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CONFORMALLY FLAT METRICS OF CONSTANT POSITIVE SCALAR CURVATURE ON SUBDOMAINS OF THE SPHERE

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Abstract

A basic problem has been to construct complete conformally flat metrics of constant positive scalar curvature on the complement of arbitrary sets $\Lambda \subset S^n$ where S^n is an n-sphere. A necessary condition for the existence of such a metric is that the Hausdorff dimension of Λ must be less than or equal to $(n-2)/2$. Examples are known when Λ is any finite collection of points, a subsphere, and also when Λ is the limit set of certain Kleinian groups. Up until now no examples have been known where Λ is a smooth (nonspherical) submanifold of positive dimension. We prove here that there are many examples whenever Λ is a small perturbation of an equatorial subsphere. A local version of this result is also proved. These theorems rely on an analysis of certain degenerate linear elliptic operators, which is complicated by the fact that these operators have infinite dimensional null-spaces. A fairly general construction of pseudodifferential right-inverses for such operators is presented.

1. Introduction

Because of the resolution of the Yamabe problem by R. Schoen in 1984 [26] (see also [13]), it is possible to divide the conformal classes of Riemannian metrics on an arbitrary compact manifold M of dimension $n \geq 3$ into three disjoint subsets containing respectively those classes which contain a metric of constant positive, zero, or negative scalar curvature, according to the sign of the first eigenvalue of the conformal Laplacian on M. This sign is well-defined within a conformal class. On a given manifold one or more of these subsets may be empty; for example, it is known that the $\widehat{A}$ genus is a topological obstruction governing the existence of metrics of positive scalar curvature when the manifold is spin. When the manifold is not assumed to be compact, these simple statements need modification. In this setting it is geometrically and analytically natural to restrict attention to complete metrics; however, a conformal class always contains both

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complete and incomplete metrics. Thus, a possible way to generalize this problem is to ask whether on a given noncompact Riemannian manifold there exist complete metrics of constant scalar curvature and, if so, is the sign of this constant well defined within the set of complete metrics within a conformal class. Another possible question is whether starting with a complete metric there exists another complete metric quasi-isometric to the given one with constant scalar curvature. Only partial answers to these questions are known, but even so it is clear that very different phenomena occur than for compact manifolds.

In a series of papers [1]–[3], Aviles and McOwen have studied the question of finding complete metrics which have constant negative scalar curvature and which are conformal to a given one. They also consider more general questions concerning prescribing scalar curvature. It is natural to study these problems on noncompact manifolds with 'simple' structure, for example those which arise by deleting a closed subset from a compact Riemannian manifold. An example of this is a result proved by Aviles and McOwen [2] to the effect that if M is an n-dimensional compact Riemannian manifold and if N is a k-dimensional submanifold, then there is a complete metric conformal to the incomplete one on $M\backslash N$ with constant negative scalar curvature if and only if $k > (n-2)/2$. This generalizes an older theorem due to Loewner and Nirenberg [14] which treats the case when M is a sphere.

Just as in the compact setting, though, the hardest analysis is required in questions concerning metrics of constant positive scalar curvature. Very deep results have been obtained by Schoen and Yau [28] and Schoen [27] in what is perhaps the most geometrically appealing instance of this problem, namely when the ambient manifold is the n-sphere S^n. One of the basic problems here is to determine which subdomains of the sphere carry complete conformally flat metrics of constant positive scalar curvature. In [28] this problem was approached through ideas of conformal geometry. One of the theorems proved there is that if a subdomain $\Omega = S^n\backslash\Lambda$ carries a complete metric with scalar curvature bounded below by a positive constant, then the Hausdorff dimension of Λ is $\leq (n-2)/2$. On the other hand, in [27] Schoen constructs many such domains Ω which carry complete conformally flat metrics of constant positive scalar curvature. In particular, he constructs such metrics on the complement of any finite set of points of cardinality larger than one in the sphere.

These results lead to many more questions. For example, Schoen's construction works only when the domain $\Omega \subset S^n$ has complement Λ of a very special type. It is natural to conjecture from the results of [28] that the complement of any subset Λ of S^n with $\mathscr{H}^{(n-2)/2}(\Lambda) < \infty$ (and of cardinality greater than one) supports a complete conformally flat metric of constant positive scalar curvature on its complement. For example, consider the simple situation when Λ is an equatorial k-sphere $\Lambda = S^k$. It is easy to find the required metric here. First stereographically project S^n from some point on S^k. Now $\Omega = \mathbf{R}^n \backslash \mathbf{R}^k$ with its flat metric. Introduce coordinates $y \in \mathbf{R}^k$, and polar coordinates (r, θ) in the orthogonal complement $\mathbf{R}^{n-k}$. In these coordinates the flat metric is $dx^2 = dr^2 + r^2 d\theta^2 + dy^2$. This is clearly conformal to

$$ds^2 = \frac{dr^2 + r^2 d\theta^2 + dy^2}{r^2} = \frac{dr^2 + dy^2}{r^2} + d\theta^2. \tag{1.1}$$

But this is nothing order than the product metric on $S^{n-k-1} \times \mathbf{H}^{k+1}$. As such it is certainly complete and has scalar curvature

$$R_{n,k} = (n-k-1)(n-k-2) - (k+1)(k) = (n-2k-2)(n-1). \tag{1.2}$$

Thus, $R_{n,k} > 0$ precisely when $n - 2k - 2 > 0$, i.e., when $k < (n-2)/2$ as expected.

A fundamental goal is to characterize those k-dimensional submanifolds $\Lambda \in S^n$ with $k < (n-2)/2$ for which $\Omega = S^n \backslash \Lambda$ supports a complete conformally flat metric of constant positive scalar curvature. No smooth examples of positive dimension other than round S^k's have been known. On the other hand, it is possible to construct complete conformally flat metrics of constant positive scalar curvature on subdomains of the sphere by lifting solutions from compact manifolds which are constructed using the solution of the Yamabe conjecture. In these constructions Λ is the limit set of a Kleinian group, hence is either a round sphere or unrectifiable with Hausdorff dimension less than $(n-2)/2$. We produce the first nonround smooth examples here by analyzing perturbations of the product metric g_0 of (1.1) on $\Omega = S^n \backslash S^k$. We also study questions of uniqueness. It is well known in the noncompact case that the same conformal class may contain different complete metrics with constant scalar curvatures of different signs. Perhaps somewhat surprisingly though, we find very many examples of complete, conformally flat constant positive scalar curvature metrics on the complement of any fixed small perturbation of S^k. In §2 we prove the following theorem:

Theorem. *Let g_0 denote the product metric* (1.1) *on $S^n \backslash S^k$ above, where $k < (n-2)/2$. Then for every $C^{3,\alpha}$ diffeomorphism τ of S^n close to the identity, there is an infinite-dimensional family of positive functions*

u such that $g = u^{4/n-2}\overline{g}$ is a complete metric on $S^n \backslash \tau(S^k)$ and has scalar curvature equal to $R_{n,k}$ where $\overline{g}$ is the usual round metric on S^n. These solutions are parametrized by functions in a certain Hölder class on the sphere S^k with small norm.

See Theorem (2.21) for the complete statement. This changes the focus of questions from those concerning existence of complete constant positive scalar curvature metrics to those concerning the variety of complete metrics on a given domain with the same constant positive scalar curvature. It is important to remark that our methods break down in all other cases, i.e., when $k \geq (n-2)/2$.

We also study the analogous question for arbitrary k-dimensional submanifolds of S^n, still assuming $k < (n-2)/2$. Although we do not produce global examples in this generality, we can construct local ones.

Theorem. *Let $\mathscr{U}$ be a tubular neighborhood of the submanifold $N \subset S^n$. Then for every $C^{2,\alpha}$ function ϕ on $\partial\mathscr{U}$ close to the identity there is a complete metric $g = u^{4/(n-2)} g_0$ on $\mathscr{U}\backslash N$ with scalar curvature $R_{n,k}$ and such that $u|_{\partial\mathscr{U}} = \phi$.*

Again, see §3 for a more precise statement.

These theorems are proved by the implicit function theorem and the technique of contraction mappings applied to the nonlinear PDE which governs the change in scalar curvature under a conformal change of metric. In either case one needs very good control of the linearized operator. This linear operator turns out to be the Laplacian plus spectral parameter $\Delta + \lambda$ on $S^{n-k-1} \times \mathbf{H}^{k+1}$ in the first case and an operator essentially equal to $\Delta + \lambda/r^2$ on the product of the ball in $\mathbf{R}^n$ with some compact manifold in the second; here r is the polar distance variable in the ball.

Degenerate operators such as these have been studied from several different perspectives. The path taken here is based on earlier work of the first author [16]–[18]), which itself is an outgrowth of a general framework for the treatment of a wide variety of degenerate operators due to Melrose ([23], [24]). Detailed analysis of operators with specific types of degeneracies using these ideas has been carried out in several cases. Beyond the references above the sources [25], [6], [21] should also be mentioned. The fourth section of this paper contains a fairly complete development of the analysis of $\Delta + \lambda(x)$ on $S^{n-k-1} \times \mathbf{H}^{k+1}$ and other related manifolds acting between natural families of weighted Sobolev and Hölder spaces. Here $\lambda(x)$ is a function converging to a constant along the set of degeneracies. In particular, we prove that $\Delta + \lambda(x)$ has closed range on these spaces for all but a discrete set of the weight parameter. We say closed range because in general there is an infinite-dimensional kernel or cokernel. However, for certain values of the weight parameter this operator is actually Fredholm. This is well known in the weight 0 case, since this is just the usual L^2 theory of the Laplacian. We are most interested in the spaces for which the operator is actually surjective, since this is necessary in the implicit function theorem and contraction mapping arguments. In any case, we can characterize the spaces for which $\Delta + \lambda(x)$ is injective, surjective, semi-Fredholm, or Fredholm. This should be compared to the analogous case on conic spaces where the Laplacian is Fredholm for all but a discrete set of weights [25], and the identical situation for the Laplacian on $\mathbf{R}^n$ (which is really a conic operator in disguise) [22].

Operators with this type of degeneracy have been studied by others. The theory developed here was actually undertaken by Melrose and Mendoza after their work in [25], although this work was never completed. Nonetheless, their work would certainly have followed a similar path to that developed here. Rempel-Schulze and Schulze [29] have developed a very complete operator calculus paralleling the one here. We have not used their approach due to the specific requirements of the problems considered. The work of Graham and Lee [7] should also be mentioned. They use the theory of "uniformly degenerate operators" developed in [17] in a nonlinear problem concerning Einstein metrics. These operators correspond to the one under consideration if there is no spherical factor. Other approaches to such operators have been taken in [4] and [15]. Concerning this last we shall say more. In that paper it is proved that if h is a smooth metric on the compact n-dimensional manifold M, and if ρ is a nonnegative function which vanishes only on the k-dimensional submanifold N which equals the distance to N in a tubular neighborhood, then the Laplacian for the metric $g = \rho^{-2}h$ is Fredholm on certain weighted Sobolev spaces for a narrow range of weights. These types of metrics generalize those considered in [17], [2]. The connection with the present work is that this Laplacian exhibits exactly the same sorts of degeneracies as $\Delta + \lambda$ on $S^{n-k-1} \times \mathbf{H}^{k+1}$. In fact, their method is to first analyze a model operator on the product manifold $\mathbf{R}^+ \times S^{n-k-1} \times N$, just as we do in §4. In fact, our methods work equally well in this more general situation. We develop our theory in §4 in this setting and extend their results to prove the following:

Theorem. *Let $P = \Delta + \lambda(x)$, where Δ is the Laplacian on $M\backslash N$ with respect to the metric g described above and $\lambda(x)$ is any smooth function on $M\backslash N$ which tends to a constant value along N. Then $P\colon r^\delta \widehat{H}^{k+2} \to r^\delta \widehat{H}^k$ is semi-Fredholm for $\delta \notin \Lambda$, where the Sobolev spaces $r^\delta \widehat{H}^k$ are* *defined precisely in* (4.4) *and* Λ *is a discrete set which is described in* (4.16). *Furthermore there is a (possibly empty) interval* $(-\delta_0, \delta_0)$ *in which* $P: r^\delta \widehat{H}^{k+2} \to r^\delta \widehat{H}^k$ *is actually Fredholm of index 0. There is an analogous assertion for* P *acting between weighted Hölder spaces.*

Thus, we not only generalize their results to the full range of weights, which is important in certain applications such as the ones in this paper, but we also extend all results to Hölder spaces.

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2. A global result

In this section we investigate the existence and variety of complete metrics on $S^n \backslash \tau(S^k)$ conformal to the standard metric and having constant positive scalar curvature; here τ is an embedding of S^k into S^n close to the identity. As noted above, it follows from [28] that for such a metric to exist, we must have $k \leq (n-2)/2$, and in fact we will assume in the rest of this section that $k < (n-2)/2$. Let $\overline{g}$ denote the standard metric on S^n. As shown in the introduction, $(S^n \backslash S^k, \overline{g})$ is conformally equivalent to the complete metric $(S^n \backslash S^k, g_0)$, where g_0 is the product metric which has scalar curvature $R(g_0) = R_{n,k} > 0$. We will show by a perturbation argument that for every $C^{3,\alpha}$ embedding τ of S^k in S^n close to the equator map there is an infinite-dimensional family of complete conformal metrics on $S^n \backslash \tau(S^k)$ of scalar curvature equal to $R_{n,k}$. In fact, we will show that the set of such metrics in a neighborhood of g_0 has a nice manifold structure parametrized by its tangent space, which may be identified with the product of an infinite-dimensional space of eigenfunctions for the Laplacian on $\mathbf{H}^{k+1}$ with the tangent space of the Banach manifold of embeddings $\tau: S^k \to S^n$ as above.

If u is a positive function on a manifold M with metric g_0 and scalar curvature $R(g_0)$, and $g = u^{4/(n-2)} g_0$ is a conformally related metric with scalar curvature $R(g)$, then it is standard (see [13]) that u satisfies the differential equation

$$\Delta u - \frac{n-2}{4(n-1)} R(g_0) u + \frac{n-2}{4(n-1)} R(g) u^{(n+2)/(n-2)} = 0 \tag{2.1}$$

on M, where $\Delta = \Delta_{g_0}$ is the (negative definite) Laplace operator acting on functions on M. The operator

$$L_{g_0} = \Delta_{g_0} - \frac{n-2}{4(n-1)} R(g_0)$$

is known as the conformal Laplacian because it satisfies the transformation rule

$$L_{g_0}(u\varphi) = u^{(n+2)/(n-2)} L_g(\varphi). \tag{2.2}$$

We shall set up the problem of finding a complete metric on $S^n \backslash \tau(S^k)$ conformal to $\overline{g}$ with scalar curvature $R_{n,k}$ by reducing it to solving a PDE on $S^n \backslash S^k$ which is a special instance of (2.1). To do this, first assume that τ is extended to be a diffeomorphism close to the identity of all of S^n to itself. We shall be more specific about its regularity and other properties later. We seek a function $\tilde{v}$ on $S^n \backslash \tau(S^k)$ such that

$$L_{\overline{g}} \tilde{v} + \frac{n-2}{4(n-1)} R_{n,k} \tilde{v}^{(n+2)/(n-2)} = 0 \tag{2.3}$$

and such that $\tilde{v}^{4/(n-2)} \overline{g}$ is complete. Transfer this equation from $S^n \backslash \tau(S^k)$ back to $S^n \backslash S^k$ by applying τ^* to both sides. We use the fact that by the naturality of the conformal Laplacian, $\tau^*(L_g w) = L_{(\tau^* g)}(\tau^* w)$ for any metric g and any function w. Setting $v = \tau^* \tilde{v}$ we get

$$L_{(\tau^* \overline{g})} v + \frac{n-2}{4(n-1)} R_{n,k} v^{(n+2)/(n-2)} = 0. \tag{2.4}$$

Next, apply the transformation rule (2.2) with u the function on $S^n \backslash S^k$ such that $u^{4/(n-2)} \overline{g}$ equals the product metric g_0 of (1.1) and with $v = u(1+w)$. The result is that

$$L_{g(\tau)}(1+w) + \frac{n-2}{4(n-1)} R_{n,k} (1+w)^{(n+2)/(n-2)} = 0. \tag{2.5}$$

Here $g(\tau)$ is the metric $u^{4/(n-2)} \tau^* \overline{g}$ on $S^n \backslash S^k$.

Clearly the process described above is reversible. Any metric on $S^n \backslash \tau(S^k)$ conformal to the standard one must correspond to a pair (τ, w), the uniqueness of which we shall examine later. Let

$$H(\tau, w) = L_{g(\tau)}(1+w) + \frac{n-2}{4(n-1)} R_{n,k} (1+w)^{(n+2)/(n-2)} \tag{2.6}$$

for $w \in C^{2,\alpha,\nu}(S^n \backslash S^k)$ and τ a $C^{3,\alpha}$ diffeomorphism of S^n (the Hölder spaces $C^{k,\alpha,\nu}$ will be defined in (2.9)). We shall determine all solutions of $H(\tau, w) = 0$ close to the known solution $(I, 0)$ by using the implicit function theorem. To do this, we must first calculate the linearization of H, then set up appropriate function spaces on which this linearization is surjective. Then an application of the (soft) implicit function theorem will complete the result.

Let us first compute the linearization. Although we only need to establish surjectivity at $(I, 0)$ and only in certain directions, it is also necessary to show that this derivative has continuous dependence on (τ, w) in a neighborhood of $(I, 0)$ in the uniform operator topology. This will be established later. At a given τ the derivative of H in w is computed as

$$\begin{aligned} d|_w H(\tau, w)\varphi &= \frac{d}{dt} H(\tau, w + t\varphi)|_{t=0} \\ &= L_{g(\tau)}\varphi + \frac{(n-2)}{4(n-1)} R_{n,k}(1+w)^{4/(n-2)}\varphi \\ &= \Delta_{g(\tau)}\varphi - \frac{(n-2k-2)(n-2)}{4}\varphi \\ &\quad + \frac{(n-2k-2)(n+2)}{4}(1+w)^{4/(n-2)}\varphi \\ &\equiv L(\tau, w)\varphi. \end{aligned} \tag{2.7}$$

Note that

$$L(I, 0)\varphi = (\Delta + (n-2k-2))\varphi. \tag{2.7$'$}$$

For convenience, we will often denote $L(I, 0)$ by L. On the other hand, to compute the derivative of H in τ at a point (τ, w) let $\tau(\varepsilon)$ denote a path of diffeomorphisms such that $\tau(0)$ is equal to the fixed map τ. Then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} H(\tau(\varepsilon), w) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_{g(\tau(\varepsilon))}(1+w). \tag{2.8}$$

The operator on the right in this formula is some second-order operator acting on $1 + w$.

Finding solutions of (2.1) will involve studying the linearized operators (2.7) and (2.8). In order to do this we must first defined the appropriate function spaces in which we will work. In $\mathbf{H}^{k+1}$ we will often use coordinates and metrics given by the Poincaré model; that is, we identity $\mathbf{H}^{k+1}$ with B^{k+1}, the unit ball in $\mathbf{R}^{k+1}$, endowed with the metric $g_{ij} = 4(1-s^2)^{-2} dx^i\, dx^j$, where $s = |x|$, $s \in [0, 1)$, and $x \in B^{k+1}$. Let $z = (s, \theta, y)$, $s > 0$, $\theta \in S^{n-k-1}$, $y \in S^k$. We also define the coordinate function $r = r(s) = 1 - s^2$. For $\nu \in \mathbf{R}$, $k \in \mathbf{N} = \{0, 1, \cdots\}$, and

$\alpha \in (0, 1)$ we define the weighted Hölder space

$$\begin{aligned} C^{k,\alpha,\nu}&(S^{n-k-1} \times \mathbf{H}^{k+1}) \\ &= \{u \in C^{k,\alpha}_{\mathrm{loc}}(S^{n-k-1} \times \mathbf{H}^{k+1}) \colon |u|_{k,\alpha,\nu} < \infty\}, \end{aligned} \tag{2.9}$$

where $|\ |_{k,\alpha,\nu}$ is the norm

$$|u|_{k,\alpha,\nu} = \sup_{z,z'}(r + r')^{-\nu} \left(\sum_{j=0}^{k} (r + r')^j |\nabla^j u| + (r + r')^{k+\alpha} [\nabla^k u]_{(\alpha)} \right). \tag{2.10}$$

Here $\nabla^j u$ denotes the jth covariant derivative of u on $S^{n-k-1} \times \mathbf{H}^{k+1}$, and $z = (r, \theta, y)$, $z' = (r', \theta', y')$; note that we might equally well have used the vector fields $r\partial_r$, $r\partial_y$, ∂_θ in place of ∇ near $r = 0$. Note also that $u \in C^{k,\alpha,\nu}$ iff $u = r^\nu v$, $v \in C^{k,\alpha,0}$. It is trivial that $C^{k,\alpha,\nu}$ is a Banach space and that for $k' \geq k$, $\nu' \geq \nu$, and $\alpha' \geq \alpha$ we have $C^{k',\alpha',\nu'} \subseteq C^{k,\alpha,\nu}$. These spaces have been used in other nonlinear problems involving degenerate or singular PDE (see [30], [7], [12] for example).

We will now discuss for which Jacobi fields φ (i.e., for which solutions of $L\varphi = 0$), $\varphi \in C^{2,\alpha,\nu}$ with $|\varphi|_{2,\alpha,\nu}$ small, and with appropriate values of ν, one expects a corresponding solution of (2.1). We first give a brief discussion of those Jacobi fields for this problem which are in $C^{2,\alpha,\nu}$. We note that there is a Poisson integral formula which represents these Jacobi fields (which are eigenfunctions of the Laplacian on $S^{n-k-1} \times \mathbf{H}^{k+1}$) in terms of their boundary values [8]. However, we shall analyze them briefly using separation of variables so as to see explicitly how they fit into the Hölder framework. First let us record the form of the operator L in the polar coordinates (s, θ), $s \in [0, 1]$, $y \in S^k$ for the ball B^{k+1}, and $\theta \in S^{n-k-1}$:

$$\begin{aligned} L = \frac{1}{4} \Bigg\{ &(1 - s^2)^2 \partial_s^2 + (1 - s^2) \left(\frac{k(1 - s^2)}{s} + (2k - 2)s \right) \partial_s \\ &+ \frac{(1 - s^2)^2}{s^2} \Delta_\theta \Bigg\} + \Delta_\theta + (n - 2k - 2) \\ = \frac{1}{4}&((1 - s^2)\partial_s)^2 + \frac{(1 - s^2)(k + (k - 2)s^2)}{4s} \partial_s \\ &+ \frac{(1 - s^2)^2}{4s^2} \Delta_y^2 + \Delta_\theta + (n - 2k - 2). \end{aligned} \tag{2.11}$$

Now let $0 = \mu_0^2 < \mu_1^2 \le \mu_2^2 \le \cdots$ be the eigenvalues for $-\Delta_{S^k}$ with corresponding orthonormal eigenfunctions $\varphi_0, \varphi_1, \cdots, \Delta_y \varphi_i = -\mu_i^2 \varphi_i$, $i \in \mathbf{N}$. Similarly, we let $0 = \lambda_0^2 < \lambda_1^2 \le \cdots$ and $\psi_0, \psi_1, \cdots$ be the eigenvalues and orthonormal eigenfunctions for $-\Delta_\theta$ on S^{n-k-1}. If $u(s, y, \theta) = \sum_{j,l} a_{j,l}(s)\varphi_j(y)\psi_l(\theta)$, then from (2.11) $Lu = 0$ implies that $L_{j,l}a_{j,l} = 0$, where

$$
\begin{aligned}
L_{j,l} &= \frac{1}{4}((1-s^2)\partial_s)^2 + \frac{(1-s^2)(k+(k-2)s^2)}{4s}\partial_s \\
&\quad - \frac{(1-s^2)^2}{4s^2}\mu_j^2 - \lambda_l^2 + (n-2k-2).
\end{aligned} \tag{2.12}
$$

The eigenvalues λ_l^2 and μ_j^2 are nonnegative integers:

$$
\begin{aligned}
\mu_j^2 &= q(q+k-1) \quad \text{for some } q \in \mathbf{N}, \\
\lambda_l^2 &= q(q+n-k-2) \quad \text{for some } q \in \mathbf{N}.
\end{aligned} \tag{2.13}
$$

$L_{j,l}$ is an operator on $[0, 1]$ for which both endpoints are regular singular points. The indicial roots at $s = 0$ are

$$
\frac{1-k}{2} \pm \sqrt{\left(\frac{k-1}{2}\right)^2 + \mu_j^2} = (1-k) - q \text{ or } +q, \qquad q \in \mathbf{N}, \tag{2.14}
$$

where we have used (2.13). It follows from this that there is a unique solution $a_{j,l}$ to $L_{j,l}a_{j,l} = 0$ such that $u_{j,l} = a_{j,l}(s)\varphi_j(y)\psi_l(\theta)$ is regular at $s = 0$. To calculate the indicial roots at $s = 1$ we shall use the expression for $L_{j,l}$ in terms of the coordinates (r, θ, y), $r = 1 - s^2$:

$$
\begin{aligned}
L_{j,l} &= (\sqrt{1-r}\, r\partial_r)^2 - \frac{k}{2}r(2-r)\sqrt{1-r}\,\partial_r \\
&\quad - \frac{r^2}{4(1-r)}\mu_j^2 - \lambda_l^2 + (n-2k-2).
\end{aligned} \tag{2.15}
$$

From this it is easily seen that the indicial roots at $r = 0$ (i.e., $s = 1$) are

$$
\gamma_l^{\pm} = \frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 + \lambda_l^2 - (n-2k-2)}. \tag{2.16}
$$

Let $D_l = (\frac{k}{2})^2 + \lambda_l^2 - (n-2k-2)$. By (2.13) $D_l = (\frac{k}{2}+q)^2 + (q-1)(n-2k-2)$ for some $q \in \mathbf{N}$. If $D_l > 0$, then $\gamma_l^{\pm}$ are both real and distinct and the two approximate solutions of $L_j a = 0$ are $r^{\gamma_l^{\pm}}$. Furthermore, $\gamma_l^- < \frac{k}{2} < \gamma_l^+$ so near $r = 0$ only $r^{\gamma_l^+}$ is in $L^2(r^{-k-1}\,dr\,dy)$. This implies that the solution of $L_{j,l}a_{j,l} = 0$ which is regular at $s = 0$ must have leading term $r^{\gamma_l^-}$ in its series expansion near $r = 0$, for otherwise the function $u_j(r, y) = a_j(r)\varphi_j(y)$ would have leading term $r^{\gamma_l^+}$ in its expansion near $r = 0$, hence would be an L^2 eigenfunction for the Laplacian on $\mathbf{H}^{k+1}$; such an eigenfunction is known not to exist. On the other hand, we are interested only in bounded solutions, which necessitates $\gamma_l^- \geq 0$. By the expression for D_l recorded after (2.16) this happens only when $q = 0$, i.e., $l = 0$, in which case we even have $\gamma_0^- > 0$. The corresponding eigenfunction is rotationally symmetric with respect to the θ variables, hence is an eigenfunction for the Laplacian on hyperbolic space. If $D_l = 0$, then the two approximate solutions are $r^{k/2}$ and $r^{k/2}\log r$. It may be shown that the solution regular at $s = 0$ behaves like $r^{k/2}\log r$, but this is unimportant for us. Finally, if $D_l < 0$, then $\gamma_l^\pm$ are both complex with real part $\frac{k}{2}$. This happens for at most finitely many values of l. Thus, in all cases the unique function $u_{j,l}(s, y, \theta) = a_{j,l}(s)\varphi_j(y)\psi_l(\theta)$ which solves $Lu_{j,l} = 0$, is bounded as $s \to 1$, and is regular at $s = 0$ satisfies:

$$(2.17) \qquad u_{j,l} \in C^{2,\alpha,\nu} \quad \text{for all } \nu \text{ such that } \begin{cases} 0 < \nu \leq \gamma_l^-, & D_l > 0, \\ 0 < \nu < \frac{k}{2}, & D_l = 0, \\ 0 < \nu \leq \frac{k}{2}, & D_l < 0. \end{cases}$$

Let ν_0 equal either the minimum of the positive values of γ_l^- for those l for which $D_l > 0$, or $\frac{k}{2}$ if whenever $D_l > 0$ the corresponding $\gamma_l^- \leq 0$. Also set

$$\mathscr{J}(\alpha, \nu) = \{u \in C^{2,\alpha,\nu} : Lu = 0\}$$

for $0 < \nu < \nu_0$, $0 < \alpha < 1$. We have proved the following:

(2.18) **Lemma.** *For $0 < \alpha < 1$ and $\nu < \nu_0$, the set $\mathscr{J}(\alpha, \nu)$ is the closure in $C^{2,\alpha,\nu}$ of the set of finite linear combinations of the functions $a_{j,0}(s)\varphi_j(y)$, and hence is infinite dimensional.*

We now must examine the map H of (2.6) and its linearization (2.7) and (2.8) more closely. In particular, we must show that H is bounded between the Hölder spaces (2.9) and that its derivative is a continuous map uniformly in (τ, w). This is not true for arbitrary diffeomorphisms τ close to the identity, but requires an extra hypothesis. Let ρ be the "defining function" for the equator $S^k \subset S^n$ given by $u^{2/(2-n)}$, where $u^{4/(n-2)}$ is the conformal factor relating the product metric g_0 and the round metric $\overline{g}$ on $S^n \backslash S^k$.

(2.19) **Proposition.** *The metric $g(\tau) = u^{4/(n-2)}\tau^*\overline{g} = \rho^{-2}\tau^*\overline{g}$ has scalar curvature $R(g(\tau)) = R_{n,k} + O(\rho)$ as $\rho \to 0$ if the metric $\tau^*\overline{g}$ has unit normal ν along S^k agreeing with the unit normal $\overline{\nu}$ for $\overline{g}$.*

Proof. This is very similar to a computation in [16]. In fact, we prove the following general fact. Let M be a compact manifold and N a compact submanifold. Let ρ be smooth on M away from N and suppose that in Fermi polar coordinates around N, ρ is a smooth nonvanishing multiple of the polar distance function r (with respect to h) to N. Define a new singular metric $g = \rho^{-2}h$. Choose now a full coordinate system (r, y, θ), where y is a coordinate along N and θ is the coordinate in the polar sphere S^{n-k-1}. Then the sectional curvature of g evaluated on any family of two-planes $P(t)$ in $T_{\gamma(t)}$, where $\gamma(t)$ is a path converging to N, has an asymptotic development in nonnegative powers of ρ, and it converges to the constant term in this expansion. In two-planes in the r, y directions it converges to $-|\nabla\rho|^2$, in two-planes in the θ directions it converges to $+|\nabla\rho|^2$, and for two-planes in the mixed (r, y), θ directions it converges to 0. This is proved by a straightforward but lengthy computation. Now sum over all two-planes to obtain the scalar curvature. Clearly, $R(g) \to R_{n,k}|\nabla\rho|^2$ as $\rho \to 0$.

To apply this to our situation, we need only compute $|\nabla\rho|^2$ along S^k with respect to the metric $\tau^*\overline{g}$. But this equals 1 if the unit normal for this metric ν equals $\overline{\nu}$. q.e.d.

To prove continuity properties of H we first introduce the space in which the diffeomorphisms τ live. Because of the previous proposition we shall require that the unit normal of $\tau^*\overline{g}$ along S^k agrees with the unit normal of $\overline{g}$. This is ensured for example by requiring that the differential of τ is an isometry between the tangent spaces T_zS^n and $T_{\tau(z)}S^n$ for $z \in S^k$. This is not a restriction, because all we care about is the image $\tau(S^k)$. Let $\mathscr{E}$ be the set of $C^{3,\alpha}$ embeddings of S^k in S^n with its natural structure as a Banach manifold. In a neighborhood of the standard equator map in $\mathscr{E}$ choose a fixed extension operator, which extends an embedding τ to a diffeomorphism of the whole sphere S^n close to the identity which has the normalization condition specified above. Then we regard this neighborhood of the equator map in $\mathscr{E}$ as a Banach submanifold of the group of all $C^{3,\alpha}$ diffeomorphisms of S^n; we shall still call this submanifold $\mathscr{E}$.

(2.20) **Corollary.** *The map H of (2.6) is a C^∞ map from a neighborhood $\mathscr{U}$ of $(I, 0)$ in $\mathscr{E}\times C^{2,\alpha,\nu}(S^{n-k-1}\times\mathbf{H}^{k+1})$ to $C^{0,\alpha,\nu}(S^{n-k-1}\times\mathbf{H}^{k+1})$ for $\nu < \nu_1 = \min(\nu_0, 1)$.*

Proof. We first check that $H: \mathscr{U} \to C^{0,\alpha,\nu}$. Because $\tau \in C^{3,\alpha}$ and $w \in C^{2,\alpha,\nu}$ it is clear that $H(\tau, w) \in C^{0,\alpha}_{\text{loc}}$. To show that it has the correct decay properties we write $H(\tau, w) = H(\tau, w) - H(I, 0)$. In this difference, we collect the terms $\Delta_{g(\tau)}(1+w)-\Delta_{g_0}1$, $cR(g(\tau))(1+w) - cR(g_0)$, and $cR(g(\tau))(1+w)^{(n+2)/(n-2)} - cR(g_0)$. The first of these obviously lies in $C^{0,\alpha,\nu}$. The second and the third may be estimated using Proposition (2.19). The point here is that $R(g(\tau)) = R_{n,k} + O(\rho)$. Using this in conjunction with Taylor's formula in the nonlinear expression in w shows that $H(\tau, w)$ is in $C^{0,\alpha,\nu}$. The proof that it is a smooth map is very similar. For example, its first Fréchet derivative is computed in (2.7) and (2.8). The differential in w is obviously bounded between these Hölder spaces, and that its differential in τ is also bounded requires an argument nearly identical to the one above. Note that the restriction $\nu_1 < 1$ follows from the fact that $R(g(\tau))$ deviates from $R_{n,k}$ by a term which is $O(\rho)$. Finally, the estimates for all higher derivatives follow similarly. q.e.d.

Finally we come to our main theorem:

(2.21) **Theorem.** *Let $0 < \nu < \nu_1$. There is a splitting of $C^{2,\alpha,\nu}$ into closed subspaces $W \oplus \mathcal{J}(\alpha, \nu)$ and a smooth map Φ from the intersection $\mathcal{U} \cap (\mathcal{E} \times \mathcal{J}(\alpha, \nu))$ to W such that $H(\tau, w_1, \Phi(\tau, w_1)) = 0$. Here we have written $w = (w_1, w_2)$ in terms of its $\mathcal{J}(\alpha, \nu)$ and W components. Furthermore, all solutions of $H(\tau, w)$ in $\mathcal{U}$ are obtained this way. The graph of Φ in $\mathcal{E} \times C^{2,\alpha,\nu} \times C^{0,\alpha,\nu}$ is a smooth Banach submanifold.*

Proof. This is proved using the implicit function theorem in the form presented, for example, in [11]. In fact, we have already shown that H is a smooth mapping from the domain into the range. It remains to identify the splitting $C^{2,\alpha,\nu} = \mathcal{J}(\alpha, \nu) \oplus W$ and to show that the differential of H in the subspace W is an isomorphism onto $C^{0,\alpha,\nu}$. This is proved in §4. In particular, this is the content of Theorem (4.55).

3. A local result

In this section we prove the rather general result that, at least locally, there is no obstruction to finding complete, conformally flat metrics of constant positive scalar curvature on $S^n \backslash \Lambda$, where Λ is a k-dimensional submanifold of S^n with $k < (n-2)/2$. For such a Λ and for $\sigma > 0$ let $N(\sigma, \Lambda)$ denote the σ-tubular neighborhood of Λ in S^n and $\overline{N}(\sigma, \Lambda)$ its closure.

(3.1) **Theorem.** *Let $k < (n-2)/2$, $0 < \alpha < 1$, and let $\Lambda \subset S^n$ be a $C^{2,\alpha}$ embedded k-dimension submanifold. Then there exists a $\sigma > 0$ such* *that* $\overline{N}(\sigma, \Lambda)\backslash\Lambda$ *admits an infinite-dimensional family of complete metrics conformal to the round metric* $\overline{g}$ *on* S^n *with scalar curvature identically equal to* $R_{n,k} > 0$.

Remarks. The proof of the theorem does not really require Λ to be a submanifold of S^n but merely a submanifold of a Riemannian manifold of constant positive scalar curvature. Also, the infinite dimensionality of the space of solutions in the theorem will be made more precise later. The boundary values of the conformal factor on $\overline{N}(\sigma, \Lambda)\backslash\Lambda$ can be prescribed as any sufficiently small $C^{2,\alpha}$ perturbation of a fixed function $\Psi_0 \in C^{2,\alpha}(\partial\overline{N}(\sigma, \Lambda))$.

We now commence the proof of Theorem (3.1). In order to simplify some of the later calculations we stereographically project $S^n\backslash\Lambda$ onto $\mathbf{R}^n\backslash\Gamma$ from some point $P \notin \Lambda$ and note that it suffices to prove the theorem with $\mathbf{R}^n\backslash\Gamma$ in place of $S^n\backslash\Lambda$ and with the flat metric g_0 on $\mathbf{R}^n$ in place of the round metric $\overline{g}$ on S^n since stereographic projection is conformal away from P. $N(\sigma, \Gamma)$ now denotes the σ-tubular neighborhood of Γ in R^n.

Let Ω be an open set in R^n. If u is a positive C^2 function on Ω such that the metric $g = u^{4/(n-2)}g_0$ on Ω has scalar curvature $R(g) = n(n-1)$, then it follows from (2.1) that u must satisfy the equation

$$\Delta u + \frac{n(n-2)}{4}u^{(n+2)/(n-2)} = 0, \qquad u > 0, \tag{3.2}$$

on Ω, where Δ is the Euclidean Laplacian on $\mathbf{R}^n$. Thus to prove Theorem (3.1), we must exhibit an infinite-dimensional family of positive solutions of (3.2) with $\Omega = N(\sigma, \Gamma)$ that blow up fact enough near Γ to guarantee completeness of the metric g.

For $x \in \overline{N}(\sigma, \Gamma)$, $\sigma > 0$, let $r = r(x) = \operatorname{dist}(x, \Gamma)$. We will consider r as a coordinate function on $\overline{N}(\sigma, \Gamma)$. We will also need to calculate the local form of Δ on $N(\sigma, \Gamma)$. First, fix σ small enough so that $\overline{N}(\sigma, \Gamma)$ is identified with the disc bundle of radius σ in the normal bundle to Γ in $\mathbf{R}^n$; that is,

$$\overline{N}(\sigma, \Gamma) \simeq \bigcup_{y\in\Gamma} \overline{N}_y(\sigma, \Gamma),$$

where $N_y(\sigma, \Gamma)$ is the ball of radius σ in the fiber of the normal bundle to Γ at y.

Let $y_0 \in \Gamma$. Then there is a neighborhood $\mathcal{U}$ of y_0 in Γ such that $\overline{N}(\sigma, \Gamma)|_{\mathcal{U}} \equiv \overline{N}_{\mathcal{U}}(\sigma, \Gamma)$ is equivalent to the trivial bundle $B^{n-k}(\sigma) \times \mathcal{U}$, where $B^{n-k}(\sigma)$ is the ball of radius σ centered at 0 in $\mathbf{R}^{n-k}$. Let us denote by F this bundle map:

$$F\colon \overline{N}_{\mathscr{U}}(\sigma, \Gamma) \to B^{n-k}(\sigma) \times \mathscr{U}.$$

F is a diffeomorphism and a linear isometry from $N_y(\sigma, \Gamma)$ to $B^{n-k}(\sigma)\times \{y\}$, $y \in \mathscr{U}$. Thus, F gives us a local coordinate system for $\overline{N}(\sigma, \Gamma)$ near y_0.

Using polar coordinates (r, θ) on B^{n-k} as in §2 so that (r, θ, y) are coordinates on $B^{n-k} \times \Gamma$, we may define Hölder spaces $C^{k,\alpha,\nu}(\overline{N}(\sigma, \Gamma))$ exactly as in (2.9). However, we shall use a slightly different, although equivalent, formulation of the norms (2.10) for these spaces. Thus, we set

$$|u|_{k,\alpha,\nu} = \sup_{0<s\leq 1/2} \sup_{s\leq r\leq 2s} s^{-\nu}|u|_{k,\alpha;[s,2s]},$$

where the norm on the right is the $C^{k,\alpha,0}$ norm restricted to the set $\{(r, \theta, y\colon s \leq r \leq 2s)\}$ with respect to the underlying metric.

(3.3) **Proposition.** *Let $(x, y) \in B^{n-k}(\sigma)\times\mathscr{U}$ be the induced coordinates for $\overline{N}(\sigma, \Gamma)$ near y_0. Then, for $u \in C^{2,\alpha}(\overline{N}(\sigma, \Gamma))$,*

$$\Delta u = \Delta_{B^{n-k}} u + \Delta_\Gamma u + e_1 \nabla^2 u + e_2 \nabla u$$

on $B^{n-k}(\sigma)\times\mathscr{U}$, where ∇^2 and ∇ denote the Hessian and gradient, respectively, on $\overline{N}(\sigma, \Gamma)$ and e_1 and e_2 are $C^{0,\alpha}$ sections of $(\mathrm{Sym}^2 N(\sigma, \Gamma))^*$ and $TN(\sigma, \Gamma)$ such that

$$|e_1|_{0,\alpha,1} + |e_2|_{0,\alpha,0} \leq C_0$$

for some constant C_0 independent of x, y, or σ and for any α, $0 < \alpha < 1$.

Proof. Let $y = (y_1, \cdots, y_k)$ be coordinates for Γ in $\mathscr{U}$, and let g_{ij} and g^{ij} denote the coefficients of the metric on $\overline{N}(\sigma, \Gamma)$ relative to the coordinates (x, y) induced by F. For convenience, concatenate the indices of x and y, so that the coordinates of x are x_i, $1 \leq i \leq n-k$, and those of y are y_i, $n-k+1 \leq i \leq n$. Then, a straightforward computation shows that

$$g_{ij} = \begin{cases} g_{ij}^{B}, & i, j \leq n-k, \\ O(r), & i \leq n-k, j > n-k, \\ g_{ij}^{\Gamma} + O(r), & i, j > n-k, \end{cases}$$

$$g^{ij} = \begin{cases} g_B^{ij} + O(r), & i, j \leq n-k, \\ O(r), & i \leq n-k, j > n-k, \\ g_\Gamma^{ij} + O(r), & i, j > n-k, \end{cases}$$

$$\sqrt{\det g_{ij}} = \sqrt{\det(g_{ij}^B)\det(g_{ij}^\Gamma)} + O(r),$$

where g_{ij}^B, g_{ij}^Γ and g_B^{ij}, g_Γ^{ij} denote the coefficient of the metric tensor and its inverse for B^{n-k} and Γ, respectively. Now from the standard formulas for Δ, ∇^2, and ∇ in local coordinates, the required estimates follow easily. q.e.d.

Since we will be finding exact solutions to (3.2) using a contraction mapping argument, we must first have an approximate solution to this equation on $\overline{N}(\sigma, \Gamma)\backslash\Gamma$ with which to start the argument. Let u_0 denote the function

$$u_0(x) = c_{n,k} r^{(2-n)/2}, \qquad x \in \overline{N}(\sigma, \Gamma)\backslash\Gamma,$$

where $c_{n,k} = ((n-2k-2)/n)^{(n-2)/4}$. Let $H(u)$ denote the quantity on the left-hand side of (3.2). Then u_0 is an approximate solution to $H(u) = 0$ in the following sense.

(3.4) **Proposition.** *Let* $0 < \alpha < 1$ *and* $(2-n)/2 < \nu < (4-n)/2$. *Then there is a constant* C_1 *independent of* α, ν, *and* σ *such that*

$$|r^2 H(u_0)|_{0,\alpha,\nu} \le C_1 \sigma^{(4-n)/2-\nu}.$$

Proof. Let $y_0 \in \Gamma$ and choose coordinates (x, y) as in Proposition (3.3) near y. Using polar coordinates (r, θ) for $x \in B^{n-k}(\sigma)$ one easily computes that

$$\Delta_{B^{n-k}} u_0 + \frac{n(n-2)}{4}(u_0)^{(n+2)/(n-2)} = 0$$

and therefore, by Proposition (3.3), we obtain

$$\begin{aligned} |r^2 H(u_0)| r^{-\nu} &\le r^{2-\nu}|e_1|\,|\nabla^2 u_0| + r^{2-\nu}|e_2|\,|\nabla u_0| \\ &\le C_2 r^{(4-n)/2-\nu} \le C_2 \sigma^{(4-n)/2-\nu} \end{aligned}$$

as $r \le \sigma$. Similarly, one can have

$$s^{\alpha-\nu}[r^2 H(u_0)]_{(\alpha), s\le r\le 2s} \le C_2 \sigma^{(4-n)/2-\nu},$$

which completes the proof.

Now, $(u_0)^{4/(n-2)} g_0$ is easily seen to be a complete metric (near Γ) and u_0 is an approximate solution to (3.2) in the sense made precise above, so the strategy is to look for solutions of (3.2) which are obtained by adding to u_0 a lower order term. Hence we linearize H about u_0, and make a Taylor expansion to get

$$(3.5) \qquad H(u+u_0) = H(u_0) + Lu + Q(u), \qquad u \in C^{2,\alpha,\nu}(\overline{N}(\sigma, \Gamma)),$$

where $\nu \in ((2-n)/2, (4-n)/2)$ and

$$(3.6)\qquad \begin{aligned} Lu &= \frac{d}{dt}H(u_0+t_u)|_{t=0} = \Delta u + \frac{(n+2)(n-2k-2)}{4}r^{-2}u, \\ Q(u) &= \int_0^1 \frac{d^2}{dt^2}H(u_0+tu)(1-t)\,dt. \end{aligned}$$

Let L_0 be the differential operator $L_0 = r^2 L$. We see that $H(u_0+u)=0$ for $u \in C^{2,\alpha,\nu}(\overline{N}(\sigma,\Gamma))$ if and only if

$$(3.7)\qquad L_0 u = -r^2 H(u_0) - r^2 Q(u).$$

We will look for solutions of (3.7) by inverting L_0 on an appropriate $C^{0,\alpha,\nu}$ and using the contraction mapping method to solve (3.7) as a fixed point problem. Before we study the operator L_0 we first estimate $r^2 Q(u)$ in $C^{0,\alpha,\nu}$.

(3.8) **Proposition.** *For $\alpha \in (0,1)$, $\nu \in ((2-n)/2, (4-n)/2)$, and $u \in C^{2,\alpha,\nu}$ we have*

$$|r^2 Q(u)|_{0,\alpha,\nu} \le C_3 |u|^2_{0,\alpha,\nu},$$

where C_3 is independent of α, ν, σ.

Proof. From (3.6) it follows that

$$|r^2 Q(u)|_{0,\alpha,\nu} \le \sup_{0\le t\le 1}\left| r^2\frac{d^2}{dt^2}H(u_0+tu)\right|_{0,\alpha,\nu},$$

and we easily compute that

$$(3.9)\qquad r^2\frac{d^2}{dt^2}H(u_0+tu) = \frac{n(n+2)}{n-2}(c_{n,k}r^{(2-n)/2}+tu)^{(6-n)/(n-2)}u^2 r^2,$$

and so, setting $\rho = \rho(\nu) = \nu - (2-n)/2$,

$$\begin{aligned} \left| r^2\frac{d^2}{dt^2}H(u_0+tu)\right| r^{-\nu} &\le C_3(r^{(2-n)/2}+tu)^{(6-n)/(n-2)}u^2 r^{(n-2)/2-\rho+2} \\ &\le C_3 u^2 r^{n-2-2\rho} \le C_3|ur^\nu|^2. \end{aligned}$$

Similarly, one can check that

$$s^{\alpha-\nu}\left[r^2\frac{d^2}{dt^2}H(u_0+tu)\right]_{(\alpha),\,s\le r\le 2s} \le C_3|u|^2_{0,\alpha,\nu},$$

thus completing the proof

Thus, for $u \in C^{2,\alpha,\nu}$, $\nu \in ((2-n)/2, (4-2)/2)$, the right-hand side of (3.7) is in $C^{0,\alpha,\nu}$ and we can estimate it from Propositions (3.4) and

(3.8). We must then understand L_0 as an operator on $C^{2,\alpha,\nu}$. From (3.6) it is clear that

$$L_0 \colon C_0^{2,\alpha,\nu} \to C^{0,\alpha,\nu}$$

is a bounded operator for any $\nu \in \mathbf{R}$, where $C_0^{2,\alpha,\nu}$ is the set $\{u \in C^{2,\alpha,\nu}(\overline{N}(\sigma, \Gamma)) \colon u \equiv 0 \text{ on } \partial\overline{N}(\sigma, \Gamma)\}$. To understand asymptotics of solutions to $L_0 u = f$ we introduce eigenfunction expansions on S^{n-k-1} and Γ. Thus, let $(\varphi_j(y), \mu_j^2)$ and $(\psi_l(\theta), \lambda_l^2)$, $j, l = 1, 2, \cdots$, be orthonormal sequences of unit eigenfunctions

$$\Delta_\Gamma \varphi_j = -\mu_j^2 \varphi_j, \qquad \Delta_{S^{n-k-1}} \psi_l = -\lambda_l^2 \psi_l.$$

By Proposition (3.3) we can regard L_0 near any point $y_0 \in \Gamma$ as an operator on $B^{n-k}(\sigma) \times \Gamma$. Let $u = \sum_{j,l=1}^{\infty} u_{jl}(r)\varphi_j(y)\psi_l(\theta)$. Then by the same procedure as in §2, we see that

$$L_0 u = \sum_{j,l} (L_{jl} u_{jl}) \varphi_j \psi_l,$$

where

$$L_{j,l} a = r^2 \frac{d^2 a}{dr^2} + (n-k-1) r \frac{da}{dr} + (\lambda - \lambda_l^2 - r^2 \mu_j^2). \tag{3.10}$$

The indicial roots of L_{jl} at the regular singular point $r = 0$ are

$$\gamma_l^{\pm} = \frac{-(n-k-2)}{2} \pm \sqrt{\frac{(n-k-2)^2}{4} + \lambda_l^2 - \lambda}.$$

Let $D = D(n, k)$ denote the value of the discriminant in this radical when $l = 0$, i.e., when $\lambda_l^2 = 0$, and define

$$\nu_0 = \begin{cases} -(n-k-2)/2, & D \le 0, \\ \gamma_0^+, & D > 0. \end{cases}$$

We note that when $D < 0$ and $\nu < \nu_0$,

$$L_0 \colon C_0^{2,\alpha,\nu} \to C^{0,\alpha,\nu}$$

has an infinite-dimensional kernel, for in this case for every j the equation $L_{j0} a = 0$ on the interval $0 \le r \le \sigma$ has a nontrivial solution $a(r)$ such that $a(\sigma) = 0$ and which blows up like r^{ν_0} as $r \to 0$; furthermore, this function is rotationally invariant on S^{n-k-1}, so it exists globally on $\overline{N}(\sigma, \Gamma)$. However, as in §2, we can still find a right inverse to L_0 on the appropriate Hölder spaces. This is the subject of §4, where we prove that there exists a bounded map $G \colon C^{0,\alpha,\nu} \to C_0^{2,\alpha,\nu}$ such that $L_0 G = I$ on

$C^{0,\alpha,\nu}$ (see Theorem (4.55)). Furthermore, given this map, it is easy to solve the Dirichlet problem.

(3.11) **Proposition.** *If* $0<\alpha<1$, $0<\nu<\nu_0$, $\varphi\in C^{2,\alpha}(\partial\overline{N}(\sigma,\Gamma))$, *and* $f\in C^{0,\alpha,\nu}$, *then there exists a solution* $u=G_\varphi f\in C^{2,\alpha,\nu}$ *to the problem* $L_0u=f$ *on* $\overline{N}(\sigma,\Gamma)\backslash\Gamma$, $u=h$ *on* $\partial\overline{N}(\sigma,\Gamma)$ *such that* $|u|_{2,\alpha,\nu}\le C(|f|_{0,\alpha,\nu}+|\varphi|_{2,\alpha})$.

Proof. Choose a fixed bounded extension operator E from $C^{2,\alpha}(\partial\overline{N})$ to compactly supported $C^{2,\alpha}$ functions on $\overline{N}(\sigma,\Gamma)\backslash\Gamma$ and set $G_\varphi(f)=E\varphi+Gf-G(L_0E\varphi)$. The boundedness of G implies the necessary estimate. q.e.d.

The following result implies Theorem (3.1).

(3.12) **Theorem.** *Fix* $\alpha\in(0,1)$ *and* $\nu\in((2-n)/2,\min((4-n)/2,\nu_0))$. *Then there exists a number* $\overline{\sigma}=\overline{\sigma}(n,\alpha,\nu,\Gamma)>0$ *such that for all* $\sigma\le\overline{\sigma}$, *and* $h\in C^{2,\alpha}(\partial\overline{N}(\sigma,\Gamma))$ *with* $|h|_{2,\alpha}\le\sigma$, *there is a solution* $u\in C^{2,\alpha,\nu}(\overline{N}(\sigma,\Gamma))$ *of equation* (3.7) *with* $u=h$ *on* $\partial\overline{N}(\sigma,\Gamma)$ *and* $|u|_{2,\alpha,\nu}\le C_4\sigma^{(4-n)/2-\nu}$ *for some constant* C_4 *independent of* σ *or* ν.

Proof. First we note that this does imply Theorem (3.1). For given the solution u provided by this theorem the function $u+u_0$ satisfies (3.2) on $\overline{N}(\sigma,\Gamma)\backslash\Gamma$ and for σ sufficiently small, $u+u_0>0$. Also, u is dominated by u_0 so that $(u+u_0)^{4/(n-2)}g_0$ is still complete.

Let $K>0$ be a number to be chosen later, and let $\overline{\sigma}$ be small enough that $\overline{\sigma}^{(4-n)/2-\nu}K\le1$. Let $B_{\sigma,K}=\{u\in C^{2,\alpha,\nu}\colon |u|_{2,\alpha,\nu}\le\sigma^{(4-n)/2-\nu}K\}$. For $u\in B_{\sigma,K}$ let $T(u)$ denote the solution of

$$\begin{aligned} L_0v&=-r^2H(u_0)-r^2Q(u)\quad\text{on }\overline{N}(\sigma,\Gamma)\backslash\Gamma,\\ v&=h\quad\text{on }\partial\overline{N}(\sigma,\Gamma)\end{aligned}$$

given by Proposition (3.11). Then $T(u)\in C^{2,\alpha,\nu}$ and by Propositions 3.4 and 3.8 we have

$$|T(u)|_{2,\alpha,\nu}\le C_5(\sigma^{(4-n)/2-\nu}+|u|_{0,\alpha,\nu}^2+\sigma)\le C_6\sigma^{(4-n)/2-\nu}$$

since $|u|_{0,\alpha,\nu}\le1$. Therefore, fixing $K=C_6=C_6(n,\Gamma,\alpha)$, we see that $T\colon B_{\sigma,K}\to B_{\sigma,K}$. Of course $B_{\sigma,K}$ is a closed ball in the Banach space $C^{2,\alpha,\nu}$, and hence is a complete metric space. Therefore, if T is a contraction, then it has a unique fixed point u, and this fixed point satisfies (3.7) and the other requirements of the theorem. So it suffices to show that $T\colon B_{\sigma,K}\to B_{\sigma,K}$ is a contraction if $\overline{\sigma}$ is sufficiently small.

Let $v_1, v_2 \in B_{\sigma,K}$ and let $w = T(v_1) - T(v_2)$. Thus,

$$L_0 w = r^2 Q(v_1) - r^2 Q(v_2) \quad \text{on } \overline{N}(\sigma, \Gamma)\backslash\Gamma,$$
$$w = 0 \quad \text{on } \partial\overline{N}(\sigma, \Gamma),$$

and $w = G(r^2(Q(v_1) - Q(v_2)))$. By the boundedness of G

$$|w|_{2,\alpha,\nu} \le C|r^2(Q(v_1) - Q(v_2))|_{0,\alpha,\nu}$$

and so by the mean value theorem (cf. [11])

$$|w|_{2,\alpha,\nu} \le \int_0^1 \left| r^2 \frac{d}{dt} Q(v_2 + t(v_1 - v_2)) \right|_{0,\alpha,\nu} dt \; |v_1 - v_2|_{2,\alpha,\nu}.$$

It therefore suffices to show that the quantity

$$\left| r^2 \frac{d}{dt} Q(v_2 + t(v_1 - v_2)) \right|_{0,\alpha,\nu}$$

can be made arbitrarily small if σ is small.

Let $v_t = v_2 + t(v_1 - v_2)$. Then from (3.6) and (3.9)

$$r^2 \frac{d}{dt} Q(v_t) = \int_0^1 r^2 C_7 (c_{n,k} r^{(2-n)/2} + s v_t)^{(8-2n)/(n-2)} v_t' v_t^2 (1-s)\, ds$$
$$+ \int_0^1 r^2 C_8 (c_{n,k} r^{(2-n)/2} + s v_t)^{(6-n)/2} v_t' v_t (1-s)\, ds,$$

where $v_t' = \frac{d}{dt} v_t$. Now, using the fact that $|v_t| \le C_9 r^\nu$ and $|v_t'| \le C_9 r^\nu$, we get

$$\left| r^2 \frac{d}{dt} Q(v_t) \right| \le C_{10} (r^{(2n-4)/2 + 3\nu} + r^{(n-2)/2 + 2\nu}),$$
$$\left| r^2 \frac{d}{dt} Q(v_t) \right| r^{-\nu} \le C_{10} (r^{(2n-4)/2 + 2\nu} + r^{(n-2)/2 + \nu})$$
$$\le C_{10} r^{(n-2)/2 + \nu} \le C_{10} \sigma^{(n-2)/2 + \nu}.$$

The last two inequalities use that $\nu > (2-n)/2$ and $r < \infty$. One can also derive the Hölder estimate similarly. Thus,

$$\left| r^2 \frac{d}{dt} Q(v_t) \right|_{0,\alpha,\nu} \le C_{11} \sigma^{(n-2)/2 + \nu},$$

which finishes the proof of Theorem (3.12).

4. Linear estimates

Because of the degenerate nature of the linearized operators (2.7), (2.8), and (3.6), the relevant surjectivity properties and estimates that we require of them for the arguments of §2,3 are not standard. In this section we derive these properties in as straightforward a manner as possible.

As discussed in the introduction, there is a general framework for treating degenerate operators such as the Laplacian on $S^p \times \mathbf{H}^q$ which developed out of the ideas in Melrose's paper [23] and his later paper with Mendoza [25] concerning elliptic operators on compact spaces with conic singularities. In this latter paper it is shown that such operators, which are called totally characteristic, are Fredholm on certain weighted Sobolev spaces for all but a discrete set of the weight parameter. We prove analogous results for the operators considered here, but as these operators frequently have infinite dimensional kernels or cokernels, the best we may hope to prove in general is that these operators, when acting between the weighted Sobolev or Hölder spaces introduced earlier, have closed range for all but a discrete set of the weights. In particular cases, either the kernel or cokernel may degenerate to a finite-dimensional space, in which case the operator is semi-Fredholm or even Fredholm.

A bit more generally, our interest in this paper is with operators of the form $\Delta + \lambda(x)$ on the manifold $\mathscr{M}^n \backslash N^k$. Here $\lambda(x)$ is a smooth function on $\mathscr{M} \backslash N$ which tends to a constant value λ on N and the metric g on $\mathscr{M} \backslash N$ is of the form $\rho^{-2} h$, where the metric h is smooth on all of $\mathscr{M}$ and ρ is a smooth function on $\mathscr{M} \backslash N$ which vanishes on N and for which $\nabla_h \rho = 1$ along N. Significant special cases of this are when $k = 0$ or when $k = n - 1$. The former case corresponds to manifold with isolated conic points which was studied in [25] and the latter corresponds to a manifold with a "conformally compact metric" [16], of which the hyperbolic space $\mathbf{H}^n$ is a special case. Of course the prototype of the more general situation is $S^n \backslash S^k$ with the metric (1.1). It is possible to develop a quite general theory for elliptic operators of arbitrary order with degeneracies of the type exhibited by $\Delta + \lambda(x)$. This will be taken up in [19]. Here we develop a simple version of this theory which is applicable in the special geometric situation described above. These operators warrant interest because of the many geometric and physical problems in which they arise.

Introduce coordinates (r, θ, y) on a neighborhood of N in $\mathscr{M}$, where (r, θ) are Riemannian polar coordinates with respect to h around N and y is a local coordinate chart in N. Introduce also the manifold M which is obtained by taking the union of $\mathscr{M} \backslash N$ with the spherical normal bundle of N with the unique minimal C^∞ structure for which the lifts of smooth functions from $\mathscr{M}$ and the polar coordinates above are smooth. Note that M has a single boundary on which $r = 0$. It is straightforward to calculate (see [16], [15]) that

$$\Delta + \lambda(x) = L + E, \tag{4.1}$$

where

$$L = r^2\partial_r^2 + (1-k)r\partial_r + \Delta_\theta + r^2\Delta_N + \lambda \tag{4.2}$$

will serve as a local model near $r = 0$ and

$$E = r \sum_{j+|\alpha|+|\beta|\le 2} a_{j,\alpha,\beta}(r, \theta, y)(r\partial_r)^j \partial_\theta^\alpha (r\partial_y)^\beta \tag{4.3}$$

is an error term that we will eventually show is negligible. In (4.2) $\Delta_N = \Delta_y$ is the Laplacian on N with respect to the metric induced by h and Δ_θ is the Laplacian with respect to the standard metric on S^{n-k-1}. (It is the special dependence of L on y that makes the analysis of this section simpler than in the general case [19].)

The ultimate goal in this section is to show that $\Delta + \lambda(x)$ is surjective on certain of the weighted Hölder spaces (2.10). However, it is much easier to approach this operator through L^2 methods. Thus, initially our discussion will concern the behavior of $\Delta + \lambda(x)$ on the weighted Sobolev spaces

$$\begin{aligned} &r^\delta \widehat{H}^k(M) \\ &\quad = \{u\colon (r\partial_r)^j (r\partial_j)^\alpha (\partial_\theta)^\beta u \in r^\delta L^2(r^{-k-1}\, dr\, d\theta\, dy),\ j + |\alpha| + |\beta| \le k\}. \end{aligned} \tag{4.4}$$

Only later will we return to the Hölder framework. The basic strategy is to first analyze the operator L, which we regard as an operator on the space $\mathbf{R}^+ \times S^{n-k-1} \times N$. We will prove that

$$L\colon r^\delta \widehat{H}^{k+2} \to r^\delta \widehat{H}^k \tag{4.5}$$

has closed range for all but a discrete set of values of the weight δ. This is accomplished by constructing a generalized inverse G for L which is bounded as a map

$$G\colon r^\delta \widehat{H}^k \to r^\delta \widehat{H}^{k+2}, \tag{4.6}$$

and such that

$$LG = I - P_1, \qquad GL = I - P_2, \tag{4.7}$$

where P_1, P_2 are orthogonal projections onto the cokernel and kernel of L in $r^\delta \widehat{H}^k$ and $r^\delta \widehat{H}^{k+2}$. G depends on δ and k, $G = G(\delta, k)$, and does not exist (or is not bounded) when $\delta \in \Lambda$, Λ a discrete set described below. We then show that each $G(\delta, k)$ is bounded on the Hölder spaces

$$G: C^{k,\alpha,\nu} \to C^{k+2,\alpha,\nu}, \tag{4.8}$$

where ν depends on δ. Finally we show that the error E is negligible on these spaces, and so we may use G as a boundary parametrix for $\Delta+\lambda(x)$ on M. In conjunction with an interior parametrix, this shows that

$$\begin{aligned} &\Delta + \lambda(x): r^{\delta}\widehat{H}^{k+2} \to r^{\delta}\widehat{H}^{k}, \\ &\Delta + \lambda(x): C^{k+2,\alpha,\nu} \to C^{k,\alpha,\nu} \end{aligned} \tag{4.9}$$

has closed range for all but a discrete set of the values δ, ν. Then it may be determined for which of these values this map is surjective, Fredholm, etc.

Introduce eigenfunction expansions in (θ, y). If

$$\Delta_y \varphi_j = -\mu_j^2 \varphi_j, \qquad \Delta_\theta \psi_l = -\lambda_l^2 \psi_l, \tag{4.10}$$

then on the (j, l) eigenspace, L is reduced by the Fuchsian operator

$$L_{j,l} = r^2\partial_r^2 + (1-k)r\partial_r - r^2\mu_j^2 - \lambda_l^2 + \lambda.$$

As recorded earlier, its indicial roots are arranged in two families:

$$s_1(l) = \frac{k}{2} + \sqrt{\frac{k^2}{4} + \lambda_l^2 - \lambda}, \qquad s_2(l) = \frac{k}{2} - \sqrt{\frac{k^2}{4} + \lambda_l^2 - \lambda}. \tag{4.11}$$

The importance of these exponents is that it may be proved that an arbitrary (locally defined) temperate solution of $(\Delta+\lambda)u = 0$ has an asymptotic expansion as $r \to 0$ with terms $r^{s_i(l)}(\log r)^p$ and distributional coefficients in (θ, y). This is proved in the extreme cases $k = 0$ in [25] and $k = n-1$ in [18], and the general case appears in [19].

However, it is precisely the difficulty of establishing mapping properties for $L_{j,l}$ which are uniform in both j, l that makes the analysis of L subtle. This approach is used in [4] and [15], but the central difficulty is that one requires knowledge of the asymptotics of Bessel functions of large argument and order, and this, as Watson puts it, is "a problem of a more recondite nature" [31]. Instead we consider the reduction of L only in the variable y. Thus, let

$$L_j = r^2\partial_r^2 + (1-k)r\partial_r + \Delta_\theta - r^2\mu_j^2 + \lambda. \tag{4.12}$$

This may be rescaled to a j-independent operator on the cylinder $C = \mathbf{R}^+ \times S^{n-k-1}$ by the substitution $R = r\mu_j$:

$$L_0 = R^2\partial_R^2 + (1-k)R\partial_R + \Delta_\theta - R^2 + \lambda. \tag{4.13}$$

We first construct an inverse for L_0, rescale to obtain an inverse for L_j, and then sum to obtain an inverse for L.

The operator L_0 is of totally characteristic type [23], but the presence of $-R^2$, as in a Bessel equation of imaginary argument, drastically affects its global behavior on C. We couple its known behavior near 0 from [25] with an obvious extension of an ODE argument which appears in [24] to describe this global behavior. Introduce the space

$$(4.14)\quad \mathscr{H}^{k,\delta,p} = \{u \colon \varphi(R)u \in R^{-\delta}H_b^k(C),\ (1-\varphi(R))u \in R^{-p}H^k(C)\},$$

where $\varphi \in C_0^\infty(\overline{\mathbf{R}_+})$, $\varphi = 1$ near $R = 0$,

$$(4.15)\qquad H_b^k(C) = \{u \colon (R\partial_R)^j\partial_\theta^\alpha u \in L^2(R^{-k-1}\,dR\,d\theta),\ j + |\alpha| \leq k\},$$

and $H^k(C)$ is the usual Sobolev space based on differentiations by ∂_R, ∂_θ and with respect to the measure $R^{-k-1}\,dR\,d\theta$.

(4.16) **Lemma.** $L_0 \colon \mathscr{H}^{k+2,\delta,p} \to \mathscr{H}^{k,\delta,p-2}$ *is Fredholm provided* $\delta \notin \Lambda$, *where* $\Lambda = \{\pm\Re\sqrt{k^2/4 + \lambda_l^2 - \lambda}\}$.

Remark. If $\delta \in \Lambda$, then $R^{s_i(l)}$ just fails to lie in $R^{-\delta}L^2(R^{-k-1}\,dR\,d\theta)$ near $R = 0$ for some $s_i(l)$ in the list (4.10). It is easy to prove that L_0 does not have closed range then.

Proof. L_0 is bounded between these spaces regardless of the value of δ. So it will suffice to construct parametrices H_1, H_2 for which $H_i \colon \mathscr{H}^{k,\delta,p+2} \to \mathscr{H}^{k+2,\delta,p}$ is bounded and such that both errors $Q_1 = I - L_0H_1$ and $Q_2 = I - H_2L_0$ are compact on $\mathscr{H}^{k,\delta,p}$ for every k, p. The H_i are obtained by joining together local parametrices near $R = 0$ and $R = \infty$. The existence of local parametrices in any bounded neighborhood of 0 is the principal result of [25] (cf. also [24]), and this step requires that $\delta \notin \Lambda$. We note however that the construction of such local parametrices is fairly easy on a "true" cone such as C, i.e., one which is a warped product. For example, in this situation it is now feasible to introduce a Fourier decomposition in θ and construct a parametrix for the family of ODE's which are induced on the eigenspaces. The reason this works is that the homogeneous solutions to these ordinary differential equations are modified Bessel functions in which the spectral parameter appears only in the order but not in the argument, so that uniform asymptotics are easily obtained. Constructions of this type appear in [5]. More sophisicated arguments using the Mellin transform in [25] seem to be required to prove the refined regularity results for totally characteristic operators which we need later.

For the other case we slightly modify an argument from [24]. Consider the partial principal symbol of L_0 in R:

$$\sigma(L_0) = \Delta_\theta - R^2(|\xi|^2 + 1). \tag{4.17}$$

As an operator on $L^2(S^{n-k-1})$, $\sigma(L_0) \leq -R^2$, hence $\|\sigma(L_0)^{-1}\| \leq 1/R^2$. So we define a parametrix for L_0 near infinity by

$$H_\infty(f) = \int e^{iR\xi} \sigma(L_0)^{-1} \hat{f}(\xi, \theta)\, d\xi. \tag{4.18}$$

Clearly

$$(1 - \varphi(R))H_\infty \colon \mathscr{H}^{k,\delta,p-2} \to \mathscr{H}^{k+2,\delta,p}$$

is bounded, and furthermore

$$K = L_0(1-\varphi)H_\infty - I \colon \mathscr{H}^{k,\delta,p} \to \mathscr{H}^{k,\delta,p}$$

is compact for any k, p. This last assertion rests on the fact that K is a finite sum of terms, each of which is both smoothing of order at least one and decaying at least like $1/R$. Thus, $K \colon \mathscr{H}^{k,\delta,p} \to \mathscr{H}^{k+1,\delta,p+1} \hookrightarrow \mathscr{H}^{k,\delta,p}$, and the latter inclusion is compact. The same construction gives a left parametrix. Finally, cut-off functions may be used in the usual way to combine these local parametrices to get a global parametrix on all of C. But the existence of bounded left and right parametrices with compact error terms implies that L_0 is Fredholm.

(4.19) **Corollary.** *If $u \in \mathscr{H}^{k+2,\delta,p_0}$ for some $k, p_0, \delta \notin \Lambda$, and $L_0 u = f$, where $f \in \mathscr{H}^{k,\delta,p}$ for every p, then $u \in \mathscr{H}^{k+2,\delta,p}$ for every p. If $f \in \mathscr{H}^{k,\delta,p_0}$ for every k, then $u \in \mathscr{H}^{k,\delta,p_0+2}$ for every k.*

Proof. Apply the parametrix H_2 to $L_0 u = f$ to conclude that

$$u = Q_2 u + H_2 f.$$

Since $Q_2 \colon \mathscr{H}^{k,\delta,p} \to \mathscr{H}^{k+1,\delta,p+1}$ and $H_2 \colon \mathscr{H}^{k+2,\delta,p+2}$, the result follows.

Because L_0 is Fredholm independently of p, the index of L_0 is independent of p. Hence by this corollary, since the kernel of L_0 is independent of p, so is the cokernel. (This may also be seen by using the argument of the corollary for the adjoint problem.) However, the index does change as δ crosses over elements of the excluded set Λ.

Standard arguments now imply the existence of a generalized inverse G_0 for L_0, i.e., an operator satisfying

$$G_0 L_0 = I - P_1, \qquad L_0 G_0 = I - P_2 \tag{4.20}$$

between any particular choice of the weighted Sobolev spaces $\mathscr{H}^{k,\delta,p-2} \to \mathscr{H}^{k+2,\delta,p}$, $\delta \notin \Lambda$. Here P_1 and P_2 are the orthogonal projections in $\mathscr{H}^{k+2,\delta,p}$ and $\mathscr{H}^{k,\delta,p-2}$ onto the kernel and cokernel of L_0. Note that although G_0 does not depend on k, it does depend on p, if only through the spaces on which the P_i are orthogonal projectors, and of course it also depends on δ. We shall regard G_0 as an integral kernel:

$$G_0 f(R, \theta) = \int G_0(R, \theta, \widetilde{R}, \widetilde{\theta}) f(\widetilde{R}, \widetilde{\theta}) \widetilde{R}^{-k-1}\, d\widetilde{R}\, d\widetilde{\theta}, \tag{4.21}$$

with similar expressions for P_1 and P_2. Note that the identity I is represented by the Schwartz kernel $R^{k+1}\delta(R - \widetilde{R})\delta(\theta - \widetilde{\theta})$, the factor R^{k+1} being included to compensate for the $\widetilde{R}^{-k-1}$ in the measure. Because G_0 is an inverse for the totally characteristic operator L_0, it has very regular behavior as either $R, \widetilde{R} \to 0$. This structure is described in great detail in [23], [24], [25] (cf. also [9, §18.3]. Thus, we merely quote results which we shall require later.

The most important property of the distribution G_0 on $C^2 = C \times C$ is that it is the pushforward of a simpler distribution on a slightly more complicated manifold. This manifold is the manifold with corners C_b^2 obtained by "blowing up" C^2 along its corner $\{R = \widetilde{R} = 0\}$. Invariantly, this process consists of replacing this corner by its interior normal bundle, which is a quarter circle bundle over $\partial C \times \partial C$, and endowing this set with the unique minimal differentiable structure for which polar coordinates in the variables $R, \widetilde{R}$ are smooth. C_b^2 is again a manifold with three hypersurface boundary faces: the lifts of the faces defined by $R = 0$ and $\widetilde{R} = 0$ on C^2 and the new front face (the spherical normal bundle). Let the lifts of $\{R = 0\}$, $\{\widetilde{R} = 0\}$ be called the left and right faces and have defining functions ρ_1, ρ_2, and let the front face have defining function ρ_{ff}. C_b^2 also has two corners (codimension two boundary faces) defined as the intersections of the left and front faces, and of the right and front faces. We shall encounter a similar construction for the full parametrix later when the variable y is reintroduced.

It is proved in [25] and [24] that the local parametrices H_1 and H_2 for L_0 near $\widetilde{R} = 0$ used earlier may be described as the pushforward under the natural "blow-down" map $C_b^2 \to C^2$ of distributions, which we also call H_i, which have classical conormal expansions at the lift of the diagonal $\{R = \widetilde{R}, \theta = \widetilde{\theta}\}$ to C_b^2 and along the boundary faces $\{\rho_1 = 0\}$, $\{\rho_2 = 0\}$. They are smooth up to the front face away from the diagonal and are extendible across the front face near the diagonal. For information about distributions with classical conormal expansions, see [9, §18.2]; these are often also called polyhomogeneous distributions, and their expansions are called polyhomogeneous. We shall frequently use these distributions here. Near a boundary with defining function ρ these expansions have the form

$$\sum_{\mathscr{R}(s_j)\to\infty}\sum_{p=0}^{P(s_j)} f_{jp}(z)\rho^{s_j}(\log\rho)^p,$$

where z is a coordinate chart in the boundary, and all coefficient functions are smooth. We shall denote by $\mathscr{A}_{phg}^{\nu,P(s_0)}$ the subclass where the exponent s_0 of lowest order equals ν. We shall also use the notation $\mathscr{A}_{phg}^{\nu}=\mathscr{A}_{phg}^{\nu,0}$ to denote the space of such distibutions with no log factors in the term of lowest order. The most important consequence of the fact that the inverting kernels live in these spaces is that they have definite rates of decay at all boundary faces. We shall deduce this structure for all the operators with which we are concerned here.

Before we proceed further, it will be useful to have the explicit formulas for the transposes L_0^t, G_0^t of the operators L_0, G_0. Of course, the transpose is taken in a particular one of the weighted L^2 spaces. Rather than determine the transpose on an arbitrary one of these spaces, we shall restrict attention to only those spaces $\mathscr{H}^{k,\delta,p}$ for which $p=\delta-2k$ for $k\in\mathbf{N}$. In particular, our fixed L^2 space will be $\mathscr{H}^{0,\delta,\delta}=R^{-\delta}L^2(R^{-k-1}\,dR\,d\theta)$. Because L_0 is symmetric when $\delta=0$, the adjoint of L_0 in $R^{-\delta}L^2$ is

$$L_0^t=R^{-2\delta}L_0R^{2\delta}. \tag{4.22}$$

On the other hand, using the expression (4.21), the adjoint of G_0 on this space is

$$G_0^t(R,\theta,\widetilde{R},\widetilde{\theta})=R^{-2\delta}G_0(\widetilde{R},\widetilde{\theta},R,\theta)\widetilde{R}^{2\delta}. \tag{4.23}$$

Of course, P_1, P_2 are orthogonal projections in $R^{-\delta}L^2$, so that

$$P_i^t(R,\theta,\widetilde{R},\widetilde{\theta})=P_i(R,\theta,\widetilde{R},\widetilde{\theta}),\qquad i=1,2. \tag{4.24}$$

Now we turn to the determination of the exact behavior of G_0 at infinity and at the boundary faces.

(4.25) **Lemma.** *$P_i(tR,\theta,t\widetilde{R},\widetilde{\theta})$ and $G_0(tR,\theta,t\widetilde{R},\widetilde{\theta})$ are rapidly decreasing as $t\to\infty$ locally uniformly in $(R,\theta,\widetilde{R},\widetilde{\theta})$; for G_0 this is only true if $(R,\theta)\neq(\widetilde{R},\widetilde{\theta})$.*

Proof. Since the P_i are finite rank projections onto the kernel and cokernel of L_0, their rapid decrease is guaranteed by Corollary (4.19) applied to both L_0 and L_0^t. To deal with G_0, fix $R \neq \widetilde{R}$ and consider the function $F(t, \theta, \widetilde{\theta}) = G_0(tR, \theta, t\widetilde{R}, \widetilde{\theta})$. Then, since $L_0^t G_0^t = \widetilde{R}^{-2\delta} L_0 G_0(\widetilde{R}j, \widetilde{\theta}) R^{2\delta} = I - P_2$ so that

$$L_{\widetilde{R}\widetilde{\theta}} G_0(R, \theta, \widetilde{R}, \widetilde{\theta}) = I - R^{2\delta} P_2(R, \theta, \widetilde{R}, \widetilde{\theta}) R^{-2\delta}, \tag{4.26}$$

where $L_{\widetilde{R}\theta}$, $L_{\widetilde{R}\widetilde{\theta}}$ represents L_0 acting in either (R, θ) or $(\widetilde{R}, \widetilde{\theta})$, it follows that

$$\begin{aligned}
(t^2\partial_t^2 &+ (1-k)t\partial_t + \Delta_\theta + \Delta_{\widetilde{\theta}} + 2\lambda - t^2R^2 - t^2\widetilde{R}^2)F(t, \theta, \widetilde{\theta}) \\
&= (L_{R\theta}G_0)(tR, \theta, t\widetilde{R}, \widetilde{\theta}) + (L_{\widetilde{R}\widetilde{\theta}}G_0)(tR, \theta, t\widetilde{R}, \widetilde{\theta}) \\
&= -P_1(tR, \theta, t\widetilde{R}, \widetilde{\theta}) - R^{2\delta}\widetilde{R}^{-2\delta}P_2(tR, \theta, t\widetilde{R}, \widetilde{\theta});
\end{aligned}$$

the delta functions, which represent the identity, are zero because $(R, \theta) \neq (\widetilde{R}, \widetilde{\theta})$. But for fixed R, $\widetilde{R}$ this is an equation of the form $L'F = G$, with $G \in \mathscr{H}^{k,\delta,M}$ for every k, M. The proof of Lemma (4.16), hence also of Corollary (4.19), work perfectly well for the operator L' and so F is rapidly decreasing in t.

(4.27) **Lemma.** *$P_i(R, \theta, \widetilde{R}, \widetilde{\theta})$ and $G_0(R, \theta, \widetilde{R}, \widetilde{\theta})$ are rapidly decreasing as $R \to \infty$ locally uniformly in $(\theta, \widetilde{R}, \widetilde{\theta})$, $\widetilde{R} \geq 0$, and as $\widetilde{R} \to \infty$ locally uniformly in $(R, \theta, \widetilde{\theta})$, $R \geq 0$. Furthermore, the coefficient functions in the asymptotic expansions for the P_i and G_0 as $R \to 0$ are rapidly decreasing as $\widetilde{R} \to \infty$, and vice versa.*

Proof. Both assertions are obvious for the P_i from Corollary (4.19) and, using this, the proof for G_0 is very similar to that in the previous lemma.

We shall wait to determine the leading exponents in these asymptotic expansions.

The next step is to rescale G_0 back to a generalized inverse for L_j, and thus obtain the full inverse for L. Recall that L_j is obtained from L_0 by the substitution $R = r\mu_j$. Since from (4.20) and (4.21)

$$L_0G_0 = R^{k+1}\delta(R - \widetilde{R})\delta(\theta - \widetilde{\theta}) - P_1(R, \theta, \widetilde{R}, \widetilde{\theta}),$$

this rescaling transforms this equation into

$$L_jG_0(r\mu_j, \theta, \tilde{r}\mu_j, \widetilde{\theta}) = \mu_j^k r^{k+1}\delta(r - \tilde{r})\delta(\theta - \widetilde{\theta}) - P_1(r\mu_j, \theta, \tilde{r}\mu_j, \widetilde{\theta});$$

here we have used that $\delta(R-\widetilde{R})$ is homogeneous of degree -1. This forces the definition

$$G_j(r,\theta,\tilde{r},\widetilde{\theta})=\mu_j^{-k}G_0(r\mu_j,\theta,\tilde{r}\mu_j,\widetilde{\theta}),\tag{4.28}$$

so that G_j satisfies

$$\begin{aligned}L_jG_j&=r^{k+1}\delta(r-\tilde{r})\delta(\theta-\tilde{\theta})-P_{1j}(r,\theta,\tilde{r},\tilde{\theta}),\\ G_jL_j&=r^{k+1}\delta(r-\tilde{r})\delta(\theta-\tilde{\theta})-P_{2j}(r,\theta,\tilde{r},\tilde{\theta}),\end{aligned}\tag{4.29}$$

where

$$P_{ij}(r,\theta,\tilde{r},\tilde{\theta})=\mu_j^{-k}P_i(r\mu_j,\theta,\tilde{r}\mu_j,\tilde{\theta}),\qquad i=1,2.$$

Finally, we define the full inverse for L to be

$$G(r,\theta,y,\tilde{r},\theta,\tilde{y})=\sum G_j(r,\theta,\tilde{r},\tilde{\theta})\varphi_j(y)\varphi_j(\tilde{y}),\tag{4.30}$$

with similar definitions for the projectors $P_i(r,\theta,y,\tilde{r},\tilde{\theta},\tilde{y})$; in other words $G=(-\Delta_y)^{-k}\times G_0(\sqrt{-\Delta_y}r,\theta,\sqrt{-\Delta_y}\tilde{r},\tilde{\theta})$. It remains to prove that G is actually bounded between the weighted Sobolev spaces (4.4).

(4.31) **Proposition.** *$G: r^{\delta}L^2\to r^{\delta}\widehat{H}^2$ is bounded provided $\delta\notin\Lambda$.*

Proof. It suffices to prove that G_j is bounded between $r^{\delta}L^2(r^{-k-1}dr\,d\theta)$ and $\{u: ur\partial_r u,\ r^2\partial_r^2u,\ \partial_\theta u,\ \partial_\theta^2u,\ r\mu_ju,\ r^2\mu_j^2u\in r^{-\delta}L^2\}\equiv r^{-\delta}H_j^2$ uniformly in j. It is easy to check that the rescaling transformation

$$f(R,\theta)\to\tilde{f}(r,\theta)=\mu_j^{\delta-k/2}f(\mu_jr,\theta)$$

is an isometry between $R^{-\delta}L^2$ and $r^{-\delta}L^2$. Then, a change of variables in (4.20) shows that $G_j\tilde{f}=\widetilde{G_0f}$, so with all norms in either $R^{-\delta}L^2$ or $r^{-\delta}L^2$:

$$\|G_j\tilde{f}\|=\|\widetilde{G_0f}\|=\|G_0f\|\le C\|f\|=C\|\tilde{f}\|,\tag{4.32}$$

with C the norm of $G_0: R^{-\delta}L^2\to R^{-\delta}L^2$, hence independent of j. The same argument can be applied to $r\partial_rG_j$, $r^2\partial_r^2G_j$, $\partial_\theta^2G_j$, $r\mu_j^2G_j$, $r^2\mu_j^2G_j$, since all of these operators are simply the rescalings of $R\partial_RG_0,\cdots,R^2G_0$, all of which are bounded on $R^{-\delta}L^2$ since $G_0:\mathscr{H}^{0,\delta,\delta}\to\mathscr{H}^{2,\delta,\delta-2}$. This sort of argument is taken from [16]. q.e.d.

An identical argument shows that P_1, P_2 are also bounded on $r^{\delta}L^2$; in fact, using Corollary (4.19), $P_i: r^{\delta}L^2\to r^{\delta}\widehat{H}^{\infty}$. In addition, by a change of variables, $P_{ij}\circ P_{ij}=P_{ij}$, $i=1,2$ for all j, so that P_1, P_2 are still projectors. Notice also that unless $P_i(R,\theta,\widetilde{R},\widetilde{\theta})\equiv0$, $P_i(r,\theta,y,\tilde{r},\tilde{\theta},\tilde{y})$ has infinite rank.

We have now proved that

$$(4.33) \qquad LG = I - P_1, \qquad GL = I - P_2.$$

By commuting derivatives through the argument of (4.31) we obtain that

$$(4.34) \quad L: r^\delta \widehat{H}^{k+2} \to r^\delta \widehat{H}^k, \quad G: r^\delta \widehat{H}^k \to r^\delta \widehat{H}^{k+2}, \quad P_i: r^\delta \widehat{H}^k \to r^\delta \widehat{H}^{k+l}$$

are all bounded for every k, l. In particular, L has closed range whenever $\delta \notin \Lambda$.

The distributions G, P_i naturally live on a "blown up" version of the product $T \times T$, where $T = C \times N$. This is defined in a manner analogous to that used earlier in defining M and C_b^2. The product T^2 is a manifold with a corner, and its corner has a natural submanifold which is defined by $S = \{r = \tilde{r} = 0, y = \tilde{y}\}$. We define T_π^2 to be the union of $T \backslash S$ and the spherical normal bundle of S in T^2, endowed with the minimal differential structure for which the lifts of smooth functions on T^2 and the polar coordinate functions about S are smooth. In fact it is convenient to use these polar coordinates from time to time. Thus, let

$$\rho_{ff} = \sqrt{r^2 + \tilde{r}^2 + |y - \tilde{y}|^2}, \qquad \omega = (r, y - \tilde{y}, \tilde{r})/\rho_{ff} = (\omega_0, \omega', \omega_{k+1}).$$

A full set of coordinates near S are $(\rho_{ff}, \omega, \tilde{y}, \theta, \tilde{\theta})$. T_π^2 has three bounding hypersurfaces which are defined by $\{\omega_0 = 0\}$, $\{\omega_{k+1} = 0\}$, and $\{\rho_{ff} = 0\}$; in analogy with C_b^2 we shall let $\rho_1 = \omega_0$ and $\rho_2 = \omega_{k+1}$. Because of the rapid decrease of G_0, P_i proved in Lemma (4.25), the distributions G, P_i are smooth in all variables in the interior of T_π^2 (although G still has a singularity along the diagonal) because of the rapid convergence of the sum (4.30). Also, G and the P_i still have asymptotic expansions as $\rho_1 \to 0$ and $\rho_2 \to 0$ because of Lemma (4.27). There are more general arguments which would guarantee the existence of these asymptotic expansions [24], but the existence of these expansions is not subtle in the present circumstances. Because we know the boundedness properties (4.34) we can determine the exponents in the leading terms of these expansions.

(4.35) **Proposition.** *P_1, P_2, and G are polyhomogeneous on all faces of T_π^2. These distributions are smooth in the normal variable ρ_{ff} to the front face and P_1, P_2 are smooth in all variables there. At the side faces these distributions have polyhomogeneous expansions, which may be described as follows: $P_1 \in \mathscr{A}^s_{phg}$ at both side faces with leading coefficient an eigenfunction for Δ_θ, or $\Delta_{\tilde{\theta}}$, and with s one of the indicial roots in the list (4.10), $\Re(s) > \frac{k}{2} + \delta$; $P_2 \in \mathscr{A}^{s-2\delta}_{phg}$ with leading coefficient again an eigenfunction* *in* θ, $\tilde{\theta}$ *and with* $\Re(s) > \frac{k}{2} + 3\delta$, s *an indicial root; on the left face either* $G \in \mathcal{A}^{s}_{phg}$, $\Re(s) > \frac{k}{2} - \delta$ *or* $G \in \mathcal{A}^{s,1}_{phg}$, $\Re(s) > \frac{k}{2} + \delta$, *and on the right face either* $G \in \mathcal{A}^{s}_{phg}$, $\Re(s) > \frac{k}{2} + \delta$ *or* $G \in \mathcal{A}^{s,1}_{phg}$, $\Re(s) > \frac{k}{2} + 3\delta$; *in each of these cases the leading coefficient is an eigenfunction in either* θ, $\tilde{\theta}$ *and* s *is an indicial root. Here we have used the variables* r, $\tilde{r}$ *instead of* ρ_1, ρ_2 *as defining functions for these side faces for simplicity.*

Proof. The existence of these expansions and the smoothness of the coefficients was commented on earlier. The leading term in any one of these expansions solves the relevant equation to first order, so the coefficient of either r or $\tilde{r}$ is an eigenfunction in either θ or $\tilde{\theta}$ with the exponent related to that eigenvalue. It remains only to determine the precise values this leading exponent may take. First of all, since P_1, P_2 are projectors on $r^{\delta}L^2$ their integral kernels are symmetric; hence for them it suffices to determine their leading terms only on the left face. Since $LP_1 = 0$ and $L^t P_2 = 0$, the leading exponent in each of these operators must be an indicial root for L or $L^t = r^{-2\delta} L r^{2\delta}$, respectively. Hence for P_1 the leading exponent is some indicial root in the list (4.10), and for P_2 it is $s - 2\delta$, s as in (4.10). But now we must use their boundedness on $r^{\delta}L^2$: it is proved in [16] (see also [17], [24]), that in the case where there are no θ-variables, an operator of this type which is in $\mathcal{A}^{a,p_l}_{phg}$ at the left face and $\mathcal{A}^{b,p_r}_{phg}$ at the right face are bounded on $r^{\delta}L^2$ provided

$$\text{(4.36)} \qquad \Re(a) + \delta > \frac{k}{2}, \quad \Re(b) - \delta > \frac{k}{2}, \quad \Re(a+b) > k.$$

The presence of the additional θ variables does not alter this criterion. Applying this to P_1 we see that the indicial root must satisfy $\Re(s) > \frac{k}{2} + \delta$, and for P_2, $\Re(t) = \Re(s) - 2\delta > \frac{k}{2} + \delta \Rightarrow \Re(s) > \frac{k}{2} + 3\delta$. Then, since by (4.33) $L_{r\theta y} G = -P_1$ and $L_{\tilde{r}\tilde{\theta}\tilde{y}} G = -r^{-2\delta}\tilde{r}^{2\delta} P_2$ near the two side faces, either the leading terms of G at these faces must be annihilated by L, in which case the leading exponent is an indicial root and there are no logarithms in the terms of lowest order or L applied to the leading term must match the leading term of either $-P_1$ or $-r^{-2\delta}\tilde{r}^{2\delta}P_2$, so that the leading exponent must be the same and the lowest order terms for G would require a logarithmic factor. Finally, using (4.36) again gives lower bounds on these leading exponents. This proves the proposition in all cases.

We come now to the Hölder estimates for L. First we prove an a priori estimate for an arbitrary solution of $Lu = f$. In this the $C^{k+2,\alpha,\nu}$ norm of u is estimated in terms of the $C^{k,\alpha,\nu}$ norm of Lu and, for example, the $C^{0,0,\nu}$ norm of u. This estimate is quite simple and uses only the scale-invariant nature of the Hölder spaces (2.5) and the scale-invariance of L. Such estimates appear also in [7]. We should note that it is impossible to prove an a priori estimate which would imply that for a fixed f an arbitrary solution u to $Lu = f$ decays at the same rate as f. In fact, it is possible to find solutions to $Lu = 0$ for which u has arbitrarily bad growth in some thin set which converges to the boundary. However, from the theory developed earlier in this section, if $u \in r^\delta L^2$ and $Lu = 0$ then $u \in r^\delta \widehat{H}^k$ for any k. A scaling argument involving the Sobolev embedding theorem now shows that $u \in C^{k,\alpha,\delta}$ for any k. The second step in dealing with the properties of L on Hölder spaces is to decide for which $f \in C^{k,\alpha,\nu}$ there exists some solution to $Lu = f$ for which $u \in C^{k+2,\alpha,\nu}$. Of course the natural choice for a solution were it to exist is the solution $u = Gf$, where G is the generalized inverse of L with respect to a particular δ. However, first we prove the a priori estimate.

(4.37) **Proposition.** *Given any k, α, ν if $Lu = f$, where $u \in C^{k+2,\alpha,\nu}$, so that $f \in C^{k,\alpha,\nu}$, then*

$$\|u\|_{k+2,\alpha,\nu} \leq C(\|f\|_{k,\alpha,\nu} + \|u\|_{0,0,\nu}). \tag{4.38}$$

Proof. Use the usual local coordinates (r, θ, y). Partition T up into boxes B_i which have the property that they approximately equal the sets $\{d_i/2 \leq r \leq 2d_i,\ |y - \tilde{y}_0| \leq d_i\}$, and choose slightly larger boxes $E_i \supset B_i$ which have dimensions in r and y larger than B_i by a fixed factor $1 + \varepsilon$. For each i choose an affine map f_i (relative to these coordinates) which carries a fixed box $E = \{\frac{1}{3} \leq r \leq 3, |y| \leq 2\}$, say, to E_i. Now it is easy to calculate that $f_i^* Lu = L f_i^* u$ for any u, and furthermore if u is a function on E_i then

$$\delta_i^\nu |f_i^* u|_{k,\alpha} = |u|_{k,\alpha,\nu}. \tag{4.39}$$

Use the notation $u_i = u|_{E_i}$, and $v_i = u|_{B_i}$. Now by standard elliptic theory

$$\begin{aligned}
|u|_{k+2,\alpha,\nu} &= \sup_i |u_i|_{k+2,\alpha,\nu} \\
&= \sup_i d_i^\nu |f_i^* u_i|_{k+2,\alpha} \\
&\leq C \sup_i (d_i^\nu L v_i|_{k,\alpha} + d_i^\nu |f_i^* v_i|_{0,0}) \\
&\leq C(|Lu|_{k,\alpha,\nu} + |u|_{0,0,\nu}),
\end{aligned} \tag{4.40}$$

which is the desired inequality.

To accomplish the second step in the analysis of the behavior of L on Hölder spaces, as discussed earlier, it will suffice to prove that the generalized inverse G and the projectors P_i are bounded on these spaces.

(4.41) **Proposition.** *For any $\delta \notin \Lambda$, the corresponding operators G, P_i are bounded between $C^{k,\alpha,\nu} \to C^{k+2,\alpha,\nu}$ and $C^{k,\alpha,\nu} \to C^{l,\alpha,\nu}$ for any k, α, l, and for $\nu = \frac{k}{2} - \delta$. Thus, the set Λ' of omitted weights is precisely the set of real parts of the indicial roots* (4.10).

Proof. We give a proof only for G, the proof for the P_i being even simpler. First write $G = G' + G''$, where G' is supported in the set

$$\{\tfrac{1}{2} \le r/\tilde{r} \le 2, |y - \tilde{y}| \le \rho_{ff}\}, \tag{4.42}$$

hence contains the singularities of G along the diagonal, and G'' is smooth on the interior of T^2_π and carries the conormal singularities of G along the sides given by (4.35). The proof that G' has the correct boundedness properties is very similar to the argument in the previous proposition.

To see this we must first represent the action of $G'u$ as an integral. For this it is most convenient to use projective coordinates $(r, \theta, y, t, \tilde{\theta}, w)$ on T^2_π, where $t = \tilde{r}/r$ and $w = (y - \tilde{y})/r$. It is easy to see that these are smooth in the interior of T^2_π and up to the front face and the side face $\rho_1 = 0$, but are singular at the other side face. The fact that G' is the pushforward of a distribution on T^2_π means that there exists a distribution $g'(r, \theta, y, t, \tilde{\theta}, w)$ such that g' is supported in

$$\{\tfrac{1}{2} \le t \le 2, |w| \le 1\} \tag{4.43}$$

and

$$G'u = \int g'(r, \theta, y, t, \tilde{\theta}, w)u(tr, y - wr, \tilde{\theta})t^{-k-1}\,dt\,dw. \tag{4.44}$$

In this formula we have used the fact that $\tilde{r} = tr$ and $\tilde{y} = y - rw$. Now, use the same partitions of T into boxes E_i and B_i and the same affine maps f_i as in Proposition (4.37). The same reasoning as in (4.40) may be used. Thus

$$\begin{aligned} |G'u|_{k,\alpha,\nu} &= \sup_i |G'u|_{k,\alpha,\nu,E_i} = \sup_i d_i^\nu |f_i^*(G'u)|_{k,\alpha} \\ &= \sup_i d_i^\nu |(f_i^*G')(f_i^*u)|_{k,\alpha} \\ &\le \sup_i C_i d_i^\nu |f_i^*u|_{k,\alpha,B} \le \sup_i C_i |u|_{k,\alpha,\nu,B_i}. \end{aligned} \tag{4.45}$$

The first inequality holds because each f_i^*G' is a pseudodifferential operator of order -2, hence is bounded between $C^{k,\alpha} \to C^{k+2,\alpha}$ on the fixed box E. The desired estimate will hold if we show that the C_i are independent of i. But this is true because the family of pseudodifferential operators $f_i^* G'$ on E is uniformly bounded in $\Psi^{-2}(E)$, which in turn is a consequence of the smoothness of G in the parameter ρ_{ff}.

For G'' we argue in two steps. First we need to show that $G''u$ is defined for any $u \in \mathscr{C}^{k,\alpha,\nu}$. Using the integral representation (4.44) it is clear that we only need check that if $u = r^\nu$ then the integration there is well defined. Since the kernel of $G \sim \tilde{r}^s$ as $\tilde{r} \to 0$ (i.e., on approach to the face $\{\rho_2 = 0\}$), with s an indicial root, $s > \frac{k}{2} + \delta$ by (4.35), i.e., $G'' \sim t^s$ as $t \to 0$, we need to check that $\int_0^1 t^{s+\nu-k-1}\, dt < \infty$. But $s + \nu - k > 0$, so this is true.

The second step is to prove that if $u \in C^{k,\alpha,\nu}$, then $G''u \in C^{k+2,\alpha,\nu}$. First note that it will suffice to merely prove that $G''u \in C^{0,0,\nu}$, because in fact any number of the derivatives $r\partial_r$, $r\partial_y$, ∂_θ, etc. may be applied to G'' and the resulting operator will always have the same form as G'' with the same asymptotics at all boundaries. Thus, we will actually have $G''u \in C^{l,\beta,\nu}$ for any l, β, ν. Now $|\int G''u| \le \int |G''||u|$. Since $|G''|$ is either itself polyhomogeneous (which is the case if G'' does not change signs near the boundary) or in any case is always dominated by a new kernel $\overline{G}$ which is polyhomogeneous with the same leading exponent, and since $|u| \le Cr^\nu$, it is clearly enough to show that $|\int \overline{G} r^\nu| \le Cr^\nu$. For this we appeal to a general proposition in the appendix to [5]. This states that if u is polyhomogeneous and if the integration in $\int \overline{G}u$ is defined, then $\int \overline{G}u$ is polyhomogeneous. Furthermore, the exponent in its leading term may be computed from the leading exponent in the expansion for u and the leading exponents in the expansions for $\overline{G}$ at the faces $\{\rho_1 = 0\}$ and $\{\rho_{ff}\} = 0$. Rather than state this proposition in full generality, we shall just record its implications in the present circumstances. Here $\overline{G} \in \mathscr{A}^s_{phg}$ with $\Re(s) > \frac{k}{2} - \delta = \nu$ on the side face, $\overline{G} \in \mathscr{A}^0_{phg}$ on the front face, and $u = r^\nu$, so this proposition implies that $\int \overline{G}\tilde{r}^\nu \in \mathscr{A}^{\min(\nu,s)}_{phg} = \mathscr{A}^\nu_{phg}$. We remark that if $\nu \in \Lambda'$, then it might occur that $\nu = s$ and this proposition would then imply that $\int \overline{G}\tilde{r}^\nu \in \mathscr{A}^{\nu,1}_{phg}$. This would not be enough to guarantee Hölder boundedness. We have proved that $G''\colon C^{k,\alpha,\nu} \to C^{0,0,\nu}$ is bounded.

Throughout this proof we have been only thinking of the behavior of G near zero, but this boundedness statement also incorporates behavior near ∞. From (4.24), (4.26), and (4.30) it is clear that Gu decreases exponentially as $r \to \infty$. Thus there are no problems here.

Finally, putting these steps together we arrive at the conclusion G: $C^{k,\alpha,\nu} \to C^{k+2,\alpha,\nu}$. This completes the argument.

(4.47) **Corollary.** *If $\nu \notin \Lambda'$ and $f \in \mathscr{A}^{\nu}_{phg}$ is in the range of L on $C^{2,\alpha,\nu}$, then there exists a solution to $Lu = f$, where $u \in \mathscr{A}^{\nu}_{phg}$.*

(4.48) **Corollary.** *If $\delta \notin \Lambda$ and $\nu \notin \Lambda'$, then since P_1 and P_2 are bounded on both the Sobolev and Hölder spaces, there are closed subspaces V, V_1, W, W_1 for which*

$$
\begin{aligned}
r^{\delta}\widehat{H}^{k+2} &= V \oplus (\ker L \cap r^{\delta}\widehat{H}^{k+2}), \\
r^{\delta}\widehat{H}^{k} &= V_1 \oplus (\operatorname{coker} L \cap r^{\delta}\widehat{H}^{k}), \\
C^{k+2,\alpha,\nu} &= W \oplus (\ker L \cap C^{k+2,\alpha,\nu}), \\
C^{k,\alpha,\nu} &= W_1 \oplus (\operatorname{coker} L \cap C^{k,\alpha,\nu})
\end{aligned}
\tag{4.49}
$$

and such that $L\colon V \to V_1$ and $L\colon W \to W_1$ are both isomorphisms.

We may finally return to the original problem.

(4.50) **Theorem.** *The maps*

$$
\begin{aligned}
\Delta + \lambda(x)&\colon r^{\delta}\widehat{H}^{k+2} \to r^{\delta}\widehat{H}^{k}, \\
\Delta + \lambda(x)&\colon C^{k+2,\alpha,\nu} \to C^{k,\alpha,\nu}
\end{aligned}
$$

on M have closed range iff $\delta \notin \Lambda$ and $\nu \notin \Lambda'$.

Proof. A suitable sequence of truncations of $u = r^{\nu}$ provides a counterexample to closed range in either case if $\nu \in \Lambda'$.

Thus, suppose $\delta \notin \Lambda$ and let $\nu = \frac{k}{2} - \delta \notin \Lambda'$. Let G be the generalized inverse for L corresponding to this choice of δ. We first note the following fact. For any $\varepsilon > 0$, let $C^{k,\alpha,\nu}_{\varepsilon}$ denote the closed subspace of functions in $C^{k,\alpha,\nu}$ with support in $r < \varepsilon$. Then

$$C^{k+2,\alpha,\nu}_{\varepsilon} \cap \ker L = 0$$

by unique continuation. Also, it is an easy exercise to check that the gap between $C^{k+2,\alpha,\nu}_{\varepsilon}$ and $\ker L$ in the sense of [10] is positive. From this it is easy to check that $L|_{C^{k+2,\alpha,\nu}_{\varepsilon}}$ has closed range. In particular,

$$\|u\|_{k+2,\alpha,\nu} \le C\|Lu\|_{k,\alpha,\nu}, \qquad u \in C^{k+2,\alpha,\nu}_{\varepsilon}, \tag{4.51}$$

and by a scaling argument C is independent of ε. A similar fact is true in the Sobolev setting.

To prove that $\Delta+\lambda(x)$ has closed range, it suffices to show that $\Delta+\lambda(x)$ restricted to functions supported close to the boundary has closed range, for standard elliptic theory ensures that this property holds for functions supported in a compact set of the interior. But functions supported in $\{r \leq \varepsilon\}$ may be regarded as living on the space T, so that the spaces $C^{k+2,\alpha,\nu}$ and the operator L make sense. By (4.1), $\Delta + \lambda(x) = L + E$, so using (4.51) we have

$$\begin{aligned}\|u\|_{k+2,\alpha,\nu} &\leq C\|Lu\|_{k,\alpha,\nu} \leq C(\|(\Delta+\lambda)u\|_{k,\alpha,\nu} + \|Eu\|_{k,\alpha,\nu}) \\ &\leq C\|(\Delta+\lambda)u\|_{k,\alpha,\nu} + \tfrac{1}{2}\|u\|_{k+2,\alpha,\nu}\end{aligned}$$

for ε small enough, $u \in C_\varepsilon^{k+2,\alpha,\nu}$, and using (4.3). Absorbing the second factor on the right into the left side of the equation gives an estimate which guarantees that $\Delta + \lambda$ has closed range on $C_\varepsilon^{k+2,\alpha,\nu}$, and hence on $C^{k+2,\alpha,\nu}$. The argument in $r^\delta \widehat{H}^{k+2}$ is identical. q.e.d.

Just as after Lemma (4.16), we may now conclude that there exist generalized inverses for $\Delta+\lambda(x)$ either from $r^\delta \widehat{H}^2 \to r^\delta L^2$, $\delta \notin \Lambda$, or from $C^{2,\alpha,\nu} \to C^{0,\alpha,\nu}$, $\nu \notin \Lambda'$. In particular, for any such δ, ν there are projectors onto the kernel or cokernel of $\Delta + \lambda(x)$ in $r^\delta \widehat{H}^2$, $C^{2,\alpha,\nu}$, or $r^\delta L^2$, $C^{0,\alpha,\nu}$, respectively.

Fix $\delta \notin \Lambda$ and set $\nu = \delta + \frac{k}{2}$. Let $\mathscr{G}$ denote the generalized inverse for $\Delta + \lambda(x)$ on $r^\delta \widehat{H}^k$, and $\mathscr{P}_i$ the two projectors. Although these projectors depend on both k and δ, by elliptic regularity if $\mathscr{P}_i$ vanishes for a particular value of δ and k then it vanishes for that δ and for every k. We shall now show that these operators have a structure identical to that of the operators G and P_i, which correspond to L. It is far simpler to determine the injectivity and surjectivity properties of $\Delta + \lambda(x)$ on the Sobolev spaces. But once we know that this operator is surjective, say, on a particular Sobolev space, then the projector on the cokernel, $\mathscr{P}_1$, must vanish. Since $L\mathscr{G} = I$ and $\mathscr{G}$ is also bounded on the appropriately weighted Hölder space, we can then conclude that $\Delta + \lambda(x)$ is surjective on this Hölder space.

As might be expected, $\mathscr{G}$ and the $\mathscr{P}_i$ are pushforwards of distributions with good regularity properties on M_π^2, the blow-up of $M \times M$ around the submanifold $S = \{r = \tilde{r} = 0, y = \tilde{y}\}$ in its corner. M_π^2 has three boundary faces with defining functions ρ_1, ρ_2, ρ_{ff} as usual.

(4.52) **Proposition.** *The distributions $\mathscr{G}$ and $\mathscr{P}_i$ are pushforwards of distributions (with the same names) on M_π^2. $\mathscr{G}$ is smooth away from the side faces and the lift of the diagonal and has polyhomogeneous expansions at these submanifolds. The $\mathscr{P}_i$ are smooth in the interior of M_π^2 and have polyhomogeneous expansions near the side faces. The expansions for all of these distributions are exactly the same as for the operators G and*

P_i corresponding to L which were obtained in Proposition (4.35). *In particular, if these operators are bounded on $r^\delta \widehat{H}^k$, then they are bounded on $C^{k,\alpha,\nu}$ for $\nu = \delta + \frac{k}{2}$.*

Proof. Although it is possible to give an argument based on the fact that $\Delta + \lambda(x)$ is very close to the operator L near $r = 0$, we shall appeal to a more general line of reasoning which will be discussed at length in [19]. It is first necessary to argue that $\mathscr{G}$ and the $\mathscr{P}_i$ have the stated regularity properties. These properties in the interior and along the diagonal are obvious; the only issue of course is whether they are polyhomogeneous along the side faces. This may be proved by combining the Mellin transform arguments of [18], [24] with the polyhomogeneous behavior of the operators G and P_i corresponding to L. Once this is known, the rest of the argument is arithmetic: the exponent of the leading term in the expansions at any one of these faces is an indicial root for either $\Delta + \lambda(x)$ or its adjoint. Since the indicial roots for $\Delta + \lambda(x)$ are exactly the same as for L, this computation is the same as in (4.35). The boundedness criteria (4.36) may then be applied to narrow the possibilities. Finally, the fact that these operators are bounded on the appropriate Hölder spaces is proved exactly as in Proposition (4.41). q.e.d.

It is not possible to give a completely general argument concerning whether either one of the projectors $\mathscr{P}_i$ vanishes for a particular value of δ. However, if for a particular δ the projector P_i corresponding to L equals zero, then it may be shown that the projector $\mathscr{P}_i$ corresponding to $\Delta + \lambda(x)$ is at most of finite rank. For example, if $P_1 = 0$ so that $LG = I$, then G may be used as a right parametrix for $\Delta + \lambda(x)$ near the boundary, hence $\Delta + \lambda(x)$ has at most a finite-dimensional cokernel. However, it is very difficult to rule out finite rank eigenvalues at 0. We shall not concentrate on this general question here. Instead we present complete results for the special cases which we required earlier in the paper.

Let us first consider the case where the operator is $\Delta + \lambda$ on the hyperbolic space $\mathbf{H}^{k+1}$. Here there are only two indicial roots:

$$s_1 = \frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 - \lambda}, \qquad s_2 = \frac{k}{2} - \sqrt{\left(\frac{k}{2}\right)^2 - \lambda}.$$

If $\lambda < k^2/4$ then s_1, s_2 are both real and $s_1 > s_2$. If $\lambda \geq k^2/4$ then s_1, s_2 both have real part $\frac{k}{2}$. Thus $\Lambda = \{\pm\sqrt{k^2/4 - \lambda}\}$ in the first case and $\Lambda = \{0\}$ in the second. Setting $\delta_0 = \Re\sqrt{(k/2)^2 - \lambda}$ we also define $\Lambda' = \{\frac{k}{2} \pm \delta_0\}$; for convenience we set $\nu_0^\pm = \frac{k}{2} \pm \delta_0$.

(4.53) **Proposition.** *$\Delta+\lambda$ is injective on $r^\delta L^2$ when $\delta > -\delta_0$ and surjective when $\delta < \delta_0$, $\delta \neq -\delta_0$. Similarly, it is injective on $C^{2,\alpha,\nu}$ when $\nu > \nu_0^-$ and surjective when $\nu < \nu_0^+$, $\nu \neq \nu_0^-$. In particular, $\Delta+\lambda$ is an isomorphism $r^\delta \widehat{H}^2 \to r^\delta L^2$ when $-\delta_0 < \delta < \delta_0$ and $C^{2,\alpha,\nu} \to C^{0,\alpha,\nu}$ when $\nu_0^- < \nu < \nu_0^+$.*

Proof. $\Delta+\lambda$ is injective on $r^\delta L^2$ when $\delta > -\delta_0$. This follows from the parametrix construction of [17] and the Mellin transform arguments of [18]. In fact, if $(\Delta+\lambda)u = 0$, $u \in r^\delta L^2$, then $u \in r^{\delta_0} L^2$ if $\delta > -\delta_0$ and $u \in r^\delta L^2$ for all δ if $\delta > \delta_0$. In either case we conclude that $u \in L^2(\mathbf{H}^{k+1})$ and hence $u = 0$ by the well-known assertion that the Laplacian on hyperbolic space has no point spectrum. Hence $\mathscr{P}_2 = 0$ when $\delta > -\delta_0$, and so $\mathscr{G}(\Delta+\lambda) = I$, where $\mathscr{G}$ is bounded on $r^\delta \widehat{H}^k$ for any k. But for $\nu = \delta + \frac{k}{2}$ these operators are also bounded on $C^{2,\alpha,\nu}$, and since $\Delta+\lambda$ has a bounded left inverse, it must be injective. Now the dual of $\Delta+\lambda$ (as an unbounded operator) on $r^\delta L^2$ is $\Delta+\lambda$ on $r^{-\delta} L^2$. $\Delta+\lambda$ is surjective on $r^\delta L^2$ precisely when it has closed range and its dual has no kernel. Therefore, $\mathscr{P}_1$ is trivial if $\delta < \delta_0$ and $(\Delta+\lambda)\mathscr{G} = I$, where $\mathscr{G}$ is bounded on $r^\delta L^2$. This $\mathscr{G}$ is bounded on $C^{0,\alpha,\nu}$, hence $\Delta+\lambda$ is surjective when $\nu < \nu_0^+$, $\nu \neq \nu_0^-$. q.e.d.

An identical argument works also for $\Delta+\lambda$ on $S^{n-k-1} \times \mathbf{H}^{k+1}$. The main fact that needs to be established is that Δ has no point spectrum when acting on $L^2(S^{n-k-1} \times \mathbf{H}^{k+1})$. Thus, for the same values of δ_0 and $\nu_0^\pm$ we have

(4.54) **Theorem.** *$\Delta+\lambda$ is injective on $r^\delta L^2$ when $\delta > -\delta_0$ and on $C^{2,\alpha,\nu}$ when $\nu > \nu_0^-$. It is surjective on $r^\delta L^2$ when $\delta < \delta_0$ and on $C^{0,\alpha,\nu}$ when $\nu < \nu_0^+$. Here $\nu_0^\pm$ and δ_0 assume the same values as in* (4.52).

Finally, we come to the last case of interest in this paper. This is the situation which arose in the analysis of §3. Changing notation slightly, the linear operator L of (3.6) is the sum of a Laplacian on a tubular neighborhood $\mathscr{U}$ of the submanifold $N \subset S^n$ and a singular term of order zero, λ/r^2. Here r is the polar distance function from N. Write Δ in polar coordinates as described there, and multiply the operator L by r^2. The resulting operator L_0 may be expressed as

$$\begin{aligned} L_0 = {} & r^2(\partial_r)^2 + (n-k-1)r\partial_r + r^2\Delta_N \\ & + \Delta_{S^{n-k-1}} + \frac{(n+2)(n-2k-2)}{4} + E, \end{aligned}$$

where E is of the form (4.3), hence is of lower order in our calculus. The indicial roots for L_0 are

$$s_l^{\pm} = \frac{2+k-n}{2} \pm \sqrt{\left(\frac{k+2}{2}\right)^2 - (n-k-2) + \lambda_l^2},$$

where as usual λ_l^2 are the eigenvalues of $-\Delta_{S^{n-k-1}}$. Let δ_l be the real part of the square root in the expression for s_l^+, and set $\nu_l^{\pm} = \Re s_l^{\pm}$.

(4.55) **Theorem.** *Let the operator L_0 act on functions on the geodesic tubular neighbourhood $\mathscr{U}_\sigma$ of radius σ around the submanifold $N \subset S^n$ which vanish on $\partial\mathscr{U}_\sigma$. Then for σ sufficiently small, L_0 is injective on $r^\delta L^2$ when $\delta > -\delta_0$ and on $C^{2,\alpha,\nu}$ when $\nu > \nu_0^-$, and is surjective on $r^\delta L^2$ when $\delta < \delta_0$, $\delta \neq -\delta_l$ and on $C^{2,\alpha,\nu}$ when $\nu < \nu_0^+$, $\nu \neq \nu_l^-$.*

Proof. By standard elliptic theory and the results of this section, we know that L_0 has closed range on $r^\delta L^2$ and $C^{k,\alpha,\nu}$ when $\delta \neq \pm\delta_l$, $\nu \neq \nu_l^{\pm}$. Thus, by the duality arguments above, all of the conclusions of the theorem will hold provided we known that L_0 is injective on $r^\delta L^2$ when $\delta > -\delta_0$. For this we argue as follows. First we reduce to the case where L_0 is equal to the operator $\widehat{L} = r^2\partial_r^2 + (n-k-1)r\partial_r + r^2\Delta_N + \Delta_\theta + \lambda$. This is accomplished by simply noting that if $\widehat{L}$ is injective, then by standard perturbation theory L_0 will also be injective if σ is small. Note that $\widehat{L}$ may be regarded as the Laplacian on the normal bundle of N in the sphere with respect to a metric on the total space of this bundle which is euclidean along the fibers. Next introduce an eigenfunction expansion in the variable $\theta \in S^{n-k-1}$. This makes sense globally on $\mathscr{U}_\sigma$ since Δ_θ is rotationally invariant. The eigenfunctions of Δ_θ on $\mathscr{U}_\sigma$ are those which are invariant under the transition maps of the normal bundle. Hence the list of eigenvalues are a subset of those which occur for Δ_θ without this extra invariance condition. Now, for each l, $\widehat{L}$ induces an operator $\widehat{L}_l$ on the eigenspace corresponding to λ_l^2. $\widehat{L}_l$ is an operator on the flat vector bundle over $(0, \sigma] \times N$ with sections the invariant eigenfunctions of Δ_θ. For each fixed $r \in (0, \sigma]$ the operator Δ_N acting on sections of this bundle has discrete spectrum $\{\mu_j^2(l)\}$ which depends on l. We still have $\mu_j^2(l) \geq 0$. Let L_{jl} denote the operator on the eigenspace corresponding to $\mu_j^2(l)$. This operator has the usual form

$$L_{jl} = r^2\partial_r^2 + (n-k-1)r\partial_r + (\lambda - r^2\mu_j^2(l) - \lambda_l^2).$$

Finally then, we need only check that there is no function u such that

$L_{jl}u = 0$ such that $u \in L^2(r^{n-k-1}dr)$ and $u = 0$ when $r = \sigma$. But this fact is true for σ sufficiently small as is easy to check since the solutions are functions of Bessel-type, and may be calculated explicitly.

Added in proof. Recently the first author [20] has established the precise regularity of the solutions found in §2 and §3.

References

[1] P. Aviles & R. McOwen, *Conformal deformations of complete manifolds with negative curvature*, J. Differential Geometry **21** (1985) 269–281.

[2] ——, *Complete conformal metrics with negative scalar curvature in compact Riemannian Manifolds*, Duke Math. J. **56** (1988) 395–398.

[3] ——, *Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds*, J. Differential Geometry **27** (1988) 225–239.

[4] J. Brüning & R. Seeley, *The expansion of the resolvent near a singular stratum of conical type*, J. Func. Anal. **95** (1991) 255–290.

[5] L. Caffarelli, R. Hardt & L. Simon, *Minimal surfaces with isolated singularities*, Manuscripta Math. 48 (1984) 1–18.

[6] C. Epstein, R. B. Melrose & G. Mendoza, *Resolvent of the Laplacian on strictly pseudoconvex domains*, to appear in Acta Math.

[7] C. R. Graham & J. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, to appear in Advances in Math.

[8] S. Helgason, *Groups and geometric analysis*, Academic Press, New York, 1986.

[9] L. Hörmander, *The analysis of linear partial differential operators*, Vol. III, Springer, Berlin, 1985

[10] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1976.

[11] S. Lang, *Real analysis*, Addison-Wesley, Reading, MA, 1983.

[12] J. Lee & R. B. Melrose, *Boundary behaviour of the complex Monge-Ampère equation*, Acta Math. **148** (1982) 159–192.

[13] J. Lee & T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987) 37–91.

[14] C. Loewner & L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, Contributions to Analysis, Academic Press, New York, 1974, 245–272.

[15] X. Ma & R. McOwen, *The Laplacian on complete manifolds with warped cylindrical ends*, preprint, 1989.

[16] R. Mazzeo, *Hodge cohomology of negatively curved manifolds*, Thesis, MIT, 1986.

[17] ——, *Hodge cohomology of a conformally compact metric*, J. Differential Geometry **28** (1988) 309–339.

[18] ——, *Unique continuation at infinity and embedded eigenvalues for asympotically hyperbolic manifolds*, Amer. J. Math., to appear.

[19] ——, *The elliptic theory of differential edge operators*. I, to appear in Comm. Partial Differential Equations.

[20] ——, *Regularity for the singular Yamabe equation*, to appear in Indiana Univ. Math. J.

[21] R. Mazzeo & R. B. Melrose, *The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration*, J. Differential Geometry **31** (1990) 185–213.

[22] R. McOwen, *The behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math. **32** (1979) 783–795.

[23] R. B. Melrose, *Transformations of boundary problems*, Acta Math. **147** (1981) 149–236.

[24] ——, *Analysis on manifolds with corners*, in preparation.

[25] R. B. Melrose & G. Mendoza, *Elliptic operators of totally characteristic type*, MSRI preprint, 1983.

[26] R. Schoen, *Conformal deformations of a Riemannian metric to constant scalar curvature*, J. Differential Geometry **20** (1984) 479–495.

[27] ——, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. **41** (1988) 317–392.

[28] R. Schoen & S. T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988) 47–71.

[29] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Symposium Partial Differential Equations, Holzau 1988, Teubner-Texte zur Mathematik, Band 112, Leipzig, 1988, 259–280.

[30] N. Smale, *Minimal surfaces with many isolated singularities*, Ann. of Math. (2) **130** (1989) 603–642.

[31] G. N. Watson, *Treatise on the theory of Bessel functions*, Cambridge University, 1952, 194.

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