr{W}_+$ and s. Thus,

$$(5.47) \qquad D^*_{+g}w = (\mathscr{R}_2 \cdot u, \Pi_E(Q + q')),$$

with D_{+g} as defined in (5.13).

As D_{+g} is elliptic, Proposition 4.12 can be invoked and it yields

$$(5.48) \qquad \|w\|_{\mathscr{L}^1, \rho} \le Z \cdot (1 + \rho^{-6}E^{-1}) \cdot (\langle Q\rangle_\rho + \tfrac{\rho}{r^2}(1 + r^2E^7) \cdot \langle Q\rangle_r + (\rho^2 + \rho^{-1}E^{-1/4}) \cdot \|u\|_{\mathscr{L}, \rho}) + Z \cdot \|w\|_{2*, \rho}.$$

This last expression gives (5.35) with the realization that $\|w\|_{2*,\rho} \leq \|u\|_{\mathscr{L}^0,\rho}$.

(h) Proof of Proposition 5.1. The h in said proposition is the same as that in (5.14) when u is given by Proposition 5.3.

The $\mathscr{L}^0$-norm of h is estimated in (5.45) by

$$\|h\|_{\mathscr{L}^0,\rho} \leq Z \cdot (1 + \rho^{-3} E^{-1/4}) \cdot \|u\|_{\mathscr{L},\rho}. \tag{5.49}$$

Meanwhile, the $\mathscr{L}^1$-norm of $\nabla_g h$ is estimated using Proposition 4.12 and gives

$$\|\nabla_g h\|_{\mathscr{L}^1,\rho} \leq Z \cdot (\|\nabla_g u\|_{\mathfrak{h},\rho} + \|h\|_{2*,\rho}). \tag{5.50}$$

Since $\|\nabla_g u\|_{\mathfrak{h},\rho} \leq \|u\|_{\mathscr{L},\rho}$ and $\|h\|_{2*,\rho} \leq \|h\|_{\mathscr{L}^0,\rho}$, these last two equations imply (5.4).

(i) The example of $\underline{\mathbb{CP}}^2$ and S^4. The manifolds S^4 (with its round metric) and $\underline{\mathbb{CP}}^2$ (with its Fubini-Study metric) provide examples for Proposition 5.4 where the projection Π_E is unnecessary.

Proposition 5.7. *Let M be a compact, oriented, Riemannian 4-manifold whose metric g obeys $\mathscr{W}_+ \equiv 0$, $B \equiv 0$, and $s > 0$. (Refer to (1.2).) There is a constant $Z > 2$ for which the following is true:*

(1) *There is a continuous map $H: C^\infty(V_+) \to C^\infty(\mathrm{Sym}^2 T^*)$ for which $h = H(Q)$ obeys $L_g h = Q$.*

(2) *If Q is as described in Proposition 5.1, then*

$$\|h\|_{\mathscr{L},\rho=1} \leq Z \cdot \langle Q \rangle_{\rho=1}. \tag{5.51}$$

Proof of Proposition 5.7. Begin with the observation that $\mathfrak{s} \cdot \mathscr{R}_1 \equiv 0$ in (5.22). This is due to the vanishing of $\mathscr{W}_+$ and B. Thus, (5.23) couples to (5.14) only through the v_- term. This term is, in fact, zero for the following reason: The metric orthogonally splits the bundle $T^* \otimes \Lambda^2_+ \simeq T^* \oplus W_+$. (Here, W_+ is the kernel of a dual to $T^* \otimes T^* \to \Lambda^2 T^* \to \Lambda^2_+$.) When $B = \mathscr{W}_+ = 0$, the components of v_- with respect to the splitting $T^* \otimes \lambda^2_+ \simeq T^* \oplus W_+$ are separately annihilated by D^*_{-g}. Use the Weitzenboch formula for $D_{-g} D^*_{-g}$ to prove this. When $s > 0$, this same Weitzenboch formula shows that the component of v_- in W_+ must vanish. As for the other component, the composition of ∂^*_{+g} followed by $\mathfrak{s}$ is identically zero. Thus, no v_- term appears in (5.23) as claimed.

As for the term $\mathfrak{s} \cdot \mathscr{R}_0 \cdot u$ in (5.23), it is a positive, constant multiple of u. Indeed, $\mathscr{R}_0$ is linear in the scalar curvature (s); as $\mathscr{W}_+ \equiv B \equiv 0$ it can only be a multiple of the identity. As the Bianchi identity insures that s is constant, one need only check that $\mathfrak{s} \cdot \mathscr{R}_0$ is a positive multiple of the identity. This task is left to the reader.

The operator $\nabla_g^* \nabla_g$ has strictly positive spectrum also. Indeed, $\nabla_g u = 0$ implies $u = 0$ which can be proved by considerations of $[\nabla_g, \nabla_g]u$.

With the preceding understood, the projector Π_E in (5.23) is the identity on $C^\infty(V_+)$ whenever E is small, say $E_0 \geq E > 0$.

Since (5.23) and (5.14) are uncoupled. One can take $E = E_0/2$ in (5.23) and still draw Proposition 5.3's conclusions. Likewise, Proposition 5.4 holds for $E = E_0/2$ and so Proposition 5.1. The conclusions of Proposition 5.1 imply those of Proposition 5.7.

6. Linear theory on connect sums

The estimates so far have involved a fixed manifold M and a fixed metric g on M. In particular, the constants in most of the inequalities of §§4–5 depend implicitly on this metric g.

However, estimates are ultimately needed for manifolds of the form $M_N \equiv M \#_N \overline{\mathbb{CP}}^2$ with N arbitrarily large. Furthermore, the sorts of metrics (see §3) that show up can have arbitrarily small injectivity radius, and arbitrarily large curvatures. This means, of course, that the results in the preceding sections cannot be directly applied on M_N. The purpose of this section is to begin the development for the analysis on M_N by considering the linear aspects of the analysis. §8 deals with the nonlinear aspects.

(a) Conformal solutions to $L_g h = Q$. Start with a compact, oriented Riemannian manifold X with smooth metric g. As remarked, $\mathscr{W}_+$ transforms covariantly under conformal transformations of g. Indeed, as a section of $\Lambda_+^2 \otimes \Lambda_+^2$,

$$\mathscr{W}_+(f \cdot g) = f \cdot \mathscr{W}_+(g) \tag{6.1}$$

for smooth $f > 0$. Equation (6.1) implies that

$$L_{f \cdot g}(f \cdot h) = f \cdot L_g h \tag{6.2}$$

for $h \in C^\infty(\mathrm{Sym}^2 T^*)$. (Warning: The inclusion in (5.6), $\iota_g : \Lambda_-^2 \otimes \Lambda_+^2 \to \mathrm{Sym}^2 T^*$, is not conformally invariant. Rather, $\iota_{f \cdot g} = f \cdot \iota_g$.)

The point of this digression on conformal transformations is that the equation $L_g h = Q$ can be made sense of without the specification of an honest metric; only a conformal metric is needed. Here is why: Let $[g]$ be a conformal metric on X, which has been specified by giving an open cover $\{U_i\}$ of X and a metric $\hat{g}_i$ on each U_i, subject to

$$\hat{g}_i = \varphi_{ij} \cdot \hat{g} \quad \text{on } U_i \cap U_j \tag{6.3}$$

for φ_{ij} a positive, smooth function on $U_i \cap U_j$.

If $[Q] \equiv \{Q_i \in C^\infty(V_+|_{U_i})\}$ has been specified, with $\{Q_i\}$ obeying (6.3) too, then it is reasonable to say that $[h] \equiv \{h_i \in C^\infty(\mathrm{Sym}^2 T^*|_{U_i})\}$ solves $L_{[g]}[h] = [Q]$ if

$$(6.4)\qquad \begin{aligned} &\text{(a)} \quad L_{\hat{g}_i} h_i = Q_i = 0 \quad \text{on } U_i, \\ &\text{(b)} \quad h_i = \varphi_{ij} \cdot h_j \quad \text{on } U_i \cap U_j. \end{aligned}$$

An $[h]$ obeying (6.4) is a *conformal solution* to $L_{[g]}[h] = [Q]$.

(b) Conformal metric on M_N. A conformal metric $[g]$ on $M_N = M\#_N\underline{\mathbb{CP}}^2$ is constructed in §3 and some of its properties are summarized in §3l. In particular, §3l describes $[g]$ à la (6.3) by giving an open covering $\{U_i\}_{i=0}^N$ of M_N with an honest metric $\hat{g}_i$ on each U_i. Please reread §3l now.

In particular, the φ_{ij} in (6.3) can be found by noting that

$$(6.5)\qquad \varphi_i^* g_M' = \frac{|x|^4}{\lambda^2} \psi_\lambda^* \varphi_0^* g_{\mathrm{FS}}$$

in $\varphi_i^{-1}(U_i \cap U_0)$.

(c) Cokernel issues. Just as there are obstructions to solving $L_g h = Q$, there will be obstructions to (6.4)'s solvability. The equation $L_g h = Q$ was generalized to (5.3) in order to postpone the discussions of obstructions. Equation (6.4) has an analogous generalization which will be described shortly. Suffice it to say now that Proposition 5.7 allows for an avoidance of this cokernel issue by a method which treats U_0 and $U_{i>0}$ differently.

(d) Conformal solutions on M_N.

In writing down and in solving (6.4)'s generalization, it proves useful to write the $\{h_i\}_{i=0}^N$ in terms of $\{\hat{h}_i\}_{i=0}^N$ which extend over M (when $i = 0$) or over $\underline{\mathbb{CP}}^2$ (when $i > 0$). When $i > 0$, $\hat{h}_i$ is a section over $\underline{\mathbb{CP}}^2$ of $\mathrm{Sym}^2 T^*$ and it obeys

$$(6.6)\qquad L_{g_{FS}} \hat{h}_i = \widehat{Q}_i$$

for a suitable section $\widehat{Q}_i$ of $V_+|_{\underline{\mathbb{CP}}^2}$.

Meanwhile, $\hat{h}_0$ is a section over M of $\mathrm{Sym}^2 T^*M$ where it obeys

$$(6.7)\qquad \Pi_E \cdot L_{g_M} \hat{h}_0 = \Pi_E \cdot \widehat{Q}_0$$

for a suitable $\widehat{Q}_0$, section of V_+ (as defined by g_M) over M. Here, Π_E is the L^2-orthogonal projection (using the metric g_M) onto the space of the eigenvectors of $\nabla_{g_M}^* \nabla_{g_M}$ which have eigenvalue E or greater.

To recover $\{h_i\}_{i=0}^N$ from $\{\hat{h}_i\}$, it is necessary to fix $r_1 \in (0, 1)$ so that the radius r_1 ball about y_0 in $\underline{\mathbb{CP}}^2$ lies in the chart φ_0.

With r_1 specified, set $h_i \equiv \hat{h}_i$ on the compliment in $\underline{\mathbb{CP}}^2$ of the ball of radius r_1 with center y_0. Inside said ball, set

$$\varphi_0^* h_i = \varphi_0^* \hat{h}_i + (1 - \beta_{r_1/4}) \cdot \frac{|x|^4}{\lambda^2} \cdot \psi_\lambda^* \varphi_i^* \hat{h}_0. \tag{6.8}$$

Meanwhile, set $h_0 \equiv \hat{h}_0$ in the compliment in M of the union of the balls about each x_i of g'_M-radius ε_1. Inside the ball about x_i of g'_M-radius ε_1, set

$$\varphi_i^* h_0 = \varphi_i^* \hat{h}_0 + (1 - \beta_{\varepsilon_1/4}) \cdot \frac{|x|^4}{\lambda^2} \cdot \psi_\lambda^* \varphi_0^* \hat{h}_i. \tag{6.9}$$

Notice from (6.8) and (6.9) that

$$\varphi_i^* h_0 = \frac{|x|^4}{\lambda^2} \psi_\lambda^* \varphi_0^* h_i, \tag{6.10}$$

where $\varepsilon_1/32 < |x| < \varepsilon_1/16$; i.e., on $\varphi_i^{-1}(U_i \cap U_0)$.

To define $\{\widehat{Q}_i\}$, it is necessary to choose $r_0 \in (0, 1)$ so as to invoke Proposition 3.15. Make $r_0 \leq r_1/8$. Its precise value will be specified in Theorem 6.3.

When $i > 0$, make $\widehat{Q}_i \equiv Q_i$ on the compliment in $\underline{\mathbb{CP}}^2$ of the radius r_1 ball about y_0. Inside this ball, set

$$\begin{aligned} \varphi_0^* \widehat{Q}_i \equiv\ & (1 - \psi_\lambda^* \beta_{16\lambda/r_0}).\varphi_0^* Q_i \\ & - \beta_{r_0/4}(P_+(g_E + m_{\mathrm{FS}}) \cdot L_{\psi_\lambda^* \varphi_i^* \overline{g}} - L_{g_E + m_{\mathrm{FS}}})(\hat{p}_0), \end{aligned} \tag{6.11}$$

where

$$\hat{p}_0 \equiv (1 - \beta_{r_1/4}) \cdot \frac{|x|^4}{\lambda^2} \cdot \psi_\lambda^* \varphi_0^* \hat{h}_0, \tag{6.12}$$

and the metric $\overline{g}$ is defined in §3m.

As for $\widehat{Q}_0$, on the compliment in M of the union of the balls with center x_i and $\overline{g}$-radius ε_1, set

$$\widehat{Q}_0 \equiv P_+(g_M) \cdot Q_0 + (L_{g_M} - P_+(g_M) L_{\overline{g}})(\hat{h}_0). \tag{6.13}$$

On the ball with center x_i and $\overline{g}$-radius equal to ε_1, set

$$\begin{aligned} \varphi_i^* \widehat{Q}_0 \equiv\ & \beta_{16\lambda/r_0} \cdot \varphi_i^* (P_+(g_M) \cdot Q_0) + \beta_{\varepsilon_1/8} \varphi_i^* P_+(g_M) L_{\varphi_i^* \overline{g}} \hat{p}_i \\ & + \varphi_i^* (L_{g_M} - P_+(g_M) L_{\overline{g}})(\hat{h}_0). \end{aligned} \tag{6.14}$$

Here,

$$\hat{p}_i \equiv (1 - \beta_{\varepsilon_1/4}) \cdot \frac{|x|^4}{\lambda^2} \psi_\lambda^* \varphi_0^* \hat{h}_i. \tag{6.15}$$

(Remark that the domain of $\varphi_i^* Q_0$'s definition has been extended to where $\lambda/r_1 \le |x| \le \varepsilon_1/32$ by defining it to be equal to $|x|^4 \psi_\lambda^* \varphi_0^* Q_i/\lambda^2$ there.)

The preceding expressions for $\{\widehat{Q}_i\}_{i=0}^N$ are justified in part by:

Proposition 6.1. *Let M be a compact, oriented 4-manifold with metric g_M. Construct the conformal metric $[g]$ on $M\#_N\overline{\mathbb{CP}}^2$ as described in* §31. *Having chosen r_0 subject to Proposition* 3.16 *'s constraints, let $\overline{g}$ be the metric on M that is given in* §3m. *Cover $M\#_N\overline{\mathbb{CP}}^2$ by the open sets $\{U_i\}_{i=0}^N$ as described above, and introduce the metrics $\hat{g}_i$ on U_i as above. Let Q_i be given sections over U_i of V_+ (defined with $\hat{g}_i$) which obey* (6.3). *Suppose that $\{\hat{h}_i\}_{i=0}^N$ obey* (6.6) *and* (6.7) *with $\{\widehat{Q}_i\}_{i=0}^N$ given by* (6.11)–(6.15). *Require additionally that*

$$(1 - \Pi_E) \cdot (L_{g_M} \hat{h}_0 - \widehat{Q}_0) \equiv 0.$$

If $\{h_i\}_{i=0}^N$ is given by (6.8) *and* (6.9), *then $L_{[g]}[h] = [Q]$; i.e., equation* (6.4) *is satisfied.*

The proof of Proposition 6.1 requires the following:

Lemma 6.2. *Orient $\mathbb{R}^4$ with an isomorphism $\Lambda^4\mathbb{R}^4 \simeq \mathbb{R}$. Let g_1 and g_2 be positive definite inner products, and use $P_\pm(g_{1,2})\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$ to denote their associated self-dual and anti-self-dual projections. If $P_+(g_1) \cdot \omega = P_-(g_2) \cdot \omega = 0$, then $\omega = 0$.*

Proof of Lemma 6.2. The first condition implies that $\omega \wedge \omega = c$ with $c < 0$ unless $\omega = 0$. The second condition implies that $c > 0$ or $\omega = 0$.

Proof of Proposition 6.1. The verification of (6.4b) is left to the reader. As for (6.4a), it is automatic on the compliment to the ball of radius r_0 about y_0 on each $\overline{\mathbb{CP}}^2$. It also holds on the compliment in M of the union of the balls with center x_i and $\overline{g}$ radius ε_1. This follows using (6.13) and Lemma 6.2.

To see (6.4a) on the remainder of U_0, focus attention on the $\overline{g}$-radius ε_1 ball about one x_i. Since $\varphi_0^*\widehat{Q}_i \equiv 0$ for $|x| < r_0/32$, it follows that $\varphi_0^* L_{g_{FS}} \hat{h}_i \equiv 0$ for $|x| < r_0/32$ and therefore $L_{\overline{g}} \hat{p}_i \equiv 0$ for $32\lambda/r_0 \le |x| \le \varepsilon_1/4$. Thus, on $\varepsilon_1/64 \le |x| \le \varepsilon_1$, the condition that $L_{g_M} \hat{h}_0 = \widehat{Q}_0$ reads

$$\varphi_i^*(P_+(g_M)) L_{\varphi_i^* \overline{g}} (\varphi_i^* \hat{h}_0 + (1 - \beta_{\varepsilon_1/4}) \tfrac{|x|^4}{\lambda^2} \psi_\lambda^* \varphi_0^* \hat{h}_i) = \varphi_i^* P_+(g_M) \cdot Q_0. \tag{6.16}$$

This is (6.4a) on the relevant part of U_0. (Use Lemma 6.2.) To see (6.4a) on the remainder of the U_i, remark that

$$L_{\overline{g}}\hat{h}_0 = \beta_{16\lambda/r_0} Q_0 \tag{6.17}$$

on the radius $\varepsilon_1/8$ ball about x_i. This means that

$$L_{\psi_\lambda^* \varphi_i^* \overline{g}} \hat{p}_0 = (\psi_\lambda^* \beta_{16\lambda/r_0}) \cdot \varphi_0^* Q_i, \tag{6.18}$$

where $8\lambda/\varepsilon_1 \leq |x| \leq r_0/4$. In turn, (6.18) allows $\varphi_0^* \widehat{Q}_i$ in (6.11) to be written as

$$\varphi_0^* \widehat{Q}_i = \varphi_0^* Q_i - L_{g_E + m_{\mathrm{FS}}} \hat{p}_0, \tag{6.19}$$

where $8\lambda/\varepsilon_1 \leq |x| \leq r_0$. Equations (6.3), (6.8), and (6.19) imply (6.4a) on the remainder of U_i.

(e) Existence and uniqueness. To make a precise existence statement about (6.6) and (6.7) it is necessary to introduce $\widetilde{C}^\infty(V_+)$ to denote the set

$$[Q] = (Q_0, Q_{i>0}) \in \mathop{\times}_{i=0}^{n} C^\infty(V_+|_{U_i})$$

for which

$$\varphi_i^* Q_0 = \frac{|x|^4}{\lambda^2} \psi_\lambda^* \varphi_0^* Q_i \tag{6.20}$$

on $\varphi_i(U_0 \cap U_i) \subset \mathbb{R}^4$. (The choice of a metric g on M_N in $[g]$ will give a natural isomorphism $\widetilde{C}^\infty(V_+) \simeq C^\infty(V_+|_{M\#_N \underline{\mathbb{CP}}^2})$.)

When measuring $[Q]$, it will be assumed that each Q_i has the following form: For $i \geq 1$, assume that

$$(\varphi_0^{-1})^*(\beta_{r_0/32}) \cdot Q_i = q_i + b_{1,i} \nabla_{g_{\mathrm{FS}}} b_{2,i}, \tag{6.21}$$

as in (4.16). For Q_0, assume that

$$\left[\prod_{i=1}^{N} (\varphi_i^{-1})^* \beta_{8\lambda/r_0}\right] \cdot Q_0 = q_0 + b_{1,0} \nabla_{g_M} b_{2,0}. \tag{6.22}$$

Given (6.21) and (6.22), set

$$\Delta_i \equiv \|q_i\|_{*,r=1} + \|b_{1,i}\|_{\mathscr{L}^0, r=1} \cdot \|b_{2,i}\|_{\mathfrak{h}, r=1}; \tag{6.23}$$

then, with $\rho > 0$ given, set

$$\Delta_{0,\rho} \equiv \|q_0\|_\rho + \|b_{1,0}\|_{\mathscr{L}^0, \rho} \cdot \|b_{2,0}\|_{\mathfrak{h}, \rho}. \tag{6.24}$$

Here, the norms for q_0, $b_{1,0}$, and $b_{2,0}$ are defined using the metric g_M on M, while those for q_i, $b_{1,i}$, $b_{2,i}$ are defined using g_{FS} on $\underline{\mathbb{CP}}^2$.

Solutions $\{\hat{h}_i\}_{i=0}^N$ to (6.6) and (6.7) will be measured using $\|\hat{h}_0\|_{\mathscr{L},\rho}$ and $\|\hat{h}_i\|_{\mathscr{L},r=1}$ with the former norm again defined by the metric g_M on M and with the latter defined by g_{FS} on $\underline{\mathbb{CP}}^2$.

If h_0, $h_{i>0}$ are defined only on U_0 and $U_{i>0}$, respectively, and obey (6.4a), extend the definition of h_0 to the complement in M of the union of the balls about x_i with $\overline{g}$-radius λ/r_1. (Use the formula $\varphi_i^* h_0 = |x|^4 \psi_\lambda^* \varphi_0^* h_i/\lambda^2$ on the ball of radius ε_1 about x_i.) With this extension implicit, measure $[h] = (h_0, h_{i>0})$ by

(6.25)

$$\|[h]\|_{\mathscr{L},\rho} \equiv \left\| \left[\prod_{i=1}^N (\varphi_i^{-1})^* \beta_{8\lambda/r_1}\right] \cdot h_0 \right\|_{\mathscr{L},\rho} + \sup_{i\geq 1} \|(\varphi_i^{-1})^* \beta_{r_1/32} \cdot h_i\|_{\mathscr{L},r=1}.$$

Here is the precise existence statement for (6.6) and (6.7):

Theorem 6.3. *Let M be a compact, oriented 4-manifold with metric g_M. There is a constant $Z \geq 1$ so that when $E \geq Z$, the following is true: The constructions of* §3 *achieve*

(1) *Given $\mu_0, \mu_1 > 0$, small $\varepsilon > 0$, and $r = Z^{-1}E^{-7/2}$, the conclusions of Theorem* 3.15 *hold.*

(2) *With $r_0 = Z^{-1}E^{-7/2}$, the conclusions of Proposition* 3.16 *hold.*

(3) *In addition, there is a continuous, linear map*

$$\widehat{H}: \widetilde{C}^\infty(V_+) \to C^\infty(\mathrm{Sym}^2 T^*M) \times_{i=1}^N C^\infty(\mathrm{Sym}^2 T^*\underline{\mathbb{CP}}^2)$$

so that $\widehat{H}([Q]) \equiv (\hat{h}_0, \hat{h}_{i>0})$ obeys (6.6) *and* (6.7).

(4) *If $\rho = Z^{-1}E^{-1/12}$, then*

$$\|\hat{h}_0\|_{\mathscr{L},\rho} + \sup_{i\geq 1} \|\hat{h}_i\|_{\mathscr{L},1} \leq Z \cdot \left(\Delta_{0,\rho} + \sup_i \Delta_i\right).$$

(5) *If $[h] = \{h_i\}_{i=0}^N$ is defined by* (6.8) *and* (6.9), *then*

$$\|[h]\|_{\mathscr{L},\rho} \leq Z \cdot \left(\Delta_{0,\rho} + \sup_i \Delta_i\right).$$

(6) *If $M = S^4$ with its standard metric, or if $M = \underline{\mathbb{CP}}^2$ with g_{FS}, then* (1)–(5) *above hold with $E \equiv 1$ and with Π_E absent in* (6.7). *In particular,* $[h]$ *obeys* (6.4).

7. Estimates on connect sums

The proof of Theorem 6.3 will employ the following strategy: Treat (6.6) as an equation which specifies $\hat{h}_i$ as a functional of two variables,

$\hat{h}_0$ and Q_i. That is, select a section $\hat{h}_0$ of $\mathrm{Sym}^2 T^*M$ and a section Q_i of $V_+|_{U_i}$ and solve (6.6) for $\hat{h}_i = \hat{h}_i[\hat{h}_0, Q_i]$, a section of $\mathrm{Sym}^2 T^*\underline{\mathbb{CP}}^2$.

Having understood $\hat{h}_i$ in this way, (6.7) becomes a linear equation for $\hat{h}_0$ with an inhomogeneous term which is a functional of Q_0 and Q_i.

But for a slight modification, Proposition 5.7 will be used to find $\hat{h}_i$, while Proposition 5.1 with some perturbation theory will find $\hat{h}_0$. Thus, the proof of Theorem 6.3 begins on $\underline{\mathbb{CP}}^2$ and ends on M.

(a) Solutions on $\underline{\mathbb{CP}}^2$. Once $\hat{h}_0 \in C^\infty(\mathrm{Sym}^2 T^*M)$ is specified, then $\widehat{Q}_i$ in (6.11) becomes well defined. Invoke Proposition 5.7 to find

$$f \equiv H(\widehat{Q}_i) \in C^\infty(\mathrm{Sym}^2 T^*\underline{\mathbb{CP}}^2). \tag{7.1}$$

Now, f is not $\hat{h}_i$; it must be suitably modified. The modification is given first. The geometry behind the modification is described afterwards.

The description of f's modification requires the following auxiliary lemma.

Lemma 7.1. *Use $p\colon \bigotimes_2 T^*\mathbb{R}^4 \to \mathrm{Sym}^2 T^*\mathbb{R}^4$ to denote the symmetrization map. Let g be a metric on $\mathbb{R}^4$. There exists a continuous, linear map*

$$v\colon C^\infty(\mathrm{Sym}^2 T^*\mathbb{R}^4) \to C^\infty(T^*\mathbb{R}^4)$$

which has the following characteristics:

(a) $(h + p\cdot\nabla_g v)|_0 = 0$.

(b) $\nabla_g(h + p\cdot\nabla_g v)|_0 = 0$.

(c) *Let φ be a Gaussian coordinate chart for the metric g which is centered at $0 \in \mathbb{R}^4$. Write $\varphi^*h = h_{\alpha\beta}\,dx^\alpha \otimes dx^\beta$ and write $\varphi^*\nabla_g h = (\nabla_g h_{\alpha\beta\gamma}\,dx^\alpha \otimes dx^\beta \otimes dx^\gamma$. Then*

$$(\varphi^{-1})_* v = h_{\alpha\beta}(0)x^\alpha\,dx^\beta + c_{\alpha\beta\gamma}x^\alpha x^\beta\,dx^\gamma, \tag{7.2}$$

where $\frac{\partial}{\partial x^\nu}c_{\alpha\beta\gamma} = 0$ and $\{c_{\alpha\beta\gamma}\}$ are linear combinations of $\{(\nabla_g h)(0)_{\alpha\beta\gamma}\}$.

Proof of Lemma 7.1. All of the assertions follow by assuming (7.2) and using linear algebra to solve for $\{c_{\alpha\beta\gamma}\}$ in terms of $\{(\nabla_g h)(0)_{\alpha\beta\gamma}\}$. q.e.d.

With Lemma 7.1 understood, here is $\hat{h}_i$: On the radius r_0 ball about y_0, set

$$\varphi_0^*\hat{h}_i \equiv \varphi_0^* f + p\cdot\nabla_{g_E + m_{\mathrm{FS}}}((1-\beta_{r_0/128})\cdot v), \tag{7.3}$$

where $v = v(\varphi_0^* f)$ is given in Lemma 7.1. On the compliment of said ball, set $\hat{h}_i \equiv f$.

Here are $\hat{h}_i$'s properties:

Proposition 7.2. *There is a constant* c *which makes the following true: Fix* r_0 *to invoke Proposition* 3.16, *and let* Q_i *be a section of* V_+ *over the compliment in* $\underline{\mathbb{CP}}^2$ *of the radius* $r_0/32$ *ball about* y_0. *Suppose* Q_i *obeys* (6.21). *Let* $\hat{h}_0 \in C^\infty(\mathrm{Sym}^2 T^*M)$ *be given and define* $\widehat{Q}_i$ *by* (6.11). *Define* f *by* (7.1) *and* $\hat{h}_i$ *by* (7.3). *Then*

(1) $L_{g_{\mathrm{FS}}} \hat{h}_i = \widehat{Q}_i$.

(2) *If* $\rho > \varepsilon_1$, *then*

$$\|\hat{h}_i\|_{\mathscr{L}, r=1} \le c \cdot (\Delta_i + \|\hat{h}_0\|_{\mathscr{L}, \rho}).$$

(3) *Let* $B \subset \underline{\mathbb{CP}}^2$ *denote the radius* $r_0/64$ *ball about* y_0. *Then*

$$\sup_B |\nabla^{\otimes 2}_{g_{\mathrm{FS}}} \hat{h}_i|_{g_{\mathrm{FS}}} \le c \cdot (r_0^{-2}\Delta_i + \ln(r_1/r_0) \cdot \|\hat{h}_0\|_{\mathscr{L}, \rho}).$$

Here, Δ_i *is defined in* (6.23).

There is a promise to keep before proving this proposition—namely a justification for f's modification in (7.3). The ability to make such a modification and still claim the proposition's first assertion follows from

Lemma 7.3. *Let* X *be a* 4*-manifold with metric* g *having* $\mathscr{W}_+ \equiv 0$. *Then* $L_g(p \cdot \nabla_g v) \equiv 0$ *for all* $v \in C^\infty(T^*X)$.

Proof of Lemma 7.3. This is the infinitesimal version of the fact that the equation $\mathscr{W}_+ \equiv 0$ is covariant under the action of X's group of diffeomorphisms. q.e.d.

Thus, Lemma 7.3 says that equation (7.3) is a *permissible* modification of f. But why change f at all? Here is a heuristic justification for this change: Remember that the conformal metic $[g]$ in Theorem 3.14 was constructed by carefully tailoring the coordinate systems in Definition 3.1 to the ambient metrics on M and $\underline{\mathbb{CP}}^2$. (Try recovering Theorem 3.14 without Gaussian coordinates.) One should expect that a change in the ambient metrics should produce a corresponding change in the connect sum coordinate systems. Now $\hat{h}_0$ and $\hat{h}_{i>0}$ are only changes in the metrics g_M on M and g_{FS} on $\underline{\mathbb{CP}}^2$. With this understood, v in (7.3) is the linearization of the predicted change in the connect summing coordinates. (Remark that (a) and (b) of Lemma 7.1 are linearizations of (3.5).)

(b) Estimates for f. The estimates for $\hat{h}_i$ in Proposition 7.2 obviously follow from estimates of $\|f\|_{\mathscr{L}, r=1}$ and of the derivatives of f in the ball B. Indeed, Proposition 7.2 is a corollary to

Proposition 7.4. *There is a constant* c *with the following significance: Make the same assumptions as in Proposition* 7.2. *Then* f *of* (7.1) *obeys*

(1) $\|f\|_{\mathscr{L}, r=1} \le c \cdot (\Delta_i + \|\hat{h}_0\|_{\mathscr{L}, \rho})$.

(2) $\sup_B |\nabla_{g_{FS}} f|_{g_{FS}} \le c \cdot (r_0^{-1}\Delta_i + \|\hat{h}_0\|_{\mathscr{L},p})$.

(3) $\sup_B |\nabla^{\otimes 2}_{g_{FS}} f|_{g_{FS}} \le c \cdot (r_0^{-2}\Delta_i + \ln(r_0/r_1) \cdot \|\hat{h}_0\|_{\mathscr{L},p})$.

The proposition's proof has two parts. The first part estimates the $\|\cdot\|_{\mathscr{L},1}$-norm of f. The second estimates f's derivatives.

Proof of Proposition 7.4. $\|f\|_{\mathscr{L},1}$: The dependence of $\|f\|_{\mathscr{L},1}$ on the Q_i-norm follows directly from Proposition 5.7. At issue is the dependence on $\|\hat{h}_0\|_{\mathscr{L},p})$.

The dependence of $\|f\|_{\mathscr{L},1}$ on $\hat{h}_0$ can be seen after observing that the second term in (6.11) has the form

$$Q' \equiv k_2 \cdot \nabla^{\otimes 2}_{g_{FS}}(\beta_{r_0/8}\hat{p}_0) + k_1 \nabla_{g_{FS}}(\beta_{r_0/8}\hat{p}_0) + \beta_{r_0/8} k_0 \hat{p}_0. \tag{7.4}$$

Here, $\{k_\alpha\}_{\alpha=0,1,2}$ are smooth with compact support where $\frac{r_0}{4} \le |x| \le r_1/2$. Furthermore, if $0 \le n \le \alpha$, then

$$|\nabla^{\otimes n}_{g_{FS}} k_\alpha|_{g_{FS}} \le c \cdot |x|^{\alpha - n}. \tag{7.5}$$

The analysis of (7.4) can be made in one swoop; even so, a different tack will be taken. (The different tack simplifies the derivative estimates in the next subsection.)

To define this new tack, introduce K as the smallest integer which makes

$$2^{-K} r_1 \le r_0/8. \tag{7.6}$$

Thus,

$$0 \le K \le c \cdot \ln(r_0/r_1 + 2). \tag{7.7}$$

Next, consider the partition of unity

$$1 = \beta_{r_1/2} + \sum_{n=2}^{K-1} (1 - \beta_{r_1/2^n})\beta_{r_1/2^{n+1}} + (1 - \beta_{r_1/2^K}). \tag{7.8}$$

Multiply Q' by (7.8) and it decomposes as

$$Q' = \sum_{n=1}^{K} Q'_{(n)}, \tag{7.9}$$

where $Q'_{(n)}$ has support where $r_1/2^{n-1} \le |x| \le r_1/2^{n+1}$.

Since the map H in Proposition 5.7 is linear, f has the corresponding decomposition

$$f = \sum_{n=1}^{K} f_{(n)} + f_{(0)}, \tag{7.10}$$

where $f_{(n)} \equiv H(Q'_{(n)})$ and $f_{(0)}$ is $H((\varphi_0^{-1})(1 - \psi_\lambda^* \beta_{16\lambda/r_0}) \cdot Q_i)$.

Lemma 7.5. *For $n \geq 1$, define $f_{(n)}$ by (7.10) with $Q_{(n)}$ given by (7.4)–(7.9). Then*

$$\|f_{(n)}\|_{\mathscr{L}, r=1} \leq c \cdot 2^{-2n} r_1^2 \cdot \|\hat{h}_0\|_{\mathscr{L}, \rho}.$$

Note that Proposition 7.4's first assertion is a direct consequence of this lemma. (Sum over n.)

Proof of Lemma 7.5. This $Q'_{(n)}$ has the same form as (7.4) but with $k_{\alpha,(n)}$ replacing k_α and with $(1 - \beta_{r_1/2^{n-1}}) \cdot \beta_{r_1/2^{n+1}} \hat{p}_0 \equiv \hat{p}_{0,(n)}$ replacing $\beta_{r_0/8} \hat{p}_0$. Here, $k_{\alpha,(n)}$ obeys (7.5) and has support where $r_1/2^{n+1} \leq |x| \leq r_1/2^{n-1}$.

With the preceding understood, write

$$Q'_{(n)} = q + b_1 \cdot \nabla_{g_{\mathrm{FS}}} b_2, \tag{7.11}$$

where

$$q \equiv k_{1,(n)} \nabla_{g_{\mathrm{FS}}} \hat{p}_{0,(n)} + k_{0,(n)} \hat{p}_{0,(n)}, \tag{7.12}$$

and

$$b_1 \equiv k_{2,(n)} \quad \text{with } b_2 \equiv \nabla_{g_{\mathrm{FS}}} \hat{p}_{0,(n)}. \tag{7.13}$$

Having written Q' as in (7.11), Lemma 7.5 follows from

Lemma 7.6. *With q, b_1, and b_2 defined above,*

(1) $\|q\|_{*, r=1} \leq c \cdot 2^{-2n} \cdot r_1^2 \cdot \|\hat{h}_0\|_{\mathscr{L}, \rho}$.

(2) $\|b_1\|_{\mathscr{L}^0, r=1} \leq c \cdot 2^{-2n} \cdot r_1^2$.

(3) $\|b_2\|_{\mathfrak{h}, r=1} \leq c \cdot \|\hat{h}_0\|_{\mathscr{L}, \rho}$.

Proof of Lemma 7.6. Assertion (2) is a useful exercise for the reader. To prove (1), note first that $|\hat{p}_0| \leq c \cdot \|\hat{h}_0\|_\infty$. Thus,

$$\int_{\mathrm{dist}_{g_{\mathrm{FS}}}(\cdot, x) \leq 1} d\,\mathrm{vol}_{g_{\mathrm{FS}}} \frac{|k_{0,(n)} \hat{p}_{0,(n)}|}{(\mathrm{dist}_{g_{\mathrm{FS}}}(\cdot, x))^2} \tag{7.14}$$

is bounded by $c \cdot 2^{-2n} r_1^2 \cdot \|\hat{h}_0\|_\infty$. Note second that

$$\|k_{1,(n)} \nabla_{g_{\mathrm{FS}}} \hat{p}_{0,(n)}\|_{*, r=1} \leq c \cdot 2^{-2n} r_1^2 \cdot \|\nabla_{g_{\mathrm{FS}}} \hat{p}_{0,(n)}\|_{2*, r=1} \tag{7.15}$$

using Holder's inequality. (The norm $\|\cdot\|_{2*, r}$ is defined in (4.12).) Now, the $\|\cdot\|_{2*, r=1}$-norm and the $\|\cdot\|_{\mathscr{L}^1, r=1}$-norm of $\nabla_{g_{\mathrm{FS}}} \hat{p}_{0,(n)}$ can be estimated by using ψ_λ to change coordinates from an integral over an annulus $\mathscr{O} \subset \mathbb{R}^4 \backslash \{0\}$ to one over $\psi_\lambda(\mathscr{O}) \subset \mathbb{R}^4 \backslash \{0\}$. This latter integral is readily bounded with the $\|\cdot\|_{\mathscr{L},p}$-norm of $\hat{h}_0$. The details provide an exercise for the reader.

(c) Derivative estimates for f. The derivative estimates for f are obtained by summing bounds for the derivatives of the $\{f_{(n)}\}_{n\geq 0}$ in (7.10). The latter are found after the following considerations: Recall that each $f_{(n)}$ is constructed from a $u_{(n)} \in C^\infty(V_+)$ using (5.14). For $\underline{\mathbb{CP}}^2$, this means that

$$\partial_{-g_{\mathrm{FS}}} f_{(n)} = \partial_{+g_{\mathrm{FS}}} u_{(n)} \tag{7.16}$$

since both v_- and φ in (5.14) necessarily vanish. (Indeed, (5.14) on $\underline{\mathbb{CP}}^2$ implies that $\nabla^*_{g_{\mathrm{FS}}}\nabla_{g_{\mathrm{FS}}}\varphi = 0$ so $\nabla_{g_{\mathrm{FS}}}\varphi = 0$.)

Meanwhile, $u_{(n)}$ is determined by the corresponding $Q'_{(n)}$ via

$$\nabla^*_{g_{\mathrm{FS}}}\nabla_{g_{\mathrm{FS}}} u_{(n)} + c\cdot u_{(n)} = Q'_{(n)} \tag{7.17}$$

with $c > 0$ a constant.

For each n, $Q'_{(n)}$ vanishes if $|x| \leq d_{(n)}$. Here, $d_{(0)} = r_0/32$ and if $n \geq 1$, $d_{(n)} = r_1/2^{n+1}$ which is no smaller than $r_0/32$. This means that $u_{(n)}$ and all of its derivatives to order k on the radius $r_0/64$ ball about y_0 are determined a priori by $d_{(n)}^{-k}\cdot\|u_{(n)}\|_\infty$ using standard techniques (see, e.g., [2]). Note that Proposition 5.4 bounds

$$\|u_{(n)}\|_\infty \leq c\cdot\langle Q_{(n)}\rangle_{r=1}. \tag{7.18}$$

Meanwhile, (7.16) determines $f_{(n)}$ and its derivatives to order k on the radius $r_0/128$ ball about y_0 in terms of $d_{(n)}^{-k}\cdot\|f_{(n)}\|_\infty$ and the derivatives of $u_{(n)}$ to order $k+1$. The argument is also classical [2]. (Or, appeal to §4, here, to use only $u_{(n)}$'s derivatives to order k.) Note that $\|f_{(n)}\|_\infty$ is bounded by $\|f_{(n)}\|_{\mathscr{L},r=1}$.

Working out the details is a straightforward exercise that gives (for $n \geq 1$)

$$\begin{aligned}
&\sup_B |\nabla_{g_{\mathrm{FS}}} f_{(n)}|_{g_{\mathrm{FS}}} \leq c\cdot 2^{-n} r_1\cdot\|\hat{h}_0\|_{\mathscr{L},p},\\
&\sup_B |\nabla^{\otimes 2}_{g_{\mathrm{FS}}} f_{(n)}|_{g_{\mathrm{FS}}} \leq c\cdot\|\hat{h}_0\|_{\mathscr{L},p}.
\end{aligned} \tag{7.19}$$

For $n = 0$, the right side above is replaced by $c\cdot r_0^{-1}\Delta_i$ and $c\cdot r_0^{-2}\Delta_i$, respectively.

Sum (7.19) over n to get the final two assertions of Proposition 7.4. (Use (7.7).)

(d) Solutions on M. Equation (6.7) will be treated by invoking Proposition 5.1. To do so, assume E is large enough to invoke said proposition. Consider now that (6.7) is solved by $\hat{h}_0$ obeying the fixed point equation

$$\hat{h}_0 = H(\hat{Q}_0[\hat{h}_0, Q_0, Q_i]). \tag{7.20}$$

Here, $\hat{Q}_0$ from (6.13)–(6.15) should be treated as a functional of Q_0, $Q_{i>0}$, and $\hat{h}_0$. The dependence on $Q_{i>0}$ and $\hat{h}_0$ is implicit through $\{\hat{h}_{i>0}\}$. Here it is assumed that r_0 has been chosen so that Proposition 7.2 can be officially invoked to construct $\{\hat{h}_{i>0}\}$.

With r_0 chosen for Proposition 7.2, let $U \subset \underline{\mathbb{CP}}^2$ denote the complement of the radius $r_0/32$ ball about y_0 and let $U' \subset M$ denote the complement of the balls about the $x_{i>0}$ with radii λ/r_0.

Proposition 7.7. *Let M be a compact, oriented 4-manifold with metric g_M. A constant $Z \geq 1$ exists so that when $E \geq Z$, then the following is true:* §3's *constructions achieve*

(1) *Given $\mu_0, \mu_1 > 0$ and small $\varepsilon > 0$, the conclusions of Theorem* 3.15 *hold using $r = Z^{-1}E^{-7/2}$.*

(2) *Using $r_0 = Z^{-1}E^{-7/2}$, Propositions* 3.16 *and* 7.2 *can be invoked.*

(3) *With E as above, Proposition* 5.1 *can be invoked.*

(4) *In addition, with $\hat{Q}_0$ as defined on* (7.20)*'s right side, a continuous, multilinear*

$$H_0 \colon C^\infty(V_+|_{U'}) \times_{i=1}^{N} C^\infty(V_+|_U) \to C^\infty(\mathrm{Sym}^2 T^*M),$$

exists with the property that $\hat{h}_0 \equiv H_0(Q_0, Q_{i>0})$ obeys (7.20).

(5) *If $\rho = Z^{-1} \cdot E^{-1/12}$, then*

$$\hat{h}_0\|_{\mathscr{L},\rho} \leq Z \cdot \left(\Delta_{0,\rho} + \sup_{i\geq 1} \Delta_i\right),$$

with $\Delta_{i>0}$ and $\Delta_{0,\rho}$ as defined in (6.23) *and* (6.24).

The proof of this proposition occupies the next three subsections.

(e) The contraction mapping theorem. Let $\mathscr{B}$ be a Banach space with norm $\|\cdot\|$ and let $\mathscr{O} \subset \mathscr{B}$ be a closed subset. A Lipschitz map $T \colon \mathscr{O} \to \mathscr{O}$ is said to be a contraction if $\alpha < 1$ exists so that $\|T(x)-T(y)\| \leq \alpha \cdot \|x-y\|$ for all $x, y \in \mathscr{O}$.

The *contraction mapping theorem* asserts that such a contraction, T, has a unique fixed point on $\mathscr{O}$ (see, e.g., [15]).

In the present case, the Banach space in question is the space $\mathscr{L}$ which is obtained by completing $C^\infty(\mathrm{Sym}^2 T^*M)$ using the norm $\|\cdot\|_{\mathscr{L},\rho}$.

(Different choices of ρ give commensurate norms on $C^\infty(\mathrm{Sym}^2 T^*M)$.) The closed domain in question will be $\mathscr{L}$ itself and the map T is $H\cdot\widehat{Q}_0$.

Since $H\cdot\widehat{Q}_0$ is a linear map, the verification that $H\cdot\widehat{Q}_0$ is a contraction requires only an appropriate estimate of $\|H\cdot\widehat{Q}_0\|_{\mathscr{L},\rho}$.

(f) The size of $H\cdot\widehat{Q}_0$. In (6.13)–(6.14), $\widehat{Q}_0$ is presented as the sum of three terms. The first term is

$$Q' \equiv \left[\prod_{i=1}^{N}(\varphi_i^{-1})^*\beta_{16\lambda/r_0}\right]\cdot P_+(g_1)Q_0. \tag{7.21}$$

According to Proposition 5.1, when ρ and r are appropriately constrained in terms of E, one has

$$\|H(Q')\|_{\mathscr{L},\rho} \le Z(\Delta_{0,\rho} + \rho^2/r^2\cdot\Delta_{0,r}). \tag{7.22}$$

The second term, Q'', of $\widehat{Q}_0$ has support only in an annulus about each $x_{i>0}$. In the radius $\varepsilon/2$-ball about x_i, this piece is

$$\varphi_i^* Q_i'' \equiv \beta_{\varepsilon_1/8}\varphi_i^* P_+(g_1)\cdot L_{\varphi_i^*\bar{g}}\hat{p}_i, \tag{7.23}$$

with $\hat{p}_i$ given in (6.15).

The $\|\cdot\|_{\mathscr{L},\rho}$-norm of $H(Q'')$ will be estimated using Proposition 5.1 by

$$\|H(Q'')\|_{\mathscr{L},\rho} \le Z\cdot\left(\|Q''\|_{*,\rho} + \frac{\rho^2}{r^2}\|Q''\|_{*,r}\right). \tag{7.24}$$

The norm $\|Q''\|_{*,\rho}$ is estimated by

Proposition 7.8. *Let M be as in Proposition 7.7 with its metric g_M. A constant Z exists so that the constructions in §3 achieve the following:*

(1) *Given r, μ_0, $\mu_1 > 0$ and small $\varepsilon > 0$, the conclusions of Theorem* 3.15 *hold.*

(2) *Choose r_0 to invoke Propositions* 3.16 *and* 7.2. *Use* (7.23) *to define Q''. Then for all $\rho \ge r$, one has, in addition, that*

$$\|Q''\|_{*,\rho} \le Z\cdot\rho^2\cdot\left\{\ln(r_1/r_0)\cdot\|\hat{h}_0\|_{\mathscr{L},\rho} + r_0^{-2}\sup_{i>0}\Delta_i\right\}.$$

This proposition is proved in §7h.

The final part of $\widehat{Q}_0$ is

$$Q''' \equiv (L_{g_M} - P_+(g_M)L_{\bar{g}})\hat{h}_0. \tag{7.25}$$

Use Proposition 5.1 to estimate $\|H(Q''')\|_{\mathscr{L},\rho}$ with the help of

Proposition 7.9. *With the same assumptions as in Proposition 7.8, one can assume the following in addition to said proposition's conclusions: Define* Q''' *by* (7.25) *and it has a decomposition as* $q + b_1 \cdot \nabla_{g_M} b_2$ *for which*

$$\langle Q''' \rangle_{*,\rho} \leq Z \cdot (\rho^2 + r_0^2) \cdot \|\hat{h}_0\|_{\mathscr{L},\rho}.$$

This proposition is proved in §7i.

(g) Proof of Proposition 7.7. Set E to invoke Proposition 5.1, and set r_0 to invoke Propositions 3.16 and 7.2. Take ρ, r to obey Proposition 5.1's constraints:

$$\rho \leq Z^{-1}, \quad \rho^{12} E > Z^{-1}, \quad E^7 r^2 < Z. \tag{7.26}$$

Now, observe that (7.22), (7.24) and Propositions 5.1, 7.7, and 7.8 imply that

$$\begin{aligned} \|H(\hat{Q}_0)\|_{\mathscr{L},\rho} \leq Z \cdot \Big\{ & \Delta_{0,\rho} + \frac{\rho^2}{r^2}\Delta_{0,r} + \frac{\rho^2}{r_0^2}\sup_{i>0}\Delta_i \\ & + \left[\rho^2 \cdot \ln\left(\frac{r_1}{r_0}\right) + (\rho^2 + r_0^2)(1 + (r_0^2/r^2))\right] \cdot \|\hat{h}_0\|_{\mathscr{L},\rho} \Big\}. \end{aligned} \tag{7.27}$$

Thus, $H(\hat{Q}_0)$ is a contraction mapping if, for instance,

$$\rho^2 \ln(r_1/r_0) + (\rho^2 + r_0^2)(1 + r_0^2/r^2) < \tfrac{1}{2}. \tag{7.28}$$

Equations (7.26) and (7.28) are readily seen to be compatible; choose $E > Z$, $\rho = (2ZE)^{-1/2}$, and $r = \frac{1}{2}(Z^{-1}E^{-1})^{2/7}$. This will satisfy (7.26). Take $r_0 = r$ and (7.28) is then satisfied for large E.

Thus, with these choices, $H(\hat{Q}_0)$ is a contraction mapping and so (7.20) has a unique solution in $\mathscr{L}$. This solution is in fact smooth because $\hat{Q}_0$ is smooth automatically. (This is because $\hat{h}_i$ is smooth on the radius $r_0/32$ ball about y_0 since $\hat{Q}_i$ vanishes there no matter what $\hat{h}_0$ is.)

(h) Proof of Proposition 7.8. First observe that $\varphi_i^* Q_i''$ has support on the annulus $\varepsilon_{1/8} \leq |x| \leq \varepsilon_1/2$. Second, observe that Proposition 7.2's third assertion implies that

$$\begin{aligned} |\varphi_i^* Q_i''|_{g_M} &\leq c \cdot \beta_{\varepsilon_1/16} \cdot (1 - \beta_{\varepsilon_1/2}) \cdot \lambda^2/\varepsilon_1^4 \\ &\quad \cdot (\ln(r_1/r_0) \cdot \|\hat{h}_0\|_{\mathscr{L},\rho} + r_0^{-2}\Delta_i). \end{aligned} \tag{7.29}$$

The next step calculates $\|Q''\|_{*,\rho}$ from (7.29). For this purpose, note that Q_i'' contributes to

$$\int_{\mathrm{dist}_{g_M}(\cdot,x)<\rho} d\,\mathrm{vol}_{g_M} \frac{|Q''|_{g_M}}{(\mathrm{dist}_{g_M}(\cdot,x))^2} \tag{7.30}$$

not at all unless

$$\mathrm{dist}_{g_M}(x, x_i) < \rho + \varepsilon_1. \tag{7.31}$$

And, when (7.31) is obeyed, then (7.29) implies that Q_i'' contributes no more than

$$c\cdot\lambda^2\cdot(\ln(r_1/r_0)\cdot\|\hat{h}_0\|_{\mathscr{L},\rho} + r_0^{-2}\Delta_i)\cdot((\mathrm{dist}_{g_M}(x, x_i))^2 + \varepsilon_1^2)^{-1}. \tag{7.32}$$

Add up these contributions from the $x_{i>0}$ by copying Proposition 3.16's proof from equation (3.70) on. The only changes are cosmetic. The result is Proposition 7.8.

(i) Proof of Proposition 7.9. This proposition is actually a corollary to Proposition 3.16 since one can write Q''' as follows with the introduction of $m \equiv \overline{g} - g_M$:

$$\begin{aligned} Q''' &= k_0(m)\cdot(m\otimes\nabla_{g_M}^{\otimes 2}\hat{h}_0) + k_1(m)\cdot(\nabla_{g_M}^{\otimes 2}m\otimes\hat{h}_0) \\ &\quad + k_2(m)\cdot(\nabla_{g_M}m\otimes\nabla_{g_M}\hat{h}_0) + k_3(m)((\nabla_{g_M}m)^{\otimes 2}\otimes\hat{h}_0). \end{aligned} \tag{7.33}$$

Here, $k_\alpha(m)$ is a tensor which has an expansion

$$k_\alpha(a) = \sum_{n=0}^{\infty} c_{\alpha,n}\cdot a^{\otimes n} \tag{7.34}$$

that is absolutely convergent for $|a|_{g_M} < 1$. The coefficient tensors $\{c_{\alpha,n}\}$ are universal, covariantly constant tensors.

With (7.33) understood, write Q'' as $q + b_1\cdot\nabla_{g_M}b_2$ with q being the last two terms in (7.33). Meanwhile, set

$$b_1 \equiv (k_0(m)\otimes m,\ k_1(m)\otimes\hat{h}_0),$$

and

$$b_2 = \begin{pmatrix} \nabla_{g_M}^{\otimes 2}\hat{h}_0 \\ \nabla_{g_M}^{\otimes 2}m \end{pmatrix}. \tag{7.35}$$

(j) Proof of Theorem 6.3. Only the theorem's final assertion needs proving, as the other assertions follow from Propositions 7.2 and 7.7.

The norms for $\{h_i\}_{i\geq 0}$ are bounded with the aid of the bounds for the norms of $\{\hat{h}_i\}_{i\geq 0}$. Start with $i \geq 1$. The $\|\cdot\|_{\mathscr{L},r=1}$-norm of $\varphi_0^{-1}\beta_{r_1/32}\cdot h_i$ is less than the sum of the norms of $\hat{h}_i$ and $\beta_{r_1/32}\cdot(1-\beta_{r_1/4})\cdot|x|^4\psi_\lambda^*\varphi_i^*\hat{h}_0/\lambda^2$. The $\|\cdot\|_{\mathscr{L},r=1}$-norm of the latter is uniformly bounded by the $\|\cdot\|_{\mathscr{L},\rho}$-norm of $\hat{h}_0$. One can prove this by mimicking Lemma 7.6's proof.

As for h_0, the $\|\cdot\|_{\mathscr{L},\rho}$-norm of $[\prod_{i=1}^N(\varphi_i^{-1})^*\beta_{8\lambda/r_1}]\cdot h_0$ can be bounded by considering the separate contributions of $\hat{h}_0$ and $\{\beta_{8\lambda/r_1}\cdot(1-\beta_{\varepsilon_1/4})\cdot |x|^4\psi_\lambda^*\varphi_0^*\hat{h}_i/\lambda^2\}_{i\geq 1}$ to the integrals which enter the $\|\cdot\|_{\mathscr{L},\rho}$-norm's definition. By considering separately the contribution of

$$\beta_{8\lambda/r_1}\cdot(1-\beta_{\varepsilon_1/4})\cdot\frac{|x|^4}{\lambda^2}\psi_\lambda^*\varphi_0^*\hat{h}_i \tag{7.36}$$

to said integrals, one arrives at the same sorts of considerations which appear in Proposition 7.8's proof. That is, one ends up summing over i expressions of the form

$$c\cdot\lambda^2\cdot(\ln(r_1/r_0)\cdot\|\hat{h}_0\|_{\mathscr{L},\rho}+r_0^{-2}\Delta_i)\cdot((\mathrm{dist}_{g_M}(x\,,\,x_i))^2+\lambda^2/r_0^2)^{-1}. \tag{7.37}$$

(One considers here those x_i for which $\mathrm{dist}_{g_M}(x\,,\,x_i)\leq\rho+\varepsilon_1$.) The evaluation of a sum over i of terms as in (7.37) is accomplished by mimicking Proposition 3.16's proof.

The details here are straightforward and omitted except to note that the derivation of (7.37) and (7.36) is facilitated by writing (7.36) as the sum of

$$\beta_{128\lambda/r_0}\cdot(1-\beta_{\varepsilon_1/4})\cdot\frac{|x|^4}{\lambda^2}\psi_\lambda^*\varphi_0^*\hat{h}_i \tag{7.38}$$

and

$$(1-\beta_{128\lambda/r_0})\cdot\beta_{8\lambda/r_1}\cdot\frac{|x|^4}{\lambda^2}\psi_\lambda^*\varphi_0^*\hat{h}_i. \tag{7.39}$$

Then, (7.38) is analyzed using Proposition 7.2 to bound the nth $(n=0\,,1\,,2)$ derivative of (7.38) by

$$c\cdot\beta_{64\lambda/r_0}\cdot(1-\beta_{\varepsilon_1/2})\cdot\frac{\lambda^2}{|x|^{2+n}}\cdot\left(\ln\left(\frac{r_1}{r_0}\right)\cdot\|\hat{h}_0\|_{\mathscr{L},\rho}+r_0^{-2}\Delta_i\right). \tag{7.40}$$

Meanwhile, integrals of (7.39) and its first derivative should be evaluated by using the strategy in Lemma 7.6's proof.

Assertion (6) of Theorem 6.3 follows by replacing references to Proposition 5.1 in this section with references to Proposition 5.7.

8. Prescribed $\mathscr{W}_+$

Invoke Theorem 3.15 to describe $M_N\equiv M\#_N\underline{\mathbb{CP}}^2$ with its conformal metric $[g]$.

The purpose of this section is to consider aspects of the prescribed curvature equation on M_N:

$$(8.1)\qquad \begin{aligned} &\text{(a)}\quad P_+(\hat{g}_i)\cdot \mathscr{W}_+(\hat{g}_i + h_i) - \mathscr{W}_+(\hat{g}_i) = q_i \quad \text{on } U_i, \\ &\text{(b)}\quad h_i = \varphi_{ij} h_j \quad \text{on } U_i \cap U_j. \end{aligned}$$

(Refer back to §31 for notation.) It is assumed that $[q] \equiv \{q_i \in C^\infty(V_+|_{U_i})\}$ in (8.1) has been a priori specified, and that

$$(8.2)\qquad \begin{aligned} &\text{(a)}\quad P_+(\hat{g}_i)q_i = q_i \quad \text{on } U_i, \\ &\text{(b)}\quad q_i = \varphi_{ij}\cdot q_j \quad \text{on } U_i \cap U_j. \end{aligned}$$

(a) Quasi-solutions to (8.1). In general, small eigenvalues obstruct (8.1) the way small eigenvalues obstructed (6.4). Remember: In §6, small eigenvalue problems were avoided at the expense of the Π_E factor in (6.7). The small eigenvalue issues in (8.1) will be avoided with a similar strategy.

Here is how to avoid small eigenvalues: First, relate (8.1) to (6.4) by rewriting (8.1) as

$$(8.3)\qquad L_{\hat{g}_i} h_i = Q_i,$$

where

$$(8.4)\qquad Q_i \equiv q_i - k_2(h_i)\cdot(h_i \otimes \nabla^{\otimes 2}_{\hat{g}_i} h_i) - k_1(h_i)\cdot(\nabla_{\hat{g}_i} h_i)^{\otimes 2}.$$

Here, $\{k_\alpha(h)\}_{\alpha=1,2}$ are tensors whose values at x are analytic functions of $h(x)$ as long as $|h(x)|_{\hat{g}_i} < 1$. (Expand $P_+(g)\mathscr{W}_+(g+h) - \mathscr{W}_+(g)$ in powers of h.)

Fix E and consider (6.6) and (6.7) with $\widehat{Q}_i$ defined in (6.11) and (6.14) using Q_i in (8.4). Here, one must consider $\{h_i\}$ in (8.4) as being given in terms of $\{\hat{h}_i\}$ by (6.8) and (6.9).

If the resulting nonlinear equation for $\{\hat{h}_i\}_{i=0}^N$ has a solution, use (6.8) and (6.9) to construct $\{h_i\}_{i=0}^N$ from said solution.

Call the resulting $[h]$ an *E-quasi-solution* to (8.1).

To be somewhat more precise, introduce $\widetilde{C}^\infty(\mathrm{Sym}^2\, T^* M_N)$ as the Frêchet space of $[h]$ in $\times_{i=0}^N C^\infty(\mathrm{Sym}^2\, T^* U_i)$ which obey (8.1b). (The choice of a metric in $[g]$'s conformal class identifies this space with $C^\infty(\mathrm{Sym}^2\, T^* M_N)$.)

Next, use (6.8)–(6.9) to define a linear map,

$$(8.5)\qquad t\colon C^\infty(\mathrm{Sym}^2\, T^* M) \times_{i=1}^N C^\infty(\mathrm{Sym}^2\, T^* \underline{\mathbb{CP}}^2) \to \widetilde{C}^\infty(\mathrm{Sym}^2\, T^* M_N).$$

With E specified, let $\widehat{H}_E$ denote the map in Theorem 6.3.

Definition 8.1. An E-quasi-solution $[h] \in \tilde{C}^\infty(\mathrm{Sym}^2 T^* M_N)$ to (8.1) solves

$$[h] = t \cdot \widehat{H}_E([Q]), \tag{8.6}$$

where $[Q] \equiv \{Q_i[h]\}_{i=0}^N$ is given by (8.2).

Note. An E-quasi-solution to (8.1) may be an honest solution under certain precise circumstances:

Proposition 8.2. *Fix E to invoke Theorem 6.3. Let $[Q]$ be given by (8.2) and let $[h]$ obey (8.6). (So, $[h]$ is an E-quasi-solution to (8.1).) Then $[h]$ is a bona-fide solution to (8.1) if*

$$(1 - \Pi_E) \cdot (L_{g_M} \hat{h}_0 - \widehat{Q}_0) = 0. \tag{8.7}$$

Here, $\{\hat{h}_i\}_{i=0}^N \equiv \widehat{H}_E([Q([h])])$ and $\widehat{Q}_0$ is given by (6.14) using Q_0 and $\{\hat{h}_i\}_{i=0}^N$ as above.

Remark that (8.7) is a finite-dimensional system of equations, and so a quasi-solution fails in only finitely many ways to solve (8.1).

Proof of Proposition 8.2. This follow from Proposition 7.1. q.e.d.

Theorem 8.3, below, is an existence theorem for quasi-solutions.

To state this theorem, reintroduce $\tilde{C}^\infty(V_+)$ as defined prior to Theorem 6.3. When $\rho > 0$ has been specified, defined a norm on $\tilde{C}^\infty(V_+)$ by assigning to $[q] = \{q_i\}_{i=0}^N$ the number

$$\|[q]\|_{*,\rho} \equiv \left\| \prod_{i=1}^N (\varphi_i^{-1}) \beta_{8\lambda/r_0} \cdot q_0 \right\|_{*,\rho} + \sup_i \|(\varphi_0^{-1})^* \beta_{r_0/32} \cdot q_i\|_{*,r=1}. \tag{8.8}$$

If $d > 0$ has been specified, set

$$\mathscr{B}_{\rho,d} \equiv \{[q] \in \tilde{C}^\infty(V_+) : \|[q]\|_{*,\rho} < d\}. \tag{8.9}$$

This space is an open subset of the Frêchet space $\tilde{C}^\infty(V_+)$ so a Frêchet manifold. k

Theorem 8.3. *Let M be a compact, oriented 4-manifold with metric g_M. There exists $Z \geq 1$ so that when $E \geq Z$, the following are true:*

(1) *Given $\mu_0, \mu_1 > 0$, small $\varepsilon > 0$, and $r = Z^{-1}E^{-7/2}$, the conclusions of Theorem 3.15 hold.*

(2) *With $r_0 = Z^{-1}E^{-7/2}$, the conclusions of Proposition 3.16 hold.*

(3) *With E as given, the conclusions of Theorem 6.3 hold.*

(4) *In addition, with $d = Z^{-1}$ and $\rho = Z^{-1}E^{-1/12}$, a smooth map $T : \mathscr{B}_{\rho,d} \to \tilde{C}^\infty(\mathrm{Sym}_2 T^* M_N)$ exists with the property that $T[q] \equiv [h]$ is an E-quasi-solution to (8.1) in Definition 8.1's sense.*

(5) *With* ρ *as above, define* $\|[h]\|_{\mathcal{L},\rho}$ *as in* (6.25) *and then*

$$\|[h]\|_{\mathcal{L},\rho} \leq Z \cdot \|[q]\|_{*,\rho}.$$

(6) *If* $M = S^4$ *with its standard metric, or if* $M = \underline{\mathbb{CP}}^2$ *with* g_{FS}, *then* (1)–(5) *hold except that* $[h]$ *is an honest solution to* (8.1).

The remainder of this section is occupied with the theorem's proof.

(b) Contraction maps again. Here is how to prove Theorem 8.1: Complete $\widetilde{C}^\infty(\mathrm{Sym}^2 T^*M_N)$ using the norm $\|[\cdot]\|_{\mathcal{L},\rho}$ of (6.25). Use $\mathcal{S}$ to denote the resulting Banach space. Find a solution in $\mathcal{S}$ using the contraction mapping theorem, and then prove that said solution is C^∞.

The contraction mapping theorem (see §7e) can be invoked on a closed domain in $\mathcal{S}$ where $t \circ \widehat{H}_E([Q(\cdot)])$ is a contraction.

Lemma 8.4. *Let* M *be a compact, oriented* 4*-manifold with metric* g_M. *Theorem* 6.3 *has one further conclusion: Suppose that* $[Q]$ *is given by* (8.2). *Then the map* $t \circ \widehat{H}_E([Q(\cdot)])$ *is smooth, and a contraction mapping on the closed domain* $\mathcal{O} = \{[h] \in \mathcal{S}: \|[h]\|_{\mathcal{L},\rho} \leq Z^{-1}\}$.

Proof of Lemma 8.4. Introduce $Q_i' \equiv Q_i - q_i$. Then write out $Q_i'(h+m) - Q_i'(h)$ as

$$\begin{aligned} &s_0(m,h)\cdot(m \otimes \nabla_{\hat{g}_i}^{\otimes 2} h) + s_1(m,h)(h \otimes \nabla_{\hat{g}_i}^{\otimes 2} m) + s_3(m,h)(\nabla_{\hat{g}_i} m \otimes \nabla_{\hat{g}_i} h) \\ &\qquad + s_4(m,h)(m \otimes (\nabla_{\hat{g}_i} h)^{\otimes 2}) + s_5(m,h)\cdot(h \otimes (\nabla_{\hat{g}_i} m)^{\otimes 2}). \end{aligned} \tag{8.10}$$

Here, $\{s_\alpha(a,b)\}$ are tensors which, at x, are analytic functions of $a(x)$ and $b(x)$ when $|a|_{\hat{g}_i} + |b|_{\hat{g}_i} < 1$. Indeed, each s_α has an absolutely convergent (for $|a|_{\hat{g}_i} + |b|_{\hat{g}_i} < 1$) expansion of the form

$$s_\alpha(a,b) = \sum_{l,l'=0}^{\infty} c_{\alpha(l,l')}\cdot(h^{\otimes l}) \otimes (m^{\otimes l'}), \tag{8.11}$$

with $c_{\alpha(l,l')}$ universal, covariantly constant tensors.

With (8.10) understood, write $Q_i'(h+m) - Q_i'(h)$ for $i > 0$ as $q_i + b_{1,i}\cdot\nabla_{\hat{g}_i} b_{2,i}$, where q_i denotes the last three terms in (8.10), while

$$b_{1,i} \equiv (s_0(m,h)\otimes m,\ s_1(m,h)\otimes h) \tag{8.12}$$

and

$$b_{2,i} \equiv \begin{pmatrix} \nabla_{\hat{g}_i} h \\ \nabla_{\hat{g}_i} m \end{pmatrix}. \tag{8.13}$$

For $i = 0$, one must write $Q_0'(h+m) - Q_0'(h)$ as $q_0 + b_{1,0}\cdot\nabla_{g_M} b_{2,0}$, where $b_{1,0}$ and $b_{2,0}$ are given by (8.12) and (8.13), but where q_0 is given by the last three terms in (8.10) plus an additional term:

$$s_0 \cdot (m \otimes (\nabla_{\hat{g}_0} - \nabla_{g_M})\nabla_{\hat{g}_0} h) + s_i(h \otimes (\nabla_{\hat{g}_0} - \nabla_{g_M})\nabla_{\hat{g}_0} m).$$

With q_i, $b_{1,i}$, and $b_{2,i}$ so defined, introduce the numbers $\Delta_{0,\rho}$ and Δ_i using (6.23) and (6.24). A direct calculation (use Lemma 3.13 and (3) of Theorem 3.15 to estimate $\nabla_{\hat{g}_0} - \nabla_{g_M}$) shows that

$$\Delta_{0,\rho} + \sup_{i\geq 1} \Delta_i \leq c \cdot \|[h]\|_{\mathscr{L},\rho} \cdot \|[m]\|_{\mathscr{L},\rho}. \tag{8.14}$$

Equation (8.14) and assertion (5) of Theorem 6.3 imply that

$$\|t \cdot \widehat{H}_E(Q([h_1])] - t \cdot \widehat{H}_E(Q([h_2]))\|_{\mathscr{L},\rho} \leq Z \cdot \nu \cdot \|[h_1 - h_2]\|_{\mathscr{L},\rho} \tag{8.15}$$

when $\|h_1\|_{\mathscr{L},\rho} + \|h_2\|_{\mathscr{L},\rho} \leq \nu < \frac{1}{2}$.

Lemma 8.4 follows from this last equation since Theorem 6.3's fifth assertion also gives

$$\|t \cdot \widehat{H}_E([q])\|_{\mathscr{L},\rho} \leq Z \cdot \|[q]\|_{*,\rho}. \tag{8.16}$$

(c) Proof of Theorem 8.3. When the conditions of Theorem 6.3 hold, Lemma 8.4 asserts that $t \circ \widehat{H}_E([Q])$, with $[Q]$ in (8.2), is a contraction mapping on the indicated closed domain $\mathscr{O} \subset \mathscr{S}$.

The contraction mapping theorem and (8.16) provide a constant Z (depending on M and g_M only) so that when (8.16)'s right side is Z^{-1} or less, the map $t \circ \widehat{H}_E([Q])$ becomes a contraction mapping on the domain $\mathscr{O}$ in Lemma 8.4.

Thus, when (8.16)'s right side is less than Z^{-1}, (8.5) has a unique solution $[h]$ in $\mathscr{O}$.

The fact that $[h]$ is smooth can be proved by applying some classical regularity theory to (5.23) and (5.14). These techniques can be found in Chapter 6 of Morrey's treatise [17] and the reader will be spared the details.

By the way, the same techniques (and almost the same argument) proves that h varies in $\widetilde{C}^\infty(\mathrm{Sym}^2 T^* M_M)$ analytically when q varies in $\widetilde{C}^\infty(V_+)$.

This establishes the first five assertions of the theorem. The last assertion follows by the identical arguments, but augmented with assertion (6) of Theorem 6.3.

9. The small eigenvalues

Theorem 3.16 constructs a conformal metric on $M_N \equiv M\#_N\underline{\mathbb{CP}}^2$ whose $\mathscr{W}_1$ is supported on the open set M'. Since $[g'_M] = [g]$ on M', $\mathscr{W}_+[g]$ can be measured using

$$(9.1) \qquad \|\mathscr{W}_+[g]\|_{*,\rho} \equiv \|\mathscr{W}_+[g'_M]\|_{*,\rho;M'},$$

where the latter norm is defined in (3.36) using g_M for g_1.

With respect to the notation in §3l, $\mathscr{W}_+[g'_M] = \mathscr{W}_+[\hat{g}_0]$ on the open set U_0, while $\mathscr{W}_+[\hat{g}_{i>0}] \equiv 0$.

Understand the preceding, and the following becomes a simple corollary of Theorems 3.15 and 8.3.

Proposition 9.1. *Let M be a compact, oriented, 4-manifold with metric g_M. There exists $Z \geq 1$ with the following significance: Fix E to invoke Theorem 8.3, and fix $\mu > 0$. If N is large and $[q]$ is given by*

$$(9.2) \qquad [q] = \{q_i \equiv -\mathscr{W}_+[\hat{g}_i]\}_{i=0}^N,$$

then (8.1) has an E-quasi-solution $[h]$ which obeys

$$(9.3) \qquad \|[h]\|_{\mathscr{L},\rho} \leq Z \cdot \|\mathscr{W}_+[g]\|_{*,\rho} < \mu.$$

Here, $\rho = Z^{-1}E^{-1/12}$.

Note. Were $[h]$ above a bona-fide solution to (8.1), then the data

$$(9.4) \qquad [g+h] = \{\hat{g}_i + h_i\}_{i=0}^N$$

would define a conformal metric on M_N $(N \geq N_\mu)$ which has $\mathscr{W}_+ \equiv 0$. Proposition 8.2 gives precise conditions when the quasi-solution is a bona-fide solution.

Proposition 8.2 suggests the following question: Can §3's constructions insure (8.7) when $[q]$ is given by (9.2)? Here is the answer:

Theorem 9.2. *Let M be a compact, oriented 4-manifold with metric g_M. There is a constant $Z_0 \geq 1$ with the following significance: Fix $E \geq Z_0$. If*

(a) *the number of Proposition 3.11's iterations is large,*

(b) *the parameters (Z_1, ε) for each iteration are chosen with Z_1 large and ε small, and*

(c) *the data for §3k's Cokernel Step is chosen appropriately,*

then Proposition 9.1 can be invoked using $[g]$ from Theorem 3.15 and the resulting $[h]$ is such that $\mathscr{W}[g+h] \equiv 0$.

Remark that Theorem 9.2 is superfluous when $M = \underline{\mathbb{CP}}^2$ and $g_M = g_{FS}$. In this case, assertion (6) of Theorem 8.3 and Theorem 3.15 yield Floer's theorem [7]:

Theorem 9.3. *For any* $N \geq 0$, $\#_N \overline{\mathbb{CP}}^2$ *has metrics with* $\mathscr{W}_+ \equiv 0$.

The remainder of this section is occupied with Theorem 9.2's proof.

(a) The strategy. The proof's first stage breaks (8.7) into a sum of pieces which are small but one. This one large piece comes from $\mathscr{W}_+[g]$.

To state the precise result with minimal notation, agree henceforth to identify $\mathscr{W}_+[g]$ for the conformal metric $[g]$ from Theorem 3.15 with $\mathscr{W}_+[g'_M]$'s restriction to M'. Likewise, after l iterations of Proposition 3.11, one can identify $\mathscr{W}_+[g^{(l)}]$ with $\mathscr{W}_+[g^{(l)}]$'s restriction to $M^{(l)}$. With this understood, one can measure $\mathscr{W}_+[g^{(l)}]$ by declaring

$$\|\mathscr{W}_+[g^{(l)}]\|_{*,\rho} \equiv \|\mathscr{W}_+[g_M^{(l)}]\|_{*,\rho;M^{(l)}}, \tag{9.5}$$

where (3.36) defines the latter norm using g_M for g_1.

The first stage of Theorem 9.2's proof results in

Proposition 9.4. *Let* M *be a compact, oriented* 4*-manifold with metric* g_M. *There is a constant* Z *with the following significance:*

(a) *Fix* E *to invoke Proposition* 9.1 *using* $\mu < E^{-8}$. *Then*

$$\|\pi_E \cdot (L_{g_M}\hat{h}_0 - \widehat{Q}_0) + \pi_E \cdot P_+(g_M) \cdot \mathscr{W}_+[g]\|_{*,\rho} \leq Z \cdot E^{-1/7} \cdot \|\mathscr{W}_+[g]\|_{*,\rho}.$$

Here, $\rho = Z^{-1} \cdot E^{-1/12}$.

This proposition will be proved shortly; accept it now so that the second stage of Theorem 9.2's proof can be described.

For the purposes of such a description, introduce

$$W \equiv P_+[g_M] \cdot (\mathscr{W}_+[g] - \mathscr{W}_+[g^{(n-1)}]). \tag{9.6}$$

Given $g^{(n-1)}$, remark that W is completely determined by the choice of parameters for the Cokernel Step. These parameters are: (1) The set $\Omega_n \subset M^{(n-1)}$, (2) a $g_M^{(n-1)}$-orthonormal frame f_* in $T^*M^{(n-1)}|_x$ at each $x \in \Omega_n$, and (3) the choice of λ, ε_1, ε_2 at each such x. With (3.61) understood, this last choice is a choice for ε, Z, and μ_0 in (3.61) plus choosing $\{\mu_x \in (0, 4\mu_0]\}_{x \in \Omega_n}$.

The second stage of Theorem 9.2's proof shows that $\pi_E \cdot W$ can be the dominant contribution to $\pi_E \cdot P_+(g_M) \cdot \mathscr{W}_+[g]$. This second stage also analyzes the dependence of $\pi_E \cdot W$ on the parameters of the Cokernel Step. To summarize the result of said second stage, introduce

$$\mathfrak{G}_d \equiv \left\{ \nu \in \operatorname{Range}(\pi_E) : \sup_M |\nu| \leq d \right\}. \tag{9.7}$$

Here is the promised summary:

Proposition 9.5. *Let M be a compact, oriented 4-manifold with metric g_M. There is a constant $Z_0 \geq 1$ with the following significance: Make E large and then make μ_0, μ_1 small. If*

(a) *there is a large number, say $(n-1)$, of iteration steps in Proposition* 3.11,

(b) *the parameters (Z_1, ε) in each step are chosen with Z_1 large and ε small, and*

(c) *in* (3.61), *Z is large and $\varepsilon = \varepsilon(n)$ is small,*

then, the points $\Omega_n \subset M^{(n-1)}$, the frames $\{f_* \in FM^{(n-1)}|_x\}_{x\in\Omega_n}$ can be chosen, and a smooth map

$$\hat{\mu}: \mathfrak{B}_{d=Z_0^{-1}} \to \bigtimes_{x\in\Omega_n} (0, 4\mu_0) \tag{9.8}$$

can be defined so that when

$$\mu_x \equiv \hat{\mu}(\nu)|_x \tag{9.9}$$

in (3.61), *then*

$$\begin{aligned} &(1)\quad \|\mathscr{W}_+[g^{(n-1)}]\|_{*,\rho} \leq \mu_1 \cdot \rho^2, \\ &(2)\quad \|W\|_{*,\rho} \leq Z_0 \cdot \mu_0 \cdot (\|\nu\|_{*,\rho} + \varepsilon(n)), \\ &(3)\quad \|\pi_E \cdot W - \mu_0 \cdot \nu\|_{*,\rho} \leq E^{-1} \cdot Z_0 \cdot \mu_0 \cdot \|\nu\|_{*,\rho}. \end{aligned} \tag{9.10}$$

Here, $\rho \equiv Z_0^{-1} E^{-1/12}$.

Accept this proposition for now. Here is the final stage of Theorem 9.2's proof.

(b) Proof of Theorem 9.2. The proof follows quite readily from Propositions 9.4 and 9.5 if one is given

Lemma 9.6. *Let M be a compact, oriented 4-manifold with metric g_M. There is a constant Z_1 such that if $E \geq 0$ and $\rho > 0$ are given, and if $\nu \in \mathrm{Range}(\pi_E)$, then*

$$\|\nu\|_\infty \leq Z_1 \cdot E^4/\rho^2 \cdot \|\nu\|_{*,\rho}.$$

Given this lemma, to prove Theorem 9.2 pick E large enough and μ_0, μ_1 small enough to invoke Propositions 9.1, 9.4, and 9.5.

Now, introduce $\rho \equiv Z_0^{-1} E^{-1/12}$ as in said propositions and set

$$\mathscr{O}_E \equiv \{\nu \in \mathrm{Range}(\pi_E) : \|\nu\|_{*,\rho} \leq \tfrac{1}{2}\rho^2 Z_0^{-1} \cdot Z_1^{-1} \cdot E^{-4}\}, \tag{9.11}$$

with $Z_{0,1}$ given by Proposition 9.5 and Lemma 9.6, respectively. Note that Lemma 9.6 insures

$$\mathscr{O}_E \subset \mathfrak{B}_{Z_0^{-1}}, \tag{9.12}$$

with $\mathfrak{B}_{Z_0^{-1}}$ as in (9.7). Thus, $\hat{\mu}$ of (9.8) is defined on $\mathcal{O}_E$.

With $g^{(n-1)}$ determined, and with the Cokernel Step's parameters determined via Proposition 9.5, with the choice of $\nu \in \mathcal{O}_E$, then the conformal metric $[g]$ from Theorem 3.15 and $[h]$ from Proposition 9.1 become continuous functions on $\mathcal{O}_E$. In particular, the assignment to $\nu \in \mathcal{O}_E$ of

$$F(\nu) \equiv \mu_0^{-1} \cdot \pi_E(L_{g_M} \hat{h}_0 - \widehat{Q}_0) \tag{9.13}$$

gives a continuous map from $\mathcal{O}_E$ to $\text{Range}(\pi_E)$.

Propositions 9.4 and 9.5 can be interpreted to say that

$$F(\nu) = \nu + R(\nu), \tag{9.14}$$

where

$$\|R(\nu)\|_{*,\rho} \le Z_2 \cdot E^{-1/7} \left(\|\nu\|_{*,\rho} + \frac{\mu_1}{\mu_0} \rho^2 \right) + \frac{\mu_1}{\mu_0} \cdot \rho^2. \tag{9.15}$$

Here, Z_2 is determined a priori from the metric g_M on M. (Take $\varepsilon(n) \ll \mu_1/\mu_0 \cdot \rho^2$).

Now, take $E \ge (8Z_2)^7$ and require that

$$\mu_1 \le \tfrac{1}{16} \cdot \mu_0 \cdot Z_0^{-1} \cdot Z_1^{-1} \cdot E^{-4}, \tag{9.16}$$

with $Z_{0,1}$ as in (9.11). These choices insure that R maps $\mathcal{O}_E$ to itself. Furthermore, if ν is on the boundary of $\mathcal{O}_E$, then

$$\frac{\|R(\nu)\|_{*,\rho}}{\|\nu\|_{*,\rho}} \le \frac{1}{2}. \tag{9.17}$$

Equation (9.17) with the Brouer Fixed Point Theorem (cf. [8]) insures that F in (9.14) has a zero on $\mathcal{O}_E$, i.e., $F^{-1}(0) \ne \varnothing$. This proves Theorem 9.2.

The current subsection ends with the

Proof of Lemma 9.6. Let $\{\nu_\alpha\}$ be an L^2-orthonormal basis for $\text{Range}(\pi_E)$. Write $\nu = \sum_\alpha c_\alpha \cdot \nu_\alpha$. Then

$$\|\nu\|_\infty \le \sum_\alpha |c_\alpha| \, \|\nu_\alpha\|_\infty \le E^2 \|\nu\|_{L^2}. \tag{9.18}$$

This uses Lemmas 4.15 and 4.16. Then,

$$\|\nu\|_{L^2}^2 \le \|\nu\|_\infty \cdot \|\nu\|_{L^1} \le Z \cdot \rho^{-2} \cdot \|\nu\|_\infty \cdot \|\nu\|_{*,\rho}. \tag{9.19}$$

These last two equations give the lemma.

(c) Proof of Proposition 9.4. Note first that Proposition 4.14 estimates the $\|\cdot\|_{*,\rho}$-norm of $\pi_E \cdot Q$ in terms of the $\|\cdot\|_{*,r}$-norm of Q.

With the preceding understood, start the proof by decomposing $\widehat{Q}_0 = Q' + Q'' + Q'''$, where Q', Q'', and Q''' are given in (7.21), (7.23), and (7.24), respectively. The estimate of $Q^\bullet$ requires bounds for $\{\Delta_{0,\rho}, \Delta_i\}$. The latter are obtained from (8.2), (8.13) (with $[m] = [-h]$), and (9.2):

$$\begin{aligned} &\Delta_i \le Z \cdot \|[h]\|^2_{\mathcal{L},\rho}, \\ &\Delta_{0,\rho} \le Z \cdot (\|\mathcal{W}_+[g]\|_{*,\rho} + \|[h]\|^2_{\mathcal{L},\rho}). \end{aligned} \tag{9.20}$$

Given (9.20), Propositions 7.8 and 7.9 bound $\|Q''\|_{*,r}$ and $\|Q'''\|_{*,r}$ by

$$Z \cdot r^2 \cdot \ln(r_1/r) \cdot (\|\mathcal{W}_+\|_{*,\rho} + (1 + 1/r^2) \cdot \|[h]\|^2_{*,r}). \tag{9.21}$$

Meanwhile,

$$\|Q' - P_+(g_M) \cdot \mathcal{W}_+[g]\|_{*,r} \le Z \cdot \|[h]\|^2_{*,r}, \tag{9.22}$$

which can be derived directly from (9.2).

Plug in Theorem 8.3's values for r and ρ ($r^2 = Z^{-1}E^{-7}$ and $\rho^2 = Z^{-1}E^{-1/12}$) to find

$$\|\pi_E \cdot \widehat{Q}_0 - \pi_E P_+(g_M)\mathcal{W}_+[g]\|_{*,\rho} \le E^{-1/6} \cdot (1 + \mu \cdot E^7) \cdot \|\mathcal{W}_+\|_{*,\rho}. \tag{9.23}$$

As for $\pi_E \cdot L_{g_M} \cdot \hat{h}_0$, one must remember first that $\hat{h}_0$ is computed from u ($\equiv \hat{u}_0$) using (5.14), where u is given by (5.22) using $\widehat{Q}_0$ for Q. Since $\pi_E \cdot u = 0$, one has $\pi_E \cdot L_{g_M} \cdot \hat{h}_0 = \pi_E \cdot q'$, where q' is given by (5.33). Lemma 5.5 estimates $\|q'\|_{*,r}$ in terms of $\|u\|_{\mathcal{L},\rho}$, while Proposition 5.4 estimates $\|u\|_{\mathcal{L},\rho}$ in terms of $\widehat{Q}_0$. Work through the details to find

$$\|\pi_E \cdot L_{g_M} \hat{h}_0\|_{*,\rho} \le E^{-1/6} \cdot (1 + \mu \cdot E^7) \cdot \|\mathcal{W}_+\|_{*,\rho}, \tag{9.24}$$

too.

Proposition 9.4 is established by (9.23) and (9.24).

(d) Choosing points Ω_n. This subsection begins the task of specifying the Cokernel Step's parameters. For this purpose, one should assume that some large number $(n-1)$ of Proposition 3.11's iteration steps have been completed. The values of (Z_1, ε) in each of these steps should be taken so that Z_1 is large and ε is small.

The set Ω_n will be determined a priori from another finite set, $\Omega' \subset M^{(n-1)}$. Choose Ω' by picking a small number $\varepsilon(n) > 0$ and then invoking Lemma 3.8 using $\varepsilon = 200 \cdot \varepsilon(n)$ (and the metric $g_M^{(n-1)}$) to obtain a set $\Omega \subset M$. Finally, set

$$\Omega' = \{x \in \Omega \colon B_{200\varepsilon}(x) \subset M^{(n-1)}\}. \tag{9.25}$$

Each $y \in \Omega'$ will spawn 30 points, $\{x_1^{(y)}, \cdots, x_{30}^{(y)}\} \subset \Omega_n$. Each $x_\alpha^{(y)}$ will lie inside the $g_M^{(n-1)}$-radius $100 \cdot \varepsilon(n)$ about y. Meanwhile, the $\{x_\alpha^{(y)}\}_{\alpha=1}^{30}$ will be separated by a $g_M^{(n-1)}$-distance $2\varepsilon(n)$ or more. Thus, ε in (3.61) is equal to $\varepsilon(n)$.

In fact, choose $\{x_\alpha^{(y)}\}_{\alpha=1}^{30}$ in any convenient manner, subject to the preceding two constraints.

(e) 3×3 **symmetric, traceless matrices.** The choice of frames for the points in Ω_n requires a preliminary digression concerning the vector space, $\mathbb{V}$, of 3×3 traceless, symmetric matrices.

To begin the digression, remark that $\mathrm{SO}(3)$ acts irreducibly on $\mathbb{V}$ by conjugation.

A convenient basis for $\mathbb{V}$ is given as

$$(9.26)\quad \begin{gathered} e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ e_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad e_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{gathered}$$

Note that this basis is orthonormal using the ($\mathrm{SO}(3)$-invariant) inner product $\frac{1}{2}\operatorname{tr}(V_1 V_2)$.

The preceding basis has the following added properties:

Lemma 9.7. *For any pair* (i, j), *the matrices* e_i *and* e_j *are conjugate by* $\mathrm{SO}(3)$. *So are* e_i *and* $-e_j$.

Proof of Lemma 9.7. This is an exercise. q.e.d.

Now, reintroduce the matrix

$$(9.27)\quad \mathbb{A}_0 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which appeared in §3 as conjugate by $\mathrm{SO}(3)$ to the value of $\mathscr{W}_-(g_{\mathrm{FS}})$ at any point of $\underline{\mathbb{CP}}^2$. Also introduce

$$(9.28)\quad a_0 \equiv \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3).$$

The $\{e_i\}_{i=1}^5$ are related to $\mathbb{A}_0$ via

Lemma 9.8. *For each* i, *both* $3 \cdot e_i$ *and* $-3 \cdot e_i$ *are conjugate by the* $\mathrm{SO}(3)$ *action to* $2 \cdot \mathbb{A}_0 + a_0 \cdot \mathbb{A}_0 \cdot a_0^{-1}$.

Proof of Lemma 9.8. Calculate that $3 \cdot e_5 = 2 \cdot \mathbb{A}_0 + a_0 \cdot \mathbb{A}_0 \cdot a_0^{-1}$ and then use Lemma 9.7.

(f) Choosing frames. The specification of Ω' required the choice of a constant $\varepsilon(n) > 0$. This $\varepsilon(n)$ will be very small, and, in particular, small enough so that $200\cdot\varepsilon(n)$ is much less than $g_M^{(n-1)}$'s injectivity radius. With this understood, the radius $200\varepsilon(n)$ ball about $y \in \Omega'$ can be assumed to lie well within a Gaussian coordinate chart with center y.

Choose a $g_M^{(n-1)}$-orthonormal frame $f_y \in FTM^{(n-1)}|_y$ to define such a coordinate system with y as center. (Use the metric $g_M^{(n-1)}$ here.) Use this coordinate system to trivialize the bundle $V_+[g_M^{(n-1)}]$ over the radius $200 \cdot \varepsilon(n)$ ball about y.

Note that if x is within this radius $200\cdot\varepsilon(n)$ ball, a $g_M^{(n-1)}$-orthonormal frame f for $TM^{(n-1)}|_x$ gives an element $a(f) \in \mathrm{SO}(3)$ by comparing f's trivialization of Λ_+^2 with f_y's. (Thus, $a(f_y) = \text{Identity}$.)

To choose frames for $TM^{(n-1)}$ at the points in $\{x_\alpha^{(y)}\}_{\alpha=1}^{30}$, one must first break this set into ten subsets, $\{\mathcal{O}_{\pm i}\}_{i=1}^5$, of three points each. Label the points in $\mathcal{O}_{\pm i}$ as $\{x_{\pm i,b}\}_{b=1}^3$.

Choose the frame $f_{\pm i,b}$ for $TM^{(n-1)}$ at $x_{\pm i,b}$ so that

$$\pm 3e_i = \sum_{b=1}^{3} a(f_{\pm i,b}) \cdot \mathbb{A}_0 \cdot a(f_{\pm i,b})^{-1}, \tag{9.29}$$

with $\{e_i\}_{i=1}^5$ as in (9.26). The solvability of (9.29) is insured by Lemma 9.8.

(g) The map $\hat{\mu}$. Before defining $\hat{\mu}$, one must note that the choice of a $g_M^{(n-1)}$-orthonormal frame g for $TM|_x$ at some x defines an isometry (use $g_M^{(n-1)}$),

$$\Lambda_f\colon \mathbb{V} \to V_+|_x. \tag{9.30}$$

If one thinks of V_+ as a subbundle of $\mathrm{End}(\bigwedge_2 TM)$, then Λ_f maps $\mathbb{V}$ into $\mathrm{End}(\bigwedge_2 TM)|_x$.

If E has been specified, define

$$\hat{\imath}\colon \mathrm{Range}(\pi_E) \to \times_{y\in\Omega'} \mathbb{V} \tag{9.31}$$

by setting

$$\hat{\imath}(\nu)_y = s(y) \cdot \sum_{i=1}^{5} (\nu(y), \Lambda_{f_y} e_i)_{g_M^{(n-1)}} \cdot e_i, \tag{9.32}$$

where $\{s(y) > 0\}_{y\in\Omega'}$ will be specified shortly.

With $\hat{t}$ understood, define $\hat{\mu}(\nu)$ by fixing $y \in \Omega'$ and then setting

$$(9.33) \qquad \hat{\mu}(\nu)_{x_{\pm i,b}} \equiv \mu_0 \cdot ((|\hat{t}(\nu)_y|^2 + \varepsilon)^{1/4} \pm \hat{t}(\nu)_{y,i})^{1/2},$$

where

$$(9.34) \qquad \hat{t}(\nu)_{y,i} \equiv \tfrac{1}{2}\operatorname{tr}(e_i \hat{t}(\nu)|_y) = s(y) \cdot (\nu(y), \Lambda_{f_y} \cdot e_i)_{g_M^{(n-1)}}.$$

Note that a bound for

$$(9.35) \qquad \sup_{y \in \Omega'} s(y) \equiv \xi$$

means that the map $\hat{\mu}$ can be defined on $\mathfrak{B}_{d=\xi^{-1}}$ in (9.7).

These mysterious $\{s(y)\}_{y\in\Omega'}$ and (Z, ε) in (3.61) are further discussed in the next proposition.

Proposition 9.9. *Let M be a compact, oriented 4-manifold with metric g_M. There is a constant $Z_0 > 1$ which has the following significance: Fix $E \geq Z_0$, $\mu_1 > 0$, $\mu_0 \in (0, 1)$, and then*

(a) *run Proposition 3.11's iteration steps a large number (say $n-1$) times using $Z_1 \geq Z_0$ in each step and ε small in each step;*

(b) *choose Z large in (3.61) (E and $g_M^{(n-1)}$ determine a lower bound) and ε small.*

A choice of $\{s(y)\}_{y\in\Omega'}$ in the interval $(0, Z_0)$ can be made so that when the Cokernel Step's parameters are as described above, and when $\hat{\mu}$ is defined as above on $\mathfrak{B}_{d=Z_0^{-1}}$, then assertions (1)–(3) of Proposition 9.5 hold.

Remark that Proposition 9.9 usurps Proposition 9.5.

(h) The size of $\|W\|_{*,\rho}$. The purpose of this subsection is to prove Proposition 9.5's first two assertions for Proposition 9.9.

With this said, remark that the proposition's first assertion is a consequence of Proposition 3.11.

The complete proof of the second assertion requires an a priori bound for ξ in (9.35). Such a bound a forthcoming. With no such bound, it will be shown here that

$$(9.36) \qquad \|W\|_{*,\rho} \leq \xi \cdot Z \cdot \mu_0 \cdot \|\nu\|_{*,\rho},$$

with Z determined a priori from g_M.

To prove (9.36), remark that $y \in \Omega'$ contributes nothing to

$$(9.37) \qquad \int_{\operatorname{dist}_{g_M}(x,\cdot)<\rho} d\operatorname{vol}_{g_M} \frac{|W|_{g_M}}{(\operatorname{dist}_{g_M}(x,\cdot))^2}$$

unless y has g_M-distance from x which is less than 2ρ. (This requires small ε in (3.61) and all $n-1$ iteration steps.)

If $\mathrm{dist}_{g_M}(x, y) < 2\rho$, then y contributes at most

$$\mu_0 \cdot \frac{c \cdot \varepsilon(n)^4 \cdot s(y) \cdot (|\nu(y)|_{g_M} + \varepsilon(n))}{((\mathrm{dist}_{g_M}(x, y))^2 + \varepsilon(n)^2)} \tag{9.38}$$

to (9.37) unless $\mathrm{dist}_{g_M}(x, y) < 200 \cdot \varepsilon(n)$. Only one y can possibly obey this last condition, and such a y contributes at most

$$\xi \cdot Z \cdot \mu_0 \cdot \varepsilon(n)^2 \tag{9.39}$$

to (9.37). Since this last can be ignored for small ε, one can estimate the value of (9.37) by summing (9.38) over all $y \in \Omega'$ obeying $\mathrm{dist}_{g_M}(x, y) \le 2\rho$. The result, for small $\varepsilon(n)$, is bounded by

$$c\mu_0 \cdot \xi \left(\varepsilon(n) + \left[\int_{\mathrm{dist}_{g_M}(x,\cdot)<2\rho} d\,\mathrm{vol}_{g_M} \frac{|\nu|_{g_M}}{(\mathrm{dist}_{g_M}(x, \cdot))^2} \right] \right). \tag{9.40}$$

Equation (9.40) follows from (9.38) using Lemmas 4.15 and 4.16 with Taylor's theorem with remainder.

(i) The evaluation of $\pi_E \cdot W$. The evaluation of $\pi_E \cdot W$ is a step-by-step procedure which begins below with Lemma 9.10. In the statement of said lemma, $\{\nu_\alpha\}$ is an L^2-orthonormal basis for $\mathrm{Range}(\pi_E)$. Also, $\mathbb{I}$ is defined in (3.15), $\mathbb{A}_0$ in (9.27), and Λ_f in (9.30).

Lemma 9.10. *There is a constant $c > 0$ with the following significance: Let M be a compact, oriented 4-manifold with metric g_M. Given $\delta > 0$ and E, then Z in (3.61) can be chosen so that when $\varepsilon \equiv \varepsilon(n) > 0$ is small, then $x \in \Omega_n$ contributes*

$$c \cdot \mu_x \cdot \varepsilon^4 \cdot \sum_\alpha (\nu_\alpha(x), \Lambda_{f_x} \cdot \mathbb{I} \cdot \mathbb{A}_0)_{g_M} \cdot \nu_\alpha + r_x \tag{9.41}$$

to $\pi_E \cdot W$. Here, the error r_x obeys

$$\left| \int_M d\,\mathrm{vol}_{g_M} (\nu_\alpha, r_x)_{g_M} \right| \le \delta \cdot \mu_x \cdot \varepsilon^4. \tag{9.42}$$

Proof of Lemma 9.10. This follows directly from Proposition 3.3 and Lemma 4.15 using Taylor's theorem again. Q.E.D.

Given this lemma, the computation of $\pi_E \cdot W$ can be done by inserting the appropriate values for $\{\mu_x\}_{x\in\Omega_n}$ and $\{f_x\}_{x\in\Omega_n}$. Here is the result:

Lemma 9.11. *Let M and g_M be as in the statement of Proposition 9.9. Given E, then Z in (3.61) can be chosen so that the following is* *true: Suppose* $\{s(y)\}_{y\in\Omega'}$ *obeys* (3.35). *For* $\varepsilon = \varepsilon(n)$ *small, define* $\hat{\mu}$ *on* $\mathfrak{B}_{d=\xi^{-1}}$ *using* (9.33) *and* (9.34) *and define the frames* $\{f_x\}_{x\in\Omega_n}$ *by* (9.29). *Then*

$$\pi_E \cdot W = \mu_0 \cdot \sum_\alpha \left(\sum_{y\in\Omega'} s(y)\cdot\varepsilon^4\cdot(\nu\,,\,\nu_\alpha)_{g_M}(y)\right)\cdot\nu_\alpha + r\,, \tag{9.43}$$

where $\|r\|_{*,\rho} \le E^{-1}\cdot\mu_0\cdot\xi\cdot(\|\nu\|_{*,\rho}+\varepsilon(n))$.

Proof of Lemma 9.11. The derivation of the first term on the right in (9.43) uses (9.41)'s first term. This is just linear algebra and is left to the reader. The error term r is estimated by summing the $\{r_x\}_{x\in\Omega_n}$ in (9.41) using (9.42). The summation yields

$$\begin{aligned}\|r\|_{*,\rho} &\le \sum_{x\in\Omega_n}\|r_x\|_{*,\rho}\\ &\le \delta\cdot\mu_0\cdot\xi\cdot\left(\sum_{y\in\Omega'}(|\nu(y)|+\varepsilon)\cdot\varepsilon^4\right)\cdot\sum_\alpha\|\nu_\alpha\|_{*,\rho}\\ &\le c\cdot\delta\cdot\mu_0\cdot\xi\cdot E^3\cdot\rho^2(\|\nu(y)\|_{L^1}+\varepsilon)\\ &\le c\cdot\delta\cdot\mu_0\cdot\xi\cdot E^3(\|\nu(y)\|_{*,\rho}+\varepsilon).\end{aligned} \tag{9.44}$$

Now, take $\delta < E^{-4}$ in the last line of (9.44) to finish the argument. q.e.d.

(j) An approximation formula. The final determination of $s(y)$ requires a digression which concerns π_E. Begin the digression by fixing E and then $d = (E+1)^{-10}$. Now, invoke Lemma 3.8 using d for ε. Let $\underline{\Omega}$ denote the resulting point set, and let $\{\psi_x\}_{x\in\underline{\Omega}}$ be a partition of unity for M which is subordinate to the cover of M which Lemma 3.8 provides.

For each $x\in\underline{\Omega}$, set

$$p(x) \equiv \int_M d\,\mathrm{vol}_{g_M}\cdot\psi_x. \tag{9.45}$$

Lemma 9.12. *Given* M *and* g_M, *there is a constant* Z *such that with* E *and* $\rho > 0$ *given, and* $d = (E+1)^{-10}$, *one has*

$$\left|\nu - \sum_\alpha\left(\sum_{x\in\underline{\Omega}} p(x)\cdot(\nu\,,\,\mu_\alpha)_{g_M}(x)\right)\cdot\nu_\alpha\right|_{g_M} \le Z\cdot(E+1)^{-4}\cdot\rho^{-2}\|\nu\|_{*,\rho}.$$

Proof of Lemma 9.12. Use Taylor's theorem with Lemmas 4.14 and 4.16 plus (9.18) and (9.19).

(k) Proof of Proposition 9.9. The proof of the proposition will be reduced here to the choice of $\{s(y)\}_{y\in\Omega'}$, a task which occupies the final subsection.

To begin, fix E large and choose $d = (E+1)^{-10}$ as in Lemma 9.12. Let $\underline{\Omega}$ be the point set as described above, and let $\{\psi_x\}_{x\in\underline{\Omega}}$ be as described above too.

Rewrite the first term on (9.43)'s right side as

$$\mu_0 \cdot \sum_\alpha \left(\sum_{x\in\underline{\Omega}} \sum_{y\in\Omega'} \psi_x(y) \cdot s(y) \cdot \varepsilon^4 \cdot (\nu, \nu_\alpha)_{g_M}(y) \right) \cdot \nu_\alpha. \tag{9.46}$$

Now, one can replace $(\nu, \nu_\alpha)_{g_M}(y)$ in (9.46) with $(\nu, \nu_\alpha)_{g_M}(x)$ for $x \in \underline{\Omega}$ for which $\psi_x(y) \neq 0$. These replacements make (9.46) equal to

$$\mu_0 \cdot \sum_\alpha \left(\sum_{x\in\underline{\Omega}} \left(\sum_{y\in\Omega'} \psi_x(y) \cdot s(y) \cdot \varepsilon^4 \right) \cdot (\nu, \nu_\alpha)_{g_M}(x) \right) \cdot \nu_\alpha + r_1, \tag{9.47}$$

where r_1 obeys

$$\|r_1\|_{*,p} \leq Z \cdot \mu_0 \cdot \xi \cdot E^{-1} \cdot \|\nu\|_{*,p}. \tag{9.48}$$

Indeed, Taylor's theorem gives

$$|r_1| \leq c\mu_0 \cdot \xi \cdot d \cdot \left[\sum_\alpha (\|\nabla\nu\|_\infty \cdot \|\nu_\alpha\|_\infty + \|\nu\|_\infty \|\nabla\nu_\alpha\|_\infty) \cdot |\nu_\alpha| \right].$$

Then use Lemmas 4.15 and 4.16 to bound this by $Z \cdot \mu \cdot \xi \cdot d \cdot E^9 \cdot \|\nu\|_{*,p}$ and plug in $d = (E+1)^{-10}$.

Note that Proposition 9.9 will follow (9.36), (9.47), (9.48) and Lemma 9.12 if $\{s(y)\}_{y\in\Omega'}$ can be found so that

$$\begin{aligned} &(1) \quad \text{for each } x \in \underline{\Omega},\ \sum_{y\in\Omega'} \psi_x(y) s(y) \cdot \varepsilon^4 = p(x), \\ &(2) \quad s(y) \in (0, Z_0), \end{aligned} \tag{9.49}$$

where $p(x)$ is given in (9.45) and Z_0 here is determined a priori only by M and g_M.

(l) Choosing $s(y)$. There is a solution to (9.49) which can be found with the help of the next lemma. The proof of this lemma is deferred to the subsection's end.

Lemma 9.13. *Let M be a compact, oriented 4-manifold with metric g_M. There is a constant Z with the following significance: In Proposition 3.11's iteration, at every step, choose Proposition 3.7's parameter $Z_0 \geq Z$*

and then, given $d > 0$, *choose the* ε *parameter small. Then, in addition to Proposition* 3.11*'s other conclusions, one can assume that*

$$\operatorname{Vol}_{g_M}(B \cap M^{(l)}) / \operatorname{Vol}_{g_M}(B) \geq \tfrac{1}{2} \tag{9.50}$$

for any ball $B \subset M$ *of* g_M*-radius* d *and any* $l = 0, 1, \cdots$.

Given this lemma, here is how to choose $\{s(y)\}$ to satisfy (9.49): First, take $d = (E+1)^{-10}$ in Lemma 9.13. Second, remark that the balls of radius d about the points in $\underline{\Omega}$ are disjoint. Third, choose $s(y) = \varepsilon(n)$ if $\operatorname{dist}_{g_M}(y, \underline{\Omega}) \geq d$. Fourth, given $x \in \underline{\Omega}$, set

$$p'(x) \equiv \operatorname{Vol}_{g_M}(B_d(x) \cap M^{(n-1)}), \tag{9.51}$$

where $B_d(x)$ is the radius d ball with center x.

It follows from Lemma 9.13 that

$$\frac{p'(x)}{p(x)} \geq c \tag{9.52}$$

for some universal constant c.

Fifth, by taking $\varepsilon = \varepsilon(n)$ small, it follows from Lemma 3.8 that one can require

$$p'(x)^{-1} \cdot \sum_{y \in \Omega' \cap B_d(x)} \varepsilon^4 \equiv c_x \geq c \tag{9.53}$$

for some universal $c > 0$.

Finally, take

$$s(y) \equiv \frac{p(x)}{c_x \cdot p'(x)} \tag{9.54}$$

when $\operatorname{dist}_{d_M}(y, x) < d$.

It is an elementary exercise for the reader to verify (9.49) from (9.54).

Proof of Lemma 9.13. To prove the lemma, note that $M^{(l)} \subset M^{(l-1)}$ as the compliment of the union over $x \in \Omega_{(l)}$ of the $g_M^{(l-1)}$-radius ε_1 ball about such an x.

This radius ε_1 ball sits inside the radius $\varepsilon = \varepsilon(l)$ ball about x, and these radius ε balls are mutually disjoint. Thus

$$\operatorname{Vol}_{g_M^{(l-1)}}(B \cap M^{(l-1)}) - \operatorname{Vol}_{g_M^l}(B \cap M^{(l)}) \leq \frac{1}{4}\pi^2 \cdot \sum_{x \in \Omega_l \cap B'} \varepsilon_1^4, \tag{9.55}$$

where B' is the radius $2 \cdot d$ ball with the same center as B. Here, it has been assumed that $\varepsilon \ll d$.

Now, Proposition 3.5 relates ε_1 to ε and allows the right side of (9.53) to be bounded by

$$\frac{\pi^2}{Z^2 c_0^2} \cdot \sum_{x \in \Omega_l \cap B'} \nu_1^2 \cdot \varepsilon^4. \tag{9.56}$$

Use (3.21) to write the preceding sum in terms of the L^2-norm of $\mathscr{W}_+[g_M^{(l-1)}]$; indeed, for small ε the right side above is bounded by

$$\frac{c}{Z^2} \cdot \int_{B'} d\,\mathrm{vol}_{g_M^{(l-1)}} |\mathscr{W}_+[g_M^{(l-1)}]|^2_{g_M^{(l-1)}}. \tag{9.57}$$

Then, use Proposition 3.11 to bound (9.57) by

$$\frac{c}{Z^2}(1-\delta)^{l-1} \cdot \|\mathscr{W}_+[g_M]\|_\infty \cdot d^4. \tag{9.58}$$

Iterate (9.53) using (9.58) to get

$$\mathrm{Vol}_{g_M^{(l)}}(B \cap M^{(l)}) \geq \mathrm{Vol}_{g_M}(B) - \frac{c}{Z^2 \delta} \cdot \|\mathscr{W}_+[g_M]\|_\infty \cdot d^4. \tag{9.59}$$

This last equation with Proposition 3.10 imply Lemma 9.13 if

$$Z^2 \geq 2 \cdot \frac{c}{\delta}(1 + \|\mathscr{W}_+[g_M]\|_\infty). \tag{9.60}$$

Acknowledgments

The author thanks the Mathematics Department of the University of California at San Diego, and M. H. Freedman and his family for their generosity, kindness, and hospitality. This work has also been supported in part by the National Science Foundation.

References

[1] M. F. Atiyah, N. J. Hitchin & I. M. Singer, *Self-duality in four dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.

[2] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampere equations*, Springer, Berlin, 1980.

[3] F. Campana, *On twistor spaces of class* $\mathscr{C}$, J. Differential Geometry **33** (1991) 541–550.

[4] S. Y. Cheng & P. Li, *Heat kernel estimates and lower bound of eigenvalues*, Comment. Math. Helv. **56** (1981) 327–315.

[5] S. D. Donaldson & R. Friedman, *Connect sums of self dual manifolds and deformations of singular spaces*, Nonlinearity **2** (1989) 197–239.

[6] S. K. Donaldson & P. B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, Oxford, 1990.

[7] A. Floer, *Self-dual conformal structures on* $l\mathbb{CP}^2$, J. Differential Geometry **33** (1991) 551–574.

[8] M. J. Greenberg & J. R. Harper, *Algebraic topology, a first course*, Benjamin/Cummings, Reading, MA, 1981.

[9] N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. (3) **43** (1981) 133–150.

[10] D. Joyce, *The hypercomplex quotient and the quaternionic quotient*, preprint, Oxford, 1990.

[11] ——, *Quotient constructions for compact, self-dual* 4 *manifolds*, preprint, Oxford, 1990.

[12] A. D. King & D. Kotschick, *The deformation theory of anti-self-dual conformal structures*, preprint, Cambridge, 1990.

[13] S. Kobayashi & K. Nomizu, *Foundations of differential geometric*. I, II, Interscience, New York, 1963.

[14] M. Kuranishi, *New proof for existence of locally complete families of complex structures*, in Proc. Conf. Complex Analysis (Minneapolis 1964), Springer, Berlin, 1965.

[15] S. Lang, *Differential manifolds*, Addison-Wesley, Reading, MA, 1985.

[16] C. LeBrun, *Explicit self-dual metrics on* $\mathbb{CP}^2\#\cdots\#\mathbb{CP}^2$, J. Differential Geometry **34** (1991), 223–254.

[17] C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer, Berlin, 1966.

[18] Y. S. Poon, *Compact, self dual manifolds of positive scalar curvature*, J. Differential Geometry **24** 97–132.

[19] ——, *Algebraic dimension of twistor spaces*, Math. Ann. **282** (1988) 621–627.

[20] C. H. Taubes, *The stable topology of self dual moduli spaces*, J. Differential Geometry **29** (1989) 163–230.

[21] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **34** (1977) 1798–1799.

HARVARD UNIVERSITY