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THE YAMABE PROBLEM ON MANIFOLDS WITH BOUNDARY

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A natural question in differential geometry is whether a given compact Riemannian manifold with boundary is necessarily conformally equivalent to one of constant scalar curvature, where the boundary is minimal. When the boundary is empty this is called the Yamabe Problem—so-called because, in 1960, Yamabe claimed to have solved this problem. In 1968, N. Trudinger found a mistake in Yamabe's paper [16] and corrected Yamabe's proof for the case in which the scalar curvature is nonpositive. In 1976, Aubin [1] showed that, if $\dim M \geq 6$ and M is not conformally flat, then M can be conformally changed to constant scalar curvature. In 1984, Richard Schoen [10] solved the Yamabe problem in the remaining cases.

In this paper, we study the problem in the context of manifolds with boundary and give an affirmative solution to the question formulated above in almost every case. In fact, we show that any compact Riemannian manifold with boundary and dimension 3, 4, or 5 is conformally equivalent to one of constant scalar curvature, where the boundary is minimal. When $n \geq 3$ and there exists a nonumbilic point at ∂M, the boundary of M, we show that the problem above has an affirmative answer. The remaining case is when $n \geq 6$ and ∂M is umbilic. Under these conditions we show that the problem above is solvable in the affirmative if either M is locally conformally flat, or the Weyl tensor does not vanish identically on the boundary.

The only case we do not consider in this paper is when $n \geq 6$, M is not locally conformally flat, ∂M is umbilic, and the Weyl tensor vanishes identically on ∂M. As a consequence of the above results we have the following theorem.

Theorem. *Any bounded domain in a Euclidean m-space $\mathbb{R}^n$, with smooth boundary and $n \geq 3$, admits a metric conformal to the Euclidean metric having constant scalar curvature and minimal boundary.*

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To study the problem above is equivalent to studying the existence of a smooth positive solution on a Riemannian manifold (M^n, g) with boundary and dimension $n \geq 3$ to the equations

$$
(1) \qquad \begin{aligned}
&\Delta u - \frac{n-2}{4(n-1)} R_g u + C u^{(n+2)/(n-2)} = 0 && \text{on } M, \\
&\frac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u = 0 && \text{on } \partial M,
\end{aligned}
$$

where R_g is the scalar curvature of M, h_g is the mean curvature of ∂M, η is the outward normal vector with respect to the metric g, and C is a constant whose sign is uniquely determined by the conformal structure. If $\overline{g} = u^{4/(n-2)} g$, then the metric $\overline{g}$ has constant scalar curvature and the boundary is minimal. We denote the linear part of the operator in (1) by L and the boundary conditions by B, thus

$$
(2) \qquad \begin{aligned}
Lu &= \Delta u - \frac{n-2}{4(n-1)} R_g u && \text{on } M, \\
Bu &= \frac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u && \text{on } \partial M.
\end{aligned}
$$

The operator L on M together with the operator B on ∂M is conformally invariant, in that it changes by a multiplicative factor when the metric of M is multiplied by a positive function. Observe that if u is a solution of (1), then u is a critical point for the Sobolev quotient $Q_g(\varphi)$ for functions φ on (M, g), which is given by

$$
(3) \qquad Q_g(\varphi) = \frac{\int_M (|\nabla \varphi|_g^2 + \frac{n-2}{4(n-1)} R_g \varphi^2)\, dv + \frac{n-2}{2} \int_{\partial M} h_g \varphi^2\, d\sigma}{(\int_M |\varphi|^{2n/(n-2)}\, dv)^{(n-2)/2}},
$$

where dv and $d\sigma$ are the Riemannian measure on M and the induced Riemannian measure on ∂M, respectively, with respect to the metric g. The Sobolev quotient $Q(M)$ is then defined by

$$
Q(M) = \inf\{Q_g(\varphi) : \varphi \in C^1(\overline{M}),\ \varphi \not\equiv 0\}.
$$

The number $Q(M)$ depends only on the conformal class of g. By choosing functions φ which are supported near a point of ∂M, it follows easily that

$$
(4) \qquad Q(M) \leq Q(S^n_+)
$$

for any n-dimensional manifold M. Here S^n_+ denotes the upper standard hemisphere.

A similar argument to the one given by Aubin in [1] shows that if $Q(M) < Q(S^n_+)$ then there exists a minimum for $Q_g(\varphi)$ over the functions

$\varphi \in H^1(M)$. It was proved by Cherrier [4] that this function is smooth. This minimizing function then becomes a positive solution of (1) on $\overline{M}$.

Since $Q(S^n_+)$ is positive, equality in (4) does not occur if $Q(M) \leq 0$. From now on we assume that $Q(M) > 0$.

In order to prove that $Q(M) < Q(S^n_+)$ for a manifold M conformally different from S^n_+, we only need to exhibit a function φ on M with $Q_g(\varphi) < Q(S^n_+)$. To do this, when $n \geq 4$, we distinguish two cases: The first case is when there exists a nonumbilic point on the boundary. The second case is when there is no nonumbilic point on ∂M, that is to say that every point is umbilic. In the first case we exploit the local geometry of a nonumbilic point. A local test function is enough to prove the inequality for the case $n > 4$ and a global one for the case $n = 4$. Our correction term, in this case, comes from the trace free part of the second fundamental form which at a nonumbilic point has positive and negative eigenvalues, and if $n \geq 4$ could have zero eigenvalues (at an umbilic point all eigenvalues are zero). To distinguish the positive, the negative, and the zero eigenvalues, we introduce the idea of breaking the symmetry by using a nonsymmetric function. This idea is motivated by the translation-invariance of the extremals on $\mathbb{R}^n_+$. In the second case, that is, when ∂M is umbilic and $n \geq 4$, the proof is parallel to the one given by Aubin [1] and Schoen [10] in the Yamabe problem on closed manifolds. To deal with this case we prove a version of the conformal normal coordinates introduced by Lee and Parker [8] for a point on the boundary. This simplifies the local analysis. When M is not locally conformally flat and $n \geq 6$, a local test function is enough to prove strict inequality in (4) provided that the Weyl tensor does not vanish on ∂M. We use the same correction term as in Aubin [1] which is, in this case, the norm of the Weyl tensor. When M is locally conformally flat (and ∂M is umbilic), we use the Green's functions method introduced by R. Schoen in [10]. We study the behavior of the positive Green's function G for the conformal Laplacian L with respect to the boundary condition B near a boundary point 0. The existence of G is guaranteed by the fact that $Q(M) > 0$. For a metric within the conformal class of g and under suitable coordinates near 0, the function G has an expansion

$$G(x) = |x|^{2-n} + A + O(|x|).$$

The sign of the constant term A in this expansion is then the crucial ingredient. If A is positive then one can find a function φ which is a small multiple of G outside a neighborhood of 0 and which satisfies $Q(\varphi) < Q(S^n_+)$. In the Appendix we prove a version of the Positive Mass

Theorem of Schoen and Yau for manifolds with boundary. Our Theorem says: $A \geq 0$ and $A = 0$ only if M is conformally equivalent to S^n_+. The metric $\overline{g} = G^{4/(n-2)}g$ on $M - \{0\}$ is scalar flat, ∂M is totally geodesic (because it is minimal and umbilic) and asymptotically Euclidean, and

$$\overline{g}_{ij} = (1 + A|y|^{2-n})^{4/(n-2)}\delta_{ij} + O(|y|^{1-n}),$$

where $y = x/|x|^2$. The end of the manifold $M - \{0\}$ is diffeomorphic to the complement of a ball centered at the origin in the half n-dimensional Euclidean space. Thus we have $Q(M) < Q(S^n_+)$ provided that M is locally conformally flat, ∂M is umbilic, and M is not conformally equivalent to S^n_+. When ∂M is umbilic and $n = 3$ or 4 we prove that the Positive Mass Theorem holds for a suitable metric within the conformal class of g and hence we have strict inequality in (4) in these cases. When $n = 5$ and ∂M is umbilic we treat this case similar to Schoen [10], using a perturbation argument. We discuss the Positive Mass Theorem for manifolds with boundary in the Appendix of this paper.

When $n = 3$ we first prove that equality in (4) does not hold if M is not conformally equivalent to S^n_+ and has an umbilic point at the boundary. To prove the inequality we show that the Positive Mass Theorem holds for these manifolds. The general case (no umbilic points at the boundary) follows by approximating our manifold by a sequence of manifolds having one umbilic point and not conformally equivalent to S^n_+. One proves then that strict inequality in (4) is preserved upon passage to the limit provided M has a nonumbilic point at the boundary.

An important case is when M is a bounded domain Ω in $\mathbb{R}^n$, with $n \geq 3$. Since S^n_+ is conformally diffeomorphic to the ball B, our result reads as $Q(\Omega) \leq Q(B)$ and equality holds if and only if Ω is the ball. We think of $Q(\Omega)$ as measuring how far Ω is from being a ball. It will be interesting to study the relation between $Q(\Omega)$ and the dilation quotients defined in the study of quasi-conformal maps.

In a forthcoming paper, we will show that, under the same hypotheses as in Theorem 6.1, a compact Riemannian manifold with boundary is conformally equivalent to one of constant scalar curvature, where the boundary has constant mean curvature. Also, we will study the case of prescribing the scalar curvature and the mean curvature of the boundary.

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1. Preliminaries: Conformal invariants

In this section, we assume that (M^n, g) is a compact Riemannian manifold with boundary and dimension $n \geq 3$. If $\tilde{g} = e^{2f} g$ is a metric conformal to g, one can compute the components of the curvature tensor $\tilde{R}$ of $\tilde{g}$ in terms of those of g. It is well known that the transformation law for the Ricci curvature is

$$(1.1) \qquad \tilde{R}_{ij} = R_{ij} - (n-2)f_{ij} + (n-2)f_i f_j - (\Delta f + (n-2)|\nabla f|^2)g_{ij}.$$

Therefore

$$(1.2) \qquad \tilde{R} = e^{-2f}(R - 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2).$$

One can compute the components of the second fundamental form $\tilde{\pi}$ in terms of the second fundamental form of g. The transformation law is

$$(1.3) \qquad \tilde{h}_{ij} = e^f h_{ij} + \frac{\partial}{\partial \eta}(e^f) g_{ij},$$

where $\frac{\partial}{\partial \eta}$ is the normal derivative with respect to the outward normal η. Hence

$$(1.4) \qquad \tilde{h} = e^{-f}\left(h + \frac{\partial}{\partial \eta} f\right).$$

The conformal Laplacian is the operator L with the boundary conditions B defined in (2). It is a conformally invariant operator. More precisely, if $\tilde{g} = u^{4/(n-2)} g$ is a metric conformal to g, and $\tilde{L}$ and $\tilde{B}$ are similarly defined with respect to the metric $\tilde{g}$, then computing $\tilde{\Delta}$ and using the transformation laws for the scalar curvature and the mean curvature, one find that

$$(1.5) \qquad \tilde{L}(u^{-1}\varphi) = u^{-(n+2)/(n-2)} L(\varphi) \quad \text{on } M$$

and

$$(1.6) \qquad \tilde{B}(u^{-1}\varphi) = u^{-n/(n-2)} B(\varphi) \quad \text{on } \partial M.$$

From the transformation laws (1.5) and (1.6) it is easy to check the following

Proposition 1.1. *Consider the metric $\tilde{g} = u^{4/(n-2)} g$ on M, where $u > 0$ is any smooth function on $\overline{M}$ (= closure of M). Then for any function $\varphi \in C^\infty(\overline{M})$ we have*

$$Q_g(\varphi) = Q_{\tilde{g}}(u^{-1}\varphi).$$

Hence $Q(M, g) = Q(M, \tilde{g})$.

Proof. Observe that $d\tilde{v} = u^{2n/(n-2)}\,dv$ and $d\tilde{\sigma} = u^{2(n-1)/(n-2)}\,d\sigma$. It follows immediately that

$$\int_M |\varphi|^{2n/(n-2)}\,dv = \int_M |u^{-1}\varphi|^{2n/(n-2)}\,d\tilde{v}. \tag{1.7}$$

From (1.5) and (1.6) it is easy to see that

$$\begin{aligned}-\int_M L_g(\varphi)\varphi\,dv + \int_{\partial M} B_g(\varphi)\varphi\,d\sigma = &-\int_M L_{\tilde{g}}(u^{-1}\varphi)u^{-1}\varphi\,d\tilde{v}\\ &+\int_{\partial M} B_{\tilde{g}}(u^{-1}\varphi)u^{-1}\varphi\,d\tilde{\sigma}.\end{aligned}$$

Integrating this equality by parts we get

$$\begin{aligned}\int_M \left(|\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)}R_g\varphi^2\right)dv + \frac{n-2}{2}\int_{\partial M} h_g\varphi^2\,d\sigma\\ = \int_M \left(|\nabla(u^{-1}\varphi)|_{\tilde{g}}^2 + \frac{n-2}{4(n-1)}R_{\tilde{g}}(u^{-1}\varphi)^2\right)d\tilde{v}\\ + \frac{n-2}{2}\int_{\partial M} h_{\tilde{g}}(u^{-1}\varphi)^2\,d\tilde{\sigma}.\end{aligned} \tag{1.8}$$

Using the last equality and (1.7) we have $Q_g(\varphi) = Q_{\tilde{g}}(u^{-1}\varphi)$. The last equality in Proposition 1.1 then follows from the fact that the set $C^\infty(\overline{M})$ is dense in $C^1(\overline{M})$ in the H_1-norm.

Definition. A point is umbilic if the tensor $T_{ij} = h_{ij} - hg_{ij}$ vanishes at the point, where h_{ij} are the coefficients of the second fundamental form and h is the mean curvature.

Another conformal invariant is the set of umbilic points of the boundary. More precisely, we have

Proposition 1.2. *Let $p \in \partial M$ be an umbilic point with respect to the metric g. If $\tilde{g} = u^2 g$ is a metric conformal to g, then p is an umbilic point with respect to the metric $\tilde{g}$.*

Proof. From the transformation law (1.3) we have

$$\tilde{h}_{ij} = uh_{ij} + \frac{\partial u}{\partial\eta}g_{ij} \quad\text{and}\quad \tilde{h} = u^{-1}h + u^{-2}\frac{\partial u}{\partial\eta}.$$

Therefore

$$\tilde{T}_{ij} = \tilde{h}_{ij} - \tilde{h}\tilde{g}_{ij} = uh_{ij} + \frac{\partial u}{\partial\eta}g_{ij} - \left(uh + \frac{\partial u}{\partial\eta}\right)g_{ij} = u(h_{ij} - hg_{ij}) = uT_{ij}.$$

Hence $\tilde{T}_{ij} = uT_{ij}$. From this identity the proposition follows.

Another important conformal invariant, as in the case of closed manifolds (see [7]), is the sign of the first eigenvalue for the conformal Laplacian with respect to the boundary conditions as in (2).

Proposition 1.3. *If $\widetilde{g} = u^{4/(n-2)}g$ is a conformal metric to g, then* $\operatorname{sign}(\widetilde{\lambda}_1(\widetilde{L})) = \operatorname{sign}(\lambda_1(L))$ *or* $\widetilde{\lambda}_1(\widetilde{L}) = \lambda_1(L) = 0$.

Proof. Let f and $\widetilde{f}$ denote the first eigenfunctions for L and $\widetilde{L}$ respectively. From the variational characterization of the first eigenvalue it follows that

$$\lambda_1 \int_M f^2\,dv = \int_M \left(|\nabla f|^2 + \frac{n-2}{4(n-1)}Rf^2\right) dv + \frac{n-2}{2}\int_{\partial M} hf^2\,d\sigma.$$

Using equality (1.8) we obtain

$$\begin{aligned}\lambda_1 \int_M f^2\,dv &= \int_M |\nabla(u^{-1}f)|^2 + \frac{n-2}{4(n-1)}\widetilde{R}(u^{-1}f)^2\,d\widetilde{v} \\ &\quad + \frac{n-2}{2}\int_{\partial M}\widetilde{h}(u^{-1}f)^2\,d\widetilde{\sigma} \\ &\geq \widetilde{\lambda}_1 \int_M (u^{-1}f)^2\,d\widetilde{v}.\end{aligned}$$

Similarly, $\widetilde{\lambda}_1 \int_M \widetilde{f}^2\,d\widetilde{v} \geq \lambda_1 \int_M (u\widetilde{f})^2\,dv$. Since f, $\widetilde{f}$, u are positive functions, the conclusion follows.

A consequence of Proposition 1.3 is the following result.

Lemma 1.1. *If (M^n, g) is a compact Riemannian manifold with boundary and $n \geq 3$, there exists a conformal metric to g whose scalar curvature does not change sign and the boundary is minimal. The sign is uniquely determined by the conformal structure, and so there are three mutually exclusive possibilities: M admits a conformal metric of* (i) *positive,* (ii) *negative, or* (iii) *identically zero scalar curvature and the boundary is minimal.*

Proof. The three possibilities are distinguished by the sign of the first eigenvalue of L with respect to the boundary conditions B. If f_1 is the first eigenfunction, it is well known that $f_1 > 0$ on M. The boundary point lemma implies that $f_1 > 0$ on $\overline{M}$. Consider the metric $f_1^{4/(n-2)}g = g_1$. From the transformation laws (1.2) and (1.4) we have

$$R_1 = \frac{-4(n-1)L_g f_1}{(n-2)f_1^{(n+2)/(n-2)}}, \qquad h_1 = \frac{2}{n-2}\frac{B_g f_1}{f_1^{n/(n-2)}}. \tag{1.9}$$

Hence

$$R_1 = \frac{-4(n-1)}{(n-2)}\lambda_1 f_1^{4/(n-2)} \quad \text{and} \quad h_1 = 0.$$

Therefore, the scalar curvature of g_1 has one sign, and it is straightforward that (i), (ii), and (iii) are mutually exclusive and exhaustive possibilities.

2. Conformal normal coordinates at the boundary

In this section we assume M is a compact Riemannian manifold with boundary and metric $\tilde{g}$, 0 is a point of ∂M, and every point at the boundary is umbilic. In this situation one can have normal coordinates at a point on the boundary; moreover, one can have conformal normal coordinates. These will be normal coordinates for some metric g conformal to $\tilde{g}$. More precisely, we have

Proposition 2.1. *For any $N > 0$, there exists a metric g conformal to $\tilde{g}$ on M such that*

$$\det g_{ij} = 1 + O(r^N),$$

where $r = |x|$ in g-normal coordinates at 0. In these coordinates, if $N \geq 5$, the scalar curvature of g satisfies $R = O(r^2)$ and $\Delta R = -\frac{1}{6}|W|^2$ at 0. Moreover, $h = O(r^2)$ and $\Delta_{\partial M} h = 0$ at 0. Here W represents the Weyl curvature tensor, and $\Delta_{\partial M}$ is the Laplacian on ∂M with respect to the induced metric.

Proof. Let φ_1 be the first eigenfunction for the conformal Laplacian with the boundary condition as in (2). The metric $g_1 = \varphi_1^{4/(n-2)}\tilde{g}$ has minimal boundary. Since every point is umbilic, the second fundamental form is zero, that is, ∂M is a totally geodesic submanifold. Hence, take normal coordinates at 0, such that $x_n = 0$ is ∂M. Now we proceed as in the proof of Theorem 5.2 of Lee and Parker [8] to get the lemma without the conclusions $h = O(r^2)$ and $\Delta_{\partial M} h = 0$ at 0. In order to verify that $h = O(r^2)$, we need to check that the second degree homogeneous polynomial f in Theorem 5.2 in [8] which satisfies that the metric $g_2 = e^{2f} g_1$ has $R_{ij}(0) = 0$ is given by a polynomial of the form $f(x) = cx_n^2 + p(x_1, \cdots, x_{n-1})$. This is a consequence of the Codazzi equation that states, for $i, j, k < n$,

$$R_{ijkn} = h_{ik;j} - h_{jk;i}. \tag{2.1}$$

Since the second fundamental form of the metric g_1 vanishes on ∂M, (2.1) implies that $R_{in}(0) = 0$ for $i < n$.

The transformation law (1.1) implies that

$$0 = R_{in} - (n-2)f_{in} + (n-2)f_i f_n - (\Delta f + (n-2)|\nabla f|^2)g_{in},$$

where the quantities on the right-hand side are taken with respect to the metric g_1. Observe that $g_{in} = 0$ for $i < n$. Since f is a second degree homogeneous polynomial, $f_k(0) = 0$, $k = 1, \cdots, n$. Hence $f_{in}(0) = 0$ for $i < n$. Thus f has the form $f(x) = cx_n^2 + \rho(x_1, \cdots, x_{n-1})$. From the transformation law (1.4) and the fact that g_1 is minimal it follows that $h = O(r^2)$ at 0. Moreover $h = 0$ on ∂M for the metric g_2.

In order to verify that $\Delta_{\partial M} h(0) = 0$ it is enough to check that the third degree homogeneous polynomial f, such that the metric $\tilde{g} = e^{2f} g_2$ at the point $0 \in \partial M$ satisfies

$$\tilde{R}_{ij;k} + \tilde{R}_{ki;j} + \tilde{R}_{jk;i} = 0, \tag{2.2}$$

also satisfies $\sum_{i=1}^{n-1} f_{nii}(0) = 0$. This is a consequence of the Codazzi equation. Differentiating (2.1) gives, for $1 \le i, j, k, l < n$,

$$R_{ijkn;l} = h_{ik;jl} - h_{jk;il}.$$

The metric g_2 satisfies $h = 0$ on ∂M. Proposition 1.2 implies that $h_{ij} = 0$ on ∂M and hence

$$R_{jn;l}(0) = 0, \qquad 1 \le j, l < n. \tag{2.3}$$

Contracting the Bianchi identity

$$\tilde{R}_{ijkl;m} + \tilde{R}_{ijlm;k} + \tilde{R}_{ijmk;l} = 0$$

on the indices i, k and again on j, l, we get

$$\tilde{R}_{;m} - 2\tilde{R}^i{}_{m;i} = 0.$$

In particular when $m = n$,

$$\tilde{R}_{;n}(0) = 2\tilde{R}_{in;i}(0).$$

Thus $2\tilde{R}_{;n}(0) = (2\tilde{R}_{in;i} + \tilde{R}_{ii;n})(0) = 0$ by (2.2) and hence $\tilde{R}_{in;i}(0) = 0$. From (2.2) it follows that $\tilde{R}_{nn;n}(0) = 0$, so that $\sum_{i=1}^{n-1} \tilde{R}_{in;i}(0) = 0$. Differentiating the transformation law (1.1) and evaluating at 0 we obtain for $1 \le i < n$

$$\tilde{R}_{in;i}(0) = R_{in;i}(0) - (n-2) f_{nii}(0). \tag{2.4}$$

Summing in (2.4) from $i = 1$ to $i = n-1$ yields that $\sum_{i=1}^{n-1} f_{nii}(0) = 0$. The transformation law (1.4) implies our result.

Now let (M, g) be a Riemannian manifold with boundary and $0 \in \partial M$. Assume that M is C^∞ up to the boundary, and extend M to a neighborhood of $0 \in \partial M$. Let $B(0, r_0)$ be a ball of radius r_0 such that $\exp_0 : B(0, r_0) \subset T_0 M \to M$ is a diffeomorphism. For later reference, we now state and prove the following fact.

Proposition 2.2. *Let $0 \in \partial M$. Then for r small, the following asymptotic formula holds:*

$$\mathrm{Vol}(M \cap B_r(0)) = \frac{1}{2} w_n r^n - \frac{1}{2} h(0) \frac{\sigma_{n-2}}{(n+1)} r^{n+1} - \frac{\sigma_{n-1}}{12n(n+2)} R(0) r^{n+2} + O(r^{n+3}),$$

where $w_n = \mathrm{Vol}(B_1(0))$, $\sigma_n = \mathrm{Vol}(S^n)$, h is the mean curvature of ∂M, and R is the scalar curvature.

Proof. Let $x_1, \cdots, x_n$ be rectangular coordinates on T_0M such that $\partial/\partial x_n$ is a unit normal vector to ∂M at $0 \in \partial M$. In a small neighborhood of 0 define $\partial\Omega = \exp_0^{-1}(\partial m)$ and $\Omega = \exp_0^{-1}(M)$. Then we can write $\partial\Omega$ as a graph $(x, f(x))$, $x = (x_1, \cdots, x_{n-1})$, where

$$f(x) = \frac{1}{2} \sum h_{ij} x_i x_j + O(|x|^3), \tag{2.5}$$

and h_{ij} is the second fundamental form of ∂M at $0 \in \partial M$. Since $x_1, \cdots, x_n$ are normal coordinates, we have $\sqrt{\det g} = 1 + O(|x|^2)$. Then

$$\mathrm{Vol}(M \cap B_r(0)) = \frac{1}{2} \mathrm{Vol}(B_r(0)) - \int_{B_r^{n-1}(0)} f(x)\, dx + \mathrm{Vol}(A(r)) + O(r^{n+3}),$$

where

$$A(r) = \{x \in \mathbb{R}^n | x_1^2 + \cdots + x_{n-1}^2 < r^2,\ x_1^2 + \cdots + x_{n-1}^2 + f^2 > x_n^2 > r^2\}.$$

Using the Taylor expansion for $f(x)$ given in (2.5) we have

$$\int_{B_r^{n-1}(0)} f(x)\, dx = \frac{1}{2} \left(\sum_{i=1}^{n-1} h_{ii} \right) \frac{\sigma_{n-2}}{(n-1)(n+1)} r^{n+1} + O(r^{n+3}),$$

where we have used the fact that

$$\int_{B_r^{n-1}(0)} x_i x_j\, dx = \frac{\sigma_{n-2}}{(n-1)(n+1)} r^{n+1} \delta_{ij}$$

and

$$\int_{B_r^{n-1}(0)} x_i x_j x_k\, dx = 0.$$

It is well known that in normal coordinates

$$\sqrt{g} = 1 - \frac{1}{6} R_{ij}(0) x_i x_j + O(|x|^3).$$

Hence using the symmetries of the ball as before, we get

$$\mathrm{Vol}(B_r(0)) = w_n r^n - \frac{1}{6} \frac{\sigma_{n-1}}{n(n+2)} \left(\sum_{i=1}^{n} R_{ii}(0) \right) r^{n+2} + O(r^{n+3}).$$

To estimate $\mathrm{Vol}(A(r))$ we observe that if

$$x_1^2 + \cdots + x_{n-1}^2 + f^2(x_1, \cdots, x_{n-1}) = r^2$$

and $x_1^2 + \cdots + x_{n-1}^2 = R^2$, then

$$R^2 + O(R^4) = r^2,$$

which implies

$$r = R + O(R^3).$$

Thus, if $|(x_1, \cdots, x_{n-1})| = O(R) = O(r)$, then $f(x) = O(r^2)$ and

$$\mathrm{Vol}(A(r)) = (r^2(f^{n-1} - R^{n-1})) = O(r^{n+3}).$$

Since $h(0) = (n-1)^{-1} \sum h_{ii}$ and $R(0) = \sum_{i=1}^n R_{ii}(0)$, we get the result.

3. The Sobolev quotient of a Riemannian manifold of dimension $n \geq 4$ with a nonumbilic point

In this section we assume M is a compact Riemannian manifold with boundary and metric g, and 0 is a nonumbilic point of ∂M. Let $Q(\varphi)$ denote the Sobolev quotient of a function φ on M, and let $E(\varphi)$ denote the energy associated with L and the boundary condition B, that is,

$$E(\varphi) = \int_M \left(|\nabla \varphi|_g^2 + \frac{n-2}{4(n-1)} R\varphi^2 \right) dv + \frac{n-2}{2} \int_{\partial M} h\varphi^2 \, d\sigma,$$

$$Q(\varphi) = \frac{E(\varphi)}{(\int_M |\varphi|^{2n/(n-2)} \, dv)^{(n-2)/n}}.$$

In this section we construct a test function φ such that $Q(\varphi) < Q(S_+^n)$. Hence we have

Theorem 3.1. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$. If M has a nonumbilic point on ∂M, then $Q(M) < Q(S_+^n)$.*

Proof. Let $\mathbb{R}_+^n = \{(x, x_n) \in \mathbb{R}^n | x \in \mathbb{R}^{n-1}, \ x_n > 0\}$ be the upper half n-dimensional Euclidean space. Observe that, for $\varepsilon > 0$ the functions

$$u_\varepsilon(x, x_n) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2 + x_n^2} \right)^{(n-2)/2}$$

are solutions of the equations

$$\begin{aligned} &\Delta u_\varepsilon + n(n-2) u_\varepsilon^{(n+2)/(n-2)} = 0 && \text{on } \mathbb{R}_+^n, \\ &\frac{\partial u_\varepsilon}{\partial \eta} = 0 && \text{on } \partial \mathbb{R}_+^n. \end{aligned} \tag{3.1}$$

Multiplying this equation by u_ε and integrating by parts, we obtain

$$\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^2 \, dx \, dx_n = n(n-2) \int_{\mathbb{R}^n_+} u_\varepsilon^{\frac{2n}{n-2}} \, dx \, dx_n.$$

From here we can express $Q(S^n_+)$ in terms of u_ε as

$$Q(S^n_+) = \frac{\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^2}{\left(\int_{\mathbb{R}^n_+} u_\varepsilon^{2n/(n-2)}\right)^{(n-2)/n}} = n(n-2)\left(\int_{\mathbb{R}^n_+} u_\varepsilon^{\frac{2n}{n-2}}\right)^{2/n}, \tag{3.2}$$

where the first equality follows from the work of Aubin [2] and Talenti [14] (see [5]).

Let $(y_1, \cdots, y_n)$ be normal coordinates around $0 \in \partial M$, such that $\eta(0) = -\frac{\partial}{\partial y_n}$ and the second fundamental form of ∂M at 0 has a diagonal form. By changing the metric g conformally and using Proposition 1.1, we can assume that g satisfies that $h = 0$ on the boundary. We further change the metric g, in a small neighborhood of 0, by the metric $e^{2f} g$ where f is a second degree homogeneous polynomial such that $R_{pq}(0) = 0$. From the transformation law 1.1 we see that $h(0) = 0$. Gluing the function e^{2f} to the function 1 with a positive function satisfying the Neumann boundary condition on ∂M, we can assume that g is a metric such that $h(0) = 0$ and $R_{pq}(0) = 0$. Let ρ_0 be a small positive number, and denote

$$B^{n-1}_{\rho_0} = B^{n-1}_{\rho_0}(0) = \{x \mid x_1^2 + \cdots + x_{n-1}^2 \le \rho_0^2\}.$$

Let $\lambda_1, \cdots, \lambda_{n-1}$ be the elements of the diagonal of the second fundamental form of ∂M at 0. Then the vectors $\frac{\partial}{\partial y_i}(0)$ are the principal directions, and the λ_i are the principal curvatures. In these coordinates, ∂M is given near 0 by the equation $y_n = f(y_1, \cdots, y_{n-1})$, where $f(y_1, \cdots, y_{n-1}) = \frac{1}{2}\sum_{i=1}^{n-1} \lambda_i y_i^2 + O(|y|^3)$. Consider the cylinder

$$C_{\rho_0} = C_{\rho_0}(0) = \{y = (x, x_n) \mid x_1^2 + \cdots + x_{n-1}^2 < \rho_0^2, -\rho_0 < x_n < \rho_0\}$$

and

$$C^+_{\rho_0} = C^+_{\rho_0}(0) = \{y = (x, x_n) \in C_{\rho_0} \mid x_n > 0\}.$$

Let $\psi(s)$ be a piecewise smooth decreasing function $|s|$ which satisfies $\psi(s) = 1$ for $|s| \le \rho_0$, $\psi(s) = 0$ for $|s| \ge 2\rho_0$, and $|\psi'(s)| \le \rho_0^{-1}$ for $\rho_0 \le |s| \le 2\rho_0$.

Let $s(y) = sa(x, x_n) = \max\{|x|, |x_n|\}$, where $|x|^2 = x_1^2 + \cdots + x_{n-1}^2$. Consider the piecewise smooth test function φ on M defined as

$\varphi = u(\psi \circ s)$ where

$$u = \left(\frac{\varepsilon}{\varepsilon^2 + |y|^2 - \delta y_1^2}\right)^{(n-2)/2}.$$

A consequence of Lemmas 3.2–3.5 and inequality (3.5) below is that there exists a constant c such that

$$\int_{C_{\rho_0}\cap M} |\nabla\varphi|_g^2\, dv \le n(n-2)\int_{C_{\rho_0}^+} \frac{\varepsilon^n\, dy}{(\varepsilon^2 + |y|^2)^n} + A\delta + B\lambda_1\varepsilon\delta + D\delta^2 + cE_3,$$

where the constants A, B, and D and the lower order term E_3 are given in Lemma 3.5. In Lemma 3.8 below, we estimate the integral on the right-hand side of the above inequality. Use of that lemma yields

$$\begin{aligned}\int_{C_{\rho_0}\cap M} |\nabla\varphi|_g^2\, dv \le Q(S_+^n)&\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\, dv\right)^{(n-2)/n}\\ &+ (n-2)^2 F\left(1 + \frac{F}{nI}\right)\\ &+ A\delta + B\lambda_1\varepsilon\delta + D\delta^2 + cE_3 + cE_4,\end{aligned}$$

where F and the lower order term E_4 are given in Lemma 3.7, and I is given in Lemma 3.8. By means of Lemmas 3.9–3.11 we get

$$\begin{aligned}E(\varphi) \le Q(S_+^n)&\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\, dv\right)^{(n-2)/n} + A_1\delta + B_1\lambda_1\varepsilon\delta + D_1\delta^2\\ &+ c\varepsilon^2|\delta| + c\varepsilon^n|\delta|\rho_0^{-n} + c|\delta|^3 + c\varepsilon^3 + c\varepsilon\delta^2 + c\varepsilon^n\rho_0^{2-n} + c\varepsilon|\delta|\rho_0\\ &+ c\varepsilon^{n-2}|\delta|\rho_0 + c\varepsilon^2\rho_0 + c\varepsilon^2|\delta|\log(\rho_0/\varepsilon) + c\varepsilon^{n-2}\rho_0^{2-n},\end{aligned}$$

where

$$A_1 = (n-2)^2 n\left[\int_{\mathbb{R}_+^n} \frac{|y|^2 y_1^2\, dy}{(1+|y|^2)^{n+1}} - \frac{2}{n}\int_{\mathbb{R}_+^n} \frac{y_1^2\, dy}{(1+|y|^2)^n} - \int_{\mathbb{R}_+^n} \frac{y^2\, dy}{(1+|y|^2)^{n+1}}\right],$$

$$\begin{aligned}B_1 = -\frac{(n-2)^2 n\sigma_{n-2}}{(n-1)(n+1)}&\left[\int_0^\infty \frac{r^{n+4}\, dr}{(1+r^2)^{n+1}} - \frac{2}{n}\int_0^\infty \frac{r^{n+2}\, dr}{(1+r^2)^n}\right.\\ &\left.- \int_0^\infty \frac{r^{n+2}\, dr}{(1+r^2)^{n+1}}\right],\end{aligned}$$

and D_1 is a constant which depends only on n. If the constant $A_1 \neq 0$, we can choose $\delta \neq 0$ such that the term $A_1\delta$ in the above expansion is negative. By first fixing ρ_0 small and then choosing ε much smaller than ρ_0 and δ such that $|\delta| = \varepsilon$, from the last equality it follows that

$$E(\varphi) < \left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dv\right)^{(n-2)/n} < \left(\int_M \varphi^{2n/(n-2)}\,dv\right)^{(n-2)/n}.$$

If $A_1 = 0$ (which actually is the case, as the reader can check easily, or just realize that if $A_1 \neq 0$, then the above proof will show that the test function φ defined in $\mathbb{R}^n_+$ has less energy than the function u_ε which is a contradiction to (3.2), then we first notice that Lemma 3.12 implies that $B_1 < 0$. Since $\sum \lambda_i = 0$ and not all λ_i are zero, because the point is not umbilic, we can assume that $\lambda_1 > 0$. Multiplying the metric by a constant and using the conformal invariance of the Sobolev quotient, expressed in Proposition 1.1, we can assume λ_1 is as large as we want. In particular if $n > 4$, we require $-\lambda_1 B_1 > D_1$. So when $n > 4$, we choose first ρ_0 small, and then ε much smaller than ρ_0, and finally we set $\delta = \varepsilon$ so that the leading term in the last asymptotic expansion is of order ε^2 and the coefficient is negative. Thus the above last inequality is true, proving our theorem.

When $n = 4$, we need to improve the last error term in the last asymptotic expansion. In order to do that we use as a cutoff function the Green's function G, associated to the conformal Laplacian but with respect to the Neumann condition but not the boundary condition given in (2). We assume that the neighborhood of 0, where we glue the function e^{2f} to the function 1, is done in a small neighborhood so that we can further assume that $h = 0$ on $\partial M - B^{n-1}_{\rho_0}$.

If $Q(M) \leq 0$, the theorem is trivial. So we assume that $Q(M) > 0$. Then Proposition 1.3 and Lemma 1.1 imply that $\lambda_1(L) > 0$, so that for $\rho_0 > 0$ small enough, the first eigenvalue for the conformal Laplacian with respect to the Neumann boundary condition is positive. Hence the operator L with respect to Neumann boundary condition has a positive Green's function G.

In the appendix we show that G has the following asymptotic expansion for small y:

$$G(y) = |y|^{-2} + N + \alpha(y),$$

where $\alpha(y) = O(|y|\log(|y|))$. We define the test function φ as follows:

$$\varphi(y) = \begin{cases} u(y) = \dfrac{\varepsilon}{\varepsilon^2 + |y|^2 - \delta y_1^2 \psi(|y|)} & \text{for } y \in B_{2\rho_0} \cap M, \\ \varepsilon_0(G(y) - \alpha(y)\psi(2|y|)) & \text{for } y \in (B_{4\rho_0} - B_{2\rho_0}) \cap M, \\ \varepsilon_0 G & \text{for } y \in M - B_{+4\rho_0}. \end{cases}$$

In order for the function φ to be continuous across $\partial B_{2\rho_0}$ we must require ε to satisfy

$$\frac{\varepsilon}{\varepsilon^2 + (2\rho_0)^2} = \varepsilon_0((2\rho_0)^{-2} + N).$$

Because of Proposition 2.2, Lemmas 3.2–3.8 hold, if we replace C_{ρ_0} by B_{ρ_0}.

Using Lemma 3.13 and arguing as in Lemma 3.5 but incorporating the boundary term (3.18), we get a constant c such that

$$\int_{B_{2\rho_0}\cap M} |\nabla\varphi|_g^2\, dv \le 8\int_{B^+_{2\rho_0}} \frac{\varepsilon^4\, dy}{(\varepsilon^2+|y|^2)^4} + \int_{\partial B^+_{2\rho_0}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta}\, d\sigma + A\delta + B\lambda_1\varepsilon\delta + D\delta^2 + cE_3,$$

where the constants A, B, and D and the lower order term E_3 are given in Lemma 3.5. By means of Lemma 3.14 and the same argument as in Lemma 3.8 we obtain

$$\int_{B_{2\rho_0}\cap M} |\nabla\varphi|_g^2\, dv \le Q(S^4_+)\left(\int_{B_{2\rho_0}\cap M}\varphi^4\, dv\right)^{1/2} + \int_{\partial B^+_{2\rho_0}} u_\varepsilon\frac{\partial u_\varepsilon}{\partial\eta}\, d\sigma + 4F + I^{-1}F^2 + A\delta + B\lambda_1\varepsilon\delta + D\delta^2 + cE_3 + cE_4,$$

where F and the lower order term E_4 are given in Lemma 3.7, and I is given in Lemma 3.8. Using Lemma 3.11 and the same estimates as in §4, after (4.9) where we replace ρ_0 by $2\rho_0$, we get

$$\begin{aligned} E(\varphi) \le{}& Q(S^4_+)\left(\int_{B_{2\rho_0}\cap M}\varphi^4\, dv\right)^{1/2} + A_1\delta + B_1\lambda_1\varepsilon\delta + D_1\delta^2 - 2N\sigma_3^+\varepsilon_0^2 \\ &+ c\varepsilon_0^2\varepsilon^2\rho_0^{-4} + v\varepsilon_0^2\rho_0 + c\varepsilon^2\rho_0 + c\varepsilon^3 \\ &+ c\varepsilon^3\log(\rho_0/\varepsilon) + c\rho_0\log(\rho_0^{-1})\varepsilon_0^2 \\ &+ c\varepsilon^2|\delta| + c\varepsilon^4|\delta|\rho_0^{-4} + c|\delta|^3 + c\varepsilon\delta^2 \\ &+ c\varepsilon|\delta|\rho_0 + c\varepsilon^2|\delta|\log(\rho_0/\varepsilon) + c\varepsilon^4\rho_0^{-2}, \end{aligned}$$

where A_1 and B_1 are given above, and D_1 is a constant that depends only on the dimension. If $A_1 \ne 0$ we argue as before. If $A_1 = 0$, since 0 is not an umbilic point, and $\sum\lambda_i = 0$, we assume as before that $\lambda_1 > 0$.

Since the Sobolev quotient of M is positive, using the double manifold of M (see Appendix) and Schoen's perturbation Lemma 1 in [10], one sees that $N \ge 0$. Multiplying the metric by a constant if necessary we can assume that $-\lambda_1 B_1 > D_1$. This is possible to do because $B_1 < 0$ (Lemma

3.12). Fixing ρ_0 small and then choosing $\delta = \varepsilon$ much smaller, from the definition of φ it follows that $\varepsilon \approx \varepsilon_0$. Since all error terms are dominated by the negative term in ε^2, we have

$$E(\varphi) < Q(S_+^4)\left(\int_{B_{2\rho_0}\cap M} \varphi^4\, dv\right)^{1/2} < Q(S_+^4)\left(\int_M \varphi^4\, dv\right)^{1/2},$$

and the proof of Theorem 3.1 is complete when $n = 4$.

In the rest of this section we prove the inequalities used in the above argument in a combination of lemmas. In order to do that we let $1 \le p, q, r, s \le n$ and $1 \le i, j, k, l \le n-1$, and let g_{pq} denotes the coefficients of the metric g with respect to the coordinates $(y_1, \cdots, y_n)$. It is well known that g^{pq}, the inverse of the metric g, in normal coordinates has the following asymptotic expansion

$$g^{pq} = \delta^{pq} - \tfrac{1}{3}R_{prsq}y_r y_s + O(|y|^3), \tag{3.3}$$

where R_{prsq} denotes the coefficients of the Riemann curvature tensor evaluated at the point 0. Also, it is well known that $\sqrt{g} = \sqrt{\det(g_{pq})}$ has the following asymptotic expansion in normal coordinates:

$$\sqrt{g} = 1 - \tfrac{1}{6}R_{pq}y_p y_q + O(|y|^3), \tag{3.4}$$

where R_{pq} denotes the coefficients of the Ricci tensor evaluated at the point 0.

Observe that

$$\int_{C_{\rho_0}\cap M} |\nabla\varphi|_g^2\, dv = \int_{C_{\rho_0}\cap M} g^{pq}\varphi_p\varphi_q\, dv,$$

where $\varphi_p = \frac{\partial\varphi}{\partial y_p}$. Then using (3.3) we obtain

$$\begin{aligned}\int_{C_{\rho_0}\cap M} |\nabla\varphi|_g^2\, dv \le \int_{C_{\rho_0}\cap M} |\nabla\varphi|^2\, dv - \frac{1}{3}R_{prsq}\int_{C_{\rho_0}\cap M} \varphi_p\varphi_q y_r y_s\, dv \\ + c\int_{C_{\rho_0}\cap M} |y|^3|\nabla\varphi|^2\, dv,\end{aligned} \tag{3.5}$$

where $|\nabla\varphi|^2 = \varphi_1^2 + \cdots + \varphi_n^2$.

A straightforward calculation shows that on $C_{\rho_0}\cap M$

$$|\nabla\varphi|^2 = \frac{(n-2)^2\varepsilon^{n-2}(|y|^2 - 2\delta y_1^2 + \delta^2 y_1^2)}{(\varepsilon^2 + |y|^2 - \delta y_1^2)^n}. \tag{3.6}$$

Lemma 3.1. *For any numbers m, k and $1 \le p \le n$ we have*

$$\int_{C_{\rho_0}} \frac{y_p^m\, dy}{(\varepsilon^2 + |y|^2)^{n-k}} = \varepsilon^{m+2k-n} \int_{C_{\rho_0/\varepsilon}} \frac{z^m\, dz}{(1_+|z|^2)^{n-k}}.$$

Proof. The change of variables $y = \varepsilon x$ yields our result.

Remark. Lemma 3.1 is also valid of we use B_{ρ_0} instead of C_{ρ_0}.

In what follows in this section, c represents a constant independent of ε, δ and ρ_0. In the next lemma we estimate the last integral in (3.5).

Lemma 3.2. *There exists a constant c such that*

$$\int_{C_{\rho_0} \cap M} |y|^3 |\nabla \varphi|^2\, dy \le E_1,$$

where $E_1 = c\varepsilon^3 + c\varepsilon^2 \rho_0$.

Proof. Using (3.6) and Lemma 3.1 we get

$$\begin{aligned}\int_{C_{\rho_0} \cap M} |y|^3 |\nabla \varphi|^2\, dy &\le c \int_{C_{\rho_0}} \frac{\varepsilon^{n-2} |y|^5\, dy}{(\varepsilon^2 + |y|^2)^n} \le c\varepsilon^2 \int_{C_{\rho_0/\varepsilon}} \frac{|y|^5\, dy}{(1 + |y|^2)^n} \\ &\le c\varepsilon^2 \int_0^{\rho_0/\varepsilon} \frac{r^{n+4}\, dr}{(1 + r^2)^n} \le E_1.\end{aligned}$$

The second integral in the right-hand side of (3.5) is estimated in

Lemma 3.3. *There exists a constant c such that*

$$R_{prsq} \int_{C_{\rho_0} \cap M} \varphi_p \varphi_q y_r y_s\, dv \le E_2,$$

where $E_2 = c\varepsilon^2 |\delta| + c\varepsilon^2 |\delta| \log(\rho_0/\varepsilon)$.

Proof. On $C_{\rho_0} \cap M$, $\varphi = u$ by definition. Thus

$$\begin{aligned}R_{prsq} \int_{C_{\rho_0} \cap M} \varphi_p \varphi_q y_r y_s\, dv &\le (n-2)^2 \varepsilon^{n-2} R_{prsq} \int_{C_{\rho_0} \cap M} \frac{y_p y_q y_r y_s\, dv}{(\varepsilon^2 + |y|^2 - \delta y_1^2)^n} \\ &\quad + c\varepsilon^{n-2} |\delta| \int_{C_{\rho_0}} \frac{|y|^4\, dy}{(\varepsilon^2 + |y|^2)^n}.\end{aligned}$$

Using Taylor's theorem and Lemma 3.1 in the above inequality we get

$$\begin{aligned}R_{prsq} \int_{C_{\rho_0} \cap M} \varphi_p \varphi_q y_r y_s\, dv &\le (n-2)^2 \varepsilon^{n-2} R_{prsq} \int_{C_{\rho_0} \cap M} \frac{y_p y_q y_r y_s\, dv}{(\varepsilon^2 + |y|^2)^n} \\ &\quad + c\varepsilon^{n-2} |\delta| \int_{C_{\rho_0} \cap M} \frac{|y|^{\varepsilon}\, dy}{(\varepsilon^2 + |y|^2)^{n+1}} + c\varepsilon^2 |\delta| \int_{C_{\rho_0/\varepsilon}} \frac{|y|^4\, dy}{(1 + |y|^2)^n}.\end{aligned}$$

The symmetries of the Riemann curvature tensor implies that the first integral on the right-hand side of the above expression vanishes. Applying Lemma 3.1 to the second integral on the right-hand side yields

$$\begin{aligned} R_{prsq}\int_{C_{\rho_0}\cap M}\varphi_p\varphi_q y_r y_s\,dv &\le c\varepsilon^2|\delta|\int_{C_{\rho_0/\varepsilon}}\frac{|y|^6\,dy}{(1+|y|^2)^{n+1}} \\ &\quad + c\varepsilon^2|\delta|\int_{C_{\rho_0/\varepsilon}}\frac{|y|^4\,dy}{(1+|y|^2)^{n}} \\ &\le c\varepsilon^2|\delta|\int_0^{\rho_0/\varepsilon}\frac{r^{n+5}\,dr}{(1+r^2)^{n+1}} \\ &\quad + c\varepsilon^2|\delta|\int_0^{\rho_0/\varepsilon}\frac{r^{n+3}\,dr}{(1+r^2)^{n}} \le E_2, \end{aligned}$$

and hence our lemma.

The first integral in the right-hand side of (3.5) is estimated in

Lemma 3.4. *The following inequality holds*:

$$\int_{C_{\rho_0}\cap M}|\nabla\varphi|^2\,dv \le \int_{C_{\rho_0}\cap M}|\nabla\varphi|^2\,dy + E_1.$$

where E_1 *is given in Lemma* 3.2.

Proof. The asymptotic formula (3.4), the fact that $R_{pq}(0) = 0$ and Lemma 3.2 yield our Lemma.

We estimate the integral on the right-hand side in Lemma 3.4 in

Lemma 3.5. *There exists a constant* c *such that*

$$\begin{aligned} \int_{C_{\rho_0}\cap M}|\nabla\varphi|^2\,dy &\le n(n-2)\int_{C^+_{\rho_0}}\frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n} \\ &\quad + A\delta + B\lambda_1\varepsilon\delta + D\delta^2 + cE_3, \end{aligned}$$

where

$$A = (n-2)^2\left[n\int_{\mathbb{R}^n_+}\frac{|y|^2y_1^2\,dy}{(1+|y|^2)^{n+1}} - 2\int_{\mathbb{R}^n_+}\frac{y_1^2\,dy}{(1+|y|^2)^n}\right],$$

$$B = -\frac{(n-2)^2\sigma_{n-2}}{(n-1)(n+1)}\left[\int_0^\infty\frac{r^{n+4}\,dr}{(1+r^2)^{n+1}} - 2\int_0^\infty\frac{r^{n+2}\,dr}{(1+r^2)^n}\right],$$

$$\begin{aligned} D = n(n-2)^2\Bigg[&\frac{n+1}{2}\int_{\mathbb{R}^n_+}\frac{|y|^2y_1^4\,dy}{(1+|y|^2)^{n+2}} - 2\int_{\mathbb{R}^n_+}\frac{y^4\,dy}{(1+|y|^2)^{n+1}} \\ &+ \frac{1}{n}\int_{\mathbb{R}^n_+}\frac{y_1^2\,dy}{(1+|y|^2)^n}\Bigg], \end{aligned}$$

and

$$E_3 = \varepsilon|\delta|\rho_0 + \varepsilon^{n-2}\rho_0^{2-n}|\delta| + |\delta|^3 + \varepsilon^3 + \varepsilon^2\rho_0 + \varepsilon\delta^2 + c\varepsilon^2|\delta|.$$

Proof. (3.6) implies that

$$\begin{aligned}\frac{1}{(n-2)^2}\int_{C_{\rho_0}\cap M}|\nabla\varphi|^2\,dy = \int_{C_{\rho_0}\cap M}&\frac{\varepsilon^{n-2}|y|^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n}\\ &-2\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n}\\ &+\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n}.\end{aligned}\tag{3.7}$$

We estimate the three integrals on the right-hand side of (3.7). By applying Taylor's theorem to the first integral we obtain

$$\begin{aligned}&\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}|y|^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n}\\ &\quad\le\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}|y|^2\,dy}{(\varepsilon^2+|y|^2)^n}+n\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}|y|^2y_1^2\,dy}{(\varepsilon^2+|y|^2)^{n+1}}\\ &\qquad+\frac{n(n+1)}{2}\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta^2|y|^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+2}}+c\varepsilon^{n-2}|\delta|^3\int_{C_{\rho_0}}\frac{|y|^8\,dy}{(\varepsilon^2+|y|^2)^{n+3}}\\ &\quad\le\int_{C^+_{\rho_0}}\frac{\varepsilon^{n-2}|y|^2\,dy}{(\varepsilon^2+|y|^2)^n}-\varepsilon^{n-2}\int_{B^{n-1}_{\rho_0}}\int_0^f\frac{(|x|^2+x_n^2)\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}\\ &\qquad+n\varepsilon^{n-2}\delta\left[\int_{C^+_{\rho_0}}\frac{|y|^2y_1^2\,dy}{(\varepsilon^2+|y|^2)^{n+1}}-\int_{B^{n-1}_{\rho_0}}\int_0^f\frac{|x|^2+x_n^2)x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}}\right.\\ &\qquad\qquad\left.+\frac{(n+1)}{2}\delta\int_{C^+_{\rho_0}}\frac{|y|^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+2}}+\frac{c\delta}{n}\int_{B^{n-1}_{\rho_0}}\frac{|x|^8\,dx}{(\varepsilon^2+x|^2)^{n+2}}\right]\\ &\qquad+c|\delta|^3\int_{C_{\rho_0/\varepsilon}}\frac{|y|^8\,dy}{(1+|y|^2)^{n+3}}.\end{aligned}\tag{3.8}$$

Observe that

$$
\begin{aligned}
&-\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{(|x|^2+x_n^2)\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}\\
&\le -\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{|x|^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}+c\int_{B^{n-1}_{\rho_0}}\frac{|x|^6\,dx}{(\varepsilon^2+|x|^2)^n}\\
&\le -\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{|x|^2\,dx\,dx_n}{(\varepsilon^2+|x|^2)^n}+c\int_{B^{n-1}_{\rho_0}}\frac{|x|^8\,dx}{(\varepsilon^2+|x|^2)^{n+1}}\\
&\quad+c\int_{B^{n-1}_{\rho_0}}\frac{|x|^6\,dx}{(\varepsilon^2+|x|^2)^n}\\
&\le \int_{B^{n-1}_{\rho_0}}\frac{|x|^2(\frac{1}{2}\lambda_i x_i^2+c_{ijk}x_ix_jx_k)\,dx}{(\varepsilon^2+|x|^2)^n}+c\int_{B^{n-1}_{\rho_0}}\frac{|x|^6\,dx}{(\varepsilon^2+|x|^2)^n}\\
&\quad+c\int_{B^{n-1}_{\rho_0}}\frac{|x|^8\,dx}{(\varepsilon^2+|x|^2)^{n+1}}
\end{aligned}
$$

The symmetries of the ball and the fact that $\sum \lambda_i = 0$ imply that the first integral on the right-hand side of the last inequality vanishes. For the other two integrals we use Lemma 3.1 to get

$$
-\varepsilon^{n-2}\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{(|x|^2+x_n^2)\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}\le c\varepsilon^2+c\varepsilon^2\rho_0. \tag{3.9}
$$

Since

$$
\begin{aligned}
&-\delta\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{(|x|^2+x_n^2)\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}\\
&\le -\delta\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{|x|^2x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}}+|\delta|\int_{B^{n-1}_{\rho_0}}\frac{|x|^8\,dx}{(\varepsilon^2+|x|^2)^{n+1}}\\
&\le -\delta\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{(|x|^2+x_n^2)\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}+\frac{c|\delta|}{\varepsilon^{n-5}}\int_{B^{n-1}_{\rho_0/\varepsilon}}\frac{|x|^8\,dx}{(1+|x|^2)^{n+1}}
\end{aligned}
$$

and Taylor's theorem and Lemma 3.1 imply

$$
\begin{aligned}
&-\delta \int_{B^{n-1}_{\rho_0}} \int_0^f \frac{|x|^2 x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}} \\
&\quad \le -\delta \int_{B^{n-1}_{\rho_0}} \int_0^f \frac{|x|^2+x_1^2\,dx}{(\varepsilon^2+|x|^2)^{n+1}} + c|\delta| \int_{B^{n-1}_{\rho_0}} \frac{|x|^{10}\,dx}{(\varepsilon^2+|x|^2)^{n+2}} \\
&\qquad - \delta \int_{B^{n-1}_{\rho_0}} \frac{(|x|^2x_1^2)(\frac{1}{2}\lambda_i x_i^2)\,dx}{(\varepsilon^2+|x|^2)^{n+1}} + c|\delta| \int_{B^{n-1}_{\rho_0}} \frac{|x|^7\,dx}{(\varepsilon^2+|x|^2)^{n+1}} \\
&\qquad + c|\delta| \int_{B^{n-1}_{\rho_0}} \frac{|x|^{10}\,dx}{(\varepsilon^2+|x|^2)^{n+2}} \\
&\quad \le -\frac{1}{2\varepsilon^{n-3}}\delta\lambda_i \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^2x_1^2x_i^2\,dx}{(1+|x|^2)^{n+1}} + \frac{1}{\varepsilon^{n-4}}c|\delta| \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^7\,dx}{(1+|x|^2)^{n+1}} \\
&\qquad + \frac{1}{\varepsilon^{n-5}}c|\delta| \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^{10}\,dx}{(1+|x|^2)^{n+2}},
\end{aligned}
$$

we have

(3.10)

$$
\begin{aligned}
&-(n-2)^2 n\varepsilon^{n-2} \int_{B^{n-1}_{\rho_0}} \int_0^f \frac{(|x|^2+x_n^2)x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}} \\
&\qquad \le -\frac{(n-2)^2}{2} n\varepsilon|\delta|\lambda_i \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^2x_1^2x_i^2\,dx}{(1+|x|^2)^{n+1}} + c\varepsilon^2|\delta|\rho_0 + c\varepsilon^2|\delta|.
\end{aligned}
$$

Now we estimate the integral on the right-hand side of (3.10).

$$
\begin{aligned}
&-\delta\lambda_i \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^2x_1^2x_i^2\,dx}{(1+|x|^2)^{n+1}} \\
&\quad \le \delta\lambda_i \int_{\mathbb{R}^{n-1}_+} \frac{|x|^2x_1^2x_i^2\,dx}{(1+|x|^2)^{n+1}} + c\varepsilon^{n-3}|\delta|\rho_0^{3-n} \\
&\quad \le -\delta\lambda_1 \int_{\mathbb{R}^{n-1}_+} \frac{|x|^2x_1^4\,dx}{(1+|x|^2)^{n+1}} \\
&\qquad -\delta\sum_{k=2}^{n-1}\lambda_k \int_{\mathbb{R}^{n-1}} \frac{|x|^2x_1^2x_k^2\,dx}{(1+|x|^2)^{n+1}} + c\varepsilon^{n-3}|\delta|\rho_0^{3-n} \\
&\quad \le -\delta \int_0^\infty \frac{r^{n+4}\,dr}{(1+r^2)^{n+1}} \left(\lambda_1 \int_{S^{n-2}} \xi_1^4\,d\xi + \sum_{k=2}^{n-1}\lambda_k \int_{S^{n-2}} \xi_1^2\xi_k^2\,d\xi\right) \\
&\qquad + c\varepsilon^{n-3}|\delta|\rho_0^{3-n}.
\end{aligned}
$$

A straightforward computation shows that

$$\int_{S^{n-2}} \xi_i^2 \xi_j^2 \, d\xi = \begin{cases} \dfrac{\sigma_{n-2}}{(n-1)(n+1)} & i \neq j, \\ \dfrac{3\sigma_{n-2}}{(n-1)(n+1)} & i = j. \end{cases} \tag{3.11}$$

Since $\lambda_1 = -\sum_{k=2}^{n-1} \lambda_k$, using (3.11) yields

$$-\frac{(n-2)^2 n}{2} \varepsilon\delta\lambda \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^2 x_1^2 x_i^2 \, dx}{(1+|x|^2)^{n+1}} \leq -\frac{(n-2)^2 n\sigma_{n-2}}{(n-1)(n+1)} \varepsilon\delta\lambda_1 \int_0^\infty \frac{r^{n+4} \, dr}{(1+r^2)^{n+1}} + c\varepsilon^{n-2}|\delta|\rho_0^{3-n}.$$

By means of (3.10), the last inequality (3.8), (3.9), and Lemma 3.1 we get

$$\begin{aligned} &(n-2)^2 \int_{C_{\rho_0} \cap M} \frac{\varepsilon^{n-2}|y|^2 \, dy}{(\varepsilon^2 + |y|^2 - \delta y_1^2)^n} \\ &\leq (n-2)^2 \left[\int_{C^+_{\rho_0}} \frac{\varepsilon^{n-2}|y|^2 \, dy}{(\varepsilon^2 + |y|^2)^n} + n\delta \int_{C^+_{\rho_0/\varepsilon}} \frac{|y|^2 y_1^2 \, dy}{(1+|y|^2)^{n+1}} \right. \\ &\qquad - \frac{n\sigma_{n-2}}{(n-1)(n+1)} \varepsilon\delta\lambda_1 \int_0^\infty \frac{r^{n+4} \, dr}{(1+r^2)^{n+1}} \\ &\qquad \left. + \frac{n(n+1)}{2} \delta^2 \int_{C^+_{\rho_0/\varepsilon}} \frac{|y|^2 y_1^4 \, dy}{(1+|y|^2)^{n+2}} \right] \\ &\quad + c\varepsilon\delta^2 \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^8 \, dx}{(1+|x|^2)^{n+2}} + c|\delta|^3 + c\varepsilon^3 + c\varepsilon^2 \rho_0 \\ &\quad + c\varepsilon^2|\delta| + c\varepsilon^{n-2}|\delta|\rho_0^{3-n}, \end{aligned}$$

from which it follows easily that

$$
\begin{aligned}
&(n-2)^2 \int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}|y|^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n} \\
&\quad \le (n-2)^2 \left[\int_{C^+_{\rho_0}} \frac{\varepsilon^{n-2}|y|^2\,dy}{(\varepsilon^2+|y|^2)^n} + n\delta \int_{\mathbb{R}^n_+} \frac{|y|^2 y_1^2\,dy}{(1+|y|^2)^{n+1}} \right. \\
&\qquad\qquad - \frac{n\sigma_{n-2}}{(n-1)(n+1)}\varepsilon\delta\lambda_1 \int_0^\infty \frac{r^{n+4}\,dr}{(1+r^2)^{n+1}} \\
&\qquad\qquad \left. + \frac{n(n+1)}{2}\delta^2 \int_{\mathbb{R}^n_+} \frac{|y|^2 y_1^4\,dy}{(1+|y|^2)^{n+2}} \right] \\
&\quad + c\varepsilon\delta^2 + c|\delta|^3 + c\varepsilon^3 + c\varepsilon^2\rho_0 + c\varepsilon^2|\delta| + c\varepsilon^{n-2}|\delta|\rho_0^{2-n}.
\end{aligned} \tag{3.12}
$$

For the second integral on the right-hand side of (3.7) we use Taylor's theorem to obtain

$$
\begin{aligned}
-\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n} &= -\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2-|y|^2)^n} \\
&\quad - n\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta^2 y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}} + O(\delta^3),
\end{aligned} \tag{3.13}
$$

where we have used that

$$
\begin{aligned}
\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}|\delta|^3|y|^6\,dy}{(\varepsilon^2 1+|y|^2)^{n+2}} &\le \int_{C_{\rho_0}} \frac{\varepsilon^{n-2}|\delta|^3|y|^6\,dy}{(\varepsilon^2+|y|^2)^{n+2}} \\
&= \int_{C_{\rho_0}/\varepsilon} \frac{|\delta|^3|y|^6\,dy}{(1+|y|^2)^n+2} \le c|\delta|^3,
\end{aligned}
$$

where the last equality follows from Lemma 3.1.

For the first integral on the right-hand side of (3.13) we observe that

$$
\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} = \int_{C^+_{\rho_0}} \frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} - \int_{B^{n-1}_{\rho_0}} \int_0^f \frac{\varepsilon^{n-2}\delta x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n},
$$

and that Lemma 3.1 and Taylor's theorem imply respectively

$$
\int_{C^+_{\rho_0}} \frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} = \int_{C^+_{\rho_0}\cap M} \frac{\delta y_1^2\,dy}{(1+|y|^2)^n} = \int_{\mathbb{R}^n_+} \frac{\delta y_1^2}{(1+|y|^2)^n} + O(\varepsilon^{n-2}\rho_0^{2-n}\delta),
$$

$$\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{\varepsilon^{n-2}\delta x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n} \le \int_{B^{n-1}_{\rho_0}}\int_0^f \frac{\varepsilon^{n-2}\delta x_1^2\,dx\,dx_n}{(\varepsilon^3 2+|x|^2)^n} + c\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}|\delta||x|^8\,dx}{(\varepsilon^2+|x|^2)^{n+1}}$$
$$\le \frac{\lambda_i}{2}\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}\delta x_1^2 x_i^2\,dx}{(\varepsilon^2+|x|^2)^n} + c\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}|\delta|\,|x|^5\,dx}{(\varepsilon^2+|x|^2)^n} + c\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}|\delta|\,|x|^8\,dx}{(\varepsilon^2+|x|^2)^{n+1}}.$$

Thus,

$$-\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} \le -\int_{\mathbb{R}^n_+} \frac{\delta y_1^2\,dy}{(1+|y|^2)^n} + \frac{\lambda_i}{2}\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}\delta x_1^2 x_i^2\,dx}{(\varepsilon^2+|x|^2)^n} + c\varepsilon^{n-2}\rho_0^{2-n}|\delta| + c\varepsilon|\delta|\rho_0, \tag{3.14}$$

where we have used that

$$\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}|\delta|\,|x|^5\,dx}{(\varepsilon^2+|x|^2)^n} \le \varepsilon|\delta|\rho_0 \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^4\,dy}{(1+|x|^2)^n} \le c\varepsilon|\delta|\rho_0,$$

and

$$\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}|\delta|\,|x|^8\,dx}{(\varepsilon^2+|x|^2)^{n+1}} \le \varepsilon|\delta|\rho_0^2 \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|x|^6\,dx}{(\varepsilon^2+|x|^2)^{n+1}} \le c\varepsilon|\delta|\rho_0^2,$$

which follows from Lemma 3.1. Using Lemma 3.1 again we estimate the second integral on the right-hand side of (3.14) as follows:

$$\frac{\lambda_i}{2}\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^{n-2}\delta x_1^2 x_i^2\,dx}{(\varepsilon^2+|x|^2)^n} = \frac{\lambda_i}{2}\int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{\varepsilon\delta x_1^2 x_i^2\,dx}{(1+|x|^2)^n}$$
$$= \frac{\lambda_i}{2}\int_{\mathbb{R}^{n-1}} \frac{\varepsilon\delta x_1^2 x_i^2\,dx}{(1+|x|^2)^n} + O(\varepsilon^{n-2}\delta\rho_0^{3-n}).$$

Since

$$\frac{\lambda_i}{2}\int_{\mathbb{R}^{n-1}}\frac{x_1^2x_i^2\,dx}{(1+|x|^2)^n}=\frac{\lambda_i}{2}\int_{\mathbb{R}^{n-1}}\frac{x_1^4\,dx}{(1+|x|^2)^n}+\sum_{i=2}^{n-1}\frac{x_1^2x_i^2\,dx}{(1+|x|^2)^n}$$
$$=\frac{1}{2}\int_0^\infty\frac{r^{n+2}}{(1+r^2)^n}\left(\lambda_1\int_{S^{n-2}}\xi_1^4\,d\xi+\sum_{i=2}^{n-1}\lambda_i\int_{S^{n-2}}\xi_1^2\xi_i^2\,d\xi\right),$$

and $\lambda_1=-\sum_{i=2}^{n-1}\lambda_i$, using (3.11) we get

$$\frac{\lambda_i}{2}\int_{\mathbb{R}^{n-1}}\frac{x_1^2x_i^2\,dx}{(1+|x|^2)^n}=\frac{\lambda_1\sigma_{n-2}}{(n-1)(n+1)}\int_0^\infty\frac{r^{n+1}\,dr}{(1+r^2)^n}.$$

Substituting this in (3.14) and the resulting equation (3.13) yields

$$\begin{aligned}
-\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n}&\le-\int_{\mathbb{R}^n_+}\frac{\delta y_1^2\,dy}{(1+|y|^2)^n}\\
&+\frac{\lambda_1\varepsilon\delta\sigma_{n-2}}{(n-1)(n+1)}\int_0^\infty\frac{r^{n+2}\,dr}{(1+r^2)^n}-n\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}}\\
&+c|\delta|^3+c\varepsilon^{n-2}\rho_0^{2-n}|\delta|+c\varepsilon|\delta|\rho_0.
\end{aligned}\tag{3.15}$$

The relation

$$-\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}}=-\int_{C^+_{\rho_0}}\frac{\varepsilon^{n-2}\delta^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}}+\int_{B^{n-1}_{\rho_0}}\int_0^f\frac{\varepsilon^{n-2}\delta^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}},$$

the inequality

$$\begin{aligned}
\int_{B^{n-1}_{\rho_0}}\int_0^f\frac{\varepsilon^{n-2}\delta^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}}&\le c\int_{B^{n-1}_{\rho_0}}\frac{\varepsilon^{n-2}\delta^2|x|^6\,dx}{(\varepsilon^2+|x|^2)^{n+1}}\\
&\le c\varepsilon\delta^2\int_{B^{n-1}_{\rho_0/\varepsilon}}\frac{|x|^6\,dx}{(1+|x|^2)^{n+1}}\le c\varepsilon\delta^2
\end{aligned}$$

obtained by means of Lemma 3.1, and

$$\begin{aligned}
-\int_{C^+_{\rho_0}}\frac{\varepsilon^{n-2}\delta^2y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}}&=-\int_{C^+_{\rho_0/\varepsilon}}\frac{\delta^2y_1^4\,dy}{(1+|y|^2)^{n+1}}\\
&=-\int_{\mathbb{R}^n_+}\frac{\delta^2y_1^4\,dy}{(1+|y|^2)^{n+1}}+O(\delta^2\varepsilon^{n-2}\rho_0^{2-n})
\end{aligned}$$

imply that

$$-\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta^2 y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}} \le -\int_{\mathbb{R}^n_+} \frac{\delta^2 y_1^4\,dy}{(1+|y|^2)^{n+1}} + c\varepsilon\delta^2 + c\varepsilon^{n-2}\delta^2\rho_0^{2-n}.$$

Multiplying (3.15) by $2(n-2)^n$ and using the last inequality we obtain

$$\begin{aligned}
&-2(n-2)^2\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n}\\
&\quad\le -2(n-2)^2\int_{\mathbb{R}^n_+}\frac{\delta y_1^2\,dy}{(1+|y|^2)^n}\\
&\qquad+\frac{2(n-2)^2\sigma_{n-2}}{(n-1)(n+1)}\lambda_1\varepsilon\delta\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^n}\\
&\qquad-2n(n-2)2\delta^2\int_{\mathbb{R}^n_+}\frac{y_1^4\,dy}{(1+|y|^2)^{n+1}}\\
&\qquad+c\varepsilon|\delta|\rho_0+c|\delta|^3+c\varepsilon\delta^2+c\varepsilon^{n-2}|\delta|\rho_0^{2-n}.
\end{aligned}\tag{3.16}$$

For the third integral on the right-hand side of (3.7) we use Taylor's theorem to get

$$\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta y_1^2\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n} \le \int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} + c\int_{C_{\rho_0}}\frac{\varepsilon^{n-2}|\delta|^3|y|^4\,dy}{(\varepsilon^2+|y|^2)^{n+1}}.$$

To estimate the first integral on the right-hand side of the above inequality we notice that

$$\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2)^n}\int_{C^+_{\rho_0}}\frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} - \int_{B^{n-1}_{\rho_0}}\int_0^f \frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2)^n},$$

and that Lemma 3.1 implies

$$\int_{C^+_{\rho_0}}\frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} = \int_{C^+_{\rho_0/\varepsilon}}\frac{\delta^2 y_1^2\,dy}{(1+|y|^2)^n} = \int_{\mathbb{R}^n_+}\frac{\delta^2 y_1^2\,dy}{(1+|y|^2)^n} + O(\varepsilon^{n-2}\rho_0^{2-n}\delta^2)$$

and

$$\int_{B^{n-1}_{\rho_0}}\int_0^f\frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} \le c\int_{B^{n-1}_{\rho_0/\varepsilon}}\frac{\varepsilon^{n-2}\delta^2|x|^4\,dx}{(\varepsilon^2+|x|^2)^n} \le c\int_{B^{n-1}_{\rho_0/\varepsilon}}\frac{\delta^2\varepsilon|x|^4\,dx}{(1+|x|^2)^n} \le c\delta^2\varepsilon,$$

so that

$$(n-2)^2\int_{C_{\rho_0}\cap M}\frac{\varepsilon^{n-2}\delta^2 y_1^2\,dy}{(\varepsilon^2+|y|^2)^n} \le (n-2)^2\int_{\mathbb{R}^n_+}\frac{\delta^2 y_i^2\,dy}{(1+|y|^2)^n} + c\delta^2\varepsilon + c\varepsilon^{n-2}\rho_0^{2-n}\delta^2.$$

Using the last inequality we obtain

$$
(3.17)\qquad \begin{aligned}
&(n-2)^2 \int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}\delta^2 y_1^2\, dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^n} \\
&\quad \le (n-2)^2 \int_{\mathbb{R}^n_+} \frac{\delta^2 y_1^2\, dy}{(1+|y|^2)^n} + c\delta^2\varepsilon + c\varepsilon^{n-2}\rho_0^{2-n}\delta^2 + c|\delta|^3,
\end{aligned}
$$

where we have used the fact

$$
\int_{C_{\rho_0}\cap M} \frac{\varepsilon^{n-2}|\delta|^3|y|^4\, dy}{(\varepsilon^2+|y|^2)^{n+1}} \le \int_{C_{\rho_0}} \frac{\varepsilon^{n-2}|\delta|^3|y|^4\, dy}{(\varepsilon^2+|y|^2)^{n+1}} \le \int_{C_{\rho_0/\varepsilon}} \frac{|\delta|^3|y|^4\, dy}{(1+|y|^2)^{n+1}} \le c|\delta|^3.
$$

To complete the proof of our lemma, multiplying (3.1) by u_ε and integrating by parts we get

$$
(3.18)\qquad \int_{C^+_{\rho_0}} |\nabla u_\varepsilon|^2\, dy = n(n-2)\int_{C^+_{\rho_0}} u_\varepsilon^{2n/(n-2)}\, dy + \int_{\partial C^+_{\rho_0}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta}\, d\sigma.
$$

Since $\frac{\partial u_\varepsilon}{\partial \eta} \le 0$ on $\partial C^+_{\rho_0}$ we have

$$
(n-2)^2 \int_{C^+_{\rho_0}} \frac{\varepsilon^{n-2}|y|^2\, dy}{(\varepsilon^2+|y|^2)^n} \le n(n-2)\int_{C^+_{\rho_0}} \frac{\varepsilon^n\, dy}{(\varepsilon^2+|y|^2)^n}.
$$

Substituting this in (3.12) and using (3.16), (3.17), (3.12), (3.7) we thus obtain our lemma.

In the next lemma we calculate the error terms introduced by the metric when we calculate the integral of the function $u^{2n/(n-2)}$, with respect to the Riemannian measure dv. More precisely, we have

Lemma 3.6. *The following asymptotic expansion holds*:

$$
\int_{C_{\rho_0}\cap M} u^{2n/(n-2)}\, dv = \int_{C_{\rho_0}\cap M} u^{2n/(n-2)}\, dy + O(\varepsilon^3).
$$

Proof. The lemma follows from the asymptotic expansion given in (3.4), the fact that $R_{pq}(0) = 0$ and the estimate

$$
\int_{C_{\rho_0}\cap M} u^{2n/(n-2)}|y|^3\, dy \le c\int_{C_{\rho_0}} \frac{\varepsilon^n|y|^3\, dy}{(\varepsilon^2+|y|^2)^n} \le c\int_{C_{\rho_0/\varepsilon}} \frac{\varepsilon^3|y|^3\, dy}{(1+|y|^2)^n} \le c\varepsilon^3.
$$

In the next lemma we express the first integral on the right-hand side of Lemma 3.5 in terms of the test function φ. For this purpose, we use our previous lemma, more precisely, the following lemma.

Lemma 3.7. *The following asymptotic expansion holds*:

$$\int_{C^+_{\rho_0}} \frac{\varepsilon^n\, dy}{(\varepsilon^2+|y|^2)^n} = \int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\, dv + F + O(E_4),$$

where

$$F = -n\int_{\mathbb{R}^n_+} \frac{\delta y_1^2\, dy}{(1+|y|^2)^{n+1}} + \frac{n\sigma_{n-2}\varepsilon\lambda_1}{(n-1)(n+1)}\int_0^\infty \frac{r^{n+2}\, dr}{(1+r^2)^{n+1}} - \frac{n(n+1)}{2}\int_{\mathbb{R}^n_+} \frac{\delta^2 y_1^4\, dy}{(1+|y|^2)^{n+2}},$$

and

$$E_4 = \varepsilon^2|\delta| + \varepsilon^n|\delta|\rho_0^{-n} + |\delta|^3 + \varepsilon^3 + \varepsilon\delta^2 + \varepsilon^n\rho_0^{2-n}.$$

Proof. Since $\varphi = u$ on $C_{\rho_0}\cap M$, Taylor's theorem implies

$$\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\, dy = \int_{C_{\rho_0}\cap M} \frac{\varepsilon^n\, dy}{(\varepsilon^2+|y|^2)^n} + n\int_{C_{\rho_0}\cap M} \frac{\delta\varepsilon^n y_1^2\, dy}{(\varepsilon^2+|y|^2)^{n+1}} + \frac{n(n+1)}{2}\int_{C_{\rho_0}\cap M} \frac{\delta^2\varepsilon^n y_1^4\, dy}{(\varepsilon^2+|y|^2)^{n+2}} + O(\delta^3),$$

where we have used the fact

$$\int_{C_{\rho_0}\cap M} \frac{|\delta|^3\varepsilon^n|y|^6\, dy}{(\varepsilon^2+|y|^2)^{n+3}} \le \int_{C_{\rho_0}} \frac{|\delta|^3\varepsilon^n|y|^6\, dy}{(\varepsilon^2+|y|^2)^{n+3}} \le c|\delta|^3.$$

Rewriting the above equation, gives

$$\begin{aligned}\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\, dy = {} & \int_{C^+_{\rho_0}} \frac{\varepsilon^n\, dy}{(\varepsilon^2+|y|^2)^n} - \int_{B^{n-1}_{\rho_0}}\int_0^f \frac{\varepsilon^n\, dx\, dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n} \\ & + n\int_{C^+_{\rho_0}} \frac{\delta\varepsilon^n y_1^2\, dy}{(\varepsilon^2+|y|^2)^{n+1}} \\ & - n\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{\delta\varepsilon^n x_1^2\, dx\, dx_n}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}} \qquad (3.19) \\ & + \frac{n(n+1)}{2}\int_{C^+_{\rho_0}} \frac{\delta^2\varepsilon^n y_1^4\, dy}{(\varepsilon^2+|y|^2)^{n+2}} + O(\delta^2\varepsilon) + O(\delta^3),\end{aligned}$$

where we have used the following estimate:

$$\int_{B^{n-1}_{\rho_0}}\int_0^{|f|}\frac{\delta^2\varepsilon^n y_1^4\,dy}{(\varepsilon^2+|y|^2)^{n+2}}\le c\int_{B^{n-1}_{\rho_0}}\frac{\delta^2\varepsilon^n|x|^6\,dx}{(\varepsilon^2+|x|^2)^{n+2}}$$
$$\le c\int_{B^{n-1}_{\rho_0/\varepsilon}}\frac{\varepsilon\delta^2|x|^6\,dx}{(1+|x|^2)^{n+2}}\le c\delta^2\varepsilon\,,$$

which can be obtained by using Lemma 3.1.

We estimate the two integrals in (3.19) that involves f. To this end, we use the Mean Value Theorem to get

$$\int_{B^{n-1}_{\rho_0}}\int_0^{f}\frac{\varepsilon^n\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}=\int_{B^{n-1}_{\rho_0}}\int_0^{f}\frac{\varepsilon^n\,dx\,dx_n}{(\varepsilon^2+|x|^2)^n}+O(\varepsilon^3)\,,$$

where we have used that

$$\int_{B^{n-1}_{\rho_0}}\frac{\varepsilon^n|x|^6\,dx}{(\varepsilon^2+|x|^2)^{n+1}}\le\int_{B^{n-1}_{\rho_0/\varepsilon}}\frac{\varepsilon^3|x|^6\,dx}{(1+|x|^2)^{n+1}}\le c\varepsilon^3.$$

By means of the Taylor's expansion for the function f, the symmetries of the ball, and the fact that $\sum\lambda_i=0$ we obtain

$$\int_{B^{n-1}_{\rho_0}}\int_0^{f}\frac{\varepsilon^n\,dx\,dx_n}{(\varepsilon^2+|x|^2)^n}=\int_{B^{n-1}_{\rho_0}}\frac{\varepsilon^n(\frac{1}{2}\lambda_i x_i^2+c_{ijk}x_ix_jx_k)\,dx}{(\varepsilon^2+|x|^2)^n}+O(\varepsilon^3)=O(\varepsilon^3)\,,$$

where we have used that

$$\int_{B^{n-1}_{\rho_0}}\frac{\varepsilon^n|x|^4\,dx}{(\varepsilon^2+|x|^2)^n}\le\int_{B^{n-1}_{\rho_0/\varepsilon}}\frac{\varepsilon^3|x|^4\,dx}{(1+|x|^2)^n}\le c\varepsilon^3\,,$$

which follows from Lemma 3.1.

Combining the last two asymptotic formulas yields

$$\int_{B^{n-1}_{\rho_0}}\int_0^{f}\frac{\varepsilon^n\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^n}=O(\varepsilon^3).\tag{3.20}$$

Similarly, for the fourth integral on the right-hand side of (3.19), we have

$$n\int_{B^{n-1}_{\rho_0}}\int_0^{f}\frac{\delta\varepsilon^n x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}}=n\int_{B^{n-1}_{\rho_0}}\int_0^{f}\frac{\delta\varepsilon^n x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2)^{n+1}}+O(\delta\varepsilon^3)$$
$$=n\int_{B^{n-1}_{\rho_0}}\frac{\delta\varepsilon^n x_1^2(\frac{1}{2}\lambda_i x_i^2)\,dx}{(\varepsilon^2+|x|^2)^{n+1}}+O(\varepsilon^2\delta)\,,\tag{3.21}$$

where we have used for the first equation, the estimate

$$\int_{B^{n-1}_{\rho_0}} \frac{|\delta|\varepsilon^n|x|^8\,dx}{(\varepsilon^2+|x|^2)^{n+2}} \le \int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{|\delta|\varepsilon^3|x|^8\,dx}{(1+|x|^2)^{n+2}} \le c|\delta|\varepsilon^3,$$

and, for the second equation,

$$\int_{B^{n-1}_{\rho_0}} \frac{\varepsilon^n|\delta|\,|x|^5\,dx}{(\varepsilon^2+|x|^2)^{n+1}} \le c\varepsilon^2|\delta|,$$

which follows from Lemma 3.1. We compute the integral on the right-hand side of (3.21) as follows:

$$\begin{aligned} n\int_{B^{n-1}_{\rho_0}} \frac{\delta\varepsilon^n x_1^2(\frac{1}{2}\lambda_i x_i^2)\,dx}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}} &= n\int_{B^{n-1}_{\rho_0/\varepsilon}} \frac{\delta\varepsilon_1^2(\frac{1}{2}\lambda_i x_i^2)\,dx}{(1+|x|^2)^{n+1}} \\ &= n\int_{\mathbb{R}^{n-1}} \frac{\delta\varepsilon x_1^2(\frac{1}{2}\lambda_i x_i^2)\,dx}{(1+|x|^2)^{n+1}} + O(\varepsilon^n\delta\rho_0^{1-n}). \end{aligned}$$

By means of polar coordinates, for the last integral we obtain

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{\delta\varepsilon x_1^2(\frac{1}{2}\lambda_i x_i^2)\,dx}{(1+|x|^2)^{n+1}} &= \frac{\varepsilon\delta}{2}\int_{\mathbb{R}^{n-1}} \frac{\lambda_1 x_1^4\,dx}{(1+|x|^2)^{n+1}} + \frac{\varepsilon\delta}{2}\sum_{i=2}^{n-1}\int_{\mathbb{R}^{n-1}} \frac{\lambda_i x_i^2 x_1^2\,dx}{(1+|x|^2)^{n+1}} \\ &= \frac{\varepsilon\delta}{2}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}} \\ &\quad\times\left(\lambda_1\int_{S^{n-2}}\xi_1^4\,d\xi + \sum_{i=2}^{n-1}\lambda_i\int_{S^{n-2}}\xi_i^2\xi_1^2\,d\xi\right) \\ &= \frac{\sigma_{n-2}\varepsilon\delta\lambda_1}{(n-1)(n+1)}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}}, \end{aligned}$$

where the last equality is obtained by using (3.11) and the fact that $\sum_{i=2}^{n-1}\lambda_i = -\lambda_1$. Combining the last equality and (3.21) yields

$$\begin{aligned} n\int_{B^{n-1}_{\rho_0}}\int_0^f \frac{\delta\varepsilon^n x_1^2\,dx\,dx_n}{(\varepsilon^2+|x|^2+x_n^2)^{n+1}} &= \frac{n\sigma_{n-2}\varepsilon\delta\lambda_1}{(n-1)(n+1)}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}} \\ &\quad + O(\varepsilon^2\delta) + O(\varepsilon^n\delta\rho_0^{1-n}). \end{aligned}$$

Substituting this and (3.20) in (3.19) and Lemma 3.1 we get

$$\begin{aligned}\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dy = &\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n} + n\int_{C^+_{\rho_0/\varepsilon}} \frac{\delta y_1^2\,dy}{(1+|y|^2)^{n+1}} \\ &- \frac{n\sigma_{n-2}\varepsilon\delta\lambda_1}{(n-1)(n+1)}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}} \\ &+ \frac{n(n+1)}{2}\int_{C^+_{\rho_0/\varepsilon}} \frac{\delta^2 y_1^4\,dy}{(1+|y|^2)^{n+2}} \\ &+ O(\delta^2\varepsilon) + O(\varepsilon^3) + O(\varepsilon^n\delta\rho_0^{1-n}) \\ &+ O(\delta^3) + O(\varepsilon^2\delta),\end{aligned}$$

from which we easily obtain

$$\begin{aligned}\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dy = &\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n} + n\int_{\mathbb{R}^n_+} \frac{\delta y_1^2\,dy}{(1+|y|^2)^{n+1}} \\ &- \frac{n\sigma_{n-2}\varepsilon\delta\lambda_1}{(n-1)(n+1)}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}} \\ &+ \frac{n(n+1)}{2}\int_{\mathbb{R}^n_+} \frac{\delta^2 y_1^4\,dy}{(1+|y|^2)^{n+2}} \\ &+ O(\delta^2\varepsilon) + O(\varepsilon^3) + O(\varepsilon^n\delta\rho_0^{-n}) + O(\delta^3) + O(\varepsilon^2\delta).\end{aligned}$$

The conclusion of our lemma thus follows from the last asymptotic formula and Lemma 3.6.

In the next lemma we relate the value of the integral on the left-hand side of Lemma 3.7 with the value of the Sobolev quotient on S^n_+. In order to do that we use Lemma 3.7 and the characterization of the Sobolev quotient given in (3.2); more precisely, we need

Lemma 3.8. *There exists a constant* c *such that*

$$\begin{aligned}n(n-2)\int\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n} \le Q(S^n_+)&\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dv\right)^{(n-2)/n} \\ &+ (n-2)^2 F\left(1+\frac{F}{nI}\right) + cE_4\end{aligned}$$

where $I = \int_{\mathbb{R}^n_+} \frac{dy}{(1+|y|^2)^n}$, *and* F *and* E_4 *are given in Lemma* 3.7.

Proof. Taylor's theorem and Lemma (3.7) imply

$$\left(\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n}\right)^{(n-2)/n} = \left(\int_{C_{\rho_0}\cap M} \varphi^{(n-2)/n}\,dv\right)^{(n-2)/n} + \frac{n-2}{n}\left(\int_{C_{\rho_0}\cap M} \varphi^{\frac{2n}{n-2}}\,dv\right)^{-2/n} F - \frac{n-2}{n^2}\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dv\right)^{-2/n-1} F^2 + O(E_4).$$

Writing

$$\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n} = \left(\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n}\right)^{2/n}\left(\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n}\right)^{(n-2)/n},$$

we get, in consequence of (3.2),

$$\begin{aligned} n(n-2)\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n} &\le Q(S^n_+)\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dv\right)^{(n-2)/n} \\ &+ (n-2)^2\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dv\right)^{-2/n}\left(\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n}\right)^{2/n} F \\ &- \frac{(n-2)^2}{n}\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dv\right)^{-2/n-1}\left(\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n}\right)^{2/n} F^2 + cE_4. \end{aligned}$$

Using Lemma 3.7 and Taylor's theorem we obtain

$$n(n-2)\int_{C^+_{\rho_0}} \frac{\varepsilon^n\,dy}{(\varepsilon^2+|y|^2)^n} \le Q(S^n_+)\left(\int_{C_{\rho_0}\cap M} \varphi^{2n/(n-2)}\,dv\right)^{(n-2)/n} + (n-2)^2 F\left(1+\frac{F}{nI}\right) + cE_4,$$

from which the lemma follows.

The next lemma shows the contribution to the energy $E(\varphi)$ of the Dirichlet integral on $(C_{2\rho_0} - C_{\rho_0})\cap M$.

Lemma 3.9. *There exists a constant c, such that*

$$\int_{(C_{2\rho_0}-C_{\rho_0})\cap M} |\nabla\varphi|^2_g\,dv \le c\varepsilon^{n-2}\rho_0^{2-n}.$$

Proof. We first observe that since the metric in Euclidean, up to second order in normal coordinates

$$|\nabla\varphi|_g^2 \le |\nabla\varphi|^2 + c|y|^2|\nabla\varphi|^2 \le 2|\nabla\varphi|^2.$$

The definition of φ and Schwartz's inequality imply

$$|\nabla\varphi|_g^2 \le 4(|\nabla\psi|^2u^2 + \psi^2|\nabla u|^2) \le 4\left(\frac{1}{\rho_0^2}u^2 + |\nabla u|^2\right).$$

Thus using (3.6) and

$$(\varepsilon^2 + |y|^2 - \delta y_1^2) \ge \frac{1}{2}(\varepsilon^2 + |y|^2) \ge \frac{1}{8}\rho_0^2,$$

on $(C_{2\rho_0} - C_{\rho_0}) \cap M$, we get

$$\int_{(C_{2\rho_0} - C_{\rho_0})\cap M} |\nabla\varphi|_g^2\, dv \le c\varepsilon^{n-2}\rho_0^{2-n}.$$

In the next lemma we calculate the integral involving the scalar curvature in the definition of $E(\varphi)$.

Lemma 3.10. *There exists a constant* c *such that*

$$\frac{(n-2)}{4(n-1)}\int_{C_{2\rho_0}\cap M} R\varphi^2\, dv \le c\varepsilon^3 + c\varepsilon^2\rho_0.$$

Proof. On $C_{2\rho_0} \cap M$ we have $\varphi \le u$. Since $R(0) = 0$, the Mean Value Theorem and the asymptotic formula (3.4) imply

$$\frac{(n-2)}{4(n-1)}\int_{C_{2\rho_0}\cap M} R\varphi^2\, dv \le c\int_{C_{2\rho_0}\cap M} |y|u^2\, dy \le c\int_{C_{2\rho_0}} \frac{\varepsilon^{n-2}|y|}{(\varepsilon^2 + |y|^2)^{n-2}}.$$

Therefore using Lemma 3.1 we get

$$\frac{(n-2)}{4(n-1)}\int_{C_{2\rho_0}\cap M} R\varphi^2\, dv \le c\int_{C_{2\rho_0/\varepsilon}} \frac{\varepsilon^3|y|\, dy}{(1 + |y|^2)^{n-2}} \le c\varepsilon^3 + c\varepsilon^2\rho_0,$$

proving the inequality.

Lemma 3.11. *There exists a constant* c *such that*

$$\int_{B_{2\rho_0}^{n-1}} h\varphi^2\, d\sigma \le c\varepsilon^3 + c\varepsilon^2\rho_0 + c\varepsilon^2|\delta| + c|\delta|\varepsilon^2\log(\rho_0/\varepsilon).$$

Proof. Since in normal coordinates the metric in Euclidean up to the second order, we have

$$\begin{aligned}\int_{B^{n-1}_{2\rho_0}} h\varphi^2\,d\sigma &\le \int_{B^{n-1}_{2\rho_0}} h\varphi^2\,dx + c\int_{B^{n-1}_{2\rho_0}} h\varphi^2|x|^2\,dx\\ &\le \int_{B^{n-1}_{2\rho_0}} h\varphi^2\,dx + c\varepsilon^2\rho_0^2 + c\varepsilon^4,\end{aligned}$$

where we have used $h(0)=0$ and Lemma 3.1 to get

$$\begin{aligned}\int_{B^{n-1}_{2\rho_0}} h\varphi^2|x|^2\,dx &\le c\int_{B^{n-1}_{2\rho_0}}\\ &\le c\varepsilon^4\int_{B^{n-1}_{2\rho_0}} \frac{|x|^3\,dx}{(1+|x|^2)^{n-2}} \le c\varepsilon^4 + c\varepsilon^2\rho_0^2.\end{aligned}$$

The Mean Value Theorem implies that

$$\begin{aligned}\int_{B^{n-1}_{2\rho_0}} h\varphi^2\,dx &\le \int_{B^{n-1}_{2\rho_0}} \frac{h\varepsilon^{n-2}\psi^2\,dx}{(\varepsilon^2+|x|^2)^{n-2}} + c\varepsilon^{n-2}\int_{B^{n-1}_{2\rho_0}} \frac{(|\delta|\,|x|^3+|x|^5)\,dx}{(\varepsilon^2+|x|^2)^{n-1}}\\ &\le \int_{B^{n-1}_{2\rho_0}} \frac{h\varepsilon^{n-2}\psi^2\,dx}{(\varepsilon^2+|x|^2)^{n-2}} + c|\delta|\varepsilon^2 + c|\delta|\varepsilon^2\log(\rho_0/\varepsilon),\end{aligned}$$

where we have used the following estimates, which are easily obtained in consequence of Lemma 3.1:

$$\int_{B^{n-1}_{2\rho_0}} \frac{\varepsilon^{n-2}|\delta|\,|x|^3\,dx}{(\varepsilon^2+|x|^2)^{n-1}} \le \int_{B^{n-1}_{2\rho_0/\varepsilon}} \frac{\varepsilon^2|\delta|\,|x|^3\,dx}{(1+|x|^2)^{n-1}} \le c\varepsilon^2|\delta| + c|\delta|\varepsilon^2\log(\rho_0/\varepsilon)$$

and

$$\int_{B^{n-1}_{2\rho_0}} \frac{\varepsilon^{n-2}|x|^5\,dx}{(\varepsilon^2+|x|^2)^{n-1}} \le \int_{B^{n-1}_{2\rho_0/\varepsilon}} \frac{\varepsilon^4|x|^5\,dx}{(1+|x|^2)^{n-1}} \le c\varepsilon^4 + c\varepsilon^2\rho_0^2.$$

Since $h(0)=0$, the Taylor's expansion for the function h yields

$$\int_{B^{n-1}_{2\rho_0}} h\varepsilon^2\,dx \le \int_{B^{n-1}_{2\rho_0}} \frac{h_i(0)x_i\varepsilon^{n-2}\psi^2(|x|)}{(\varepsilon^2+|x|^2)^{n-2}}\,dx + c\int_{B^{n-1}_{2\rho_0}} \frac{\varepsilon^{n-2}|x|^2\,dx}{(\varepsilon^2+|x|^2)^{n-2}}.$$

By the symmetries of the ball and the fact that $\psi=\psi(|x|)$ we see that the first integral on the right-hand side of the above inequality vanishes. An easy computation shows that

$$\int_{B^{n-1}_{2\rho_0}} \frac{\varepsilon^{n-2}|x|^2\,dx}{(\varepsilon^2+|x|^2)^{n-2}} = \int_{B^{n-1}_{2\rho_0/\varepsilon}} \frac{\varepsilon^3|x|^2\,dx}{(1+|x|^2)^{n-2}} \le c\varepsilon^3 + c\varepsilon^2\rho_0.$$

The above estimates hence imply our lemma.

Lemma 3.12. *The number*

$$B_1 = -\frac{(n-2)^2 n\sigma_{n-2}}{(n-1)(n+1)}\left[\int_0^\infty \frac{r^{n+4}\,dr}{(1+r^2)^{n+1}} - \frac{2}{n}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^n}\right.$$
$$\left.\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}}\right]$$

is negative.

Proof. Integration by parts shows that

$$\int_0^\infty \frac{r^{n+4}\,dr}{(1+r^2)^{n+1}} = \frac{n+3}{2n}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^n}.$$

Since

$$\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^n} = \int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}} + \int_0^\infty \frac{r^{n+4}\,dr}{(1+r^2)^{n+1}},$$

a computation yields that

$$\int \frac{t^m\,dt}{(1+t^2)^n} = -\frac{1}{(2n-m-1)}\frac{t^{m-1}}{(1+t^2)^{n-1}} + \frac{m-1}{2n-m-1}\int \frac{t^{m-2}\,dt}{(1+t^2)^n}.$$

Thus

$$\int_0^\infty \frac{r^{n+4}\,dr}{(1+r^2)^{n+1}} = \frac{n+3}{n-3}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}},$$

and

$$\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^n} = \frac{2n}{n-3}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}}.$$

Hence

$$\begin{aligned} B_1 &= \frac{(n-2)^2\sigma_{n-2}}{(n-1)(n+1)}\left(-n\int_0^\infty \frac{r^{n+4}\,dr}{(1+r^2)^{n+1}} + 2\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^n}\right. \\ &\qquad \left. + n\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}}\right) \\ &= -\frac{2n(n-2)^2\sigma_{n-2}}{(n-3)(n-1)(n+1)}\int_0^\infty \frac{r^{n+2}\,dr}{(1+r^2)^{n+1}}. \end{aligned}$$

The next two lemmas deal with the four-dimensional case only, and the first one is an estimate on the Dirichlet integral.

Lemma 3.13. *There exists a constant* c *such that*

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} |\nabla\varphi|_g^2\,dv \le 4\int_{B^+_{2\rho_0}-B^+_{\rho_0}} \frac{\varepsilon^2|y|^2\,dy}{(\varepsilon^2+|y|^2)^4} + c\varepsilon^2|\delta|\rho_0^{-2} + E_1 + E_2,$$

where E_1 and E_2 are given in Lemma 3.2 and Lemma 3.3 respectively.

Proof. A similar calculation as in Lemmas 3.2–3.4 shows that

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} |\nabla\varphi|_g^2\, dv \le \int_{(B_{2\rho_0}-B_{\rho_0})\cap M} |\nabla\varphi|^2\, dy + E_1 + E_2,$$

where E_1 and E_2 are given in Lemma 3.2 and Lemma 3.3 respectively. By the definition of φ and the inequalities $|\nabla(\psi y_1^2)| \le c\rho_0$ and $\varepsilon^2+|y|^2-\delta\psi y_1^2 \ge c\rho_0^2$ on $(B_{2\rho_0}-B_{\rho_0})\cap M$ we get

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} |\nabla\varphi|_g^2\, dv \le 4\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^2|y|^2\, dy}{(\varepsilon^2+|y|^2-\delta\psi y_1^2)^4} + c\varepsilon^2|\delta|\rho_0^{-2} + E_1 + E_2.$$

The Mean Value Theorem implies

$$\begin{aligned}\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} |\nabla\varphi|_g^2\, dv &\le 4\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^2|y|^2\, dy}{(\varepsilon^2+|y|^2)^4} \\ &\quad + c\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^2|\delta|\,|y|^2\psi y_1^2}{(\varepsilon^2+|y|^2)^4}\, dy \\ &\quad + c\varepsilon^2|\delta|\rho_0^{-2} + cE_1 + cE_2 \\ &\le 4\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^2|y|^2\, dy}{(\varepsilon^2+|y|^2)^4} \\ &\quad + c\varepsilon^2|\delta|\rho_0^{-2} + E_1 + E_2.\end{aligned}$$

Using Proposition 2.2 and the fact that $h(0)=0$ and $R(0)=0$, we obtain

$$4\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^2|y|^2\, dy}{(\varepsilon^2+|y|^2)^4} \le 4\int_{B^+_{2\rho_0}-B^+_{\rho_0}} \frac{\varepsilon^2|y|^2\, dy}{(\varepsilon^2+|y|^2)^4} + c\varepsilon^3.$$

Combining the above two inequalities yields our lemma.

In the following lemma we deal with the integral of the function $\varphi^{2n/(n-2)}$, when $n=4$.

Lemma 3.14. *The following asymptotic formula holds*

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \varphi^4\, dv = \int_{B^+_{2\rho_0}-B^+_{\rho_0}} \frac{\varepsilon^4\, dy}{(\varepsilon^2+|y|^2)^4} + O(\varepsilon^3) + O(\varepsilon^4\delta\rho_0^{-4}).$$

Proof. A similar calculation as in Lemma 3.6 shows that

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \varphi^4\, dv = \int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \varphi^4\, dy + O(\varepsilon^3).$$

The Mean Value Theorem implies

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \varphi^4\,dy = \int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^4\,dy}{(\varepsilon^2+|y|^2)^4} + O(\varepsilon^4\delta\rho_0^{-4})$$

where we have used that

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^4|\delta|y_1^2\psi\,dy}{(\varepsilon^2+|y|^2-\delta y_1^2)^5} \le c\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \varepsilon^4|\delta|\rho_0^{-10}y_1^2\,dy$$
$$\le c\varepsilon^4|\delta|\rho_0^{-4}.$$

By Proposition 2.2, the fact that $h(0)=0$ and $R(0)=0$, and the estimate in the proof of Lemma 3.6 we obtain

$$\int_{(B_{2\rho_0}-B_{\rho_0})\cap M} \frac{\varepsilon^4\,dy}{(\varepsilon^2+|y|^2)^4} = \int_{B^+_{2\rho_0}-B^+_{\rho_0}} \frac{\varepsilon^4\,dy}{(\varepsilon^2+|y|^2)^4} + O(\varepsilon^3).$$

Hence the lemma follows from the above estimates.

4. The Sobolev quotient of a Riemannian manifold with umbilic boundary

In this section we assume (M^n, g) is a compact Riemannian manifold with boundary of dimension $n \ge 3$, 0 is a point of ∂M, and every point at the boundary is umbilic. We use the coordinates introduced in Proposition 2.1 and the Positive Mass Theorem, which will be discussed in the Appendix of this paper, to prove the following.

Theorem 4.1. *Let (M^n, g) be a compact Riemannian manifold with boundary. Assume that ∂M is umbilic and that M is not conformally diffeomorphic to S^n_+. Then*

$$Q(M) < Q(S^n_+), \tag{4.1}$$

if either

(i) *the Weyl tensor does not vanish identically on ∂M and $n \ge 6$,*
(ii) *M is locally conformally flat, or*
(iii) *$n = 3, 4,$ or 5.*

Proof. It is enough to construct a function φ such that $Q(\varphi) < Q(S^n_+)$. By Proposition 1.1 we can assume that g is the metric of Proposition 2.1. In case (i), let $0 \in \partial M$ such that the Weyl tensor does not vanish. Let $(x_1, x_2, \cdots, x_n)$ be conformal normal coordinates at $0 \in \partial M$. Consider, for $\varepsilon > 0$, the functions

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{(n-2)/2}.$$

Set $\varphi = u_\varepsilon \psi$, where ψ is the same cutoff function supported in $B_{2\rho_0}$ as defined in §3. Taking N in Proposition 2.1 arbitrarily large, we can make dv_g as close to dx as we want. Therefore we will assume that $dv_g = dx$ in conformal normal coordinates. Since φ is a radial function and $g^{rr} = 1$ in normal coordinates, we have $|\nabla \varphi|_g^2 = |\partial_r \varphi|^2$. Thus

$$\int_{B^+_{\rho_0}} |\nabla \varphi|_g^2 \, dx \leq Q(S^n_+)\left(\int_{B^+_{\rho_0}} \varphi^{2n/(n-2)}\right)^{(n-2)/n} + \int_{\partial B^+_{\rho_0}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta}. \tag{4.2}$$

A straightforward computation shows that $\partial u_\varepsilon / \partial \eta \leq 0$ on $\partial B^+_{\rho_0}$.

In conformal normal coordinates $R = O(r^2)$ and $\Delta R = -\frac{1}{6}|W(0)|^2$, so

(4.3)
$$\begin{aligned}\int_{B^+_{\rho_0}} R\varphi^2 \, dx &= \int_0^{\rho_0} \int_{S^+_r} \left(\frac{1}{2} R_{;ij} x_i x_j + O(r^3)\right) u_\varepsilon^2 \, dw_r \, dr \\ &= \int_0^{\rho_0} (-cr^2 |W(0)|^2 + O(r^3)) \left(\frac{\varepsilon}{\varepsilon^2 + r^2}\right)^{n-2} r^{n-1} dr \leq E_n,\end{aligned}$$

where

$$E_n = \begin{cases} -c|W(0)|^2 \varepsilon^4 + c\varepsilon^5 & \text{if } n > 6, \\ -c|W(0)|^2 \varepsilon^4 \log(\varepsilon^{-1}\rho_0) + c\varepsilon^4 & \text{if } n = 6, \end{cases}$$

and in the last inequality we have used the change of variables $r = \varepsilon t$.

A straightforward computation shows that

$$\int_{B^+_{2\rho_0} - B^+_{\rho_0}} R\varphi^2 \, dv \leq c\varepsilon^{n-2}. \tag{4.4}$$

Since $h = O(r^2)$ at $0 \in \partial M$ we have, by the Taylor expansion for h,

$$\begin{aligned}\int_{B_{\rho_0} \cap \partial M} h\varphi^2 \, d\sigma \leq& \int_{B^{n-1}_{\rho_0}} \frac{1}{2} h_{;ij} x_i x_j \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{n-2} dx \\ &+ \int_{B^{n-1}_{\rho_0}} \frac{1}{6} h_{;ijk} x_i x_j x_k \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{n-2} dx \\ &+ c\int_{B^{n-1}_{\rho_0}} |x|^4 \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{n-2} dx.\end{aligned}$$

Because of the symmetries of the ball and the fact that $\Delta_{\partial M} h(0) = 0$, the first and second integrals on the right-hand side vanish. For the third integral we use polar coordinates and then the change of variable $r = \varepsilon t$ to get

$$\int_{B_{\rho_0} \cap \partial M} h\varphi^2 \, d\sigma \leq c\varepsilon^5 \int_0^{\rho_0/\varepsilon} \frac{t^{n+2}}{(1+t^2)^{n-2}} dt \leq \widetilde{E}_n,$$

where

$$\widetilde{E}_n = \begin{cases} c\varepsilon^5 + c\varepsilon^4\rho_0 & \text{if } n = 6, \\ c\varepsilon^5 \log(\varepsilon^{-1}\rho_0) & \text{if } n = 7, \\ c\varepsilon^5 & \text{if } n \geq 8. \end{cases} \tag{4.5}$$

It is easy to check that

$$\int_{\partial M \cap B2\rho_0 - B_{\rho_0}} h\varphi^2 \, d\sigma \leq c\varepsilon^{n-2}.$$

Using the last two inequalities, (4.2), (4.3), and (4.4) we obtain

$$E(\varphi) \leq Q(S^n_+)\left(\int_M \varphi^{2n/(n-2)} \, dv\right)^{(n-2)/n} + E_n + \widetilde{E}_n + c\varepsilon^{n-2}.$$

Since the Weyl tensor does not vanish at 0, we have $|W(0)|^2 > 0$. Fix ρ_0 small and choose ε smaller to get

$$E(\varphi) < Q(S^n_+)\left(\int_M \varphi^{2n/(n-2)} \, dv\right)^{(n-2)/n}.$$

Thus (4.1) follows in this case.

For the second case, we assume $Q(M) > 0$, otherwise the inequality (4.1) is trivial. We will use a global test function and the idea introduced by R. Schoen in [10], that is, the Green's function of the associated linear operator. Since M is locally conformally flat, by Proposition 1.1 we can assume that near 0 the metric is the flat metric and the boundary is minimal. Let $(x_1, \cdots, x_n)$ be rectangular coordinates around $0 \in \partial M$. Since ∂M is umbilic and minimal, it is a hyperplane. We can assume that ∂M is given by $x_n = 0$ in the coordinates $(x_1, \cdots, x_n)$. Let G be the positive solution of $LG = 0$ on $M - \{0\}$ and $BG = 0$ on $\partial M - \{0\}$. Since G is a harmonic function near 0 which satisfies $\frac{\partial G}{\partial x_n} = 0$ on ∂M near 0, it has an expansion for $|x|$ small:

$$G(x) = |x|^{2-n} + A + \alpha(x), \tag{4.6}$$

where $\alpha(x)$ is a smooth harmonic function near 0, with $\alpha(0) = 0$.

Let ρ_0 be a small radius, and $\varepsilon_0 > 0$ a number to be chosen small relative to ρ_0. Let $u_\varepsilon(x)$ and $\psi(s)$ be as before. We now construct a piecewise smooth test function φ on M as follows:

$$\varphi(x) = \begin{cases} u_\varepsilon(x) & \text{for } x \in M \cap B_{\rho_0}, \\ \varepsilon_0(G(x) - \alpha(x)\psi(|x|)) & \text{for } x \in M \cap (B_{2\rho_0} - B_{\rho_0}), \\ \varepsilon_0 G & \text{for } x \in M - B_{2\rho_0}. \end{cases} \tag{4.7}$$

In order for the function φ to be continuous across ∂B_{ρ_0} we must require ε to satisfy

$$\left(\frac{\varepsilon}{\varepsilon^2+\rho_0^2}\right)^{(n-2)/2} = \varepsilon_0(\rho_0^{2-n}+A). \tag{4.8}$$

We compute $E(\varphi)$ as a sum of the energy in $B_{\rho_0}\cap M$ and the energy on $M-B_{\rho_0}$. By means of (3.1) for u_ε, after integration by parts we have

$$\int_{B^+_{\rho_0}} |\nabla u_\varepsilon|^2\,dx = n(n-2)\int_{B^+_{\rho_0}} u_\varepsilon^{2n/(n-2)}\,dx + \int_{\partial B^+_{\rho_0}\cap M} u_\varepsilon\frac{\partial u_\varepsilon}{\partial r}.$$

Using (3.2), the definition of φ, and the fact that $dv = dx$ we get

$$\begin{aligned}(4.9)\quad &\int_{B_{\rho_0}\cap M} |\nabla\varphi|^2\,dv\\ &\le Q(S^n_+)\left(\int_{B_{\rho_0}\cap M}\varphi^{2n/(n-2)}\right)^{(n-2)/n} + \int_{\partial B^+_{\rho_0}\cap M} u_\varepsilon\frac{\partial u_\varepsilon}{\partial r}.\end{aligned}$$

Evaluating the energy of φ on $M-B_{\rho_0}$ yields

$$\begin{aligned}&\int_{M-B_{\rho_0}} |\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)}R\varphi^2\,dv + \frac{n-2}{2}\int_{\partial M-B_{\rho_0}} h\varphi^2\,d\sigma\\ &\quad= \varepsilon_0^2\Bigg[\int_{M-B_{\rho_0}}\left(|\nabla G|_g^2 + \frac{n-2}{4(n-1)}RG^2\right)dv + \frac{n-2}{2}\int_{\partial M-B_{\rho_0}} hG^2\,d\sigma\\ &\qquad+\int_{M\cap B_{2\rho_0}-B_{\rho_0}}\Big(|\nabla\psi\alpha|^2 - 2\nabla G\cdot\nabla(\psi\alpha)\Big)\,dv\\ &\qquad+\frac{n-2}{4(n-1)}\int_{M\cap B_{2\rho_0}-B_{\rho_0}}\Big(-2RG\alpha\psi + R\alpha^2\psi^2\Big)dv\\ &\qquad\qquad+\frac{n-2}{2}\int_{\partial M\cap(B_{2\rho_0}-B_{\rho_0})}(-2G\alpha\psi+\alpha^2\psi^2)h\,d\sigma\Bigg].\end{aligned}$$

Since $|\alpha(x)|\le c|x|$ and $|\nabla\alpha|\le c$ we see that $|\nabla(\psi\alpha)|\le c$ for $\rho_0\le|x|\le 2\rho_0$. Using this and the fact that $h=0$ in a neighborhood of 0 we get

$$\begin{aligned}&\int_{M-B_{\rho_0}}(|\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)}R\varphi^2)\,dv + \frac{n-2}{2}\int_{\partial M-B_{\rho_0}} h\varphi^2\,d\sigma\\ &\quad\le \varepsilon_0^2\left[\int_{M-B_{\rho_0}}\left(|\nabla G|_g^2 + \frac{n-2}{4(n-1)}RG^2\right)dv + \frac{n-2}{2}\int_{\partial M-B_{\rho_0}} hG^2\,d\sigma\right]\\ &\quad+c\rho_0\varepsilon_0^2.\end{aligned}$$

Since G satisfies $LG = 0$ on M and $\frac{\partial G}{\partial \eta} + \frac{n-2}{2} hG = 0$ on $\partial M - \{0\}$, the first two terms on the right-hand side of the above inequality become a boundary integral:

$$\begin{aligned}(4.10)\qquad &\int_{M-B_{\rho_0}} \left(|\nabla\varphi|_g^2 + \frac{n-2}{4(n-1)} R\varphi^2\right) dv + \frac{n-2}{2}\int_{\partial M - B_{\rho_0}} h\varphi^2\, d\sigma \\ &\leq -\varepsilon_0^2 \int_{M\cap\partial B_{\rho_0}} G\frac{\partial G}{\partial r} + c\rho_0\varepsilon_0^2.\end{aligned}$$

Since (M^n, g) satisfies that $R = 0$ and $h = 0$ in a neighborhood of 0, we have (4.9) and (4.10), using in consequence of

$$\begin{aligned}(4.11)\qquad E(\varphi) \leq\, &Q(S_+^n)\left(\int_M \varphi^{2n/(n-2)}\, dv\right)^{(n-2)/n} \\ &+ \int_{\partial B_{\rho_0}^+ \cap M} \left(u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G\frac{\partial G}{\partial r}\right) + c\rho_0\varepsilon_0^2.\end{aligned}$$

If M is not conformally equivalent to S_+^n, then in the Green's function expansion (4.6), A is a positive constant. In the Appendix of this paper we prove that the Positive Mass Theorem holds in this case, that is, $A \geq 0$ and $A = 0$ if and only if M is conformally equivalent to S_+^n. We use this to show that the boundary integral is negative. For $|x| = \rho_0$ from (4.6) and (4.8) it follows that

$$\frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0 \frac{\partial G}{\partial r} \leq -(n-2)\varepsilon_0\rho_0^{-1}\left[(\rho_0^{2-n} + A)\left(\left(\frac{\varepsilon}{\rho_0}\right)^2 + 1\right)^{-1} - \rho_0^{2-n}\right] + c\varepsilon_0.$$

Using the inequality $(1+t^2)^{-1} \geq 1 - t^2$, we get

$$\frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0\frac{\partial G}{\partial r} - (n-2)AE_0\rho_0^{-1} + c\rho_0^{-1}\varepsilon_0^2\varepsilon^2 + c\varepsilon_0.$$

Thus

$$\int_{M\cap B_{\rho_0}} \left(u_\varepsilon\frac{\partial u_\varepsilon}{\partial r} - \varepsilon_0^2 G\frac{\partial G}{\partial r}\right) \leq -(n-2)A\sigma_{n-1}^+\varepsilon_0^2 + c\varepsilon_0^2\varepsilon^2\rho_0^{-n} + c\varepsilon_0^2\rho_0.$$

Substituting this in (4.11) gives

$$\begin{aligned}(4.12)\qquad E(\varphi) \leq\, &Q(S_+^n)\left(\int_M \varphi^{(2n/(n-2)}\, dv\right)^{(n-2)/n} \\ &- (n-2)A\sigma_{n-1}^+\varepsilon_0^2 + c\varepsilon_0^2\varepsilon^2\rho_0^{-n} + c\varepsilon_0^2\rho_0.\end{aligned}$$

Fixing ρ_0 small and then choosing ε much smaller we have from (4.8) that $\varepsilon^{(n-2)/2} \approx \varepsilon_0$. Since $A > 0$, we have

$$E(\varphi) < Q(S_+^n)\left(\int_M \varphi^{2n/(n-2)}\, dv\right)^{(n-2)/n}.$$

Hence the theorem follows.

For the case (M^n, g), where $n = 3, 4, 5$, we outline the necessary changes in the above proof. Set φ_1 to be the first eigenfunction for the conformal Laplacian with respect to the boundary conditions as in (2). Lemma 1.1 states that the metric $g_1 = \varphi_1^{4/(n-2)} g$ has minimal boundary, and hence Proposition 1.2 implies that the second fundamental form vanishes on ∂M. Let $(x_1, \cdots, x_n)$ be geodesic normal coordinates at $0 \in \partial M$ so that the boundary is given by $x_n = 0$.

When $n = 3$, the Green's function G for the conformal Laplacian with the boundary conditions as in (2) with respect to the metric g_1 has the expansion for x small as

$$G(x) = |x|^{-1} + A + O''(|x|), \tag{4.13}$$

where A is a constant. Here we write $f = O'(r^m)$ to mean $f = O(r^m)$ and $\nabla f = O(r^{m-1})$. O'' is defined similarly.

When $n = 4$, we let f be the second degree homogeneous polynomial such that $g_2 = e^{2f} g_1$ satisfies $R_{ij}(0) = 0$, $h = 0$ on ∂M. This was proved in Proposition 2.1. It is proved in the Appendix that the Green's function G for the conformal Laplacian with the boundary conditions as in (2) with respect to the metric g_1 has the expansion for x small as

$$G(x) = |x|^{-2} + A + O''(|x| \log |x|). \tag{4.14}$$

Using Proposition 1.1 we can assume that g is the metric g_1 for $n = 3$ and g_2 for $n = 4$. Let φ be as above, and apply the same argument. Correction terms must be introduced in $B^+_{2\rho_0}$ to account for the difference between g and the Euclidean metric. It follows from Proposition 2.2 and the fact $\varphi = u_\varepsilon$ on $B^+_{\rho_0}$ is radial that

$$\begin{aligned}\int_{B^+_{\rho_0}} |\nabla \varphi|_g^2 \, dv &\le \int_{B^+_{\rho_0}} |\nabla u_\varepsilon|^2 \, dx + c \int_{B^+_{\rho_0}} |x|^{n-1} |\nabla u_\varepsilon|^2 \, dx \\ &\le \int_{B^+_{\rho_0}} |\nabla u_\varepsilon|^2 \, dx + \begin{cases} c\varepsilon\rho_0, & n = 3, \\ c\varepsilon^3 + c\varepsilon\rho_0, & n = 4, \end{cases}\end{aligned}$$

and

$$\int_{B^+_{\rho_0}} u_\varepsilon^{2n/(n-2)} \, dv \le \int_{B^+_{\rho_0}} \varphi^{2n/(n-2)} \, dx + \begin{cases} c\varepsilon^2, & n = 3, \\ c\varepsilon^3 + c\varepsilon^3 \log(\rho_0 \varepsilon^{-1}) & n = 4. \end{cases}$$

Using the definition of φ and the fact that $R(0) = 0$ for $n = 4$ we get

$$\int_{B^+_{\rho_0}} R\varphi^2 \, dv \le \begin{cases} c\varepsilon\rho_0, & n = 3, \\ c\varepsilon^2\rho_0, & n = 4. \end{cases}$$

For $n = 4$ one easily checks that the inequality (4.10) holds with the last term being $c\rho_0 \log \rho_0^{-1} \varepsilon_0^2$ instead of $c\rho_0 \varepsilon_0^2$. This error term comes from the term $O''(|x| \log |x|)$ in the Green's function expansion (4.14).

We finally have

$$E(\varphi) \leq Q(S_+^n) \left(\int_M \varphi^{2n/(n-2)} \, dv \right)^{(n-2)/n} - (n-2) A \sigma_{n-1}^+ \varepsilon_0^2 + E_n ,$$

where E_n is the error term,

$$E_3 = c\varepsilon_0^2 \varepsilon^2 \rho_0^{-3} + c\varepsilon_0^2 \rho_0 + c\varepsilon \rho_0 + c\varepsilon^2 ,$$
$$E_4 = c\varepsilon_0^2 \varepsilon^2 \rho_0^{-4} + c\varepsilon_0^2 \rho_0 + c\varepsilon^2 \rho_0 + c\varepsilon^3 + c\varepsilon^3 \log(\rho_0 \varepsilon^{-1}) + c\rho_0 \log \rho_0^{-1} \varepsilon_0^2 .$$

Fixing ρ_0 small and then choosing ε much smaller, we have from (4.8) that $\varepsilon^{(n-2)/2} \approx \varepsilon_0$. Since $A > 0$ (see the Appendix), we have that all error terms are dominated by the negative term, and the proof of Theorem 4.1 is complete when $n = 3$ or 4.

If $n = 5$, by the transformation law 1.1, it is easy to check that there exists a constant c such that the metric $g_3 = e^{2cx_5^3} g_2$ satisfies all the conditions which g_2 does and $R_{nn,n}(0) = 0$. Similar considerations as in Proposition 2.1 show that for the metric g_3, $R_{in,j}(0) = 0$ for $1 \leq i, j \leq 4$ and $\sum_{i=1}^n R_{ii,n}(0) = 0$. Now it is straightforward to check that $\mathrm{Vol}(B_r(0) \cap M)$ is asymptotic Euclidean up to fourth order. With the metric g_3 as above we can proceed as in the proof of R. Schoen [10, pp. 484–493] to get the theorem.

Remark 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth umbilic boundary and $n \geq 3$. It is elementary to see that if Ω is not conformally equivalent to the ball, then Ω is a ball with a finite number of balls deleted. If one considers Ω as a domain in S_+^n such that $\partial S_+^n \subset \partial \Omega$ for the constant function 1 on Ω, we have $Q(1) < Q(S_+^n)$. Thus we do not need the Positive Mass Theorem in this case to prove inequality (4.1).

Remark 2. It is clear from the proof of Theorem 4.1 that the hypothesis on the boundary being umbilic is not necessary. The proof uses only the fact that it is umbilic in a small neighborhood.

In the case $n = 3$ we will weaken the hypothesis of umbilicity of the boundary and only assume the existence of one umbilic point on the boundary. The following theorem will be needed in the proof of Theorem 5.1 in the next section.

Theorem 4.2. *Let (M, g) be a three-dimensional manifold. Assume there exists an umbilic point on ∂M and that M is not conformally diffeomorphic to S_+^3. Then $Q(M) < Q(S_+^3)$.*

Proof. We outline the necessary changes in the above proof. Change the metric g to the metric $g_1 = \varphi_1^4 g$ as before. By Proposition 1.1 it is enough to establish the above inequality for the metric g_1. Let (x_1, x_2) be normal coordinates at the boundary around $0 \in \partial M$. Let (x_1, x_2, x_3) be Fermi coordinates (see the Appendix for the definition and discussion on these coordinates). The boundary condition defined in (2) is the Neumann condition because $h = 0$ on ∂M. Moreover, in these coordinates $\frac{\partial}{\partial \eta} = -\frac{\partial}{\partial x_3}$. Since $0 \in \partial M$ is an umbilic point and minimal, Proposition 1.2 implies that the second fundamental form vanishes at 0. Thus the Fermi coordinates are normal coordinates. We show in the Appendix that the Green's function for the conformal Laplacian has the following expansion for x small:

$$G(x) = |x|^{-1} + A + O''(|x|).$$

We show in Appendix that $A > 0$ if M is not conformally equivalent to S^3_+ with the standard metric.

To prove the above estimate define the function φ as in (4.7). Now observe that in the 3-dimensional case the same estimates apply in any normal coordinate system, not necessarily geodesic normal coordinates.

5. The Sobolev quotient of a three-dimensional manifold with a nonumbilic point

Let (M^3, g) be a three-dimensional compact Riemannian manifold with boundary. In this section we prove the 3-dimensional version of Theorem 3.1.

Theorem 5.1. *Let (M, g) be a 3-dimensional Riemannian manifold with boundary having a nonumbilic point. Then*

$$Q(M) < Q(S^3_+). \tag{5.1}$$

In order to prove Theorem 5.1 we let $p \in \partial M$ be a nonumbilic point. Let δ be a small positive number. Choose $0 \in \partial M$ such that it is not near p. We perturb the boundary of M in a small neighborhood of 0. That is, we construct manifolds M_δ that coincide with M outside a ball around 0 of radius 2δ, and $0 \in \partial M_\delta$ is an umbilic point. Because $p \in \partial M_\delta$ is a nonumbilic point, M_δ is not conformally equivalent to S^3_+. Thus, Theorem 4.2 implies that $Q(M_\delta) < Q(S^3_+)$. The correction term in the expansion of the Sobolev quotient $Q(M_\delta)$ is the mass A_δ. In Lemma 5.2 we show that the positivity of the masses is preserved under passage to the limit. Using this we are able to show that the strict inequality $Q(M_\delta) < Q(S^3_+)$ is preserved when one lets δ go to zero.

Proof of Theorem 5.1. We assume that $Q(M) > 0$, as otherwise, the inequality is trivial. Using Proposition 1.1 we can assume that g is the metric given by Lemma 1.1, that is, $R_g > 0$ and the boundary is minimal. Let $p \in \partial M$ be a nonumbilic point and $0 \in \partial M$ be any other point. Let (x_1, x_2, x_3) be normal coordinates around $0 \in \partial M$ such that $\eta(0) = -\frac{\partial}{\partial x_3}$ and the second fundamental form of ∂M at 0 has a diagonal form. Let λ_1, λ_2 be the elements of the diagonal. Then for $i = 1, 2$ the vectors $\frac{\partial}{\partial x_i}(0)$ are the principal directions and the λ_i are the principal curvatures. In these coordinates, ∂M is given near 0 by the equation $x_3 = f(x_1, x_2)$, where $f(x_1, x_2) = \frac{1}{2}\lambda_i x_i^2 + O(|x|^3)$. Let $\psi(x_1, x_2)$ be a piecewise smooth nondecreasing function of $|x|$ which satisfies $\psi(x) = 0$ for $|x| \le \delta$, $\psi(x) = 1$ for $|x| \ge 2\delta$, and $|\nabla^j \psi| \le c\delta^{-j}$, $j = 1, 2$, for $\delta \le |x| \le 2\delta$. Consider the manifold M_δ with defined as the perturbation of M in a neighborhood of 0 boundary ∂M_δ given by the equation $x_3 = f\psi$. Thus ∂M_δ agrees with ∂M outside $B^2_{2\delta}$. Observe that $0 \in \partial M_\delta$ is an umbilic point because (x_1, x_2, x_3) are geodesic normal coordinates at the point 0. It is easy to check that the second fundamental form of ∂M_δ is bounded independent of δ. Let B_δ denote the linear boundary operator defined by (2) on M_δ. Let λ_δ denote the lowest eigenvalue of L with respect to the boundary conditions B_δ, and λ the lowest eigenvalue of L with respect to the boundary condition B. Let G be the Green's function of L with pole at 0 normalized so that $\lim_{|x|\to 0} |x|^{n-2} G(x) = 1$. Then we have

Lemma 5.1. *The eigenvalues λ_δ converge to λ as δ tends to* 0, *and hence $\lambda_\delta > 0$ for δ sufficiently small. Thus L with respect to the boundary condition B_δ has a positive Green's function with pole at* 0 *normalized so that* $\lim_{|x|\to 0} |x|^{n-2} G_\delta(x) = 1$. *The functions G_δ converge uniformly to G in C^2 norm on compact subsets of $M - B_{\delta_1}(0)$ for some $\delta_1 > 0$.*

Proof. Since $|\psi''(r)| \le cr^{-2}$, by the definition of $h_\delta(g)$ it is uniformly bounded. From the variational characterization of the first eigenvalue it follows that $\lambda_\delta \to \lambda$ when $\delta \to 0$. Let $\delta_1 > 0$ be small enough. Integrating the equation which G_δ satisfies, and then using integration by parts and the fact that we have normalized the functions G_δ, we get

$$\int_{M - B_{\delta_1}(0)} R_g G_\delta \, dv \le c,$$

where c is a constant that does not depend on δ. Since $R_g > 0$ on M, we have a uniform integral bound for the Green's function G_δ. Elliptic theory gives a uniform bound on the sup norm for G_δ. Theorem (6.30) in [6] implies that on $M - B_{\delta_1}(0)$ the functions G_δ are uniformly bounded in $C^{2,\alpha}$ norm, because G_δ is uniformly bounded on $C^{1,\alpha}$ norm on $\partial B_{\delta_1} \cap M$, and on $\partial M - B_{\delta_1}(0)$ the function G_δ satisfies the boundary condition $\frac{\partial G_\delta}{\partial \eta} = 0$. The convergence of G_δ to G now follows because we have a uniform upper bound on G_δ and its derivatives on compact subsets of $M - B_{\delta_1}(0)$. This completes a sketch of the proof of Lemma 5.1.

We show in the Appendix that the Green's function G_δ for the conformal Laplacian has the following expansion for $|x|$ small:

$$G_\delta(x) = |x|^{-1} + A_\delta + O''(|x|).$$

Moreover from the discussion of the above expansion in the Appendix and elliptic theory, it follows that $O''(|x|) = A_\delta O''(|x|)$, where $O''(|x|)$ on the right-hand side does not depend on δ. Observe that M_δ is not conformally equivalent with S^3_+, because $p \in \partial M_\delta$ is a nonumbilic point. The Positive Mass Theorem implies that $A_\delta > 0$ (see the Appendix). We define the test function φ as in (4.7), constructed from M_δ where $\rho_0 > 2\delta$. Let c be a positive constant independent of δ. The function φ on $B_{\rho_0} \cap M$ depends only on the geodesic distance r to 0; it follows from Gauss's lemma $|\nabla\varphi|^2_g = |\varphi'|^2$. Since $h(0) = 0$ and $h_\delta(0) = 0$, by Proposition 2.2 we have

$$\begin{aligned}(5.2)\quad \int_{B_{\rho_0}\cap M} |\nabla\varphi|^2_g\, dv &\le \int_{B_{\rho_0}\cap M_\delta} |\nabla\varphi|^2_g\, dv + c\int_{B_{\rho_0}} |x|^2|\nabla\varphi|^2(x)\, dx \\ &\le \int_{B_{\rho_0}\cap M_\delta} |\nabla\varphi|^2\, dx + c\varepsilon\rho_0.\end{aligned}$$

Using the definition of φ we get

$$\int_{B_{\rho_0}\cap M} R\varphi^2\, dv \le c\int_{B_{\rho_0}} u_\varepsilon^2\, dx \le c\varepsilon\rho_0.$$

The same estimate as above is also true if we replace M by M_δ. Thus

$$(5.3)\qquad \int_{B_{\rho_0}\cap M} R\varphi^2\, dv \le \int_{B_{\rho_0}\cap M_\delta} R\varphi^2\, dv + c\varepsilon\rho_0.$$

Note that in the argument of Theorem 4.2, we use the fact that $h = 0$ on ∂M. For ∂M_δ, $h_\delta(g)$ is different from zero on $B_{2\delta}$. Since we want to apply the estimates of the previous section, we estimate the boundary terms involving $h_\delta(g)$. On the one hand

$$(5.4)\qquad \int_{B_{2\delta}\cap\partial M_\delta} h_\delta(g)u_\varepsilon^2\,d\sigma_\delta(g) = \int_{B_\delta\cap\partial M_\delta} h_\delta(g)u_\varepsilon^2\,d\sigma_\delta(g) + \int_{B_{2\delta}-B_\delta\cap\partial M_\delta} h_\delta(g)u_\varepsilon^2\,d\sigma_\delta(g).$$

On $B_\delta\cap\partial M_\delta$ the boundary of M_δ is given by the equation $x_3 = 0$. The fact that (x_1, x_2, x_3) are normal coordinates implies that $h_\delta(0) = 0$. Hence $h_\delta(x) \le c|x|$ on B_δ. Therefore

$$(5.5)\qquad \int_{B_\delta\cap\partial M_\delta} h_\delta(g)u_\varepsilon^2\,d\sigma_\delta(g) \le c\varepsilon\delta.$$

For the second integral on the right-hand side of (5.3) we first observe that since in normal coordinates the metric g is Euclidean up to the second order, for $|x|$ small we have

$$(5.6)\qquad h_\delta(g) = h_\delta(\delta_{ij}) + O(|x|),$$

where $O(|x|)$ does not depend on δ, and $h_\delta(g)$ and $h_\delta(\delta_{ij})$ denote the mean curvature of ∂M_δ with respect to the metric g and the Euclidean metric respectively. Also since the metric g is euclidean up to the second order it is easy to check that for $|x|$ small

$$(5.7)\qquad d\sigma_\delta(g) = d\sigma_\delta(\delta_{ij}) + O(|x|^2)\,dx,$$

where $O(|x|^2)$ does not depend on δ, and $d\sigma_\delta(g)$ and $d\sigma_\delta(\delta_{ij})$ are the induced Riemannian measure on ∂M_δ with respect to the metric g and the Euclidean metric respectively.

Using (5.5), (5.6), and the fact that h_δ is uniformly bounded we get

$$\begin{aligned}\int_{B_{2\delta}-B_\delta\cap\partial M_\delta} & h_\delta(g)u_\varepsilon^2\,d\sigma_\delta(g)\\ &\le \int_{B_{2\delta}-B_\delta\cap\partial M_\delta} h_\delta(\delta_{ij})u_\varepsilon^2\,d\sigma_\delta(\delta_{ij}) + c\int_{B_{2\delta}\cap\partial M_\delta} |x|u_\varepsilon^2\,dx\\ &\le \int_{B_{2\delta}-B_\delta\cap\partial M_\delta} h_\delta(\delta_{ij})u_\varepsilon^2\,d\sigma_\delta(\delta_{ij}) + c\varepsilon\delta.\end{aligned}$$

Substituting this and (5.5) in (5.4) yields

$$(5.8)\qquad \int_{B_{2\delta}\cap\partial M_\delta} h_\delta(g)u_\varepsilon^2\,d\sigma_\delta(g) \le \int_{B_{2\delta}-B_\delta\cap\partial M_\delta} h_\delta(\delta_{ij})u_\varepsilon^2\,d\sigma_\delta(\delta_{ij}) + c\varepsilon\delta.$$

From now on we will denote $h_\delta(\delta_{ij})$ by h_δ and $d\sigma_\delta(\delta_{ij})$ by $d\sigma_\delta$. We estimate the integral on the right-hand side of (5.8) as

$$(5.9)\quad \int_{B_{2\delta}-B_\delta\cap\partial M_\delta} h_\delta u_\varepsilon^2\,d\sigma_\delta = \int_{B_{2\delta}-B_\delta} h_\delta\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)d\sigma_\delta + \int_{B_{2\delta}-B_\delta} h_\delta\left[u_\varepsilon^2-\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)\right]d\sigma_\delta.$$

The second integral on the right-side of the above equality can be estimated as

$$\int_{B_{2\delta}-B_\delta} h_\delta\left[u_\varepsilon^2-\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)\right]d\sigma_\delta \le c\int_{B_{2\delta}-B_\delta}\left[\left(\frac{\varepsilon}{\varepsilon+|x|^2}\right)-\left(\frac{\varepsilon}{\varepsilon^2+|x|^2+f_\delta^2}\right)\right]d\sigma_\delta,$$

where c, in this case, is the uniform bound on the sup norm of h_δ and $f_\delta=\psi_\delta f$. By observing that $|f_\delta|\le|f|\le c|x|^2$ so that

$$\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)-\left(\frac{\varepsilon}{\varepsilon^2+|x|^2+f_\delta^2}\right)\le\frac{c\varepsilon|x|^4}{(\varepsilon^2+|x|^2)^2},$$

the above integral inequality is reduced to

$$\int_{B_{2\delta}-B_\delta} h_\delta\left[u_\varepsilon^2-\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)\right]d\sigma_\delta\le c\varepsilon\delta^2.$$

The first integral on the right-hand side of (5.9) is

$$(5.10)\quad \int_{B_{2\delta}-B_\delta} h_\delta\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)d\sigma_\delta=\int_\delta^{2\delta}\left(\frac{\varepsilon}{\varepsilon^2+s^2}\right)\frac{d}{ds}\left(\int_{B_s}h_\delta\,d\sigma_\delta\right)ds.$$

The definition of the mean curvature and integration by parts yield

$$(5.11)\quad \begin{aligned}\int_{B_s}h_\delta\,d\sigma_\delta&=\int_{B_s}\partial_i\left(\frac{\partial_i(f_\delta)}{\sqrt{1+|\nabla f_\delta|^2}}\right)\sqrt{1+|\nabla f_\delta|^2}\,dx\\&=-\int_{B_s}\frac{(\partial_i f_\delta)(\partial_j f_\delta)(\partial_i\partial_j f_\delta)}{(1+|\nabla f_\delta|^2)}+\int_{\partial B_s}\nabla f_\delta\cdot\eta.\end{aligned}$$

On the one hand, since ψ_δ is radially symmetric,

$$\int_{\partial B_s}\nabla f_\delta\cdot\eta=\frac{\psi_\delta(s)}{s}\int_{\partial B_s}\nabla f\cdot x+\psi_\delta'(s)\int_{\partial B_s}f.$$

Using the symmetries of the circumference and the fact that $f = \frac{1}{2}\sum \lambda_i x_i^2 + \sum c_{ijk}x_i x_j x_k + O(|x|^4)$, where $\sum \lambda_i = 0$, we obtain

$$\int_{\partial B_s} \nabla f_\delta \cdot \eta = O(s^4).$$

On the other hand, by the definition of f_δ it is easy to check that $|\partial_i f_\delta| \le c|x|$ and $|\partial_i \partial_j f_\delta| \le c$ which implies that the first integral on the right-hand side of (5.11) has the order of s^4. Hence

$$\int_{B_s} h_\delta \, d\sigma_\delta = O(s^4). \tag{5.12}$$

Therefore, using integration by parts we get

$$\begin{aligned}&\int_\delta^{2\delta} \left(\frac{\varepsilon}{\varepsilon^2 + s^2}\right) \frac{d}{ds}\left(\int_{B_s} h_\delta \, d\sigma_\delta\right) ds \\ &\quad = \left(\frac{\varepsilon}{\varepsilon^2 + (2\delta)^2}\right) \int_{B_{2\delta}} h_\delta \, d\sigma_\delta + 2\varepsilon \int_\delta^{2\delta} \frac{s}{(\varepsilon^2 + s^2)^2} \left(\int_{B_{2s}} h_\delta \, d\sigma_\delta\right) ds.\end{aligned}$$

An easy calculation and (5.12) imply

$$\int_\delta^{2\delta} \left(\frac{\varepsilon}{\varepsilon^2 + s^2}\right) \frac{d}{ds}\left(\int_{B_s} h_\delta \, d\sigma_\delta\right) ds \le c\varepsilon\delta^2.$$

Substituting this first in (5.10) and then in (5.9) we get

$$\int_{B_{2\delta} - B_\delta \cap \partial M_\delta} h_\delta u_\varepsilon^2 \, d\sigma_\delta \le c\varepsilon\delta^2.$$

Substituting this in (5.8) yields

$$\int_{B_{2\delta} \cap \partial M_\delta} h_\delta(g) u_\varepsilon^2 \, d\sigma_\delta(g) \le c\varepsilon\delta. \tag{5.13}$$

On the other hand when we estimate the integral on the right-hand side of (5.2), using integration by parts as in (4.9) we get a nonzero term that can be estimated as follows:

$$\int_{\partial M_\delta \cap B^3_{\rho_0}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta_\delta} d\sigma_\delta \le \int_{B^2_{2\delta}(0) - B^2_\delta(0)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta_\delta} d\sigma_\delta + c\varepsilon\rho_0^2.$$

Substituting the definition of u_ε, we have

$$u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta_\delta} = -\frac{\varepsilon(x_1, x_2, x_3) \cdot \eta_\delta}{(\varepsilon^2 + |x|^2 + x_3^2)^2},$$

where η_δ is the unit outward normal vector to ∂M_δ with respect to the Euclidean metric. A straightforward computation shows that for $\delta \leq |x| \leq 2\delta$,

$$(x_1, x_2, x_3) \cdot \eta_\delta = \left\{ \sum_{i=1}^{2} \frac{1}{2}\lambda_i x_i^2 \psi + fr\psi'(r) + O(|x|^3) \right\} \frac{1}{\sqrt{1 + |\nabla f_\delta|^2}},$$

where $O(|x|^3)$ does not depend on δ. Thus

$$\begin{aligned}
&\int_{B_{2\delta}^2(0)-B_\delta^2(0)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta_\delta}\, d\sigma_\delta \\
&\quad \leq \int_{B_{2\delta}^2(0)-B_\delta^2(0)} \frac{\varepsilon}{(\varepsilon^2 + |x|^2)^2} \left(\sum_{i=1}^{2} \frac{1}{2}\lambda_i x_i^2 \psi + fr\psi'(r) + O(|x|^3) \right) dx \\
&\quad + c\varepsilon \int_{B_{2\delta}^2(0)-B_\delta^2(0)} \frac{|x|^6}{(\varepsilon^2 + |x|^2)^3}\, dx,
\end{aligned}$$

where we have used the fact $(x_1, x_2, x_3) \cdot \eta_\delta \leq c|x|^2$ and

$$\left(\frac{1}{\varepsilon^2 + |x|^2} \right)^2 - \left(\frac{1}{\varepsilon^2 + |x|^2 + f_\delta^2} \right)^2 \leq c \frac{|x|^4}{(\varepsilon^2 + |x|^2)^3}.$$

Using the Taylor's expansion for the function f, the symmetries of the annulus, and the facts that $\sum \lambda_i = 0$ and ψ is symmetric we obtain

$$\int_{B_{2\delta}^2(0)-B_\delta^2(0)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta_\delta} \leq c\varepsilon\delta^2.$$

Thus

$$\int_{\partial M_\delta \cap B_{\rho_0}} u_\varepsilon \frac{\partial u_\varepsilon}{\partial \eta_\delta}\, d\sigma_\delta \leq c\varepsilon\delta^2 + c\varepsilon\rho_0^2.$$

Using this, (5.2), (5.3), (5.13), arguing as in §4, and then Proposition 2.2 we get

$$\begin{aligned}
E(\varphi) \leq Q(S_+^n) \left(\int_M \varphi^{2n/(n-2)}\, dv \right)^{(n-2)/n} - (n-2) A_\delta \sigma_{n-1}^+ \varepsilon_0^2 \\
+ cA_\delta \varepsilon_0^2 \varepsilon^2 \rho_0^{-3} + cA_\delta \varepsilon_0^2 \rho_0 + c\varepsilon\rho_0 + c\varepsilon\delta.
\end{aligned}$$

If

$$\varliminf_{\delta \to 0} A_\delta > 0, \tag{5.14}$$

then choosing ρ_0 small enough and δ so that $2\delta < \rho_0$, and further choosing ε small enough (hence $\varepsilon \approx \varepsilon_0^2$) we get

$$E(\varphi) < Q(S_+^n)\left(\int_M \varphi^{2n/(n-2)}\,dv\right)^{(n-2)/n}.$$

The following gives a condition under which (5.14) holds.

Lemma 5.2. *If* ∂M *has a nonumbilic point, then* (5.14) *holds.*

Proof. Let $p \in \partial M$ be a nonumbilic point. Since nonumbilicity is an open property, there exists a neighborhood of p, $U \subset \partial M$, where every point is nonumbilic. Let K be a compact subset of U where every point is nonumbilic. Let χ be a smooth nonnegative function with compact support in $M - B_{\delta_1}(0)$ with $\chi \equiv 1$ on K. For a tensor $V = v_{ij}$ with compact support in $M - B_{\delta_1}(0)$ we define for δ fixed

$$g^t = G_\delta^{4/(n-2)} g_{ij} + t v_{ij} \quad \text{on } M_\delta.$$

Denote $R^t{}_{ij} = \mathrm{Ric}(g^t)$ and $h_{ij}{}^t = \pi(g^t)$, and let R^t and h^t denote the scalar curvature of g^t and the mean curvature of ∂M_δ with respect to g^t. Let u_t denote the solutions of

$$(5.15)\qquad \begin{cases} \Delta_t u_t - \frac{n-2}{4(n-1)} R^t u_t = 0 & \text{on } M_\delta - \{0\}, \\ \dfrac{\partial u_t}{\partial \eta_t} + \dfrac{n-2}{2} h^t u_t = 0 & \text{on } \partial M_\delta - \{0\}, \\ u_t(0) = 1. \end{cases}$$

Such solutions exists for $|t| < \beta$, with β depending only on g and v. In fact, we can write $u_t = F_t G_\delta^{-1}$, where F_t is the normalized Green's function for the metric $g + tG_\delta^{-4/(n-2)} V$ which exists for t small and depends smoothly on t. Since at $0 \in \partial M_\delta$ the second fundamental form vanishes, in the Appendix we show that for $|x|$ small

$$F_t(x) = |x|^{2-n} + A_t + O'(|x|),$$
$$G_\delta(x) = |x|^{2-n} + A_\delta + O'(|x|),$$

from which it follows that for $|x|$ small

$$u_t(x) = 1 + (A_t - A_\delta)|x|^{n-2} + O'(|x|^{n-1}).$$

Integrating (5.15) with respect to dv^t and using the divergence theorem we find

$$\sigma_{n-1}(A_\delta - A_t) = -\int_{\partial M_\delta - \{0\}} h^t u_t \, d\sigma^t + \frac{1}{2(n-1)} \int_{M_\delta - \{0\}} R^t u_t \, dv^t.$$

Differentiating the integrals on the right-hand side of the above equation and evaluating at $t = 0$ we find

$$\begin{aligned}\frac{d}{dt}\int_{M_\delta-\{0\}} R^t u_t\, dv^t\Big|_{t=0} &= \int_{M_\delta-\{0\}} \frac{d}{dt}(R^t\, dv^t)\Big|_{t=0}\\ &= -\int_{M_\delta}(\langle \mathrm{Ric}(g^0),\, V\rangle + \nabla^*\xi)\, dv^0,\end{aligned}$$

where $\xi = -\nabla^* V + \nabla\, \mathrm{tr}_g V$ and

$$\begin{aligned}\frac{d}{dt}\int_{\partial M_\delta-\{0\}} h^t u_t\, d\sigma^t\Big|_{t=0} &= \int_{\partial M_\delta-\{0\}} \frac{d}{dt}(h^t\, d\sigma^t)\Big|_{t=0}\\ &= \frac{1}{2}\int_{\partial M_\delta-\{0\}} \mathrm{tr}_g(\nabla_\eta V)\, d\sigma^0,\end{aligned}$$

η being the outward normal vector. Here, we have used the fact that $R_0 = 0 = h_0$ and $u_0 \equiv 1$. Assume that χ has compact support in a thin neighborhood of the boundary set K. Denote also by $\pi_{\partial M_\delta}$ the tensor which coincides on the boundary with the second fundamental form of the boundary, and assume that it is zero in any other component. Taking $V = -\chi\pi_{\partial M_\delta}$ we get the term

$$\text{(5.16)}\quad \begin{aligned}&\frac{d}{dt}\left(-\int_{\partial M_\delta-\{0\}} h^t u_t\, d\sigma^t + \frac{1}{2(n-1)}\int_{\partial M_\delta-\{0\}} R^t u_t\, dv^t\right)\Big|_{t=0}\\ &\geq \int_K \|\pi^0\|^2\, d\sigma^0.\end{aligned}$$

Since M_δ is minimal with respect to the metric g^0, $\|T^0\| = \|\pi^0\|$. Using Proposition 1.2 we have

$$\text{(5.17)}\qquad \|T^0\|^2\, d\sigma^0 = \|T_g\|^2 G_\delta^{2(n-3)/(n-2)}\, d\sigma.$$

Since $n = 3$, the right-hand side of (5.17) does not depend on δ. Moreover, on K, $\|T_g\|^2 > 0$ because every point is nonumbilic. By Lemma 5.1, the metric g^t varies smoothly in t up to any order on the support of χ uniformly in δ. Thus, there exists t_0 small so that $A_\delta - A_{t_0} \geq \gamma$ with $\gamma > 0$ independent of δ. Therefore for δ small we have

$$A_\delta = A_{t_0} + (A_\delta - A_{t_0}) \geq \gamma.$$

This establishes (5.14) because the Sobolev quotient depends smoothly on t. Then for t small it will be positive, and by the Positive Mass Theorem $A_{t_0} > 0$.

The following corollary is a special case of Theorems 3.1, 4.1, 4.2, and 5.1 when M is a bounded domain.

Corollary 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $n \geq 3$. Then $Q(\Omega) \leq Q(B)$, where B is the ball, and the equality holds only if Ω is the ball.*

Proof. Since because B and S^n_+ are conformally diffeomorphic and from Proposition 1.1that $Q(B) = Q(S^n_+)$. Since Ω is flat, the conclusion of the corollary follows from Theorems 3.1, 4.1, and 5.1.

6. Conformal deformation to constant scalar curvature and minimal boundary

We now prove our main theorem concerning conformal deformation.

Theorem 6.1. *Let (M^n, g) be a compact Riemannian manifold with boundary and $n \geq 3$. Assume that M^n satisfies any of the following three conditions:*

(i) $n = 3, 4,$ *or* 5,

(ii) *M has a nonumbilic point on ∂M,*

(iii) *∂M is umbilic and either M is locally conformally flat or $n \geq 6$ and the Weyl tensor does not vanish identically on ∂M.*

Then there exists a conformally related metric $u^{4/(n-2)}g$, $u > 0$ on the closure $\overline{M}$ of M, of constant scalar curvature on M and zero mean curvature on ∂M.

Proof. For $n = 4, 5$, if there exists a nonumbilic point, Theorem 3.1 implies that $Q(M) < Q(S^n_+)$. If there is not a nonumbilic point, then the boundary is umbilic. Thus Theorem 4.1 implies $Q(M) < Q(S^n_+)$ provided that M is not conformally equivalent to S^n_+. If $n = 3$, and M^n has an umbilic point, then from Theorem 4.2 it follows that $Q(M) < Q(S^n_+)$ if M is not conformally equivalent to S^n_+. If M does not have an umbilic point, then it has a nonumbilic point. Theorem 5.1 implies that $Q(M) < Q(S^n_+)$ in this case. Thus, if M is not conformally equivalent to S^n_+, we have $Q(M) < Q(S^n_+)$ when $n = 3, 4,$ or 5. If M satisfies (ii), Theorems 3.1 and 5.1 show that $Q(M) < Q(S^n_+)$. If M satisfies (iii), Theorem 4.1 implies that $Q(M) < Q(S^n_+)$ provided that M is not conformally equivalent to S^n_+. Let $\alpha \in [1, (n+2)/(n-2)]$, $\alpha_0 = (n+2)/(n-2)$, and consider the ratio

$$Q_\alpha(\varphi) = \frac{E(\varphi)}{(\int_M |\varphi|^{\alpha+1}\, dv)^{2/\alpha+1}}, \qquad \varphi \in H_1(M),$$

where

$$E(\varphi) = \int_M (|\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R\varphi^2)\, dv + \frac{n-2}{2} \int_{\partial M} h\varphi^2\, d\sigma ,$$

and $H_1(M)$ is the Sobolev space of functions with L^2 first derivatives. By the Sobolev embedding theorem it is elementary to show that there exists, for any $\alpha \in (1, \alpha_0)$ satisfying smooth functions $u_\alpha > 0$, $\int_M u_\alpha^{\alpha+1}\, dv = 1$ and

$$Q_\alpha(u_\alpha) = \min\{Q_\alpha(\varphi) : \varphi \in H_1(M),\ \varphi \neq 0\}.$$

We denote this value by $Q_\alpha(M)$ so that $Q_{\alpha_0}(M) = Q(M)$. Moreover, u_α satisfies the Euler-Lagrange equations

$$(6.1) \qquad \begin{cases} \Delta u_\alpha - \frac{n-2}{4(n-1)} R u_\alpha + Q_\alpha(M) u_\alpha^\alpha = 0 & \text{on } M, \\ \dfrac{\partial u_\alpha}{\partial \eta} + \dfrac{n-2}{2} h u_\alpha = 0 & \text{on } \partial M. \end{cases}$$

One attempts to take the limit as $\alpha \uparrow \alpha_0$. Since we have a uniform bound on the H_1 norm of u_α, by weak compactness we can find a weakly convergent sequence $\{u_{\alpha_i}\}$. The weak form of (6.1) is

$$\int_M \left(\nabla\phi \cdot \nabla u_\alpha + \frac{n-2}{4(n-1)} R u_\alpha \phi - Q_\alpha(M) u_\alpha^\alpha \phi \right) dv + \frac{n-2}{2} \int_{\partial M} h u_\alpha \phi\, d\sigma = 0$$

for any $\phi \in C^\infty(M)$. Since $H_1(M)$ is compactly contained in $L^p(M)$ for any $p < 2n/(n-2)$ and $L^2(\partial M)$, it follows easily that the weak H_1 limit u of the sequence u_{α_i} satisfies the limiting equation. (Note that one sees immediately that $\lim_{\alpha \to \alpha_0} Q_\alpha(M) = Q(M)$.) A regularity result of Cherrier [4] then implies that u is smooth. One needs only show that u is nonzero, and this is where the fact $Q(M) < Q(S^n_+)$ enters. Given $P \in M$ and $\rho > 0$ small, let ϕ be a smooth function on M which is equal to one in $B_\rho(P)$ and zero outside $B_{2\rho}(P)$. Multiplying (6.1) by $\phi^2 u_\alpha$ and integrating by parts we get

$$\begin{aligned} \int_M \phi^2 |\nabla u_\alpha|^2\, dv \le & -2 \int_M \phi u_\alpha \nabla\phi \cdot \nabla u_\alpha\, dv + c \int_M \phi^2 u_\alpha^2\, dv \\ & + c \int_{\partial M} \phi^2 u_\alpha^2\, d\sigma + Q_\alpha(M) \int_M \phi^2 u_\alpha^{\alpha+1}\, dv. \end{aligned}$$

which easily implies, for any $\varepsilon > 0$,

$$\begin{aligned} &(1-\varepsilon) \int_M |\nabla \phi u_\alpha|^2\, dv \\ &\quad \le c(\varepsilon)\rho^{-2} \int_M u_\alpha^2\, dv + c \int_{\partial M} \phi^2 u_\alpha^2\, d\sigma + Q_\alpha(M) \int_M \phi^2 u_\alpha^{\alpha+1}\, dv , \end{aligned}$$

where $c(\varepsilon)$ depends on ε and M. The Sobolev inequality in $B_{2\rho} \subset M - \partial M$ for functions in $H_0^1(B_{2\rho})$ holds with the Euclidean Sobolev constant $Q(S^n)$ plus an error term which is of order ρ^2 because the metric is Euclidean up to the second order. The Sobolev inequality in $B_{2\rho}$ such that $B_{2\rho} \cap \partial M \neq \varnothing$, for functions in $H^1(B_{2\rho})$, holds with the constant $Q(S^n_+)$ plus an error term of order ρ because the metric is Euclidean up to the first order. Since

$$Q(S^n) = \frac{n(n-2)}{4} \operatorname{Vol}(S^n)^{2/n} > \frac{n(n-2)}{4} \operatorname{Vol}(S^n_+)^{2/n} = Q(S^n_+),$$

we have

$$\begin{aligned}
&(1-\varepsilon)(Q(S^n_+) - c\rho)\left(\int_M \phi u_\alpha^{2n/(n-2)}\, dv\right)^{(n-2)/n} \\
(6.2) \qquad &\leq c(\varepsilon)\rho^{-2} \int_M u_\alpha^2\, dv \\
&\quad + c\int_{\partial M} \phi^2 u_\alpha^2\, d\sigma + Q_\alpha(M) \int_M \phi^2 u_\alpha^{\alpha+1}\, dv.
\end{aligned}$$

Now observe that $\phi^2 u_\alpha^{\alpha+1} = (\phi u_\alpha)^\alpha u\alpha - 1_\alpha$ so that

$$\begin{aligned}
\int_M \phi^2 u_\alpha^{\alpha+1}\, dv &\leq \left(\int_M (\phi u_\alpha^{2n/(n-2)}\, dv\right)^{(n-2)/n} \left(\int_M u_\alpha^{(\alpha-1)n/2}\, dv\right)^{2/n} \\
&\leq \left(\int_M (\phi u_\alpha)^{2n/(n-2)}\, dv\right)^{(n-2)/n},
\end{aligned}$$

where we have used Hölder's inequality twice, normalized g so that $\operatorname{Vol}_g(M) = 1$, $\int_M u_\alpha^{\alpha+1}\, dv = 1$, and used the fact that $(\alpha - 1)\frac{n}{2} \leq \alpha + 1$. Since our theorem is trivial if M is conformally diffeomorphic to S^n_+, we assume this is not the case, and hence we have $Q(M) < Q(S^n_+)$. In particular, we have $Q_\alpha(M) < Q(S^n_+)$ for α near α_0. Then by fixing ε, ρ small enough to absorb the last term on the right-hand side of (6.2) to the left get

$$\left(\int_M (\phi u_\alpha)^{2n/(n-2)}\, dv\right)^{(n-2)/n} \leq c\int_M u_\alpha^2\, dv + c\int_{\partial M} u_\alpha^2\, d\sigma.$$

Since ϕ is one on $B_\rho(P)$, we can take a finite covering of M by balls of radius ρ and sum these inequalities to obtain

$$\left(\int_M u_\alpha^{2n/(n-2)}\, dv\right)^{(n-2)/n} \leq c\left(\int_M u_\alpha^2\, dv + \int_{\partial M} u_\alpha^2\, d\sigma\right).$$

Since $\alpha + 1 \leq 2n/(n-2)$, this implies

$$1 \leq c\left(\int_M u_\alpha^2 \, dv + \int_{\partial M} u_\alpha^2 \, dv\right).$$

Since H_1 is compactly contained in $L^2(M)$ as well as in $L^2(\partial M)$, the same lower bound holds on u and hence u is a nonzero, nonnegative solution of the Euler-Lagrange equations

$$\Delta u - \frac{n-2}{4(n-1)} R_g u + Q(M) u^{(n+2)/(n-2)} = 0 \quad \text{on } M,$$

$$\frac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u = 0 \quad \text{on } \partial M,$$

Since u does not vanish, the maximum principle implies that $u > 0$ on M. From the boundary point lemma we have that $u > 0$ on $\overline{M}$.

As an immediate consequence of Theorem 6.1 we have the following.

Corollary 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $n \geq 3$. Then there exists a metric conformal to the euclidean metric with constant scalar curvature on Ω and with minimal boundary.*

Appendix: The Positive Mass Theorem for manifolds with boundary

Here we give a brief discussion of the Positive Mass Theorem for manifolds with boundary. We first discuss the locally conformally flat case for an n-dimensional manifold with $n \geq 3$. Then we will discuss the three- and four-dimensional cases.

The situation we cover here is when the asymptotic "end" of the manifold is diffeomorphic to the complement of a ball centered at the origin in the half n-dimensional Euclidean space. This situation is not covered in the previous work on the Positive Mass Theorems. We refer the reader to the work of Lee and Parker [8] for the history on this problem. We show that the theorem holds for the relevant cases which we need in this paper.

In this Appendix we assume (M^n, g) is a compact Riemannian manifold with boundary and dimension $n \geq 3$, and also that the Sobolev quotient $Q(M)$ is positive. Let us first consider the case where (M^n, g) is locally conformally flat and the boundary ∂M of M is umbilic. Let φ_1 be the first eigenfunction for the conformal Laplacian with respect to the boundary condition as in (2). Lemma 1.1 states that the metric $g_1 = \varphi_1^{4/(n-2)} g$ satisfies that $R_{g_1} > 0$ and $h_{g_1} = 0$ on ∂M. Since (M^n, g) is locally conformally flat, there exists a locally defined positive function u such that near $0 \in \partial M$ the metric $g_2 = u^{4/(n-2)} g_1 = \delta_{ij}$.

The umbilicity of the boundary implies that ∂M near 0 is either a piece of sphere or a hyperplane. Since both neighborhoods in consideration are conformally equivalent through the inversion map $f(x) = x|x|^{-2}$, we can assume that near 0 the boundary is a hyperplane. Gluing the function u and the constant function 1 with a function satisfying the Neumann condition we can assume that the metric g_2 above is globally defined and satisfies that in a neighborhood of 0 it is the Euclidean metric and $h_{g_2} = 0$ on ∂M. So we can take rectangular coordinates $(x_1, \cdots, x_n)$ near 0, such that the metric is the Euclidean metric and the boundary is $x_n = 0$. In these coordinates the conformal Laplacian is the standard Laplacian, and the boundary condition is the Neumann condition. Therefore, the Green's function has an expansion for x small

$$G(x) = |x|^{2-n} + A + \alpha(x), \tag{1}$$

where $\alpha(x)$ is a harmonic function such that $\alpha(0) = 0$ and $\frac{\partial \alpha}{\partial \eta} = 0$.

The Positive Mass Theorem implies that $A \geq 0$. Moreover $A = 0$ if and only if M is conformally equivalent to (S^n_+, g_0), where g_0 is the standard round metric. We claim that this theorem can be reduced to the analogous theorem for manifolds without boundary due to Schoen-Yau [11]–[13] or see Lee-Parker [8]. In order to see this, we consider the double $\widetilde{M} = M \cup \partial M \cup M$ of the manifold M, with the standard metric $\tilde{g}$. For a general manifold $\widetilde{M}$ the metric $\tilde{g}$ is defined near the boundary as follows: Let $0 \in \partial M$, and $x_1, \cdots, x_{n-1}$ be coordinates at the boundary. Let $\gamma(x_n)$ be a geodesic leaving from $(x_1, \cdots, x_{n-1})$ in the orthogonal direction to ∂M and parametrized by arc length. Then $(x_1, \cdots, x_{n-1}, x_n)$ are the so-called Fermi coordinates at $0 \in \partial M$. In these coordinates the arc length ds^2 is written as

$$ds^2 = dx_n^2 + g_{ij}(x)dx_i dx_j,$$

where $1 \leq i, j \leq n-1$. The metric $\tilde{g}$ is then defined as

$$\tilde{g}(x_1, \cdots, x_n) = \begin{cases} g(x_1, \cdots, x_n), & x_n \geq 0, \\ g(x_1, \cdots, -x_n), & x_n \leq 0. \end{cases}$$

It is clear that $\tilde{g}$ is continuous. Moreover,

$$\begin{aligned} g_{ij,n} &= \partial_n(\partial_i, \partial_j) = (\nabla_{\partial_n}\partial_i, \partial_j) + (\partial_i, \nabla_{\partial_n}\partial_j) \\ &= (\nabla_{\partial_i}\partial_n, \partial_j) + (\partial_i, \nabla_{\partial_j}, \partial_n). \end{aligned}$$

Since $-\partial_n$ is the outward normal vector and the second fundamental form is symmetric, we get $g_{ij,n} = -2h_{ij}$. Thus if the second fundamental form vanishes, the metric $\widetilde{g}$ is C^1. In this case, since

$$\frac{\partial^2}{\partial x_n^2}\widetilde{g}(x, x_n) = \frac{\partial^2}{\partial x_n^2}\widetilde{g}(x, -(x_n)),$$

we have that the metric $\widetilde{g}$ is actually C^2. Moreover, if the initial metric g is smooth up to the boundary, then the metric $\widetilde{g}$ has Lipschitz second derivatives, and is $C^{2,\alpha}$ in particular.

In our case, the metric for (M, g_2) near 0 is clearly C^∞. When we consider the Green's function $\widetilde{G}$, for the conformal Laplacian on the closed manifold $(\widetilde{M}, \widetilde{g}_2)$, since near 0 the metric is the Euclidean metric δ_{ij}, then for x small we have

$$\widetilde{G}(x) = |x|^{2-n} + \widetilde{A} + \widetilde{\alpha}(x), \tag{2}$$

where $\widetilde{\alpha}(x)$ is a harmonic function, such that $\widetilde{\alpha}(0) = 0$.

The metric $\widetilde{G}^{4/(n-2)}\widetilde{g}$ is an asymptotically flat metric defined on the manifold $\widetilde{M} - \{0\}$.

Consider the inverted coordinates $z^i = x^i/r^2$, where $r^2 = x_1^2 + \cdots + x_n^2$. In these coordinates, using the expansion (2) one checks easily that $\widetilde{G}^{4/(n-2)}\widetilde{g}$ has the following expansion near infinity:

$$\widetilde{G}^{4/(n-2)}\widetilde{g}(z) = (1 + \widetilde{A}\rho^{2-n} + O''(\rho^{1-n}))^{4/(n-2)}\delta_{ij},$$

where $\rho = |z| = r^{-1}$. Moreover, from the transformation formula (1.2) we get that the scalar curvature of $\widetilde{G}^{4/(n-2)}\widetilde{g}$ is zero. It follows from the Positive Mass Theorem, [11]–[13], and [8] that $\widetilde{A} \geq 0$ where the equality holds if and only if $(\widetilde{M} - \{0\}, \widetilde{G}^{4/(n-2)}\widetilde{g})$ is isometric to $(\mathbb{R}^n, \delta_{ij})$.

To finish this case, observe that the Green's function G is obtained as

$$G(x, x_n) = \frac{1}{2}[\widetilde{G}(x, x_n) + \widetilde{G}(x, -x_n)]. \tag{3}$$

Hence $A = \widetilde{A}$. Therefore $A \geq 0$ and equality holds if and only if $(M - \{0\}, G^{4/(n-2)}g)$ is isometric to $(\mathbb{R}^n_+, \delta_{ij})$.

When $n = 4$ we study the case where (M^4, g) is a compact Riemannian manifold with boundary and ∂M is umbilic. Consider as before the metric $g_1 = \varphi_1^2 g$ which has the property that $R_{g_1} > 0$ and $h_{g_1} = 0$ on ∂M. Therefore, the umbilicity of the boundary implies that ∂M is totally geodesic. Take geodesic normal coordinates $(x_1, \cdots, x_4)$ near 0 such that ∂M is given by the equation $x_4 = 0$. Near 0 we further change the metric g_1 by the metric $g_2 = e^{2f} g_1$, where f is a homogeneous polynomial of second degree. It was proved in Proposition 2.1 that there exists

f such that the $h_{g_2} = 0$ in a neighborhood of 0. Gluing the function e^{2f} to the function 1 with a function satisfying the Neumann boundary condition on ∂M we can assume that the metric g_2 is globally defined and satisfies that $R_{ij}(0) = 0$ and $h_{g_2} = 0$ on ∂M.

Now we want to show that for the metric g_2 the Green's function for the conformal Laplacian has the expansion for x small as

$$G(x) = |x|^{-2} + A + O''(|x| \log |x|).$$

Since for the metric g_2 the boundary is minimal, the boundary condition for the conformal Laplacian is the Neumann condition. To study the above expansion, it is enough to consider the expansion for the Green's function for the conformal Laplacian on the double manifold $(\widetilde{M}, \widetilde{g})$, and note as before that G is given by formula (3).

Let $(x^1, \cdots, x^4)$ be normal coordinates at $x = 0$. It is well known that in normal coordinates the metric has the following asymptotic expansion:

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{iklj} x^k x^l + O(|x|^3). \tag{4}$$

Using this expansion one proves easily that

$$\sqrt{g} = 1 - \frac{1}{6} R_{ij}(0) x^i x^j + O(|x|^3). \tag{5}$$

Assume that at $x = 0$, $R_{ij}(0) = 0$. We will show that in normal coordinates, if $R_{ij}(0) = 0$ the Green's function for the conformal Laplacian L has the following asymptotic expansion around 0, for x small:

$$G(x) = |x|^{-2} + A + O''(|x| \log |x|). \tag{6}$$

In order to do that, write $G = r^{-2}(1 + \psi)$, where $r = |x|$. We want to study the equation

$$LG = 2\sigma_3 \delta_0, \tag{7}$$

where $L = \Delta_g - \frac{1}{6} R_g$ and $\sigma_3 = \mathrm{Vol}(S^3)$. The Laplacian on radial functions is given by

$$\Delta_g = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{(\sqrt{g})'}{\sqrt{g}}\frac{\partial}{\partial r}. \tag{8}$$

Thus

$$\Delta_g r^{-2} = 2\sigma_3 \delta_0 + \frac{(\sqrt{g})'}{\sqrt{g}} \frac{\partial}{\partial r}(r^{-2}).$$

(7) then is equivalent to

$$L(r^{-2}\psi) - \frac{1}{6} R_g (r^{-2}) + \frac{(\sqrt{g})'}{\sqrt{g}} \frac{\partial}{\partial r}(r^{-2}) = 0. \tag{9}$$

From the definition of the Laplacian one has that

$$\Delta_g = \Delta + P, \tag{10}$$

where $P = \frac{\partial_i(\sqrt{g})}{\sqrt{g}} g^{ij}\partial_j + \partial_i((g^{ij} - \delta^{ij})\partial_j)$ and Δ is the Euclidean Laplacian.

From (8) and (10) it follows that the operator P acting on radial functions is $\frac{(\sqrt{g})^1}{\sqrt{g}} \frac{\partial}{\partial r}$.

Multiplying by r^4, writing $T = r^2\Delta - 4r\partial_r$, and assuming ψ is continuous, we see that (7) is equivalent to

$$D\psi = T\psi + r^2([r^2P(r^{-2}) - \tfrac{1}{6}R] + [r^2P(r^{-2}) - \tfrac{1}{6}R]\psi + r^2P(\psi)) = 0. \tag{11}$$

We will compute a formal asymptotic solution to (11). Let $\overline{\psi} = \psi_1 + \psi_2 + \psi_3 + \psi_4$, with $\psi_k \in \mathscr{P}_k$; where $\mathscr{P}_k$ denotes the space of homogeneous polynomials in x of degree k.

From (5) and the fact that $R_{ij}(0) = 0$ we conclude that $\sqrt{g} = 1 + O(|x|^3)$, so that

$$P(r^{-2}) = \frac{(\sqrt{g})'}{\sqrt{g}} \frac{\partial}{\partial r} r^{-2} = O(r^{-1}).$$

Since $R_{ij}(0) = 0$, $R = O(r)$ and hence we start by setting $\psi_1 = \psi_2 = 0$.

Let $\mathscr{C}_k$ denote the set of smooth functions that vanish to order k at 0. The operator T is not invertible on P_k for $k \geq 2$. However, $\mathscr{P}_k = \operatorname{im} T \oplus \ker T$ because T is selfadjoint with respect to Euclidean inner product

$$\langle \sum a_I x^I, \sum b_j x^j \rangle = \sum a_I b_I$$

on $\mathscr{P}_k$. To find ψ_3 we observe that if we let $\overline{\psi} = \psi_1 + \psi_2 = 0$, we have $D\overline{\psi} \in \mathscr{C}_3$. Set the right-hand side as $b_3 + \mathscr{C}_4$ with $b_3 \in \mathscr{P}_3$. We try $\psi_3 = p_3 + q_3 \log r$, with $p_3, q_3 \in \mathscr{P}_3$. By direct computation,

$$T(p_3 + q_3 \log r) = Tp_3 - 4q_3 + T(q_3)\log r.$$

Thus we can solve $T\psi_3 + b_3 = 0$ by writing $-b_3 = Tp_3 + q_3$, $Tq_3 = 0$, and setting

$$\psi_3 = p_3 - \tfrac{1}{4} q_3 \log r.$$

If $\overline{\psi} = \psi_1 + \psi_2 + \psi_3$,

$$D\overline{\psi} \in \mathscr{C}_4 + \mathscr{C}_6 \log r.$$

Write the right-hand side as $b_4 + \mathscr{C}_5 + \mathscr{C}_6 \log r$. Consider $\psi_4 = p_4 + q_4 \log r$ with $p_4, q_4 \in \mathscr{P}_4$. By direct computation we have

$$T(p_4 + q_4 \log r) = T(p_4) - 6q_4 + (Tq_4)\log r.$$

We can solve $T\psi_4 + b_4 = 0$ with $b_4 \in \mathscr{P}_4$ by writing $-b_4 = Tp_4 + q_4$, $Tq_4 = 0$ and setting $\psi_4 = p_4 - \frac{1}{6}q_4 \log r$.

If $\overline{\psi} = \psi_1 + \psi_2 + \psi_3 + \psi_4$,

$$D\overline{\psi} \in \mathscr{C}_5 + \mathscr{C}_6 \log r.$$

Therefore $\overline{\psi}$ satisfies

$$L(r^{-2}\overline{\psi}) - \frac{1}{6}R(r^{-2}) + \frac{(\sqrt{g})'}{\sqrt{g}}\frac{\partial}{\partial r}(r^{-2}) \in r^{-4}(\mathscr{C}_5 + \mathscr{C}_6 \log r). \tag{12}$$

Now write $\psi = \varphi + \overline{\psi}$. By (9) and (12) we have $L(r^{-2}\varphi) \in C^\alpha$. Elliptic regularity theory asserts that $r^{-2}\varphi \in C^{2,\alpha}$. Since $r^{-2}\overline{\psi} = O''(r\log r)$, we have the expansion for G as in (6).

Now we want to calculate the expansion of the metric $\widetilde{G}^2\widetilde{g}$ on $\widetilde{M} - \{0\}$ near infinity on inverted coordinates. Since

$$\frac{\partial}{\partial z^i} = \rho^{-2}(\delta_{ij} - 2\rho^{-2}z_i z_j)\frac{\partial}{\partial x^j},$$

we have

$$\widetilde{g}\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = \rho^{-4}(\delta_{ik} - 2\rho^{-2}z_i z_k)(\delta_{jl} - 2\rho^{-2}z_j z_l)\widetilde{g}_{kl}(\rho^{-2}z).$$

Using the expansion (6) for $\widetilde{G}$, we get for z near infinity

$$\widehat{g} = \widetilde{g}(z) = \widetilde{G}^2\widetilde{g}(z) = (1 + \widetilde{A}\rho^{-2} + O''(\rho^{-3}\log(\rho)))^2(\delta_{ij} + O''(\rho^{-2})).$$

The metric $\widehat{g}$ has zero scalar curvature. From the above expansion for the metric $\widehat{g}$ we easily conclude that

$$\widehat{g}_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\tau}(\widetilde{M}_\infty),$$

where $\widetilde{M}_\infty$ is a neighborhood of infinity, $\tau > 1$, and $C^{1,\alpha}_{-r}$ is the weighted Hölder space of C^1 functions with weight $-\tau$ and Hölder exponent α. (See [8] for a precise definition of $C^{1,\alpha}_{-\tau}$.)

In this case we have that the mass is well defined, and Lemma 9.7 in [8] shows that in this case $m(\widehat{g}) = c(n)\widetilde{A}$, where $c(n)$ is a positive constant. Now apply the Positive Mass Theorem 10.1 as in [8] to conclude that $\widetilde{A} \geq 0$, and equality holds if and only if $(\widetilde{M} - \{0\}, \widehat{g})$ is isometric to $\mathbb{R}^4$ with its Euclidean metric. This clearly implies using (3) that $A \geq 0$ and equality holds if (M^4, g) is conformally equivalent to (S^4_+, g_0).

Let (M^3, g) be a 3-dimensional compact Riemannian manifold with boundary, and $0 \in \partial M$ be an umbilic point. We also assume that $Q(M) > 0$. We change the metric g by the metric $g_1 = \varphi_1^{4/(n-2)}g$, where φ_1

is the first eigenfunction for the conformal Laplacian. Let (x_1, x_2) be normal coordinates at 0 on ∂M, and (x_1, x_2, x_3) be Fermi coordinates. Since ∂M is minimal and the point $0 \in \partial M$ is umbilic, we have that (x_1, x_2, x_3) are normal coordinates at 0.

Now consider the doubling manifold $(\widetilde{M}, \tilde{g})$. In the coordinates (x_1, x_2, x_3), which are normal at 0, it is well known that the Green's function $\widetilde{G}$ for the conformal Laplacian has an expansion for x small

$$\widetilde{G}(x) = |x|^{-1} + \widetilde{A} + O''(|x|). \tag{13}$$

Observe that the metric $\tilde{g}$ is only Lipschitz continuous. Therefore we will use the version of the Positive Mass Theorem (Theorem 6.3 in [3]) due to Bartnik which applies to spin manifolds with weak regularity assumptions on the metric. We remark that the proof extends easily to spin manifolds with several asymptotic ends (see [9]). Since a 3-dimensional orientable manifold admits a spin structure, the above remark allows us to extend the theorem to nonorientable 3-dimensional manifolds applying the theorem to one of the ends on the orientable double cover, which has two ends. Now consider the metric $\widetilde{G}^4\tilde{g}$ on $\widetilde{M}-\{0\}$. From the expansion for the Green's function, in inverted coordinates $z^i = r^{-2}x^i$, where $r^2 = x_1^2 + x_2^2 + x_3^2$ if $\rho = |z| = r^{-1}$, we have near infinity that

$$\hat{g}(z) = \widetilde{G}^4\tilde{g}(z) = (1 + \widetilde{A}\rho^{-1} + O''(\rho^{-2}))^4(\delta_{ij} + O''(\rho^{-2})).$$

Hence it is clear that

$$g_{ij} - \delta_{ij} \in W^{2,q}_{-r}(\widetilde{M}_\infty),$$

where $\tau \geq 1/2$, and $W^{2,q}_{-\tau}$ denotes the weighted Sobolev space of weight $-\tau$ (see [3] for a precise definition). Since the scalar curvature of $\hat{g}$ is zero, we apply the Positive Mass Theorem to conclude that $m(\hat{g}) = c(n)\widetilde{A} \geq 0$ where the equality holds if and only if $(\widetilde{M} - \{0\}, \hat{g})$ is isometric to $\mathbb{R}^3$ with the Euclidean metric. The constant $c(n)$ is a normalization constant which is positive. Since the Green's function for the conformal Laplacian on (M, g_1) is given by the formula (3), the above result implies that $A \geq 0$ where the equality holds if and only if (M, g_1) is conformally equivalent to (S^3_+, g_0).

Finally we would like to remark that if the second fundamental form vanishes at $0 \in \partial M$, and (x_1, x_2, x_3) are normal coordinates at 0, such that $\eta(0) = -\frac{\partial}{\partial x_3}$, then for $|x|$ small the expansion for the Green's function G_g for the conformal Laplacian is as in (13), and the Positive Mass Theorem holds. In order to see this let g_1 be as before. Multiplying φ_1

by a positive constant we can assume that $\varphi_1(0) = 1$. From the transformation law (1.5) and (1.6) we see that $G_g = \varphi_1 G_{g_1}$. Since at 0 the second fundamental form vanishes with respect to the metric g and the boundary is minimal with respect to the metric g_1, Proposition 1.2 implies that at 0 the second fundamental form with respect to the metric g_1 vanishes. Hence (x_1, x_2, x_3) are normal coordinates at 0 with respect to the metric g_1, and the theorem holds for (M, g_1), because the metric in Fermi coordinates and in normal coordinates coincides up to second order near 0. Moreover, the transformation law (1.4) yields that $\frac{\partial}{\partial x_3}\varphi_1(0) = 0$. For any vector fields Y and Z we have

$$D^1_Y Z - D_Y Z = \tfrac{1}{2}[Y(f)Z + Z(f)Y - g(Y, Z)\nabla f],$$

where D^1 and D are the Riemannian connections with respect to the metrics g_1 and g respectively, and $e^f = \varphi_1^{4/(n-2)}$. Setting $Z = \frac{\partial}{\partial x_3}$ and $Y = \frac{\partial}{\partial x_i}$ for $i = 1, 2$ in the above equality and evaluating at 0 we get $\frac{\partial}{\partial x_i}\varphi_1(0) = 0$. Thus, Taylor's theorem implies that $\varphi_1(x) = 1 + O(|x|^2)$ for $|x|$ small. Since $G_g = \varphi_1 G_{g_1}$, we have that for $|x|$ small G_g has the expansion as in (13), and the Positive Mass Theorem holds.

Bibliography

[1] T. Aubin, *Équations différentielles non linéaires et probléme de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976) 269–296.

[2] ——, *Problèmes isopérimétrique et espaces de Sobolev*, J. Differential Geometry **11** (1976) 573–598.

[3] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure. Appl. Math. **39** (1986) 661–693.

[4] P. Cherrier, *Problèmes de Neumann non linéaires sur les variétés Riemanniennes*, J. Funct. Anal. **57** (1984) 154–206.

[5] J. Escobar, *Positive solutions for some semilinear elliptic equations with Neumann boundary conditions*, in preparation.

[6] D. Gilbarg & N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1977.

[7] J. Kazdan & F. Warner, *Scalar curvature and conformal deformations of Riemannian structure*, J. Differential Geometry **10** (1975) 113–134.

[8] J. Lee & T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987) 37–91.

[9] T. Parker & C. Taubes, *On Witten's proof of the positive energy theorems*, Comm. Math. Phys. **84** (1982) 223–237.

[10] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geometry **20** (1984) 479–495.

[11] R. Schoen & S. T. Yau, *On the proof of the positive mass conjecture in General Relativity*, Comm. Math. Phys. **65** (1979) 45–76.

[12] ——, *Proof of the positive action conjecture in quantum relativity*, Phys. Rev. Lett. **42** (1979) 547–548.

[13] ——, *Conformally flat manifolds, Kleinian groups, and scalar curvature*, Invent. Math. **92** (1988) 47–71.

[14] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976) 353–372.

[15] N. Trudinger, *Remarks concerning the conformal deformation of a Riemannian structure on compact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **22** (1968) 165–274.

[16] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960) 21–37.

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