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THE PICARD GROUP OF THE UNIVERSAL PICARD VARIETIES OVER THE MODULI SPACE OF CURVES

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1. Introduction

We denote by $\mathcal{M}_g^0$ the moduli space of smooth curves of genus g ($g \geq 3$) without automorphisms, and by $\mathcal{C}_g \xrightarrow{\pi} \mathcal{M}_g^0$ the universal curve over $\mathcal{M}_g^0$. For any integer d, we denote by $\psi_d\colon \mathcal{T}_g^d \to \mathcal{M}_g^0$ the universal Picard (Jacobian) variety of degree d; the fiber $J^d(C)$ over a point $[C]$ of $\mathcal{M}_g^0$ parametrizes line bundles on C of degree d, modulo isomorphism. The construction of these bundles can be found for example in [9]. Note that although for a fixed curve C the varieties $J^d(C)$ are all isomorphic to the Jacobian variety of the curve, it is not true that this isomorphism can be carried out over $\mathcal{M}_g^0$: For $d_1 \neq d_2$ the isomorphism $J^{d_1}(C) \cong J^{d_2}(C)$ depends on the choice of a line bundle on C of degree $d_1 - d_2$; on the other hand, except in the case where $d_1 - d_2$ is a multiple of $2g-2$, there is no "uniform" choice of a line bundle of degree $d_1 - d_2$ on the fibers of the universal curve (see Theorem 2). In this work we describe the Picard group of the $\mathcal{T}_g^d$'s; first a definition.

Definition. We define the relative Picard group of $\mathcal{T}_g^d$, denoted by $\mathcal{R}\operatorname{Pic}(\mathcal{T}_g^d)$, to be the cokernel of the map $\psi_d^*\colon \operatorname{Pic}(\mathcal{M}_g^0) \to \operatorname{Pic}(\mathcal{T}_g^d)$.

Lemma 1. *Two line bundles on $\mathcal{T}_g^d$ define the same element in $\mathcal{R}\operatorname{Pic}(\mathcal{T}_g^d)$ if and only if their restrictions to the fibers of the map ψ_d define isomorphic line bundles.*

Proof. This is a restatement of the see-saw principle (see [10]). q.e.d.

Since the Picard group of $\mathcal{M}_g^0$ is known (see [1]), we are going to describe the groups $\mathcal{R}\operatorname{Pic}(\mathcal{T}_g^d)$. As the first step for this, we shall describe a "weaker" group $\mathcal{N}(\mathcal{T}_g^d)$ (which we call the relative Neron-Severi group of $\mathcal{T}_g^d$) defined to be the group of line bundles on $\mathcal{T}_g^d$, modulo the rela-

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tion that two line bundles are equivalent if their restrictions to the fibers of the map ψ_d are algebraically equivalent, i.e., if they define the same element in the Neron-Severi group of the fibers. We do this and, at the last section of this paper, we prove that actually we have an isomorphism $\mathscr{R}\,\mathrm{Pic}(\mathscr{T}_g^d) \cong \mathscr{N}(\mathscr{T}_g^d)$ and so this leads to the description of the relative Picard groups.

Lemma 2. *The Neron-Severi group of the Jacobian of a curve C with general moduli is generated by the class θ of its theta divisor.* (*The expression "general moduli" means that there is a countable union of subvarieties of $\mathscr{M}_g^0$ where the above property fails.*)

Proof. See Lemma on p. 359 in [2]. q.e.d.

If $\mathscr{L}$ is a line bundle on $\mathscr{T}_g^d$, the class of its restriction to a fiber $J^d(C)$, where C is a curve with general moduli, will be a multiple, say $m\theta$, of the class of the theta divisor of the Jacobian of C. On the other hand, the above condition is an open condition on $\mathscr{M}_g^0$ and so, since $\mathscr{M}_g^0$ is connected, we get that the restriction of $\mathscr{L}$ to every fiber has class $m\theta$. We are going to refer to $m\theta$ as the "class" of the line bundle $\mathscr{L}$. We can define an embedding of groups

$$\varphi_d\colon \mathscr{N}(\mathscr{T}_g^d) \hookrightarrow \mathbf{Z}.$$

To describe the group $\mathscr{N}(\mathscr{T}_g^d)$ is equivalent to finding the generator k_g^d of the image of the map φ_d. This is exactly the content of Theorem 1.

Before we state our main theorem, let us make the following remark: For given g, there are some obvious relations among the various numbers k_g^d: $k_g^d = k_g^{2g-2+d} = k_g^{2g-2-d}$. This follows from the fact that $\mathscr{T}_g^d \cong \mathscr{T}_g^{2g-2+d} \cong \mathscr{T}_g^{2g-2-d}$, where the isomorphisms are constructed using the relative dualizing sheaf ω_π of the family π: $L \mapsto L\otimes\omega_\pi$ and $L \mapsto L^{-1}\otimes\omega_\pi$. It is enough therefore to restrict in the range $0 \le d \le g-1$. It is also clear that $k_g^{g-1} = 1$. The bundle $\mathscr{T}_g^{g-1}$ has a natural line bundle Θ with "class" equal to θ. This is the image of the $(g-1)$th universal symmetric product bundle $\mathscr{C}_g^{(g-1)}$ over $\mathscr{M}_g^d$ by the natural map sending $D_C \in C^{(g-1)}$ to the bundle $\mathscr{O}(D_C)$ in $J^{g-1}(C)$. Our main theorem is:

Theorem 1. *For $d = 0, \cdots, g-1$ we denote by $\mathscr{T}_g^d$ the universal Picard varieties over $\mathscr{M}_g^0$. Then the numbers k_g^d* (*see definition above*) *are given by the following formula*:

$$k_g^d = \frac{2g-2}{\mathrm{g.c.d.}(2g-2,\, g+d-1)}.$$

The organization of this paper goes as follows: first we prove the theorem in the case $d = 0$, and as an application we give another proof of the strong Franchetta's conjecture (first proved by Mestrano, see [6]). Then we complete the proof of the theorem for the other d's.

2. The case $d = 0$

We start with the following lemma.

Lemma 3. *Let $\mathscr{C}_g \xrightarrow{\pi} \mathscr{M}_g^0$ denote the universal curve over $\mathscr{M}_g^0$, and ω_π the relative dualizing sheaf of π. Then there is a nonempty Zariski open subset $\mathscr{U}$ of $\mathscr{M}_g^0$ such that there is a holomorphic section of ω_π on $\pi^{-1}(\mathscr{U})$.*

Proof. Let $\mathscr{L}$ to be an ample line bundle on $\mathscr{M}_g^0$ and assume that $\mathscr{L} = \mathscr{O}(D)$, where D is an effective divisor on $\mathscr{M}_g^0$. By the projection formula and the ampleness of $\mathscr{L}$, there exists a positive integer n such that $\mathbf{h}^0(\mathscr{C}_g, \omega_\pi \otimes \pi^*\mathscr{L}^n) = \mathbf{h}^0(\mathscr{M}_g^0, \pi_*\omega_\pi \otimes \mathscr{L}^n) > 0$. Over the set $\mathscr{U} = \mathscr{M}_g^0 \setminus \operatorname{supp}(D)$ we have that $\mathbf{h}^0(\pi^{-1}(\mathscr{U}), \omega_\pi) > 0$, and so we get on the Zariski open subset $\pi^{-1}(\mathscr{U})$ of $\mathscr{C}_g$ a holomorphic section of ω_π of relative degree $2g-2$ over $\mathscr{M}_g^0$. q.e.d.

From the above Lemma 3 we can cover the Zariski open subset $\mathscr{U}$ by open analytic subsets $\{U_a\}$ such that over each U_a there are $2g-2$ sections s_a^i of the map π (we can choose $\mathscr{U}$ such that the restriction of the map π to the above holomorphic section gives an unramified covering of $\mathscr{U}$ of degree $2g-2$). Therefore locally over each U_a we can construct a collection of $2g-2$ different isomorphisms

$$\begin{aligned}\varphi_a^i\colon \mathscr{T}_g^0(U_a) &\to \mathscr{T}_g^{g-1}(U_a),\\ L_C &\mapsto L_C \otimes (g-1)\mathscr{O}(s_a^i([C])),\end{aligned}$$

where $\mathscr{T}_g^d(U_a)$ denotes the restriction of the bundle $\mathscr{T}_g^d$ to U_a, and L_C is an element of $\mathscr{T}_g^0(U_a)$ sitting over $[C] \in U_a$. As we saw in the introduction, we have on $\mathscr{T}_g^{g-1}$ a natural line bundle Θ with "class" θ. Pulling this back by the above local isomorphisms, we get on the open neighborhood $\psi_0^{-1}(U_a) = \mathscr{T}_g^0(U_a)$ of $\mathscr{T}_g^0$ a collection of $2g-2$ line bundles whose restriction to the fibers over U_a has class θ. Consider now for each U_a the tensor product of all these line bundles. We get on each $\mathscr{T}_g^0(U_a)$ a line bundle $\mathscr{L}_a$. Since the above construction of the $\mathscr{L}_a$ remains invariant under the action of the monodromy group of this covering at a point of U_a, these $\mathscr{L}_a$'s fit together and give rise to a line bundle $\mathscr{L}$ on $\psi_0^{-1}(\mathscr{U}) = \mathscr{T}_g^0(\mathscr{U})$, and so by extension to a line bundle over $\mathscr{T}_g^0$ with "class" $(2g-2)\theta$. Therefore k_g^0 must divide $2g-2$. On the other hand there is a map

$$\psi\colon \mathscr{T}_g^{g-1} \to \mathscr{T}_g^{2g-2} \cong \mathscr{T}_g^0, \qquad L \mapsto L^{\otimes 2}.$$

The push forward $\psi_*(\Theta)$ of the effective divisor Θ defines a line bundle on $\mathscr{T}_g^0$ with "class" $2^{2g-2}\theta$. So the generator k_g^0 must divide g.c.d.$(2^{2g-2}, 2g-2)$. If $g-1 = \text{odd}$, we get that k_g^0 must divide 2. On the other hand if $g-1 = \text{even}$, say $g-1 = 2^k N$ with g.c.d.$(2, N) = 1$, we do the following:

Over $\mathscr{M}_g^0$, consider the universal symmetric product bundle $\mathscr{C}_g^{(2^k)}$ of degree 2^k, i.e., over a point $[C]$ of $\mathscr{M}_g^0$ the fiber is the (2^k)th symmetric product $C^{(2^k)}$ of the curve C. Over $\mathscr{U}$ we can define a covering of degree

$$\binom{2(g-1)}{2^k} = \binom{2^{k+1}N}{2^k}$$

in $\mathscr{C}_g^{(2^k)}$: just consider the covering of degree $2g-2$ on $\mathscr{C}_g$ (see Lemma 3), and over each point $[C]$ of $\mathscr{U}$ take in $C^{(2^k)}$ all the possible 2^k-sums of the $2g-2$ points lying over $[C]$ in $\mathscr{C}_g$. Observe now that the above number is $2n$ where n is *odd*. We define *locally* maps

$$\varphi_a^{i_1, \dots, i_{2^k}}\colon \mathscr{T}_g^0(U_a) \to \mathscr{T}_g^{g-1}(U_a),$$
$$L_C \mapsto L_C \otimes \tfrac{(g-1)}{2^k}\mathscr{O}(s_a^{i_1}([C]) + \cdots + s_a^{i_{2^k}}([C])).$$

As before we construct a line bundle over $\mathscr{T}_g^0(\mathscr{U})$ with "class" $2n\theta$, and so we get again that k_g^0 divides 2. Hence $k_g^d = 1$ or 2. In order to prove that $k_g^0 = 2$ we have to work a little bit more: In what follows in this section we prove this and illustrate in general the technique which we use to determine the numbers k_g^d.

Remark. In the case of the Jacobian $\mathscr{T}_g^0$ there is a better way of constructing a line bundle on the total space whose restriction to the fibers has class 2θ (see [11, pp. 419–420]). The construction depends on the fact that $\mathscr{T}_g^0$ acts on $\mathscr{T}_g^{(g-1)}$, and the author has not succeeded in carrying out a similar construction for the general case of $\mathscr{T}_g^d$'s. On the other hand, as we will see later, the above method can be generalized for the Jacobian varieties of any degree.

We denote by C a smooth curve of genus g, and by $C^{(d)}$ its dth symmetric product. Let $\theta_{(d)}$ be the class of the pullback of the theta divisor from the Jacobian by the Abel-Jacobi map u_d: $C^{(d)} \to J(C)$. We denote by $x_{(d)}$ the class in $C^{(d)}$ of the divisor $p_0 + C_{d-1} = \{D \in C^{(d)}, D - p_0 \geq 0\}$ for a fixed point p_0 in C; this class is independent of the choice of the point p_0. In other words, the class $x_{(d)}$ is the class of the image of a coordinate plane from the dth ordinary product $C^{\times d}$ to $C^{(d)}$ by the natural map. We denote also by $\delta_{(d)}$ the class of the diagonal divisor $\{D + 2p, D \in C^{(d-2)}, p \in C\}$ in $C^{(d)}$. The following lemma expresses the class $\theta_{(d)}$ in $C^{(d)}$ in terms of $x_{(d)}$ and $\delta_{(d)}$.

Lemma 4 (*MacDonald*). *The class $\theta_{(d)}$ in the dth symmetric product $C^{(d)}$ of a smooth curve of genus g is given by*

$$\theta_{(d)} = (d + g - 1)x_{(d)} - \delta_{(d)}/2.$$

Proof. This is a special case of Proposition 5.1 on p. 358 in [2]. Following the notation of [2], one has to take $n_1 = 1$, $n_2 = d - 2$, $a_1 = 2$, $a_2 = 1$. q.e.d.

The essential tool for this paper is the result of Harer-Arbarello-Cornalba (see [1]) about the Picard groups of the moduli stack of pointed curves. We denote by $\mathscr{M}^0_{g,d}$ the moduli space of d-pointed curves over $\mathscr{M}^0_g$, and by $\mathscr{C}_{g,d}$ the universal curve over this. We denote by s_i, $i = 1, \cdots, d$, the sections of the map π_d: $\mathscr{C}_{g,d} \to \mathscr{M}^0_{g,d}$, and by ω_{π_d} the relative dualizing sheaf of π_d. Given a line bundle on $\mathscr{M}^0_{g,d}$, it induces a line bundle on the pointed moduli stack, and so by Theorem 1 in [1] it is a linear integral combination of the line bundles $s_i^*(\omega_{\pi_d})$ and the Hodge bundle on $\mathscr{M}^0_{g,d}$. On the other hand, there are inclusions i_d, j_d

$$\begin{tikzcd}
\mathscr{C}_{g,d} \arrow[r, "j_d"] \arrow[d, shift left, "\pi_d"] & \underbrace{\mathscr{C}_g \times_{\mathscr{M}^0_g} \cdots \times_{\mathscr{M}^0_g} \mathscr{C}_g}_{d \text{ factors}} \times_{\mathscr{M}^0_g} \mathscr{C}_g \arrow[r, "p_{d+1}"] \arrow[d, shift left, "p"] & \mathscr{C}_g \\
\mathscr{M}^0_{g,d} \arrow[r, "i_d"] \arrow[u, shift left, "s_i"] & \underbrace{\mathscr{C}_g \times_{\mathscr{M}^0_g} \cdots \times_{\mathscr{M}^0_g} \mathscr{C}_g}_{d \text{ factors}} \arrow[u, shift left, "\delta_{i,d+1}"] \arrow[ur, "p_i"'] &
\end{tikzcd}$$

where the image of the map i_d avoids exactly the 2-diagonals D_{ij}, $1 \leq i < j \leq d$, and the image of j_d avoids exactly the 2-diagonals D^{un}_{ij},

$1 \leq i < j \leq d$. Note that the diagonal maps $\delta_{i,d+1}$ restrict to the sections s_i on the images of $\mathscr{M}^0_{g,d}$ and $\mathscr{C}_{g,d}$. Therefore from the exact sequence of the open image of $\mathscr{M}^0_{g,d}$ inside $\mathscr{C}_g^{\times d} \stackrel{\text{def}}{=} \mathscr{C}_g \times_{\mathscr{M}_g^0} \cdots \times_{\mathscr{M}_g^0} \mathscr{C}_g$ (d factors), we get that

$$\operatorname{Pic}(\mathscr{C}_g^{\times d}) = \mathbf{Z}[\tilde{\lambda}, \delta^*_{i,d+1}\omega_p, D_{i,j}],$$

where $\tilde{\lambda}$ is the Hodge bundle. If $\mathscr{L}$ is a line bundle on $\mathscr{C}_g^{\times d}$, then the restriction $\mathscr{L}|_{C^{\times d}}$ of $\mathscr{L}$ to the fiber $C^{\times d}$ has class

$$\mathscr{L}|_{C^{\times d}} \sim (2g-2)\sum_{i=1}^{d} a_i f_i + \sum_{0 \leq i < j \leq d} b_{ij}\Delta_{ij},$$

where f_i denotes the class of the ith coordinate plane in $C^{\times d}$, Δ_{ij} denotes the class of the ij-diagonal in $C^{\times d}$, and the numbers a_i, b_{ij} are integers. Indeed, it is easy to see that the restriction of the Hodge bundle $\tilde{\lambda}$ to the fibers is trivial and also that $\delta^*_{i,d+1}\omega_p = \delta^*_{i,d+1}p^*_{d+1}K_C = p_i^*K_C \sim (2g-2)f_i$. In addition, since the curve C is not rational, we have that the classes f_i and Δ_{ij} are linearly independent over the integers.

Say now that $\mathscr{L}$ is a line bundle on $\mathscr{T}_g^d$ with "class" equal to $n\theta$. Consider the pullback of $\mathscr{L}$ by the maps

$$\mathscr{C}_g^{\times d} \xrightarrow{q_d} \mathscr{C}_g^{(d)} \xrightarrow{u_d} \mathscr{T}_g^d,$$

where u_d is the Abel-Jacobi map and q_d is the canonical map. Let $\mathscr{L}' = q_d^* u_d^* \mathscr{L}$ on $\mathscr{C}_g^{\times d}$. We define $f \stackrel{\text{def}}{=} f_1 + \cdots + f_d$ and $\Delta \stackrel{\text{def}}{=} \sum_{ij}\Delta_{ij}$. Since $q_d^* x_{(d)} = f$ and $q_d^*\delta_{(d)} = 2\Delta$, from Lemma 4 it follows that the restriction $\mathscr{L}'|_{C^{\times d}}$ of $\mathscr{L}'$ to the product $C^{\times d}$ has class $n(d+g-1)f - n\Delta$. Therefore from the above discussion we must have that

$$(*) \qquad 2g-2 \mid n(d+g-1).$$

This is the basic relation we use in the following.

Let us now complete the proof of the case $d = 0$ (i.e., that $k_g^0 = 2$): If $k_g^d = 1$, then according to $(*)$ we must have that $2g-2 \mid g-1$; a contradiction. Therefore $k_g^0 = 2$.

Another consequence of the formula $(*)$ is that it leads to a proof of the strong Franchetta's conjecture which we recall in the following section.

3. A proof of the strong Franchetta's conjecture

Theorem 2 (*strong Franchetta's conjecture*). *The only rational sections of the universal Picard varieties are those "coming" from a multiple of the canonical bundle. In other words, if the variety $\mathscr{T}_g^d$ admits a rational section, then $2g-2|d$ and the section is the trivial one.*

Remark. The above theorem implies that if we have a canonical way of choosing a line bundle on the general fiber of the universal curve (i.e., on each fiber over a nonempty Zariski open subset of $\mathscr{M}_g^0$), then this must be a multiple of the canonical bundle. Notice that if $\mathscr{X} \to \mathscr{B}$ is a family of smooth curves, then a canonical choice of a line bundle of degree d on the general curve gives rise to a rational section in the dth Picard variety $\mathscr{T}_{\mathscr{X}}^d$ of $\mathscr{X}$ over $\mathscr{B}$, but in general not to a line bundle over a Zariski open subset of $\mathscr{B}$. A sufficient condition for this to happen is the existence on $\mathscr{T}_{\mathscr{X}}^d \times \mathscr{X}$ of a Poincaré bundle, i.e., a line bundle $\mathscr{L}$ such that $\mathscr{L}|_{\{L_b\}\times\mathscr{X}_b} \cong L_b$ on $\mathscr{X}_b$, where L_b is a line bundle of degree d on the fiber $\mathscr{X}_b$ over $b \in \mathscr{B}$. In our case of the universal family of curves over $\mathscr{M}_g^0$, it has been shown in [7] that this happens if and only if $\mathrm{g.d.c.}(2g-2, d-g+1) = 1$. If this is the case, the Enriques and Chisini's theorem (namely: If $\mathscr{L}$ is a line bundle on the universal curve $\mathscr{C}_g$ over $\mathscr{M}_g^0$, then the restriction of $\mathscr{L}$ to the fibers has degree a multiple of $2g-2$; see [4]) implies that the $\mathscr{T}_g^d$ has no rational section.

Let us now prove the strong Franchetta's conjecture. We mention first the following lemma.

Lemma 5. *The only rational section of the Jacobian bundle $\mathscr{T}_g^0$ is the trivial one.*

Proof. This is a consequence of Theorem 1 in [1] and of the fact that the Deligne-Mumford covering of $\mathscr{M}_g^0$ by the n-torsion points of the Jacobians has only trivial section (see [3]). For a complete proof of the lemma see Theorem 2.8 in [8]).

Proof of Theorem 2. Let us say that for some d with $1 \le d \le g-1$ the variety $\mathscr{T}_g^d$ has a rational section σ. Then there exists a birational isomorphism

$$\begin{aligned}\Phi\colon \mathscr{T}_g^d &\to \mathscr{T}_g^0,\\ L_C &\mapsto L_C \otimes \sigma([C])^{-1}.\end{aligned}$$

Say first that $d = g-1$: If $\mathscr{T}_g^{g-1}$ has a rational section, then by the above map we get a line bundle with "class" θ on the Jacobian bundle $\mathscr{T}_g^0$; a contradiction. This result was first proved by Mestrano and Ramanan (see [8]). For d with $1 \leq d \leq g-2$, if the bundle $\mathcal{T}_g^d$ has a rational section, then, since it is birationally isomorphic to $\mathcal{T}_g^0$, it will have a line bundle with "class" 2θ. In this case the basic relation $(*)$ implies that

$$2g-2|2(d+g-1), \quad \text{i.e., } g-1|d+g-1;$$

a contradiction, since $1 \leq d \leq g-2$. This was first proved by Mestrano (see [6]).

4. Proof of Theorem 1

In this section we complete the proof of Theorem 1. We essentially use the relation

$$(*) \qquad 2(g-1)|k_g^d(d+g-1)$$

of the previous section. Note that the number

$$(2g-2)/\,\mathrm{g.c.d.}(2g-2\,,\,d+g-1)$$

is the minimum integer n such that $2g-2|n(d+g-1)$. Therefore $k_g^d = (2g-2)|\,\mathrm{g.c.d.}(2g-2\,,\,d+g-1)\gamma$, γ an integer, and so we have to show that $\gamma = 1$. The proof is split into two parts. In the first part we prove that $\gamma|2g-2$ and in the second that $\gamma = 1$. We write

$$\left\{\begin{matrix} g-1=\alpha m \\ d=\beta m \end{matrix}\right\} m = (d\,,\,g-1), \quad (\alpha\,,\,\beta) = 1,$$

where the parentheses above denote g.c.d.'s.

Part (I). We have two cases:

$\alpha+\beta = odd$. In this case we have $(d+g-1\,,\,2g-2) = m(\alpha+\beta\,,\,2\alpha) = m(\alpha+\beta\,,\,\alpha) = m$, and so $k_g^d = (2g-2)\gamma/m = 2a\gamma$.

Consider now the map

$$\varphi\colon \mathcal{T}_g^d \to \mathcal{T}_g^{(g-1)\beta}, \\ L \mapsto L^{\otimes a}.$$

The target of φ has a line bundle with "class" 2θ and so, by pulling back we get a line bundle on $\mathcal{T}_g^d$ with "class" $2\alpha^2\theta$. Therefore $k_g^d|2\alpha^2$, i.e., $2\alpha\gamma|2\alpha^2$, i.e., $\gamma|\alpha|2g-2$.

$\alpha+\beta = even$. In this case α and β are odd numbers and we have that $(d+g-1\,,\,2g-2) = m(\alpha+\beta\,,\,2\alpha) = 2m$, and so $k_g^d = (2g-2)\gamma|2m = a\gamma$.

Considering again the above map φ, the target has a line bundle with "class" θ (since β is odd) and so, as before, we conclude that $k_g^d|\alpha^2$, i.e., $\alpha\gamma|\alpha^2$, i.e., $\gamma|\alpha|2g-2$.

Part (II). We show now that $\gamma = 1$. We have seen in all the cases that this constant divides $2g-2$ and so it is enough to prove that for each prime p dividing $2g-2$, we have $\mathrm{g.c.d.}(p, \gamma) = 1$. Let $m_p = \{\max$ power of p dividing $(2g-2)/\mathrm{g.c.d.}(2g-2, d+g-1)\}$. For each prime p that divides $2g-2$, we will construct a line bundle with "class" $p^{m_p}A\theta$, where $\mathrm{g.c.d.}(A, p) = 1$. Then, since the k_g^d's are the generators, this implies that $\gamma = 1$. The idea for this construction is the same of that of constructing the line bundle with "class" 2θ on $\mathcal{T}_g^0$:

For each odd prime p as above write

$$\left\{\begin{matrix} g-1=p^u U \\ d=p^w W \end{matrix}\right\} (U, p) = 1 = (W, p).$$

We have two cases:

$u \geq w$. Then

$$\begin{aligned}\frac{2g-2}{\mathrm{g.c.d.}(2g-2, d+g-1)} &= \frac{2p^u U}{\mathrm{g.c.d.}(2p^u U, p^u U + p^w W)} \\ &= \frac{2p^u U}{p^w\,\mathrm{g.c.d.}(2p^{u-w} U, p^{u-w} U + W)},\end{aligned}$$

and so $m_p = u - w$.

Consider now a holomorphic section of the relative dualizing sheaf of the universal curve over a nonempty Zariski open subset of $\mathcal{M}_g^0$ as in Lemma 3. Then as in §2 we define *locally* maps

$$\begin{aligned}&\mathcal{T}_g^d(U_a) \to \mathcal{T}_g^{(g-1)W}(U_a), \\ &\quad L_{[C]} \mapsto L_C \otimes (p^{u-w}UW - W)\mathcal{O}(q_{i_1}^C + \cdots + q_{i_{p^w}}^C,\end{aligned}$$

where the points q_i^C are the $2g-2$ points of the above section over the point $[C]$. The number of these maps is

$$\binom{2(g-1)}{p^w} = \binom{2p^u U}{p^w}$$

and this number is $p^{u-w}A = p^{m_p}A$, where $\mathrm{g.c.d.}(A, p) = 1$. Since $\mathcal{T}_g^{(g-1)W}$ has a line bundle with "class" 2θ, we can construct as in §2 a line bundle on $\mathcal{T}_g^d$ with "class" $2p^{m_p}A\theta$. Since p is an odd prime, we obtain what we were looking for.

$u < w$. In this case $m_p = 0$. The method is the same. The only difference is that instead of a section of the relative dualizing sheaf ω_π we have to consider a section of $\omega_\pi^{p^{w-u}}$. The rest of the proof of this case goes as before.

If $p = 2$, then, since we saw in the first part of the proof that $\gamma|\alpha$, we have to examine only the case where $\alpha = even$ and so $\beta = odd$; therefore, in this case we have that $u \geq w+1$ (using the above notation). The rest of the proof is similar to that of the first case above. q.e.d.

Notation. For each d, we denote by $\mathscr{L}_g^d$ a line bundle on $\mathscr{T}_g^d$ with "class" $k_g^d\theta$ (we have just constructed such line bundles).

5. The description of the Picard group of the $\mathscr{T}_g^d$'s

Since we have described the relative Neron-Severi group of $\mathscr{T}_g^d$'s, the following theorem leads to the description of the relative Picard group.

Theorem 3. *The relative Picard group $\mathscr{R}\,\mathrm{Pic}(\mathscr{T}_g^d)$ is isomorphic to the group $\mathscr{N}(\mathscr{T}_g^d)$.*

We start with some lemmas.

Lemma 6. *We denote by A an abelian variety, and by θ its principal polarization. For each point L in A we denote by T_L the translation map in A defined by L. We have the following*:

1. *If $\mathscr{L}$ is a line bundle on A with class equal to $m\theta$, where m is a nonzero integer, then the set of points L_i in A such that $T_{L_i}^*\mathscr{L} = \mathscr{L}$ is exactly the subgroup $A_m = \langle L_i,\ i = 1, \cdots, m^{2g}\rangle$ of m-torsion points of A.*

2. *If $\mathscr{L}$, $\mathscr{L}'$ are two line bundles on A with class equal to $m\theta$, then there exists a point M in A such that $T_M^*\mathscr{L} = \mathscr{L}'$. Furthermore, the set $G_m = \{M_i,\ i = 1, \cdots, m^{2g}\}$ of all such M's is a coset of A_m in A.*

Proof. See [10] or [5].

Lemma 7. *We denote by $\mathscr{T}_n$ the subvariety of $\mathscr{T}_g^0$ consisting of the n-torsion points of $\mathscr{T}_g^0$. Then the only rational section of the map $\tau_n\colon \mathscr{T}_n \to \mathscr{M}_g^0$ is the trivial one.*

Proof. It is known [3] that $\mathscr{T}_n \cong \mathscr{M}_g^0 \times_{\mathrm{Sp}(\mathbf{Z}_n^{2g})} \mathbf{Z}_n^{2g}$, with the group $\mathrm{Sp}(\mathbf{Z}_n^{2g})$ acting transitively on $\mathbf{Z}_n^{2g}\backslash\{0\}$. Therefore it has no nontrivial rational section (see Corollary 2.6 in [8]).

Proof of Theorem 3. We first do the case $d = 0$. Consider two line bundles $\mathscr{L}$, $\mathscr{L}'$ on $\mathscr{T}_g^0$ with "classes" equal to $m\theta$, m a nonzero integer. We denote by $\mathscr{T}_m(C) = \langle L_i^C,\ i = 1, \cdots, m^{2g}\rangle$ the group of m torsion points of $\mathscr{T}_g^0(C)$, and by $\mathscr{G}_m(C) = \{M_i^C,\ 1, \cdots, m^{2g}\}$ the coset of points such that $T^*_{M_i^C}\mathscr{L}|_{\mathscr{T}_g^0(C)} = \mathscr{L}'|_{\mathscr{T}_g^0(C)}$ (as in Lemma 6). We claim the following:

Claim. $\mathscr{G}_m(C) \subseteq \mathscr{T}_{m^{2g}}(C)$.

Proof of Claim. Take $M^C \in \mathscr{G}_m(C)$. Then $\mathscr{G}_m(C) = \{M^C \otimes L_i^C,\ i = 1, \cdots, m^{2g}\}$. The product $\bigotimes_{i=1}^{m^{2g}}(M^C \otimes L_i^C) = \bigotimes_{i=1}^{m^{2g}} M^C = (M^C)^{\otimes m^{2g}}$ (since $\bigotimes_{i=1}^{m^{2g}} L_i^C = \mathscr{O}_C$) gives a canonical way of choosing a line bundle on the fiber C. Therefore this induces a section on $\mathscr{T}_g^0$ over $\mathscr{M}_g^0$, and so by the strong Franchetta's theorem we get that $(M^C)^{\otimes m^{2g}} = \mathscr{O}_C$. This proves the claim.

Consider now the exact sequence

$$0 \to \mathscr{T}_m(C) \to \mathscr{T}_{m^{2g}}(C) \to \mathscr{T}_{m^{2g-1}}(C) \to 0,$$
$$L \mapsto L^{\otimes m}.$$

From the above claim and Lemma 6 the coset $\mathscr{G}_m(C)$ defines a point in $\mathscr{T}_{m^{2g}}(C)/\mathscr{T}_m(C)$, i.e., a point in $\mathscr{T}_{m^{2g-1}}(C)$. Therefore Lemma 7 implies that $\mathscr{G}_m(C) = \mathscr{T}_m(C)$ and so $\mathscr{L}|_{\mathscr{T}_g^0(C)} \cong \mathscr{L}'|_{\mathscr{T}_g^0(C)}$. This proves the theorem in the case $d = 0$.

For the case $d \neq 0$, choose an identification of $\mathscr{T}_g^d(C)$ and $\mathscr{T}_g^0(C)$ and reduce to the case $d = 0$. This proves Theorem 3.

Theorem 4. *The Picard group of the universal Picard variety* $\psi_d\colon \mathscr{T}_g^d \to \mathscr{M}_g^0$ *is freely generated over* $\mathbf{Z}$ *by the line bundles* $\mathscr{L}_g^d$, *and* $\psi_d^*(\lambda)$, *where* $\mathscr{L}_g^d$ *is the line bundle defined at the end of the previous section, and* λ *is the Hodge bundle on* $\mathscr{M}_g^0$.

Proof. This is a consequence of Theorem 3 and the fact that $\mathrm{Pic}(\mathscr{M}_g^0) = \mathbf{Z}[\lambda]$ (see Theorem 1 in [1]).

Remark 1. Using exactly the same method as above, we can actually describe the Picard group of the universal Picard stacks over the moduli space $\mathscr{M}_g$ of smooth curves of genus g.

Remark 2. For any smooth curve C of genus g, there is a canonical way of choosing a line bundle on $J^d(C)$ with class $k_g^d\theta$: If $m = \mathrm{g.c.d.}(2g-2, d+g-1)$, consider $L_i \in J^m(C)$, $i = 1, \cdots, (k_g^d)^{2g}$, with

$L_i^{\otimes k_g^d} = K_C$. If $s = (2g-2-(d+g-1))/m = (g-d-1)/m$, then the line bundle

$$\mathscr{O}(\{D - L_i^{\otimes s}, D \in C^{(g-1)}\})^{\otimes k_g^d}$$

has class $k_g^d \theta$ and so it remains invariant under translations by $L_j^{-1} \otimes L_i$ (see Lemma 6). This means that the above line bundle is independent of the choice of L_i and so it is a canonical choice of a line bundle on $J^d(C)$. Moreover, these canonical choices are the restrictions of the generator bundles $\mathscr{L}_g^d$ to the fibers of ψ_d: $\mathscr{T}_g^d \to \mathscr{M}_g^0$. To see this, observe that the proof of Theorem 3 works if, instead of two line bundles on the total space, we just have two canonical choices of line bundles on the fibers of ψ_d. Therefore, since the restriction of $\mathscr{L}_g^d$ to a fiber has class $k_g^d \theta$, which is the same as the class of the above canonical choice of a line bundle on that fiber, these two are isomorphic line bundles.

Remark 3. It might be possible to show directly that the above canonical choices of line bundles on the fibers of $\mathscr{T}_g^d$ are actually restrictions of a line bundle on the total space. This will give another way of constructing the generator line bundles for the Picard groups.

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