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GRAUERT TUBES AND THE HOMOGENOUS MONGE-AMPÈRE EQUATION

VICTOR GUILLEMIN & MATTHEW STENZEL

1. Introduction

Let X be a compact real-analytic manifold of dimension n. A theorem of Bruhat and Whitney [2] states that there exists a complex "thickening" of X: a complex n-dimensional manifold, M, and a real-analytic imbedding of X in M with the property that, as a submanifold of M, X is totally real. (Meaning that if $p \in X$, and J_p is the defining map for the complex structure on T_pM, then $J_p(T_pX)$ intersects T_pX in $\{0\}$.) In addition one can arrange that there exists on M an antiholomorphic involution

$$\sigma: M \to M \tag{1.1}$$

whose fixed point set is X. Suppose now that X is equipped with a Riemannian metric. What we will be concerned with in this paper is the following question: Can one find a Kaehler structure on M which is in some way "intrinsically associated" with the Riemannian structure on X? Before posing this question in a more precise form we will first say a few words about Grauert tubes; In [4] Grauert showed that there exist a neighborhood M_1 of X in M and a smooth strictly plurisubharmonic function

$$\rho: M_1 \to [0, 1) \tag{1.2}$$

with $X = \rho^{-1}(0)$ and $\rho(\sigma(m)) = \rho(m)$ for all $m \in M_1$. The fact that ρ is strictly plurisubharmonic implies that the open sets

$$M_\varepsilon = \rho 6-1([0, \varepsilon)), \qquad 0 < \varepsilon < 1, \tag{1.3}$$

are strictly pseudoconvex and hence possess lots of globally defined holomorphic functions. From this Grauert was able to deduce that X itself possesses a lot of globally defined real-analytic functions. (The fact that a real-analytic manifold possesses a lot of globally defined real-analytic

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functions had been originally proved by Morrey, but his proof was much more complicated.)

The question we will be interested in is: Given a Riemann metric ds^2 on X, can one choose the ρ above *in a canonical way* compatible with ds^2? In particular, from the Kaehler form $\partial\overline{\partial}\rho/\sqrt{-1}$ one gets a Kaehler metric ds^2_M on M, and one property that one would like ρ to have is

$$\iota(ds^2_M) = ds^2, \tag{1.4}$$

where $\iota\colon x \to M$ is the inclusion map. In other words one would like ι to be an *isometric imbedding* of X into M. This condition in itself, however, is clearly not sufficient to determine ρ uniquely. From some recent work of Burns and Patrizio-Wong (of which we will say more below) we were led to impose a second condition on ρ: namely that its square root satisfy the homogeneous Monge-Ampère equation

$$\det\left(\frac{\partial^2}{\partial z_i \partial \overline{z}_j}(\sqrt{\rho})\right) = 0 \tag{1.5}$$

on $M - X$. The main result of this paper is the following:

Theorem. *Assume the metric ds^2 is real-analytic. Then there exists a neighborhood U of X in M and, on U, a unique real-analytic solution ρ of the equation* (1.5) *satisfying the initial conditions* (1.4).

A few comments about this result: For special metrics, solutions of the homogeneous Monge-Ampère equation have been known for a long time. For instance, for the rank one symmetric spaces, explicit formulas for ρ can be found in Patrizio-Wong [8, §2]. More generally, the geometry of the solutions of (1.5) has been studied extensively by Burns [3], Wong [11], and Patrizio-Wong [8] (and, in a slightly different context from the above, by Stoll [10], Lempert [6], Bedford-Kalka [1], and others). A couple of years ago we noticed a property of (1.5) that has no doubt been notices by others (since it is very easy to check). Namely, for every metric ds^2, (1.4)–(1.5) possess a *formal* solution: One can find a smooth strictly plurisubharmonic function $\rho\colon M \to [0, \infty)$ which satisfies the initial conditions (1.4) and satisfies (1.5) in the sense that the left-hand side of (1.5) vanishes to infinite order on X. Moreover, this solution is unique in the sense that if ρ_1 is another solution, $\rho - \rho_1$ vanishes to infinite order on X. This prompted us to try to prove that, if ds^2 is real-analytic, this formal solution converges in a small neighborhood of X. This turns out to be a rather hard problem for the following reason: The Cauchy data for the equation above are defined on a codimension n submanifold of

M rather than on a hypersurface. To convert (1.4)–(1.5) into a standard nonlinear Cauchy problem one has to "blow up" X, i.e., write the normal coordinates to X in M in polar coordinate form. This has the effect, however, of making the hypersurface on which the initial data of the equation are defined highly characteristic. We were able to surmount these difficulties by a suitable modification of the "hodograph" techniques used by Hans Lewy in his classical work on the real two-dimension Monge-Ampère problem [7]; we succeeded a few months ago in obtaining a rather long and complicated proof of the theorem above. We will not attempt to present an account of this proof here since, as we realized recently, our efforts were completely misguided: In the real-analytic case, the equations (1.4)–(1.5) have an extremely simple solution which (though we did not have the perspecality to notice so earlier) is implicit in the work of Dan Burns cited above. We will present this solution in §3, and in §4 show how to derive from it the formulas of Patrizio-Wong for the rank one symmetric spaces. The motivating idea behind our result is described in §2 where we discuss the solution of a real Monge-Ampère problem that has many similarities to the problem above. Finally in §5 we show that the problem (1.4)–(1.5) is intimately related to a problem in symplectic geometry which, in view of some recent results of Eliashberg on the symplectic geometry of Stein manifolds, is itself quite interesting.

We would like to thank David Jerison for listening patiently to expositions of various version of this material and suggesting a number of improvements. We would also like to thank Charles Epstein and Richard Melrose for keeping us abreast of their recent work on Grauert tubes. We hope at some future date to be able to show that this work is not entirely unrelated to what we have described here.

Added in proof. We have learned recently of some results of Lempert and Szöke which are closely related to ours.

2. The real homogeneous Monge-Ampère problem

Let $\Delta\colon X \to X \times X$ be the diagonal embedding $\Delta(x) = (x, x)$. Henceforth we will identify X with its image in $X \times X$ via this embedding. Notice that, with this identification, X also becomes the fixed-point set of the involution

$$\tau\colon X \times X \to X \times X$$

sending (x, y) to (y, x). Given a Riemannian metric ds^2 on X we consider the following variant of the Monge-Ampère problem discussed in

§1: Find a τ-invariant neighborhood W of X in $X \times X$ and a smooth τ-invariant function

$$f: W \to [0, \infty) \tag{2.1}$$

such that:

1. $f^{-1}(0) = X$.
2. f is "strictly plurisubharmonic" in the sense that the symmetric two-tensor

$$\sum \frac{\partial^2 f(x, y)}{\partial x_i \partial y_j} dx_i \circ dy_j \tag{2.2}$$

is exactly of signature (n, n) at all points of W.

3. The restriction of the tensor (2.2) to X is the metric ds^2.
4. On $W - X$ the square root of f satisfies the homogeneous Monge-Ampère equation

$$\det\left(\frac{\partial^2}{\partial x_i \partial y_j}\sqrt{f}\right) = 0. \tag{2.3}$$

We claim that this problem has a very simple solution: Namely, let $f(x, y)$ be the square of the geodesic distance from x to y measured with respect to ds^2. Restricted to a sufficiently small neighborhood W of the diagonal f is a smooth function which clearly has all the properties listed above except, perhaps, for the property (2.3). We verify that it has this property as well. Fix $(x, y) \in W - X$, let $r = \sqrt{f(x, y)}$, and let ξ be the x-derivative of $\sqrt{f}$ at (x, y). By Gauss's lemma, ξ is the unit outward-pointing covector to the geodesic sphere $S(r, y)$ at the point x. Now let U be a small neighborhood of x in X and let $\mathring{U} = U - \{x\}$. Leaving x fixed and varying y we get a map

$$\mathring{U} \ni y \to (d_x\sqrt{f})(x, y) \in T_x^*$$

which maps $\mathring{U}$ into the unit sphere in T_x^*. Therefore, in particular, this map is of rank $\leq n - 1$ and so the determinant of its Jacobi matrix at $y \in \mathring{U}$,

$$\det\left(\frac{\partial^2}{\partial x_i \partial y_j}\sqrt{f}\right)(x, y),$$

has to be zero.

3. The complex homogeneous Monge-Ampère equation

Suppose now that the metric ds^2 is real-analytic. Then the function $f = f(x, y)$ defined in the previous section is real-analytic, so it extends to a holomorphic function $\tilde{f} = \tilde{f}(z, w)$ on a small (connected) neighborhood U of X in $M \times M$. Let S be the hypersurface defined by the equation $\tilde{f} = 0$. On $U - S$, the square root of $\tilde{f}$ is a (double-valued) holomorphic function. Since $U - S$ is connected and $\sqrt{f}$ satisfies (2.3) on $W - X$, each branch of the square-root of $\tilde{f}$ has to satisfy the complex analogue of (2.3), viz.

$$\det\left(\frac{\partial^2}{\partial z_i \partial w_j}(\tilde{f})^{1/2}\right) = 0 \tag{3.1}$$

on $U - S$. We embed M in $M \times M$ via the mapping $z \to (z, \bar{z})$, where $\bar{z} = \sigma(z)$. This embedding is consistent with the diagonal embedding of X into $X \times X$ and enables us to think of M as a submanifold of $M \times M$. We will denote by g the restriction of $\tilde{f}$ to $M \cap U$, and claim that g is *real-valued* and is *strictly negative* except on the set X where it is equal to zero by definition. To prove the first of these claims, note that, since f is real-valued on W, its holomorphic extension to U satisfies

$$\overline{\tilde{f}}(z, w) = \tilde{f}(\bar{z}, \bar{w}).$$

On the other hand, $f(x, y) = f(x, y)$, so $\tilde{f}(z, w) = \tilde{f}(w, z)$. Thus, in particular,

$$g(z) = \tilde{f}(z, \bar{z}) = \overline{\tilde{f}}(\bar{z}, z) = \overline{\tilde{f}}(z, \bar{z}) = \bar{g}(z),$$

which shows that g is real-valued.

To prove the second assertion fix $p \in X$ and let $(x_1, \cdots, x_n)$ (resp. $(y_1, \cdots, y_n)$) be geodesic coordinates on X centered at p. Then for x and y near p,

$$f(x, y) = \sum a_{ij}(x_i - y_i)(x_j - y_j) + O((x - y)^3),$$

where $a_{ij} = a_{ij}(p) =$ the (i, j)th coefficient of the metric tensor at p. Let $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_n)$ be the holomorphic extensions of $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ to M. Then in a neighborhood of p in $M \times M$,

$$\tilde{f}(z, w) = \sum a_{ij}(z_i - w_i)(z_u - w_y) + O((z - w)^3),$$

so, near p, g has the form

$$g(z) = \sum a_{ij}(z_i - \bar{z}_i)(z_j - \bar{z}_j) + O((z - \bar{z})^3) \tag{3.2}$$

or, by setting $z_i = u_i + \sqrt{-1}v_i$,

$$g(z) = (-4)\sum a_{ij}v_iv_j + O(v^3). \tag{3.3}$$

Since the quadratic term on the right-hand side is negative definite, it is clear that for U sufficiently small, g is strictly negative on $M \cap U$ except on X (i.e., on the set, $v = 0$) where it is zero by definition.

We will define a single-valued branch of $\sqrt{g}$ by setting

$$\sqrt{g} = i\sqrt{\rho}, \tag{3.4}$$

where $\rho = -f$, and the square root of the right is the positive square root. Let us now show that ρ is a solution to the Monge-Ampère problem of §1. In terms of the coordinates above $\tilde{f}(z, w)$ has a convergent power series expansion

$$\sum c_{\alpha\beta} z^\alpha w^\beta \tag{3.5}$$

in a neighborhood of p, and the power series satisfies the nonlinear equation (3.1). We claim that $g(z) = \tilde{f}(z, \overline{z})$ satisfies the corresponding equation

$$\det\left(\frac{\partial^2}{\partial z_i \partial \overline{z}_j}\sqrt{g}\right) = 0. \tag{3.6}$$

Indeed this is clear just by substituting $\overline{z}$ for w in the series (3.5) and treating z and $\overline{z}$, in the traditional nineteenth century way, as independent coordinates. Since g satisfies (3.6), so does ρ; so ρ is indeed a solution of the Monge-Ampère equation (1.5). We must still check that it satisfies, at p, the initial condition (1.4). This, however, follows from (3.3) since the a_{ij}'s in (3.3) are just the coefficients of the metric tensor at p.

4. The rank-one symmetric spaces

For the compact rank-one symmetric spaces Patrizio and Wong have obtained explicit formulas for the solution ρ of the Monge-Ampère problem (1.4)–(1.5). We will describe here how to obtain their formula of $M = S^n$ (the standard n-sphere) by the methods of §3. (Their formulas for the other rank-one symmetric spaces can also be computed quite easily by these methods. For details see [9].)

The standard n-sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation

$$x_1^2 + \cdots + x_{n+1}^2 = 1, \tag{4.1}$$

and, therefore, its Bruhat-Whitney "thickening" is the complex hypersurface

$$(4.2)\qquad M = \{(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1},\ z_1^2 + \cdots + z_{n+1}^2 = 1\}$$

in $\mathbb{C}^{n+1}$ (inside of which S^n sits as the fixed point set of the involution $z \to \bar{z}$). Fox x and y in S^n, the geodesic distance between x and y is:

$$2\sin^{-1}\frac{|x-y|}{2} \quad \text{or} \quad 2\sin^{-1}\left(\frac{1}{2}\sqrt{h(x,y)}\right),$$

where $h(x, y) = (x - y)^2$. The analytic continuation of this function to $M \times M$ is the doubly-branched holomorphic function $2\sin^{-1}\frac{1}{2}\sqrt{(z-w)^2})$. Setting $w = \bar{z}$ in this expression we get the function

$$2\sin^{-1}(\pm i|\mathrm{Im}\, z|).$$

Noting that $\sin^{-1}(it) = i\sin h^{-1}t$, we can also write this function as $\pm 2i(\sin h^{-1})(|\mathrm{Im}\, z|)$. Note, however, that $z^2 = 1$ or $|z|^2 = 1 + 2|\mathrm{Im}\, z|^2$, so this function can also be written in the form $\pm i\cosh^{-1}(|z|^2)$. Thus the function $\cosh^{-1}(|z|^2)$ is the solution of the Monge-Ampère problem for S^n (see [8, Theorem 1.2].

5. Cotangent bundles

If X is a differentiable manifold, then its cotangent bundle T^*X is a symplectic manifold, and X has a natural imbedding in T^*X as the zero section. The question we want to investigate in this section is: Is there a natural Kaehler structure on T^*X compatible with its symplectic structure and with the choice of a given Riemannian structure on X? One condition we would like this Kaehler structure to have, if it exists, is the analogue of the condition (1.4), namely that the imbedding of X into T^*X be an isometric imbedding. This, however, is not enough to determine the Kaehler structure uniquely: one clearly needs a condition analogous to (1.5) for this. We will show below that here is such a condition: namely that the potential function which defines the Kaehler form be *quadratic* on each cotangent fiber. The following are the details:

We will denote by $\sigma: T^*X \to T^*X$ the involution which maps (x, ξ) onto $(x, -\xi)$ and by $g: T^*X \to \mathbb{R}$ the symbol of the Laplace operator associated with the given metric on X. (In other words, $g(x, \xi) = |\xi|^2$.) Also, without further mention, we will assume from now on that all data are real-analytic.

Theorem. *There exist a σ-invariant neighborhood U of X in T^*X and a unique complex structure on U with the following two properties:*

(i) *σ is an antiholomorphic involution.*

(ii) *The one-form $\alpha = \operatorname{Im}\overline{\partial} g$ is the standard symplectic one-form $\sum \xi_i \, dx_i$.*

Proof. Let M be, as in §1, a complex analytic manifold, and $\sigma\colon M \to M$ an antiholomorphic involution with X as its fixed point set. Also, as in §1 let $\rho\colon M \to [0, \infty)$ be a strictly plurisubharmonic function satisfying $\rho^{-1}(0) = X$ and $\sigma^*\rho = \rho$. We will set $\alpha = \operatorname{Im}\overline{\partial}\rho$ and $\omega = d\alpha = \sqrt{-1}\,\overline{\partial}\partial\rho$. Since ρ is strictly plurisubharmonic, ω is a symplectic form. In particular, for every point $p \in X$, ω_p is nondegenerate as an alternating bilinear form on the tangent space to M at p. In other words the mapping of T_pM into T_p^*M mapping v onto $\iota(v)\omega_p$ is bijective. Let Ξ be the vector field on M defined by the identity

$$\iota(\Xi)\omega = \alpha. \tag{5.1}$$

We will deduce the existence of a complex structure on T^*X with the properties described above from the following lemma and a theorem of Kostant and Sternberg which we will state below.

Lemma. *The function $\sqrt{\rho}$ satisfies the Monge-Ampère equation* (1.5) *if and only if it satisfies the equation*

$$\Xi\rho = 2\rho. \tag{5.2}$$

Proof. Let $f = f(t)$ be a smooth function of the real variable t. Then

$$\overline{\partial}\partial f(\rho) = f'(\rho)\overline{\partial}\partial\rho + f''(\rho)\overline{\partial}\rho \wedge \partial\rho \tag{5.3}$$

and hence

$$(\overline{\partial}\partial f(\rho))^n = (f')^n(\overline{\partial}\partial\rho)^n + nf''(f')^{n-1}(\overline{\partial}\rho \wedge \partial\rho)(\overline{\partial}\partial\rho)^{n-1}. \tag{5.4}$$

Noting that

$$\overline{\partial}\rho \wedge \partial\rho = i\,d\rho \wedge \alpha \quad \text{and} \quad \overline{\partial}\partial\rho = -i\omega$$

we can rewrite the right-hand side of (5.4) in the form:

$$(-if')^n\left(\omega^n - n\frac{f''}{f')}\,d\rho \wedge \alpha \wedge \omega^{n-1}\right).$$

However, $\iota(\Xi)\omega = \alpha$ so this is also equal to

$$(-if')^n\left(\omega - n\frac{f''}{f'}\,d\rho \wedge (\iota(\Xi)\omega^n)\right).$$

Moreover, since $d\rho \wedge \omega^n = 0$, we have

$$\iota(\Xi)(d\rho \wedge \omega^n) = 0 = (\Xi\rho)\omega^n - d\rho \wedge \iota(\Xi)\omega^n,$$

so we can finally rewrite (5.4) in the form:

$$(-if')^n \left(1 + \frac{f''}{f'}\Xi\rho\right)\omega^n.$$

In particular if $f(\rho) = \rho^{1/2}$, then (5.4) is zero if and only if $\Xi\rho = 2\rho$. q.e.d.

The theorem of Kostant and Sternberg that we mentioned above is the following:

Theorem (see [5, p. 228]). *Let (M, ω) be a symplectic manifold, X a Lagrangian submanifold of M, and α a one-form on M with the property that $d\alpha = \omega$ and that $\alpha_p = 0$ at all points $p \in X$. Then there exist a neighborhood U of X in T^*X, a neighborhood V of X in M, and a unique diffeomorphism $\psi\colon U \to V$ such that ψ is the identity on X and such that $\psi^*\alpha$ is the standard symplectic one-form $\sum \xi_i\, dx_i$ restricted to U.*

Let us apply this theorem to the M and the X in the paragraph above. We will show that the pullback by ψ of the complex structure on M is a complex structure on U which satisfies all the hypotheses of the theorem above. Since $\psi^*\alpha = \sum \xi_i\, dx_i$ and $\psi^*\omega = \sum d\xi_i \wedge dx_i$, ψ^{-1} maps the vector field Ξ onto the vector field $\sum \xi_i \partial/\partial\xi_i$. Thus if $g = \psi^*\rho$, (5.2) becomes the Euler equation

$$\sum \xi_i \frac{\partial}{\partial \xi_i} g = 2g, \tag{5.5}$$

which implies that g has to be a homogeneous polynomial of degree 2 in the ξ-coordinates. If ρ also satisfies the initial conditions (1.4), it is easy to see that g has to be the symbol of the Laplace operator associated with the metric on X. To show that the complex structure that we have just defined on U has the property that the involution $(x, \xi) \to (x, -\xi)$ is an antiholomorphic mapping, we note that, composing ψ on the left by this involution and on the right by the complex conjugation mapping σ, we get another mapping satisfying the hypotheses of the Kostant-Sternberg theorem. Therefore the uniqueness part of this theorem states that ψ has to intertwine these two involutions.

Finally we have to show that the complex structure in the theorem above is unique. This is easily deduced from the following.

Proposition. *Let M be a $2n$-dimensional real-analytic manifold equipped with two different complex structures, J_1 and J_2, and let X be an*

n-dimensional real-analytic submanifold which is totally real with respect to both J_1 and J_2. Then there exist neighborhoods U_1 and U_2 of X in M and a real-analytic diffeomorphism $\kappa: U_1 \to U_2$ which is the identity on X and is a holomorphic mapping of (U_1, J_1) onto (U_2, J_2) (see [2]).

This theorem can be interpreted as saying that the Bruhat-Whitney "thickening" of the manifold X is unique up to complex isomorphism. Suppose now that J_1 and J_2 are two complex structures on T^*X satisfying hypotheses (i) and (ii). Let U_1, U_2 and $\kappa: U_1 \to U_2$ be as in the proposition above. Then g and $\kappa^* g$ are both solutions of the Monge-Ampère problem relative to the complex structure J_1. Hence $g = \kappa^* g$ and, therefore, by hypothesis (ii) κ preserves the one-form $\sum \xi_i \, dx_i$. Since κ is the identity mapping on X, this forces κ to the identity mapping everywhere.

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