ALMOST RIEMANNIAN SPACES

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Introduction

We call a complete metric space (X, d) *almost Riemannian* if X is finite dimensional and d is a geodesically complete inner metric of (metric) curvature locally bounded below. This paper is an investigation of the local and global properties of these and more general inner metric spaces. Our two global results are generalizations of Toponogov's Comparison Theorem and Maximal Diameter Theorem. The latter is used to prove our main local result: that an almost Riemannian space is a topological manifold, and that its metric structure has an infinitesimal approximation by a Euclidean geometry (hence the name "almost Riemannian"). We also prove a precompactness theorem (cf. [6]) for any class of n-dimensional almost Riemannian spaces with fixed bounds on diameter and curvature.

In order to state these theorems precisely we need a few definitions (for more details see [17] and [18]). Throughout this paper X denotes a metrically complete inner metric space which is *convex* in the sense that every pair of points is jointed by a minimal curve. Convexity is implied by local compactness (and metric completeness). S_k will denote the simply connected, two-dimensional space form of curvature k. By *monotonicity* we mean the well-known fact that the angle between two minimal curves of fixed length in S_k is a monotone increasing function of the distance between the endpoints opposite the angle. A *geodesic terminal* is a point in X beyond which some geodesic cannot be extended. An open subset U of X is *geodesically complete* if it has no geodesic terminals.

Definition A. An open set U in X is said to be a *region of curvature* $\geq k$ if for every triangle $(\gamma_{ab}, \gamma_{bc}, \gamma_{ca})$ of minimal curves in U,

(a) there exists a representative $(\tilde{\gamma}_{AB}, \tilde{\gamma}_{BC}, \tilde{\gamma}_{CA})$ in S_k (i.e., $\tilde{\gamma}_{AB}, \tilde{\gamma}_{BC}, \tilde{\gamma}_{CA}$ are minimal of the same length as their correspondent curves) and

(b) for any y on γ_{AB} and Y on $\tilde{\gamma}_{AB}$ such that $d(y, a) = d(Y, A)$, we have $d(y, c) \geq d(Y, C)$.

If x is contained in a region of curvature $\geq k$, let $c_k(x) = \sup\{r : B(x, r)$ is a region of curvature $\geq k\}$, and put $c_k(x) = 0$ otherwise. Then c_k is continuous or $c_k = \infty$. If for all $x \in X$ there is a k such that $c_k(x) > 0$, then we say X has *curvature locally bounded below*. If for some fixed $p \in X$ and k, c_k has a positive lower bound on $B(p, r)$ for all r, we say X has *curvature uniformly* $\geq k$. If X is locally compact, then curvature uniformly $\geq k$ is equivalent to $c_k > 0$ on X.

Monotonicity implies that Definition A is equivalent to that of *Gebiet der Riemannscher Krümmung* $\geq k$ in [18], so the *angle* $\alpha(\gamma_1, \gamma_2)$ between two geodesics exists and is a bona fide metric on the *space of directions* S_p (unit geodesics) at a point $p \in X$. For $\gamma \in S_p$ we let $\mathscr{E}(\gamma) = \sup\{t : \gamma|_{[0, t]}$ is minimal$\}$.

Definition B. We say that a (geodesic) triangle $(\gamma_1, \gamma_2, \gamma_3)$ in X is A1 if there exists a representative triangle $(\bar\gamma_1, \bar\gamma_2, \bar\gamma_3)$ in S_k and $\alpha(\bar\gamma_i, \bar\gamma_2) \leq \alpha(\gamma_i, \gamma_2)$ for $i = 1, 3$. We say that a (geodesic) wedge $(\gamma_{ab}, \beta_{ac})$ is A2 if there is a representative wedge $(\tilde\gamma_{AB}, \tilde\beta_{AC})$ in S_k (i.e., whose sides are minimal with $L(\tilde\gamma_{AB}) = L(\gamma_{ab})$, $L(\tilde\beta_{AC}) = L(\beta_{ac})$, $\alpha(\tilde\gamma_{AB}, \tilde\beta_{AC}) = \alpha(\gamma_{ab}, \beta_{ac}))$ and $d(B, C) \geq d(b, c)$. If $d(B, C) = d(b, c)$, we say $(\gamma_{ab}, \beta_{ac})$ is A2 *with equality*.

Remark. By monotonicity, the conditions A1 and A2 are equivalent in the sense that $(\gamma_1, \gamma_2, \gamma_3)$ is A1 if and only if (γ_1, γ_2) and (γ_2, γ_3) are A2.

A wedge (γ_1, γ_2) or triangle $(\gamma_1, \gamma_2, \gamma_3)$ in X is *proper* if γ_1 and γ_3 are minimal and $L(\gamma_2) \leq \pi/\sqrt{k}$. If $c_k(x) = r > 0$, then in $B(x, r/2)$ every proper triangle is A1, and every proper wedge is A2 (cf. Lemma 1.1 and [17]). We prove

Theorem C. *If X is geodesically complete of curvature uniformly $\geq k$, then every proper triangle in X is A1, and every proper wedge in X is A2.*

Corollary D. *If $k > 0$, and X is geodesically complete of curvature uniformly $\geq k$ then $\operatorname{diam}(X) \leq \pi/\sqrt{k}$.*

We say X has *rigid curvature* k if every wedge of minimal curves in X is A2 with equality.

Theorem E. *If $k > 0$, X is geodesically complete of curvature $\geq k$, and $\operatorname{diam}(X) = \pi/\sqrt{k}$, then X has rigid curvature k. In particular, if X is locally compact, X is isometric to S_k^n for some n.*

For Riemannian manifolds of sectional curvature $\geq k$, the Rauch Comparison Theorem implies that Definition A is satisfied. Nonetheless, all prior proofs of Toponogov's Theorem require further applications of Rauch's Theorem or its generalizations. In other words, the proof of

Theorem C shows for the first time, even in the Riemannian case, that the powerful improvement from local comparisons to global comparisons is purely a metric phenomenon.

The notion of a "completeness" in the Riemannian case can be generalized to the metric case either as metric or, more strongly, geodesic completeness. For our proof of Theorem C we require geodesic completeness (although essentially in only one place), and we do not know if it is true with the weaker assumption. On the other hand, we give two examples to show that Theorem E fails without geodesic completeness.

Theorem F. *If (X, d) is almost Riemannian, then X is a topological manifold. For each $p \in X$ there exist an n-dimensional vector space $\overline{T}_p$ ($n = \dim X$) with inner product $\langle \cdot, \cdot \rangle$, a continuous map $\exp_p : \overline{T}_p \to X$, and a dense subset T_p of $\overline{T}_p$, having the following properties:*

  (a) *if $v \in T_p$, then $tv \in T_p$ for all $t \in \mathbf{R}$,*

  (b) *the correspondence $v \leftrightarrow \gamma_v$, where $\gamma_v(t) = \exp_p(tv)$, is one-to-one between unit vectors in $(T_p, \langle \cdot, \cdot \rangle_p)$ and unit geodesics starting at p, and*

  (c) *$\exp_p$ preserves angles on T_p (i.e., $\alpha(v, w) = \alpha(\gamma_v, \gamma_w)$ for all $v, w \in T_p$).*

Note that (b) implies that $\exp_p$ restricted to T_p is surjective. On the other hand, $\exp_p$ need not be locally one-to-one (so there may not be "normal coordinates"); but very short geodesics starting at p are "almost minimal" in the following sense: the ratio of the length of a short geodesic to the distance between its endpoints is uniformly close to 1 (Lemma 2.8). In particular, there are not arbitrarily short geodesic loops starting at p.

Theorem F is the last "manifold theorem" having as its hypothesis only finite dimensionality and some combination of the three fundamental metric conditions, (1) geodesic completeness, (2) curvature locally bounded below, and (3) curvature locally bounded above. In [2] and [15] (cf. also [1]) it is shown that a space satisfying (1), (2), and (3) is a smooth manifold with a $C^{1,\alpha}$ Riemannian metric. This theorem leads to a short, entirely "metric" proof of the Convergence Theorem for Riemannian manifolds ([16], [5]). The main theorem of [17] is that a space satisfying (2) and (3) is a smooth manifold with boundary, with failure of geodesic completeness occurring precisely on the boundary. Theorem F covers the case of (1) and (2), and examples show that there are finite dimensional nonmanifolds satisfying any other combination of the above properties.

Theorem F is also a little progress toward solving the conjecture that limits in the Grove-Petersen class of Riemannian manifolds [9] are topological manifolds. These spaces have curvature bounded below, but are not

always geodesically complete. Theorem F proves the geodesically complete case, and reduces the general problem to considering neighborhoods of singularities (i.e., points where there is no upper curvature bound) which also lie in the closure of the set of geodesic terminals.

We do not, at present, know of a geometrically meaningful topology for the union $\overline{TX}$ of the spaces $\overline{T}_p$, except when X is *locally strictly convex* in the following sense: For any $p \in X$ and small $r > 0$, any pair of points in $B(p, r)$ can be joined by a unique minimal curve. We denote by $\pi: \overline{TX} \to X$ the projection which takes elements of $\overline{T}_p$ to p, and by $\exp: \overline{TX} \to X$ the function given by $\exp(v) = \exp_p(v)$, then $v \in T_p$.

Theorem G. *If (X, d) is a locally strictly convex, almost Riemannian space, then $\overline{TX}$ has a topology such that $\exp$ is continuous, $(\overline{TX}, \pi)$ has the structure of a vector bundle isomorphic to the topological tangent bundle of X [13, p. 251], and $\langle \cdot, \cdot \rangle_*$ is a continuous fiber metric. In particular, except possibly for $\dim X = 4$, X admits a smooth structure.*

Theorem G completely generalizes the main theorem of [2] (except for $\dim X = 4$), by removing the upper curvature bound from the hypothesis.

In light of Theorems F and G we can ask whether almost Riemannian spaces in general admit smooth structures. If some do not, then one must ask how large is the class of topological manifolds admitting an almost Riemannian metric.

Theorem H. *For any fixed k, n, and $D > 0$, the set of all n-dimensional almost Riemannian spaces of curvature $\geq k$ and diameter $\leq D$ is precompact in the Gromov-Hausdorff metric.*

Note that the above class of spaces properly contains the class of Riemannian n-manifolds of sectional curvature $\geq k$ and diameter $\leq D$.

1. The Generalized Toponogov Theorem

If X has curvature locally bounded below, and there are at most two directions at some point p, then X is isometric to a circle or an interval. Some of the lemmas below fail in this trivial case, and to avoid special exceptions in the statements *we assume for the rest of this paper that S_p has at least three elements*. The next lemma formulates a standard technique in proofs of Toponogov's Theorem (see, e.g., [3], [7] for an argument).

Lemma 1.1. *Let $\gamma_{ab}: [0, 1] \to X$ be a geodesic with $L(\gamma_{ab}) \leq \pi/\sqrt{k}$, γ_{ac} be minimal, and $0 = t_0 < t_1 < \cdots < t_i = 1$. Let γ_j denote γ_{ab} restricted to $[t_j, t_{j+1}]$, and suppose α_j is minimal from c to t_j, with $\alpha_0 = \gamma_{ac}$. If the triangles $(\alpha_j, \gamma_j, \alpha_{j+1})$ are A1 for $0 \leq j < i$, then $(\gamma_{ab}, \gamma_{ac})$ is A2.*

Lemma 1.2. *Let* $\varepsilon > 0$ *and* k *be arbitrary. Then*

(a) *there exists a number* $\delta > 0$ *such that if* γ_{xa} *and* γ_{xb} *are minimal curves in* S_k *of length* $L \leq 1$ *with* $d(a, b)/L < \delta$, *then* $\alpha(\gamma_{xa}, \gamma_{xb}) \leq \varepsilon$, *and*

(b) *there exists a* $\nu > 0$ *such that if* γ_{xa} *and* γ_{xb} *are minimal curves in* S_k *of length* $L \leq 1$ *with* $\alpha(\gamma_{xa}, \gamma_{xb}) \leq \nu$, *then* $d(a, b)/L < \varepsilon$.

Proof. For $a \neq x$, let $\psi(a)$ be the smallest number such that $\alpha(\gamma_{xa}, \gamma_{xb})$ $= \varepsilon$ if $d(a, b)/d(a, x) = \psi(a)$. The map ψ is easily seen to be continuous (in fact dependent only on $d(x, a)$) and positive, with $\lim_{a \to x} \psi(a) =$ $2 \cdot \sin(\varepsilon/2)$, and so has some positive minimum δ on $\overline{B}(x, 1)$. This proves part (a), and the proof of part (b) is similar.

Lemma 1.3. *Suppose* $B = B(p, r)$ *is a region of curvature* $\geq k$ *in* X. *Let* $\{\gamma_i\}$ *and* $\{\eta_i\}$ *be Cauchy sequences in* S_p. *For any positive* $s_i \to 0$ *and* $t_i \to 0$ *such that* $s_i \leq C(\gamma_i)$, $t_i \leq C(\eta_i)$, *and* $c_1 \leq s_i/t_i \leq c_2$ *for some* $c_1, c_2 \in (0, \infty)$, *if* $d_i = d(\gamma_i(s_i), \eta_i(t_i))$, *then*

$$\lim_{i \to \infty} \alpha(\gamma_i, \eta_i) = \lim_{i \to \infty} \cos^{-1}[(s_i^2 + t_i^2 - d_i^2)/(2s_i t_i)].$$

Proof. For any positive $s \leq C(\gamma_i)$ and $t \leq C(\eta_i)$, define $d_i(s, t) = d(\gamma_i(s), \eta_i(t))$ and

$$\varphi_i(s, t) = \cos^{-1}[(s^2 + t^2 - d_i(s, t)^2)/(2st)].$$

If φ_i is continuously extended to $(0, 0)$, then $\varphi_i(0, 0) = \alpha(\gamma_i, \eta_i)$. We have

$$\cos \varphi_j(s, t) - \cos \varphi_i(s, t)$$
$$= [(d_i(s, t) - d_j(s, t))(d_i(s, t) + d_j(s, t)]/(2st).$$

Assuming $0 < c_1 \leq s/t \leq c_2 < \infty$, we have

$$(d_i(s, t) - d_j(s, t))/s$$
$$\leq d(\gamma_i(s), \gamma_j(s))/s + d(\eta_i(t), \eta_j(t))/s$$
$$\leq d(\gamma_i(s), \gamma_j(s))/s + d(\eta_i(t), \eta_j(t))/(c_1 t).$$

By Lemma 1.2(b) and A2, the last amount is arbitrarily small for sufficiently large i and j, *independent of* s *and* t. By a similar argument we obtain that $(d_i(s, t) + d_j(s, t))/t$ is bounded, and conclude that for any $\zeta > 0$ there exists an m such that for all $i, j > m$, $|\varphi_i(s, t) - \varphi_j(s, t)| \leq$ $\zeta/2$. If m is also chosen large enough that $|\varphi_j(0, 0) - \lim_{i \to \infty} \alpha(\gamma_i, \eta_i)| \leq$ $\zeta/2$ for all $j > m$, then for $s \leq C(\gamma_j)$ and $t \leq C(\eta_j)$, $|\varphi_j(s, t) -$ $\lim_{i \to \infty} \alpha(\gamma_i, \eta_i)| \leq \zeta$, and the lemma follows.

Lemma 1.4. *If either S_p is precompact or p is not a geodesic terminal, then for any $\alpha, \beta \in S_p$, $\delta > 0$, and $a_1, a_2 > 0$ such that $a_1 + a_2 = \alpha(\alpha, \beta)$, there exists $\gamma \in S_p$ such that $|\alpha(\alpha, \gamma) - a_1| < \delta$ and $|\alpha(\alpha, \gamma) - a_2| < \delta$.*

Proof. Assume first that $c = \alpha(\alpha, \beta) < \pi$. We need only consider the case $a_1 = a_2 = c/2$. Let $\eta_i : [0, 1] \to X$ be minimal from $\alpha(2^{-i})$ to $\beta(2^{-i})$ and γ_i be minimal from p to $\eta_i(1/2)$; we denote by α_i the restriction of α to $[0, 2^{-i}]$, with similar notation for β. Let $a = \lim_{i \to \infty} \alpha(\alpha, \gamma_i)$ and $b = \lim_{i \to \infty} \alpha(\beta, \gamma_i)$. By the triangle inequality, $a + b \geq c$. Let $(\tilde{\alpha}_i, \tilde{\mu}_i, \tilde{\gamma}_i)$ and $(\tilde{\gamma}_i, \tilde{\mu}_i, \tilde{\beta}_i)$ represent $(\alpha_i, \eta_i|_{[0, 1/2]}, \gamma_i)$ and $(\gamma_i, \eta_i|_{[1/2, 1]}, \beta_i)$, respectively, so that $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ do not coincide (all of these curves are assumed parametrized on $[0, 1]$). If S_p is precompact, we can assume $\{\gamma_i\}$ is Cauchy, and by Lemma 1.3, $a = \lim_{i \to \infty} \alpha(\tilde{\alpha}_i, \tilde{\gamma}_i) = \lim_{i \to \infty} \alpha(\tilde{\alpha}_i, \tilde{\beta}_i)/2$ and $b = \lim_{i \to \infty} \alpha(\tilde{\beta}_i, \tilde{\gamma}_i) = \lim_{i \to \infty} \alpha(\tilde{\alpha}_i, \tilde{\beta}_i)/2$. On the other hand,

$$d(\tilde{\alpha}_i(1), \tilde{\beta}_i(1)) \leq d(\alpha(2^{-i}), \eta_i(1/2)) + d(\beta(2^{-i}), \eta_i(1/2))$$

and so $\alpha(\alpha, \beta) \leq \lim_{i \to \infty} \alpha(\tilde{\alpha}_i, \tilde{\beta}_i)$, and the case $c < \pi$ follows.

If p is not a geodesic terminal, let $T > 0$ and α' be such that the restrictions α_T and α'_T to $[0, T]$ together form a minimal curve. Let ζ_i be minimal from $\alpha(T)$ to $\eta_i(1/2)$ and $(\tilde{\alpha}_T, \tilde{\zeta}_i, \tilde{\gamma}_i)$ represent $(\alpha_T, \zeta_i, \gamma_i)$. Then $\alpha(\tilde{\alpha}_T, \tilde{\gamma}_i) \leq \alpha(\tilde{\alpha}_i, \tilde{\gamma}_i)$ by Definition A and if $AB_i = \alpha(\tilde{\alpha}_T, \tilde{\gamma}_i)$, $\limsup_{i \to \infty} A_i \leq a$. If $a' = \lim_{i \to \infty} \alpha(\alpha', \gamma_i)$ and $A'_i = \alpha(\tilde{\alpha}'_T, \tilde{\gamma}_i)$ then $\limsup_{i \to \infty} A'_i \leq a'$. Since $\lim_{i \to \infty} A_i + A'_i = \pi$, we obtain $\lim_{i \to \infty} \alpha(\tilde{\alpha}_i, \tilde{\gamma}_i) = \lim_{i \to \infty} A_i = a$. A similar argument proves $\lim_{i \to \infty} \alpha(\tilde{\alpha}_i, \tilde{\gamma}_i) = b$, and the proof of $c < \pi$ is complete as in the case of S_p compact.

If $c = \pi$, we can choose a direction distinct from α and β and apply the above special case.

Remark. Since a Riemannian manifold has *positive* cut radius, one of the few simplifications of the proof of Theorem C in the Riemannian case is that Lemma 1.4 is true for $\delta = 0$.

Lemma 1.5. *Suppose $\alpha, \beta : [0, 1] \to X$ are minimal starting at p with $L(\alpha) \leq \pi/\sqrt{k}$, $L(\beta) < \pi/\sqrt{k}$, and $0 < a = \alpha(\alpha, \beta) < \pi$. Suppose also that $\tilde{\alpha}, \tilde{\beta} : [0, 1] \to S_k$ are minimal, and $(\tilde{\alpha}, \tilde{\beta})$ represents (α, β). Let $a_1, a_2 > 0$ satisfy $a_1 + a_2 = \alpha(\alpha, \beta)$, $\tilde{\gamma}$ be minimal from $\tilde{\alpha}(1)$ to $\tilde{\beta}(1)$, and t be such that if $\tilde{\nu}$ is minimal from $\tilde{\alpha}(0)$ to $\tilde{\gamma}(t)$ then $\alpha(\tilde{\nu}, \tilde{\alpha}) = a_1$. If for every $\delta > 0$ there is a geodesic μ starting at p with $L(\mu) = L(\tilde{\nu})$ so that $|\alpha(\mu, \alpha) - a_1| < \delta$, $|\alpha(\nu, \alpha) - a_2| < \delta$, and both (α, μ) and (β, μ) are A2, then (α, β) is A2.*

Proof. Let $\zeta > 0$ be arbitrary. For sufficiently small δ, there is a representative $(\tilde{\alpha}, \tilde{\mu})$ of (α, μ) such that $d(\tilde{\mu}(1), \tilde{\gamma}; (t)) \le \zeta$. We assume both $\tilde{\mu}$ and μ are parameterized on $[0, 1]$; by A2 and the triangle inequality, $d(\alpha(1), \mu(1)) \le d(\tilde{\gamma}(0), \tilde{\gamma}(t)) + \zeta$. Since a similar argument applies to $d(\beta(1), \mu(1))$, we have

$$\begin{aligned}
d(\alpha(1), \beta(1)) &\le d(\alpha(1), \mu(1)) + d(\beta(1), \mu(1)) \\
&\le d(\tilde{\alpha}(1), \tilde{\gamma}(t)) + d(\tilde{\beta}(1), \tilde{\gamma}(t)) + 2\zeta \\
&= d(\tilde{\alpha}(1), \tilde{\beta}(1)) + 2\zeta.
\end{aligned}$$

Lemma 1.6. *Given k and $0 < D < \pi/\sqrt{k}$, for all sufficiently small $\chi > 0$, if $\tilde{\gamma}_{AB}$ and $\tilde{\gamma}_{AC}$ are unit minimal in S_k with $0 < \alpha(\tilde{\gamma}_{AB}, \tilde{\gamma}_{AC}) < \pi$, $L(\tilde{\gamma}_{AB}) \le D$, and $d(B, C) \le 4\chi$, then $\max\{d(A, \tilde{\alpha}(s))\} < t + \chi$ for any $0 < t \le \min\{L(\tilde{\gamma}_{AB}), L(\tilde{\gamma}_{AC})\}$ and no minimal curve $\tilde{\alpha}$ from $\tilde{\gamma}_{AB}(t)$ to $\tilde{\gamma}_{AC}(t)$.*

Proof. Since metric balls are convex for $k \le 0$, we need only to consider $k > 0$; by scaling the metric we reduce to $k = 1$, and clearly now we can assume $t > \pi/2$. Let $\chi > 0$ be small enough that

$$\cos D - (\cos 2\chi)(\cos(D + \chi)) > 0.$$

We fix curves $\tilde{\gamma}_{AB}$ and $\tilde{\gamma}_{AC}$ as above, assume $\tilde{\alpha}$ is parametrized on $[0, 1]$, and let $\tau = d(A, \tilde{\alpha}(1/2)) = \max\{d(A, \gamma(s))\}$. Letting $\lambda = L(\tilde{\alpha})$ and applying the cosine law to $\alpha(\tilde{\gamma}_{AB}, \tilde{\alpha})$ we obtain

$$\frac{\cos \tau - (\cos t)(\cos \lambda/2)}{\sin \lambda/2} = \frac{\cos t - (\cos t)(\cos \lambda)}{\sin \lambda},$$

which reduces to $\cos \tau = \cos t / \cos \lambda/2$.

Applying the sum formula to $\cos(\tau - t)$ we see that $\tau - t$ is maximized when $d(A, B) = d(A, C) = t = D$ and $\lambda = 4\chi$. Thus we only need to prove $\cos^{-1}(\cos D / \cos 2\chi) \le \cos(D + \chi)$, and this follows from the way χ was chosen.

Definition 1.7. If $\alpha, \beta : [0, 1] \to X$ are minimal curves starting at p, we call a proper triangle (α, γ, β) *p-based*.

Proposition 1.8. *Let $0 < D < \pi/\sqrt{k}$, and suppose $c_k \ge c > 0$ on $B(p, 2D)$ and $B(p, D)$ is geodesically complete. Then every p-based triangle in $B = B(p, D)$ is A1.*

Proof. Let $\chi < D$ be as in Lemma 1.6 and also less than $c/12$. Let τ be small enough that if $\tilde{\alpha}$ and $\tilde{\gamma}$ are geodesics in S_k with $\alpha(\tilde{\alpha}, \tilde{\gamma}) \le \tau$, then $d(\tilde{\alpha}(t), \tilde{\gamma}(t)) \le \chi$ for all $0 \le t \le D$. We call a p-based triangle (α, γ, β) *thin* if $\alpha(\alpha, \beta) \le \tau$ and γ is minimal. Note that $\chi < D$ implies γ lies in $B(p, 2D)$. Consider the following statements:

S1(n, m). If (α, γ, β) is thin such that $(n-1)\cdot\chi \le L(\alpha) \le n\cdot\chi$ and $(m-1)\cdot\chi \le L(\beta) \le m\cdot\chi$, then (α, γ, β) is A1.

S2(n, m). If (α, γ, β) is thin such that $(n-1)\cdot\chi \le L(\alpha) \le n\cdot\chi$ and $(m-1)\cdot\chi \le L(\beta) \le m\cdot\chi$, then (α, β) is A2.

S3(n). If (α, γ, β) is p-based and lies in $\overline{B}(p, n\cdot\chi)$, then (α, γ, β) is A1.

Note that by monotonicity S1(n, m) and S3(n) state equivalently that (α, γ) and (β, γ) are A2. S1(6,6), S2(6, 6), and S3(6) are true by the way χ was chosen. We will prove by induction that S3(n) holds for $n \le (D-3\chi)/\chi$. This will show that every p-based triangle in $\overline{B}(p, D-3\chi)$ is A1, and the proposition follows from letting $\chi \to 0$.

If $\alpha(\alpha, \beta) = 0$, the proof is trivial in each step; we will always assume $\alpha(\alpha, \beta) > 0$ without further mention.

Step 1. S1(n, n) *and* S2(n, n) *imply* S2$(n, n+1)$.

Proof. Fix a thin triangle (α, γ, β) such that $n\cdot\chi \le L(\alpha) \le (n+1)\cdot\chi$ and $(n-1)\cdot\chi \le L(\beta) \le n\cdot\chi$. Let q lie on α such that $d(p, q) = L(\beta)$, let $x = \alpha(1)$, $y = \beta(1)$, and η be minimal from y to q. If ν is the segment of α from p to q, we obtain from S2(n, n) that (β, ν) is A2, and from Sl(n, n) that (ν, η) is A2. S2(n, n) implies $\{\text{diam } x, y, q\} \le 3\chi$; if ζ is the segment of α from q to x, then both (η, ζ) and (ζ, γ) are A2, and that (α, β) is A2 follows from Lemma 1.1.

Step 2. S3(n) *implies that if* ω *is minimal from* p *to a point* $a \in B(p, (n-1)\cdot\chi)$, *and* ξ *is minimal starting with* a *with* $L(\xi) \le 4\chi$, *then* (ω, ξ) *is* A2.

Proof. Let $R' = L(\omega)$, assume both ω and ξ are unit, and let $x = \xi(L(\xi))$. Choose a representative $(\tilde{\omega}, \tilde{\xi})$ in S_k, denoting the corresponding points with capitals. Let $\tilde{\mu}$ be the unit minimal from P to X, $R = \min\{R', L(\tilde{\mu})\}$, and $\tilde{\kappa}$ be minimal from A to $\tilde{\mu}(R)$. Since $n \le (D-3\chi)/\chi$, $L(\tilde{\mu})+L(\tilde{\xi}) \le D$, and by Lemma 1.6, for all s, $d(P, \tilde{\kappa}(s)) < R+\chi \le n\cdot\chi$. For any sufficiently small $\delta > 0$, by Lemma 1.4 and geodesic completeness there exists a geodesic $\kappa: [0, 1] \to X$ starting at a of length $L = L(\tilde{\kappa})$ with $|\alpha(\kappa, \omega) - \alpha(\tilde{\kappa}, \tilde{\omega})| < \delta$ and $|\alpha(\kappa, \xi) - \alpha(\tilde{\kappa}, \tilde{\xi})| < \delta$. For small enough δ, S3(n) implies that $d(p, \kappa(s)) < n\cdot\chi$ for all s and (κ, ω) is A2. On the other hand, by the triangle inequality, $L(\tilde{\kappa}) \le 8\chi$ and $\text{diam}\{\kappa(1), a, x\} < 12\chi$; thus (κ, ξ) is A2. Lemma 1.5 now implies (ω, ξ) is A2.

Step 3. S1(m, m), S2(m, m), *for all* $m \le n$, *and* S3(n) *imply* S1$(n, n+1)$.

Proof. Let (α, γ, β) be as above. The proof that (α, γ) is A2 is similar to the argument in Step 1. Let a be the point on β such that

$d(a, y) = \chi$, $R = d(p, a)$, ω denote the segment of β from p to a, and ξ be minimal from a to x. By the triangle inequality (and the fact that $\alpha(\alpha, \beta) \leq \tau$), $L(\xi) \leq 4\chi$, and Step 2 implies (ω, ξ) is A2. By a proof similar to that of Step 1, S1(n, n) and S2(n, n) imply (α, ω) is A2. If λ denotes the segment of β from a to y, (ξ, λ, γ) is also A1, and the proof is complete by Lemma 1.1.

Step 4. S1$(n, n + 1)$ *and* S2$(n, n + 1)$ *imply* S1$(n + 1, n + 1)$ *and* S2$(n + 1, n + 1)$.

Proof. This is an easy application of Lemma 1.1.

Step 5. S1(m, m), S2(m, m), *for all* $m \leq n + 1$, *and* S3(n) *imply* S3$(n + 1, n + 1)$ *(and the induction is complete)*.

Proof. Let (α, γ, β) be p-based, with $\gamma: [0, 1] \to \overline{B}(p, (n+1) \cdot \chi)$, and assume first the $\alpha(\alpha, \beta) < \pi$. We claim the following: If ζ is minimal from p to $q = \gamma(t)$, for some t, $t_i \to t$, and η_i is minimal from p to $\gamma(t_i)$, then for all sufficiently large i, $(\zeta_i, \gamma_i, \eta_i)$ is A1, where γ_i is γ restricted to the interval between t_i and t. By passing to a subsequence, if necessary, we can assume $\lim_{i \to \infty} \alpha(\eta_i, \zeta)$ exists and is either 0 or $2\varepsilon > 0$. In the first case the proof is complete by Sl(m, m) for $m \leq n + 1$. In the second case $\alpha(\eta_i, \zeta) > \varepsilon$ for all large i. Let a be the point on ζ such that $d(a, q) = 2 \cdot \chi$, ω denote the segment of ζ from p to a, ν that from a to q, and μ_i be minimal from a to $\gamma(t_i)$. Since $L(\omega) + L(\mu_i) \to L(\eta_i)$, if $(\tilde{\zeta}, \tilde{\eta}_i)$ represents (ζ, η_i) in S_k then $\alpha(\tilde{\zeta}, \tilde{\eta}_i) \to 0$. Now $\alpha(\eta_i, \zeta) > \varepsilon$ implies (ζ, η_i) in S_k then $\alpha(\tilde{\zeta}, \tilde{\eta}_i) \to 0$. Now $\alpha(\eta_i, \zeta) > \varepsilon$ implies (ζ, η_i) is A2 for large i. By Step 2, (ω, μ_i) is A2. Since γ_i is minimal for large enough i and (μ_i, ν), (ν, γ_i) are A2, it follows from Lemma 1.1 that (ζ, γ_i) is A2. By a similar argument we obtain that (η_i, γ_i) is A2 for all sufficiently large i, and the proof of the claim is complete.

For $s > 0$, let γ_s denote $\gamma|_{[0, s]}$, and denote by A1(s) the statement: *for every minimal* β_s *from* p *to* $\gamma(s)$, $(\alpha, \gamma_s, \beta_s)$ *is* A1. The above claim implies that A1(δ) is true for sufficiently small $\delta > 0$, and the claim and Lemma 1.1 prove that if A1(T) is true for some T, then A1$(T + \delta)$ is true. Likewise, if A1(s) is true for all $s < T$ then A1(T) is true; it follows that A1(T) holds for all T. This completes the proof for $\alpha(\alpha, \beta) < \pi$.

For $\alpha(\alpha, \beta) = \pi$, either α and β together form a minimal curve, in which case the proof is trivial, or the minimal curve between their endpoints does not pass through p, in which case we proceed as above.

Proof of Theorem C. By Proposition 1.8 the proof is complete for $k \leq 0$. For $k > 0$ we have that every proper triangle (α, γ, β) in X such that $d(\alpha(0), \gamma) < \pi/\sqrt{k}$ is A1. Applying Lemma 1.1 we have that (α, γ, β)

is A1 if α and γ are minimal of length $< \pi/\sqrt{k}$ and β is minimal of length $\pi/\sqrt{k}$. A limiting argument shows that if α and β are minimal starting at p of length $\pi/\sqrt{k}$, then α and β have both endpoints in common (this also proves Corollary D). The theorem now follows by an easy application of Lemma 1.1.   q.e.d.

In order to prove Theorem E we first reconcile the conclusion of Theorem C with Definition A (1.9) and then prove a rigidity result (1.10).

Proposition 1.9. *If X is geodesically complete of curvature uniformly $\geq k$, then all of X is a region of curvature $\geq k$; in other words, $c_k = \infty$.*

Proof. Let $(\gamma_{ab}, \gamma_{bc}, \gamma_{ca})$ be a triangle of minimal curves in X. By monotonicity we need only show that for any x strictly between a and b on γ_{ab}, the following holds: if γ_{ax} is the segment of γ_{ab} from a to x, γ_{xc} is minimal, and $(\tilde{\gamma}_{AX}, \tilde{\gamma}_{XC}, \tilde{\gamma}_{CA})$ represents $(\gamma_{ax}, \gamma_{xc}, \gamma_{ca})$ in S_k, then by extending $\tilde{\gamma}_{AX}$ to a minimal curve $\tilde{\gamma}_{AB}$ with $L(\tilde{\gamma}_{AB}) = d(a, b)$ it will follow that $d(B, C) \geq d(b, c)$. But by A1, $\alpha(\gamma_{ax}, \gamma_{xc}) \geq \alpha(\tilde{\gamma}_{AX}, \tilde{\gamma}_{XC})$ and so $\alpha(\gamma_{bx}, \gamma_{xc}) \geq \alpha(\tilde{\gamma}_{BX}, \tilde{\gamma}_{XC})$; the proof is complete by A2.

Proposition 1.10. *Suppose X is geodesically complete of curvature uniformly $\geq k$ and (γ_1, γ_2) is proper and A2 with equality, with representative $(\overline{\gamma}_1, \overline{\gamma}_2)$, $i = 1, 2$. Then for all $0 \leq t \leq L_2$, $d(\gamma_1(L_1), \gamma_2(t)) = d(\overline{\gamma}_1(L_1), \overline{\gamma}_2(t))$. In addition, if γ_2 is minimal, then $d(\gamma_1(s), \gamma_2(t)) = d(\overline{\gamma}_1(s), \overline{\gamma}_2(t))$ for all $0 \leq s \leq L_1$.*

Proof. The proof is trivial when $\alpha(\gamma_1, \gamma_2) = 0$ or π; we assume otherwise below. The second statement of the proposition follows immediately from Proposition 1.9 and A2. If γ_2 is not minimal, partition the domain of γ_2 into finitely many intervals $[t_i, t_{i+1}]$ such that the restriction α_i of γ_2 to $[t_i, t_{i+1}]$ is minimal. Let β_i be minimal from $\gamma_1(L_1)$ to $\alpha_i(t_i)$ (e.g., $\beta_1 = \gamma_1$). Then by an argument similar to the proof of Lemma 1.1 we see that (β_i, α_i) is A2 with equality for all i, and that if $\tilde{\beta}_i$ is minimal in S_k from $\tilde{\gamma}_1(L_1)$ to $\tilde{\gamma}_2(t_i)$, then $L(\tilde{\beta}_i) = L(\beta_i)$. The proof is now finished by the special case proved above.

Proof of Theorem E. We will show that every wedge (γ_1, γ_2), such that γ_1 and γ_2 are minimal of length $< \pi/\sqrt{k}$, is A2 with equality. Let p be any point such that there exists a q with $d(p, q) = \pi/\sqrt{k}$. Choosing a minimal curve from p to q we can apply A2 (via Theorem C) to conclude that every geodesic of length $\pi/\sqrt{k}$ starting at p is minimal from p to q, and geodesics starting at q behave likewise. Using geodesic completeness we can extend any minimal curve α from p to q to a geodesic γ passing through q and returning to p. For any $\varepsilon > 0$ small enough, if $c = \pi/\sqrt{k}$, $a = c - \varepsilon$, and $b = c + \varepsilon$, then $\mu = \gamma|_{[a,b]}$ is minimal. If $\eta = \gamma|_{[0,a]}$ and

$\nu = \gamma|_{[b,2c]}$, then applying A1 to (η, μ, ν) we obtain $\alpha(\eta, \nu) = \pi$ (i.e., γ is a closed geodesic). Thus, (η, ν) is A2 with equality.

Suppose β is minimal starting at p of length a. Since $\alpha(\alpha, \beta) + \alpha(\beta, -\nu) = \pi$, from the triangle inequality and A2 we obtain that (α, β) is A2 with equality. From Proposition 1.8 we conclude that if γ_1 and γ_2 are *any* minimal curves starting at p of length $< \pi/\sqrt{k}$, then (γ_1, γ_2) is A2 with equality.

The proof will now be complete if we can show that for any point $x \in X$ there exists a point $y \in X$ such that $d(x, y) = \pi/\sqrt{k}$. Let γ_1 be minimal from p to x and γ_2 be minimal starting at p of length $L = \pi/\sqrt{k} - L(\gamma_1)$, such that $\alpha(\gamma_1, \gamma_2) = \pi$. Then since (γ_1, γ_2) is A2 with equality, $d(x, \gamma_2(L)) = \pi/\sqrt{k}$.

If X is locally compact, we know from [2] that X is a manifold. Satz (p. 361 of [18]) states that a manifold of constant curvature is isometric to a space form; since X has diameter $\pi/\sqrt{k}$, X must be a sphere.

Examples 1.11. A standard hemisphere shows Theorem E is false without geodesic completeness. For a more interesting example, one can "suspend" $\mathbf{RP}^n$ with the metric of constant curvature 1 in the following way: Let $W_{\sin}\mathbf{RP}^n$ be the warped product of $\mathbf{RP}^n$ with $(0, \pi)$ as the base space and sine as the warping function. We can complete the space by attaching two "endpoints." One can show (cf. [8, Example 2.5]) that the resulting space X satisfies the conclusion of Theorem C with $k = 1$. On the other hand, diam $X = \pi$, but X is not a manifold, let alone a sphere. Of course, X is not geodesically complete at the "endpoints."

2. Local consequences of curvature bounded below

We assume throughout this section that X has curvature locally bounded below. Briefly, we recall the definitions of spaces and maps associated with S_p. The *tangent space* T_p at a point $p \in X$ is the metric space obtained from $S_p \times \mathbf{R}^+$ by identifying all points of the form $(\gamma, 0)$ (and denoting the resulting point 0) with the following metric, where the class of (γ, t) in the identification space is denoted $t\gamma$:

$$\delta(t\gamma, s\beta) = (t^2 + s^2 - 2st \cdot \cos\alpha(\gamma, \beta))^{1/2}.$$

We denote by $\overline{S}_p$ the metric completion of S_p; then elements of $\overline{T}_p$ can clearly be written in the form $t\overline{\gamma}$, where $\overline{\gamma} \in \overline{S}_p$, $t \in \mathbf{R}^+$, and $0\overline{\gamma} = 0$. For $v \in T_p$, we let $C(v) = C(v/\|v\|)$. We define the *exponential map* $\exp_p: T_p \to X$ by $\exp_p(s \cdot \gamma) = \gamma(s)$ wherever this makes sense (if X is

geodesically complete $\exp_p$ is defined on all of T_p). $\exp_p$ is (locally) a radial isometry and preserves the angle between radial geodesics.

Proposition 2.1. *For any $p \in X$, $\exp_p$ is continuous.*

Proof. It suffices to prove that, if γ and γ_i are geodesics starting at p such that $\lim_{i \to \infty} \alpha(\gamma_i, \gamma) = 0$, then $\lim_{i \to \infty} \gamma_i(t) = \gamma(t)$ whenever $\gamma_i(t)$ and $\gamma(t)$ are defined. We can assume $t > 0$; let

$$T = \sup\{t \colon \lim_{i \to \infty} \gamma_i(t) = \gamma(t)\}.$$

If

$$T < W = \sup\{t \colon \gamma(t) \text{ and } \gamma_i(t) \text{ are defined}\},$$

we can find $0 < c < T < d < W$ such that $\gamma|_{[c,d]}$ is minimal. Let α_i be minimal from $\gamma(d)$ to $\gamma_i(T)$, and let η_i denote the segment of γ_i backwards from $\gamma_i(T)$ to $\gamma_i(c)$, and $\nu_i|_{[T,d]}$. By the triangle inequality, $L(\eta_i) + L(\alpha_i) - d(\gamma_i(c), \gamma(t)) \leq d(\gamma_i(T), \gamma(T)) + d(\gamma_i(c))$. The latter quantity is arbitrarily small for large i. Since $L(\eta_i)$ and $L(\alpha_i)$ have positive lower bounds, it follows from the cosine law for S_k and A1 that $\lim_{i \to \infty} \alpha(\alpha_i, \eta_i) = \pi$. Then $\lim_{i \to \infty} \alpha(\alpha_i, \nu_i) = 0$, and we obtain from Lemma 1.2(b) and A2 that $\lim_{i \to \infty} \gamma_i(d) = \gamma(d)$, a contradiction.   q.e.d.

There is a continuous extension of $\exp_p$ onto the closure in $\overline{T}_p$ of its domain of definition; this extension is also denoted by $\exp_p$.

For the remainder of this paper we let $B = B(p, r) \subset X$ be a region of curvature $\geq k$. We call a minimal curve *strictly minimal* if it is the unique minimal curve between its endpoints.

Lemma 2.2. *Suppose $\overline{B}$ is compact an let γ_{pb} and γ_{pc} be strictly minimal in B. Then for any $a_i \to p$ and minimal curves γ_i and η_i from a_i to b and c, respectively, $\lim_{i \to \infty} \alpha(\gamma_i, \eta_i) \geq \alpha(\gamma_{pb}, \gamma_{pc})$.*

Proof. Let $\zeta > 0$. Choose $T > 0$ so that if γ_{PB} and γ_{PC} are minimal curves in S_k with $d(P, B) = d(P, C) = T$ and $d(B, C) = d(\gamma_{pb}(T), \gamma_{pc}(T))$, then $\alpha(\gamma_{pb}, \gamma_{pc}) - \alpha(\gamma_{PB}, \gamma_{PC}) \leq \zeta$. $\overline{B}$ is compact and therefore $\gamma = \lim_{i \to \infty} \gamma_i$ and $\eta = \lim_{i \to \infty} \eta_i$ exist and are minimal from p to b and c, respectively. But γ_{pb} and γ_{pc} are strictly minimal, hence $\gamma = \gamma_{pb}$, $\eta = \gamma_{pc}$, and $\lim_{i \to \infty} \alpha(\gamma_i \gamma_{bp}) = \lim_{i \to \infty} \alpha(\eta_i, \gamma_{cp}) = 0$. By A2, $\lim_{i \to \infty} d(\gamma_{pb}(T), \gamma_i(T)) = \lim_{i \to \infty} d(\gamma_{pc}(T), \eta_i(T)) = 0$. If C_i is the point closest to C in S_k such that $d(P, C_i) = T$ and $d(B, C_i) = d(\gamma_i(T), \eta_i(T))$, then applying A1 and Lemma 1.2, we obtain $\lim_{i \to \infty} \alpha(\gamma_i, \eta_i) \geq \lim_{i \to \infty} \alpha(\gamma_{PB}, \gamma_{PC_i}) = \alpha(\gamma_{PB}, \gamma_{PC}) \geq \alpha(\gamma_{pb}, \gamma_{pc}) - \zeta$.

Lemma 2.3. *Suppose either p is not a geodesic terminal or $\overline{B}$ is compact. If $\alpha_0, \beta_0 \in S_p$ are such that $\alpha(\alpha_0, \beta_0) \geq \pi - \varepsilon$ for some $\varepsilon > 0$, then $\alpha(\alpha_0, \gamma_0) + \alpha(\gamma_0, \beta_0) \leq \pi + \varepsilon$, for any $\gamma_0 \in S_p$.*

Proof. If p is not a geodesic terminal then we can find $\alpha'_0 \in S_p$ such that $\alpha(\alpha_0, \alpha'_0) = \pi$ and the proof follows from the triangle inequality.

Suppose $\overline{B}$ is compact. Let $p_i \to p$ be points on α and a, b, c be points on $\alpha_0, \beta_0, \gamma_0$ such that the segments of $\alpha_0, \beta_0, \gamma_0$ from p to a, b, c, respectively, are all strictly minimal. For $i > 0$ let α_i be the segment of α_0 from p_i to a, α'_i be the segment of α from p_i to p, and β_i, γ_i be minimal from p_i to b, c, respectively. If for all i we set $a_i = \alpha(\alpha_i, \gamma_i)$ and $b_i = \alpha(\beta_i, \gamma_i)$, then by Lemma 2.2 $a_0 \le \lim_{i \to \infty} a_i$ and $b_0 \le \lim_{i \to \infty} b_i$. Therefore,

$$a_0 + b_0 \le \lim_{i \to \infty} a_i + b_i \le \lim_{i \to \infty} a_i + (\pi - a_i) + \alpha(\alpha'_i, \beta_i)$$
$$= 2\pi - \lim_{i \to \infty} \alpha(\alpha_i, \beta_i) \le \pi + \varepsilon,$$

since $\lim_{i \to \infty} \alpha(\alpha_i, \beta_i) \ge \alpha(\alpha, \beta)$.

Proposition 2.4. *If p is not a geodesic terminal or S_p is precompact, then $(\overline{S}_p, \alpha)$ is an inner metric space. Suppose, in addition, that either B is geodesically complete or $\overline{B}$ is compact. If $\overline{S}_p$ is convex, then $c_1(\gamma) = \pi$ for all $\gamma \in \overline{S}_p$ (in particular, $\overline{S}_p$ has curvature uniformly ≥ 1).*

Proof. Let $\overline{\alpha}, \overline{\beta} \in \overline{S}_p$ and $A = \alpha(\overline{\alpha}, \overline{\beta})$. For any $\varepsilon > 0$, by Lemma 1.4 we can choose $\gamma_1 \in S_p$ such that $\alpha(\overline{\alpha}, \gamma_1) + \alpha(\gamma_1, \overline{\beta}) \le (1 + \varepsilon/2) \cdot A$. We can then choose $\gamma_2, \gamma_3 \in S_p$ such that

$$\alpha(\overline{\alpha}, \gamma_2) + \alpha(\gamma_2, \gamma_1) + \alpha(\gamma_1, \gamma_3) + \alpha(\gamma_3, \overline{\beta}) \le (1 + 3\varepsilon/4) \cdot A.$$

Repeating this procedure we can construct a map from the dyadic rationals of $[0, 1]$ into $\overline{S}_p$ whose continuous extension to $[0, 1]$ is a curve from $\overline{\alpha}$ to $\overline{\beta}$ of length $(1 + \varepsilon) \cdot A$. Therefore α is an inner metric on $\overline{S}_p$.

To prove the second statement it suffices, by monotonicity and Definition A, to show the following: Suppose $\overline{\alpha}_1, \cdots, \overline{\alpha}_4 \in \overline{S}_p$ satisfy $0 < \alpha(\overline{\alpha}_i, \overline{\alpha}_j) < \pi$ for all i, j, $\overline{\alpha}_2$ is between $\overline{\alpha}_1$ and $\overline{\alpha}_3$, and Γ_i are unit geodesics starting at $P \in S_k^3$ so that $\alpha(\overline{\alpha}_i, \overline{\alpha}_j) = \alpha(\Gamma_i, \Gamma_j)$ for all pairs (i, j) except $(3, 4)$ and $(4, 3)$. Then $\alpha(\overline{\alpha}_4, \overline{\alpha}_3) \le \alpha(\Gamma_4, \Gamma_3)$.

Let $\alpha_{ij} \in S_p$ be such that $\lim_{i \to \infty} \alpha_{ij} = \overline{\alpha}_j$, and choose $t_i \to 0$ such that (1) $0 < t_i \le C(\alpha_{ij})$ and (2) if Q_i is the intersection of the minimal curve from $\Gamma_1(t_i)$ to $\Gamma_3(t_i)$ and $s_i = d(P, Q_i)$, then $s_i \le C(\alpha_{i2})$. Now let Γ_{ij} be unit geodesics starting at P in S_k^3 such that the following holds for $A_{ij} = \Gamma_{ij}(t_i)$ if $j \ne 2$ and $A_{i2} = \Gamma_{i2}(s_i)$: If $a_{ij} = \alpha_{ij}(t_i)$ for $j \ne 2$ and $a_{i2} = \alpha_{i2}(s_i)$, then $d(a_{ij}, a_{ik}) = d(A_{ij}, A_{ik})$ for all $(j, k) \notin \{(3, 4), (4, 3)\}$ and $d(A_{i3}, A_{i4})$ is the closest to $d(a_{i3}, a_{i4}(t_i))$ of its two

possible values. Choosing a subsequence if necessary, we can assume $\{\Gamma_{ij}\}$ is convergent for all j, and by Lemma 1.3 $\lim_{i\to\infty}\alpha(\Gamma_{ij},\Gamma_{ik})=\alpha(\Gamma_j,\Gamma_k)$ for all $j>k$ with $j\neq 3$ and $k\neq 4$. From the way $\{\Gamma_{ij}\}$ was chosen it follows that $\lim_{i\to\infty}\alpha(\Gamma_{i3},\Gamma_{i4})=\alpha(\Gamma_3,\Gamma_4)$ as well. Therefore, by Lemma 1.3 it suffices to show that

$$\lim_{i\to\infty}d(A_{i3},A_{i4})/t_i\geq\lim_{i\to\infty}d(a_{i3},a_{i4})/t_i.$$

Let β_{ijk} be minimal in X from a_{ij} to a_{ik} and Ψ_{ijk} be minimal in S_k^3 from A_{ij} to A_{ik}. Then since $\alpha(\Gamma_1,\Gamma_2)+\alpha(\Gamma_2,\Gamma_3)=\alpha(\Gamma_1,\Gamma_3)$, given any $\varepsilon>0$, for all sufficiently large i we have by A1 that $\alpha(\beta_{i12},\beta_{i23})\geq\alpha(\Psi_{i12},\Psi_{i23})>\pi-\varepsilon$. The triangle inequality and Lemma 2.3 now imply that $\alpha(\Psi_{i23},\Psi_{i24})\geq\alpha(\beta_{i23},\beta_{i24})-2\varepsilon$ for all large i. In other words, by A2 and the triangle inequality, $d(A_{i3},A_{i4})\geq d(a_{i3},a_{i4})-K_i$, where K_i is the length of the side of a triangle in S_k with its opposite angle 2ε and adjacent sides of length $d(a_{i2},a_{i3})$. In other words, $\lim_{i\to\infty}K_i=2\cdot d(a_{i2},a_{i3})\cdot\sin\varepsilon$. On the other hand, $\lim_{i\to\infty}d(a_{i2},a_{i3})/t_i=2\sin\alpha(\Gamma_2,\Gamma_3)$, and the proof is complete by letting $\varepsilon\to 0$.

Proposition 2.5. *Suppose that either $\overline{B}$ is compact and p is not a geodesic terminal, or B is geodesically complete. Then $\overline{S}_p$ is convex, geodesically complete, and has rigid curvature 1.*

Proof. If a *single* geodesic passes through p, then $\overline{S}_p$ has diameter π. The proof will therefore be complete by Proposition 2.4 and Theorem E if we show that $\overline{S}_p$ is convex and geodesically complete.

Let $\overline{\eta}_1,\overline{\eta}_2\in\overline{S}_p$ be distinct and $a=(\pi-\alpha(\overline{\eta}_1,\overline{\eta}_2))/2$. Since p is not a geodesic terminal, there exists an $\overline{\eta}_1'\in\overline{S}_p$ such that $\alpha(\overline{\eta}_1,\overline{\eta}_1')=\pi$. Because $\overline{S}_p$ is an inner metric space and S_p is dense in $\overline{S}_p$, we can choose $\gamma_i\in S_p$ such that

$$|\alpha(\overline{\eta}_2,\gamma_i)-a|\leq 2^{-i}\quad\text{and}\quad|\alpha(\overline{\eta}_1,\gamma_i)-a|\leq 2^{-i}.$$

We claim that any such sequence $\{\gamma_i\}$ is Cauchy. If not, we can find subsequences, which we reindex and denote by $\{\gamma_{1i}\}$ and $\{\gamma_{2i}\}$, such that $\alpha(\gamma_{1i},\gamma_{2i})>\delta>0$ for all i. Let $\eta_{1i},\eta_{2i}\in S_p$ such that $\eta_{1i}\to\overline{\eta}_1$ and $\eta_{2i}\to\overline{\eta}_2$. In the plane, choose points X,A,B, and T such that A,B, and T are collinear, $XA=1$, $XB=1$, and $\alpha(\overline{XA},\overline{XB})=\alpha(\overline{XB},\overline{XT})=a$. Choose $t_i\to 0$ such that $t_i\leq\min\{C(\gamma_{1i}),C_{\gamma_{2i}}\}$, $r_i=t_i\cdot XA/XT\leq C(\eta_{1i})$, and $s_i=t_i\cdot XB/XT\leq C(\eta_{2i})$. Let β_i,ζ_{1i}, and ζ_{2i} be minimal curves from $\eta_{2i}(s_i)$ to $\eta_{1i}(r_i)$, $\gamma_{1i}(t_i)$, and $\gamma_{2i}(t_i)$, respectively. By Lemma 1.3 for $k=1,2$ and any $\lambda>0$ there exists a j such that for all $i<j$, $L(\beta_i)+L(\zeta_{ki})\leq(1+\lambda)\cdot d(\eta_{1i}(r_i),\gamma_{ki}(t_i))$; it

follows that the angle of a wedge W_i in S_k representing the wedge formed by β_i and ζ_{ki} tends to π. A1 then implies that $\lim_{i \to \infty} \alpha(\beta_i, \zeta_{ki}) = \pi$. By Lemma 2.3 and the triangle inequality, $\lim_{i \to \infty} \alpha(\zeta_{1i}, \zeta_{2i}) = 0$. Let Z_{1i} and Z_{2i} be unit minimal curves in S_K, with common endpoint y and other endpoints z_{1i} and z_{2i}, respectively, such that $L(Z_{1i}) = L(\zeta_{1i})$, $L(Z_{2i}) = L(\zeta_{2i})$, and $\alpha(Z_{1i}, Z_{2i}) = \alpha(\zeta_{1i}, \zeta_{2i})$. Then

$$
\begin{aligned}
0 &= \lim_{i \to \infty} d(z_{1i}, z_{2i})/L(Z_{1i}) \\
&\geq \lim_{i \to \infty} d(\gamma_{1i}(t_i), \gamma_{2i}(t_i))/L(\zeta_{i1}) \\
&= \lim_{i \to \infty} d(\gamma_{1i}(t_i), \gamma_{2i}(t_i)) \cdot (XT/BT)/t_i.
\end{aligned}
$$

This last limit being 0 implies that $\lim_{i \to \infty} \alpha(\gamma_{1i}, \gamma_{2i}) = 0$, a contradiction.

We now have shown that for any distinct $\overline{\eta}_1, \overline{\eta}_2 \in \overline{S}_p$ there exists a *unique* $\overline{\eta}_3 \in \overline{S}_p$ such that $\alpha(\overline{\eta}_1, \overline{\eta}_2) = \alpha(\overline{\eta}_2, \overline{\eta}_3) = \alpha(\overline{\eta}_1, \overline{\eta}_3)/2$. Applying this result to $\overline{\eta}_2$ and $\overline{\eta}'_1$ we can also find a unique $\overline{\eta}_4 \in \overline{S}_p$ such that $\alpha(\overline{\eta}_1, \overline{\eta}_4) = \alpha(\overline{\eta}_4, \overline{\eta}_2) = \alpha(\overline{\eta}_1, \overline{\eta}_2)/2$. Using these two results and an argument similar to the first part of the proof of Proposition 2.4 (without the ε's!) we can construct a minimal curve between any two points and extend any minimal curve.

Proposition 2.6. *If X is geodesically complete of rigid curvature 1 and $\dim X \geq n > 0$, then X contains a convex subset isometric to S^n.*

Proof. It is immediate from A2 (with equality) that any geodesic in X of length π is minimal. In particular, X has diameter π. If X is compact we know already from Theorem E that X is itself a sphere. Suppose X is noncompact; then X is infinite dimensional and we need to show that for all n, X contains a convex subset isometric to S^n. By Proposition 2.5, for any $p \in X$, $\overline{S}_p$ is again geodesically complete of rigid curvature 1. Furthermore, since $C(\gamma) = \pi$ for all $\gamma \in S_p$, $\overline{S}_p = S_p$ and S_p is noncompact. Suppose that S_p has a convex subset S isometric to S^n for some $n > 0$. Then it follows from A2 (with equality) that $S' = \{\gamma(t): \gamma \in S \text{ and } t \leq \pi\}$ is a convex subset of X isometric to S^{n+1} The proof is now complete by induction, since any geodesically complete space of rigid curvature 1 contains a copy of S^1, namely any geodesic of length 2π.

Lemma 2.7. *Let $A_i, B_i, C_i \in S_k$, with A_i, C_i distirct,*

$$
\lim_{i \to \infty} d(A_i, C_i) = 0,
$$

and $d(A_i, B_i) \geq D$ for some $D > 0$ and all i. Suppose γ is minimal from A_i to B_i, and β_i is minimal from A_i to C_i. Then $\varphi = \lim_{i \to \infty} \alpha(\gamma_i, \beta_i)$

exists if and only if

$$L = \lim_{i \to \infty} [d(A_i, B_i) - d(B_i, C_i)]/d(A_i, C_i)$$

exists. If φ and L exist, $L = \cos \varphi$.

Proof. If $d(A_i, B_i) = d(B_i, C_i)$ for all i, then $\lim_{i \to \infty} \alpha(\gamma_i, \beta_i) = \pi/2$ follows from the cosine laws for S_k. In the general case, let α'_i be unit minimal of length $\max\{d(A_i, B_i), d(A_i, C_i)\}$ starting at B_i and containing the point C_i; let $D_i = \alpha'_i(d(A_i, B_i))$ and α_i be the segment of α'_i from D_i to B_i. If $D_i = A_i$ for all large i, then α_i and γ_i coincide, and the lemma is trivial. Otherwise, applying the above special case we obtain that if ζ_i is minimal from D_i to A_i, then $\lim_{i \to \infty} \alpha(\zeta_i, \beta_i) = \lim_{i \to \infty} \alpha(\zeta_i, \gamma_i) = \pi/2$. The lemma now follows from the cosine laws and the definition of angle.

Lemma 2.8. *If $S \subset \overline{S}_p$ is compact, then for every small $\varepsilon > 0$ there exists a $\rho > 0$ such that for all $\overline{\gamma} \in S$ and $0 < t < \rho$, $1 - \varepsilon \le d(p, \exp_p(t\overline{\gamma}))/t \le 1$. In particular, $\exp_p(t\overline{\gamma}) \ne p$ for all $0 < t < \rho$, $\overline{\gamma} \in S$, and elements of $S_p \cap S$ do not form loops at p of length $< \rho$.*

Proof. Given any i we can find a finite set $Y_i \subset S_p$ such that for every $\overline{\alpha} \in S$ there is a $\gamma \in Y_i$ such that $\alpha(\overline{\alpha}, \gamma) \le 2^{-i}$. Then $Y = \bigcup_{i=1}^{\infty} Y_i$ is precompact and $S \subset \overline{Y}$. Let $\delta = \sin^{-1}(1 - \varepsilon/2)$ and $\alpha_1, \cdots, \alpha_M \in S_p$ be δ-dense in Y. Choose $R > 0$ small enough that $\alpha_i|_{[0, R]}$ is minimal for all i. Let Γ_{ab} be minimal in S_k of length R and Γ_{ac} be unit minimal, with $\alpha(\Gamma_{ab}, \Gamma_{ac}) = \delta$. Then Lemma 2.7 implies that $\lim_{t \to 0}(R - d(b, \Gamma_{ac}(t)))/t = 1 - \varepsilon/2$; let $\rho > 0$ be such that for all $t < \rho$, $(R - d(b, \Gamma_{a;c}(t)))/t \ge 1 - \varepsilon$. For any $\alpha \in Y$, there exists some α_i such that $\alpha(\alpha, \alpha_i) < \delta$. By the triangle inequality, $d(p, \alpha(t)) \ge R - d(\alpha(t), \alpha_i(R))$, and A2 implies $1 - \varepsilon \le d(p, \alpha(t))/t \le 1$ for all $t < \rho$. The lemma now follows from Proposition 2.1.

Lemma 2.9. *If $S \subset \overline{S}_p$ is compact, then for any $\varepsilon > 0$ there are δ, $R > 0$ so that if $\overline{\alpha}, \overline{\beta} \in S$ and $d(\exp_p(t\overline{\alpha}), \exp_p(s\overline{\beta}))/s < \delta$ for some $0 < s < t < R$, then $\alpha(\overline{\alpha}, \overline{\beta}) < \varepsilon$.*

Proof. As in the proof of Lemma 2.8 we let $Y \subset S_p$ be precompact such that $S \subset \overline{Y}$. Suppose, contrary to the lemma, there exist $\overline{\alpha}_i, \overline{\beta}_i \in S$ and $0 < s_i < t_i < 2^{-i}$ such that $\alpha(\overline{\alpha}_i, \overline{\beta}_i) > 2\varepsilon$ and, letting $\overline{d}_i = d(\exp_p(s_i\overline{\alpha}_i), \exp_p(t_{+i}\overline{\beta}_i))$, $\overline{d}_i/s_i < 2^{-i-1}$. By Proposition 2.1 we can find $\alpha_i, \beta_i \in Y$ such that $\alpha(\alpha_i, \beta_i) < \varepsilon$ and, letting $d_i = d(\alpha_i(s_i), \beta_i(t_i))$, $d_i/s_i < 2^{-i}$. Choosing a subsequence if necessary, we can assume that both $\{\alpha_j\}$ and $\{\beta_i\}$ are Cauchy. Let $\zeta, \eta \in S_p$ be such that for all

sufficiently large i, $\alpha(\alpha_i, \zeta) < \varepsilon/4$ and $\alpha(\beta_i, \eta) < \varepsilon/4$. In S_k, let $\Gamma_{xa}, \Gamma_{xb}, \Gamma_{xc}$ be unit minimal such that $\alpha(\Gamma_{xa}, \Gamma_{xb}) = \alpha(\Gamma_{xb}, \Gamma_{xc}) = \varepsilon/4$ and $\alpha(\Gamma_{xa}, \Gamma_{xc}) = \varepsilon/2$. Define

$$a_i' = d(\Gamma_{xa}(s_i), \Gamma_{xb}(s_i)), \qquad b_i' = d(\Gamma_{xb}(t_i), \Gamma_{xc}(t_i)),$$
$$c_i = d(\zeta(s_i), \eta(t_k)), \qquad c_i' = d(\Gamma_{xa}(s_i), \Gamma_{xc}(t_i)).$$

By A2 and the triangle inequality

$$c_i \le a_i' + b_i' + d_i \le c_i' + (t_i - s_i) + d_i.$$

Lemma 2.8 implies that if $\delta > 0$, then for all sufficiently large i,

$$1 - \delta \le d(p, \beta_i(t_i))/t_i \le (s_i + d_i)/t_i$$
(*) $$\Leftrightarrow t_i - s_i \le \delta \cdot t_i + d_i \le \delta \cdot (s_i + (t_i - s_i)) + d_i$$
$$\Leftrightarrow (t_i - s_i)/s_i \le (\delta + d_i/s_i)/(1 - \delta).$$

Combining these inequalities we obtain $\lim_{i \to \infty}(c_i' - c_i)/s_i \ge 0$. From (*) it follows that $\lim_{i \to \infty} s_i/t_i = 1$. By Lemma 1.3, we obtain

$$\cos \alpha(\zeta, \eta) = \lim_{i \to \infty}(s_i^2 + t_i^2 - c_i^2)/2s_i t_i$$
$$= \lim_{i \to \infty}[(s_i^2 + t_i^2 - c_i'^2) + (c_i'^2 - c_i^2)]/2s_i t_i$$
$$= \cos(\varepsilon/2) + \lim_{i \to \infty}(c_i' + c_i)(c_i' - c_i)/2s_i t_i$$
$$\ge \cos(\varepsilon/2),$$

since $\lim_{i \to \infty} c_i'/t_i$ is bounded. From the triangle inequality we have, for all sufficiently large i, $\alpha(\alpha_i, \beta_i) < \varepsilon$, a contradiction.

Proposition 2.10. *If B is geodesically complete, then* $\dim \overline{T}_p = \dim B(p, \rho)$ *for all sufficiently small $\rho > 0$.*

Proof. We first prove that $\dim B(p, r) \ge \dim \overline{T}_p$. Suppose S is a convex subset of $\overline{S}_p$ isometric to S^n. Let T be the cone on S in $\overline{T}_p$. T is isometric to $\mathbf{R}^{n+1}$. Let $U = B(0, 1) \cap T$ and consider the maps $\varphi_\rho : U \to B(p, \rho)$ given by $\varphi_\rho(v) = \exp_p(\rho v)$. We claim that for any $\varepsilon > 0$ there exists a $\rho > 0$ such that φ_ρ is an ε-mapping, i.e., for all $x \in \varphi_\rho(U)$, $\operatorname{diam}(\varphi_\rho^{-1}(x)) < \varepsilon$. Let $\zeta = \cos^{-1}(1 - \varepsilon^2/2)$. By Lemma 2.8 and 2.9 there exists a $\rho > 0$ such that for all $0 < s < t < \rho$ and $\overline{\alpha}, \overline{\beta} \in S$: (1) $1 - \varepsilon/2 \le d(p, \exp_p(t\overline{\beta}))/t$, and (2) if $\exp_p(s\overline{\alpha}) = \exp_p(t\overline{\beta})$, then $\alpha(\overline{\alpha}, \overline{\beta}) < \zeta$. Then by (1), if $\exp_p(s\overline{\alpha}) = \exp_p(t\overline{\beta})$,

$$d\left(\frac{s}{\rho\overline{\alpha}}, \frac{t}{\rho\overline{\alpha}}\right) = (t - s)/\rho \le (t - d(p, \exp_p(s\overline{\alpha})))/\rho$$
$$= (t - d(p, \exp_p(t\overline{\beta})))/\rho \le t\varepsilon/(2\rho) < \varepsilon/2.$$

From (2) we have $d(\frac{t}{\rho\alpha}, \frac{t}{\rho\beta}) \leq \varepsilon/2$, and $d(\frac{s}{\rho\alpha}, \frac{t}{\rho\beta}) < \varepsilon$ follows from the triangle inequality.

Combining this result with Proposition 2.6, we have that for any $n \leq \dim \overline{S}_p + 1$ there exist ε-maps from the n-dimensional set U into $B(p, \rho)$ for arbitrarily small $\varepsilon > 0$. By [14, IV.5.A], $\dim B(p, \rho) \geq n$ and hence $\dim B(p, \rho) \geq \dim \overline{T}_p$.

On the other hand, for $\rho < \pi/\sqrt{k}$, the set $B' = B(0, \rho) \subset T$ can be naturally identified with $B(0, \rho)$ in the tangent space at a point z in S_k^{n-1}; we define a new metric δ on B' by $\delta(v, w) = d(\exp_z(v), \exp_z(w))$. Since B' is then isometric to $B(z, \rho) \subset S_k^m$, if $K = \{v \in B' \cap T_p : C(v) \geq \|v\|)\}$, then $\exp_p|_K$ is surjective onto $B(p, \rho)$ and distance decreasing by A2. Since K is a closed subset of B', K has Hausdorff dimension $\leq n + 1$, and since a distance decreasing map cannot increase Hausdorff dimension, $\dim B(p, \rho) \leq n + 1$.

Lemma 2.11. *Suppose B is geodesically complete, and $\overline{B}$ is compact. Let γ_{pb} and γ_{pc} be strictly minimal in B. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $a \in B(p, \delta)$ and minimal curves γ_{ab} and γ_{ac},*

$$|\alpha(\gamma_{pb}, \gamma_{pc}) - \alpha(\gamma_{ab}, \gamma_{ac})| < \varepsilon.$$

Proof. Let $a_i \to p$ and suppose γ_i and η_i are minimal curves from a_i to b and c, respectively. By Lemma 2.2, $\lim_{i \to \infty} \alpha(\gamma_i, \eta_i) \geq \alpha(\gamma_{pb}, \gamma_{pc})$.

To prove the opposite inequality let d be a point on the geodesic extension of γ_{bp} beyond p such that γ_{pd} is strictly minimal. If β_i is a minimal curve from a_i to d, and ν_i is minimal starting at a_i such that $\alpha(\gamma_i, \nu_i) = \pi$, then, by the above argument, $\lim_{i \to \infty} \alpha(\gamma_i, \beta_i) = \pi$, hence $\lim_{i \to \infty} \alpha(\beta_i, \nu_i) = 0$. Also by the above argument, $\lim_{i \to \infty} \alpha(\beta_i, \eta_i) \geq \alpha(\gamma_{pd}, \gamma_{pc}) = \pi - \alpha(\gamma_{pb}, \gamma_{pc})$. By the triangle inequality, $\alpha(\gamma_i, \eta_i) \leq \pi - \alpha(\eta_i, \beta_i) + \alpha(\beta_i, \nu_i)$, and we obtain the necessary inequality by passing to the limit.

Corollary 2.12. *If X is geodesically complete, then the following are equivalent:*

(a) *X has dimension $n < \infty$,*

(b) *at one point $p \in X$, $\overline{S}_p$ is compact, with $\dim \overline{S}_p = n - 1$, and*

(c) *at every point $q \in X$, $\overline{T}_q$ is isometric to $\mathbf{R}^n$.*

Proof. If $\overline{S}_p$ is compact, then by Proposition 2.5 $\overline{S}_p$ is a sphere, and, equivalently, $\overline{T}_p$ is isometric to Euclidean space of one higher dimension. Proposition 2.10 proves (a) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) is obvious. By Proposition 2.10, (b) $\Rightarrow$ (a) will follow if we show that the mapping $q \mapsto \dim \overline{T}_q$ is upper semicontinuous. We will show first that if (b) holds then X is

locally compact. Let $\{x_i\}$ be a sequence of points in $B(p, R)$ for some $R > 0$. Let α_i be minimal from p to x_i. Then $\{\alpha_i\}$ has a Cauchy subsequence, which by Proposition 2.1 corresponds to a convergent subsequence of $\{x_i\}$.

By elementary linear algebra, for any m there exists a $\delta - m > 0$ so that $\overline{S}_q$ contains a convex subset S isometric to S^m if and only if for every $\delta > \delta_m$ there exist $\gamma_1, \cdots, \gamma_{m+1} \in S_q$ such that $|\alpha(\gamma_i, \gamma_j) - \pi/2| < \delta$ for all $i \neq j$. In particular, we can let $\delta = \delta_m/3$ and choose $r > 0$ such that $\gamma_i|_{[0, R]}$ is strictly minimal for all i. If $z \in X$ is sufficiently close to q and α_i is minimal from z to $\gamma_i(r)$, by Lemma 2.11 $|\alpha(\alpha_i, \alpha_j) - \pi/2| < \delta_m/2$ and so $\dim \overline{S}_z \geq m$.

Proof of Theorem F. Suppose $\dim X = n$, let $p \in X$ be arbitrary, and let $\gamma_1, \cdots, \gamma_n \in T_p$ be such that $|\alpha(\gamma_i, \gamma_j) - \pi/2| < \delta_{n+1}/2$ (see the proof of Corollary 2.12). Choose $R > 0$ small enough that $\gamma_i|_{[-R, R]}$ is strictly minimal for all i, and $B(p, 3R)$ is a region of curvature $\geq k$. Define $u: X \to \mathbf{R}^n$ by $u^i(x) = d(x, z_i)$, where $z_i = \gamma_i(R)$ (cf. [2]). We will show first that u is injective, and hence a homeomorphism, near p. Suppose, to the contrary, there exist points $x_j, y_j \to p$ in B such that $u(x_j) = u(y_j)$. Let η_j be minimal from x_j to y_j, and γ_{ij} be minimal from x_j to z_i. We will show that, passing to a subsequence if necessary, $\lim_{j \to \infty} \alpha(\eta_j, \gamma_{ij}) = \pi/2$ for all i, and thereby obtain a contradiction. For then by Lemma 2.11, for all large j we have $|\alpha(\gamma_{ij}, \gamma_{kj}) - \pi/2| < \delta_{n+1}$ and $|\alpha(\gamma_{ij}, \eta_j) - \pi/2| < \delta_{n+1}$. In other words, the space of directions at x_j has dimension $\geq n$, which is not possible by Corollary 2.12.

Let α_{ij} be minimal from x_j to $w_i = \gamma_i(-R)$, δ_{ij} be minimal from y_j to z_i, and β_{ij} be minimal from y_j to w_i. Set $A_{ij} = d(x_j, w_i)$, $B_{ij} = d(y_j, w_i)$, and $C_j = d(x_j, y_j)$. Passing to a subsequence if necessary we can assume all of the following limits exist:

$$a = \lim_{j \to \infty} \alpha(\eta_j, \alpha_{ij}), \quad b = \lim_{j \to \infty} \alpha(\eta_j, \beta_{ij}), \quad c = \lim_{j \to \infty} \alpha(\gamma_j, \alpha_{ij}),$$
$$d = \lim_{j \to \infty} \alpha(\eta_j, \delta_{ij}), \quad K = \lim_{j \to \infty} (A_{ij} - B_{ij})/C_j.$$

From Lemma 2.7 and A1 we know $c, d \geq \pi/2$. By Lemma 2.11, $\lim_{j \to \infty} \alpha(\alpha_{ij}, \gamma_{ij}) = \lim_{j \to \infty} \alpha(\beta_{ij}, \delta_{ij}) = \pi$. Applying Lemma 2.3, Lemma 2.7, and A1 we obtain

$$K \geq \sin(\pi/2 - a) - \sin(c - \pi/2) \geq 0,$$
$$-K \geq \sin(\pi/2 - b) - \sin(d - \pi/2) \geq 0.$$

In other words, $K = 0$ and $c = \pi/2$.

Let $B = B(0, R) \subset T_p$ and define functions $\varphi, \psi: B \to \mathbf{R}^n$ by

$$\varphi(v) = u(\exp_p(v)) - u(p),$$

$$\psi(v) = (\|v - R\gamma_1\|^{1/2} - R, \cdots, \|v - R\gamma_n\|^{1/2} - R).$$

Since $\mathbf{R}^n$ can be identified with its own tangent space, invariance of domain and the above argument that u is a homeomorphism imply that, for small r, $\psi|_{B(0,r)}$ is a homeomorphism onto a neighborhood of 0 in $\mathbf{R}^n$. Let $S(0, \varepsilon)$ denote the sphere of radius $\varepsilon > 0$ in $\mathbf{R}^n$, and $p_\varepsilon: \mathbf{R}^n\backslash\{0\} \to S(0, \varepsilon)$ be the radial projection. Then

$$\psi_\varepsilon = p_\varepsilon \circ \psi|_{S(0,\varepsilon)}: S(0, \varepsilon) \to S(0, \varepsilon)$$

is defined and has degree ± 1 for small ε. On the other hand, Lemma 2.8 implies that, for small ε, $\varphi^{-1}(0) \cap B(0, 2\varepsilon) = \{0\}$, and the map

$$\varphi_\varepsilon = p_\varepsilon \circ \varphi|_{S(0,\varepsilon)}: S(0, \varepsilon) \to S(0, \varepsilon)$$

is defined. If we suppose $\varphi(B(0, 2\varepsilon))$ contains no neighborhood of 0, 0 must be a topological boundary point of $\varphi(B(0, \varepsilon))$. But then φ_ε has a continuous extension over $\overline{B}(0, \varepsilon)$ (cf. [10, p. 96]), and so $\deg(\varphi_\varepsilon) = 0$. Therefore, to obtain a contradiction and prove that X is a manifold, we need only show that for small $\varepsilon > 0$, φ_ε and ψ_ε are homotopic. Choose $\delta > 0$ such that for all $v \in \overline{T}_p$, there is some γ_i with $|\alpha(v, \gamma_i) - \pi/2| > \delta$. By Lemma 2.7 (with $\varphi = \pi/2 - \delta$) and A2, there exists a $\rho > 0$ such that for any $v \in S_p$, $\varepsilon < \rho$, and i as above, $d(\exp_p(\varepsilon v), \gamma_i(R))$ and $\|\varepsilon v - (R)\gamma_i\|^{1/2}$ are either both $< R - \zeta$ or both $> R + \zeta$, where $\zeta = (\varepsilon/2) \cdot \sin \delta$. Since $\varphi(\overline{B}(0, \varepsilon))$ and $\psi(\overline{B}(0, \varepsilon))$ are both bounded, we obtain that $\alpha(\varphi_\varepsilon(\varepsilon v), \psi_\varepsilon(\varepsilon v)) \leq \pi - \nu$ for some $\nu > 0$. Since S_p is dense in $\overline{S}_p$, the same inequality holds on $\overline{S}_p$, and φ_ε and ψ_ε are homotopic.

For $v \in T_p$ we define $\|v\|_p = L(\gamma_v|_{[0,1]})$, and extend $\|\cdot\|_p$ continuously to a norm on $\overline{T}_p$. Using a result of Fréchet [14] we can obtain a compatible vector space structure on $\overline{T}_p$ and inner product $\langle \cdot, \cdot \rangle_p$. The remainder of Theorem F follows from Proposition 1 and the discussion of the tangent space at the beginning of this section.

Proof of Theorem G. If X is locally strictly convex, $\overline{T}_p = T_p$ for all p, and so we omit the "bar" notation. For any $p \in X$, let $\overline{B}(p, R)$ be strictly convex and small enough that it is contained in a region of curvature $\geq k$ and homomorphic to an open subset of $\mathbf{R}^n$. Let $\gamma_1, \cdots, \gamma_n$ be a basis for T_p; then by strict convexity $\gamma|_{[0, R]}$ is strictly minimal for all i. By Lemma 2.11 there is an $r > 0$ small enough that for all $q \in B(p, r)$, if γ_i^q

is minimal from q to $\gamma_i(R)$, then $\gamma_1^q, \cdots, \gamma_n^q$ lie in $B(p, R)$ and form a basis for T_q. We define $\varphi: \pi^{-1}(B(p, r)) \to B(p, r) \times T_p$ by $\varphi(\Sigma c_i \gamma_i^q) = (q, \Sigma c_i \gamma_i)$. It is easy to verify that φ is a vector bundle chart, and that the collection of all such charts is a vector bundle atlas. The proof that exp and $\langle \cdot, \cdot \rangle_*$ are continuous is also straightforward.

To prove that TX is isomorphic to the topological tangent bundle of X, we need only find an embedding $\psi: TX \to X \times X$ whose image is an open subset containing the diagonal, whose restriction to the 0-section is the diagonal map, and such that $\psi(T_p) = p \times U$ for some open set U of X containing p. For simplicity, we assume first that X is compact, and choose $r > 0$ such that for all $p \in X$, $B(p, r)$ is strictly convex. We let $\rho: \mathbf{R} \to (-r, r)$ be a homeomorphism fixing 0 and define $\psi: TX \to X \times X$ by $\psi(v) = (p, \exp_p([\rho(\|v\|)/\|v\|] \cdot v))$ for $v \in T_p$. The continuity of ψ is easily proved using the above local trivializations and the continuity of exp; the injectivity of ψ follows from the convexity of $B(p, r)$. For the noncompact case, we fix a base point, replace r by a possibly decreasing function of the distance of a point p to x, and continuously deform the homeomorphism ρ accordingly.

Since the topological tangent bundle of X is a vector bundle, it follows from [12, Theorem 5.12] that the product $X \times \mathbf{R}^q$ can be given a smooth structure for some q. By the Product Structure Theorem [11], this implies that X has a smooth structure for $\dim X \geq 5$. Since every manifold is smooth in dimensions ≤ 3, the proof of Theorem 2 is complete.

Proof of Theorem H. For any compact metric space Y we denote by $N(\varepsilon, r, Y)$ the maximum number of disjoint balls of radius ε that can be put in a ball of radius r in Y. Suppose $\dim X = n$. By [5, Proposition 5.2], it suffices to prove that $N(\varepsilon, r, X) \leq N(\varepsilon, r, S_L^n)$, where for simplicity we use $L = \min\{0, k\}$ instead of k. Let $B(x, r)$ be given, and endow $B(0, r) \subset \overline{T}_x$ with the metric δ defined in the proof of Proposition 2.10. Let $B_i = B(i(p_i, \varepsilon)$ be a collection of N disjoint balls in $B(x, r)$. Let $d_i = d(x, p_i) \leq r - \varepsilon$, and $v_i \in S_x$ be minimal from x to p_i. Then since $\exp_x$ is distance decreasing, $\exp_x(B_\delta(v_i, \varepsilon)) \subseteq B_k$, and the balls $B_\delta(v_i, \varepsilon)$ are N disjoint ε-balls in $B(0, r) \cong B(z, r) \subset S_k^n$. q.e.d.

The above argument is easily modified to obtain a "pointed" precompactness theorem without an upper bound on the diameter (cf. [5]).

Example 2.13. The "squashed sphere" Q, due to K. Grove and P. Petersen, is obtained is a limit of Riemannian manifolds of positive curvature by flattening the upper and lower hemispheres of S^2, while allowing curvature along the equator to go to infinity. Q may also be obtained by

gluing together flat disks along their boundaries. Q is easily verified to be almost Riemannian. If $p \in Q$ lies on the interior of either disk, $T_p = \mathbf{R}^2$ and $\exp_p$ is an isometry on $B(0, r)$ for small r. If p lies on the equator, T_p can be identified with $\mathbf{R}^2 \backslash \{(t, 0): t \neq 0\}$, i.e., S_p is S^1 minus two antipodal points. The missing points correspond to the two "directions" of the equator, which is not a geodesic (but is a limit of geodesics). Points along the equator are joined by pairs of minimal curves, Euclidean segments crossing each disk. The space $X = Q \times S^1$ can be given a natural "product" inner metric so that geodesics are "products" of geodesics in Q and S^1. If $p \in Q$ is on the equator, then at $x = (p, z) \in X$, T_p consists of $\mathbf{R}^3$ with two coplanar open half-planes removed. The cut radius function C is not continuous at the two points in S_p corresponding to the S^1-directions.

Example 2.14. Let S_i^n, $n > 0$, be the standard sphere of radius 2^{-i}, and $X_k = S_1^n \times \cdots \times S_k^n$. Then $\{X_k\}$ has a unique compact Gromov-Hausdorff limit X which is easily verified to be a geodesically complete inner metric space with curvature uniformly ≥ 0. Since X is infinite dimensional, X is not almost Riemannian. Note that if $n = 0$, X is a limit of flat manifolds, but does to itself have an upper curvature bound.

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