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A GEOMETRIC CHARACTERIZATION OF NEGATIVELY CURVED LOCALLY SYMMETRIC SPACES

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Introduction

Let M be a compact connected Riemannian manifold of nonpositive sectional curvature. In [1]-[3] the rank of M is defined as follows: For $v \in T^1M$ let the rank of v be the dimension of the vector space of parallel Jacobi fields along the geodesic γ_v with initial velocity v, and let $\operatorname{rank}(M)$ be the infimum of the rank of the elements of T^1M. Ballmann, Brin, Eberlein, and Spatzier showed [1]-[3] (compare also [5], [6]) that if M is irreducible (i.e., the de Rham decomposition of the universal covering of M is trivial), and $\operatorname{rank}(M) \geq 2$, then M is locally symmetric of higher rank.

As the rank of M measures its flatness, we can define a notion of rank for general manifolds of nonpositive curvature which measures the distribution of the curvature maximum in the following way: Let $-a^2 \leq 0$ be the maximum of the curvature of M. For an element v of the unit tangent bundle T^1M of M define the *hyperbolic rank* of v to be the dimension of the vector space of parallel vector fields J along the geodesic γ_v with initial velocity $\gamma_v'(0) = v$ with the following properties:

(1) J is orthogonal to the tangent of γ_v.

(2) For every $t \in \mathbf{R}$ the curvature of the plane spanned by $\gamma_v'(t)$ and $J(t)$ equals $-a^2$.

Let the *hyperbolic rank* $h\text{-}\operatorname{rank}(M)$ of M be the minimum of the hyperbolic ranks of the vectors $v \in T^1M$. It is not difficult to see that $\operatorname{rank}(M) = h\text{-}\operatorname{rank}(M) + 1$ for manifolds with curvature maximum 0.

With this notion of rank the result of Ballmann, Brin, Eberlein, and Spatzier holds for every manifold of nonpositive curvature:

Theorem. *If $h\text{-}\operatorname{rank}(M) > 0$ and M is irreducible, then M is locally symmetric.*

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The purpose of this paper is to provide the proof of the above theorem in the case $a > 0$. By rescaling the metric of M we may thus assume that the maximum of the curvature K equals -1. The idea of proof is to show that the universal covering $\widetilde{M}$ of M is homogeneous. Since $\widetilde{M}$ admits a compact quotient, this implies by a result of Heintze [10] that $\widetilde{M}$ is symmetric, and hence M is locally symmetric.

The organization of the paper is as follows: In §1 we recall from [9] the definition and basic properties of a family of distances which are defined on (the complement of one point of) the ideal boundary $\partial\widetilde{M}$ of the universal covering $\widetilde{M}$ of M. We show that these distances define a class of rectifiable curves on $\partial\widetilde{M}$. In §2 we define a distribution E' on an open subset of the unit tangent bundle T^1M of M, which is invariant under the geodesic flow and tangent to the strong unstable foliation W^{su} of T^1M. We show that curves which are tangent to E' give rise to rectifiable curves on $\partial\widetilde{M}$. §3 is devoted to the investigation of the space of rectifiable curves in $\partial\widetilde{M}$. This is used in §4 to show that the distribution E' generates the whole tangent bundle of the foliation W^{su}. In §5 we study the Carnot-Carathéodory metrics on the leaves of W^{su}, which are induced by E'. These metrics give rise to a generalized conformal structure on $\partial\widetilde{M}$ whose associated group G of 1-quasiconformal transformations is investigated in §6. In §7 we show that G acts transitively on $T^1\widetilde{M}$ as a topological transformation group commuting with the geodesic flow. These transformations preserve the fibers of the fibration $T^1\widetilde{M} \to \widetilde{M}$, and hence G acts as a group of isometries transitively on $\widetilde{M}$, i.e., $\widetilde{M}$ is homogeneous.

We assume that our methods can also be used to show the analogous result for manifolds of finite volume, but we did not check this.

Before we proceed it will be useful to fix some notation which will be used throughout the paper (for definitions see [4], [12]). M denotes an $(m+1)$-dimensional compact connected Riemannian manifold of negative curvature $-\infty < -b^2 \le K \le -1 < 0$ and fundamental group Γ.

The *geodesic flow* Φ^t acts on the unit tangent bundle T^1M (resp. $T^1\widetilde{M}$) of M (resp. the universal covering $\widetilde{M}$ of M). $T^1\widetilde{M}$ admits foliations W^{su}, W^{u} which are invariant under Φ^t and the action of the isometry group of $\widetilde{M}$ on $T^1\widetilde{M}$. The leaves of W^{su} (resp. W^{u}) are called the *strong unstable* (resp. *unstable*) *manifolds* of $T^1\widetilde{M}$.

For $v \in T^1\widetilde{M}$ let γ_v be the geodesic line in $\widetilde{M}$ with initial direction $\gamma_v'(0) = v$. v also determines a *Busemann function* θ_v at the point $\gamma_v(-\infty)$ of the *ideal boundary* $\partial\widetilde{M}$ of $\widetilde{M}$, which is normalized by $\theta_v(\gamma_v(0)) = 0$. The leaf of W^u containing v then consists of all $w \in T^1\widetilde{M}$ such that $\gamma_w(-\infty) = \gamma_v(-\infty)$. Thus every $\xi \in \partial\widetilde{M}$ determines a unique leaf $W^u(\xi)$ of the foliation W^u. The restriction of the canonical projection $P: T^1\widetilde{M} \to \widetilde{M}$ maps $W^u(\xi)$ diffeomorphically onto $\widetilde{M}$.

The *horosphere* $\theta_v^{-1}(t)$ ($t \in \mathbf{R}$) is the image under P of the leaf $W^{\mathrm{su}}(\Phi^t v)$ of W^{su} containing $\Phi^t v$. The restriction of the Riemannian metric of $\widetilde{M}$ to $\theta_v^{-1}(t)$ induces a distance $d_{v,t}$ on $\theta_v^{-1}(t)$. Let $\pi_{v,t}: \widetilde{M} \cup \partial\widetilde{M} - \gamma_v(-\infty) \to \theta_v^{-1}(t)$ be the projection along the geodesics which are asymptotic to $\gamma_v(-\infty)$. Clearly $\theta_w = \theta_v$ and $\pi_{w,t} = \pi_{v,t}$ for all $w \in W^{\mathrm{su}}(v)$ and all $t \in \mathbf{R}$.

Finally denote by d the distance on $\widetilde{M}$ induced by the Riemannian metric.

1. A class of distances of $\partial\widetilde{M}$

Choose $v \in T^1\widetilde{M}$, $R > 0$, and define

$$\alpha(x, y) = \sup\{t \in \mathbf{R} \mid d_{v,t}(\pi_{v,t}(x), \pi_{v,t}(y)) \leq R\}$$

and $\eta_{v,R}(x, y) = e^{-\alpha(x,y)}$ for $x, y \in \partial\widetilde{M} - \gamma_v(-\infty)$. Then $\eta_{w,R} = \eta_{v,R}$ for all $w \in W^{\mathrm{su}}(v)$, $\eta_{\Phi^t v,R} = e^t\eta_{v,R}$ for all $t \in \mathbf{R}$, and $\eta_{d\Psi v,R} = \eta_{v,R} \circ \Psi^{-1}$ for each isometry Ψ of $\widetilde{M}$ (recall that the isometries of $\widetilde{M}$ act on $\partial\widetilde{M}$ in a natural way). Moreover (Corollary 3 of [9]) we have

Lemma 1.1. *$\eta_{v,R}: (x, y) \to \eta_{v,R}(x, y)$ is a distance on $\partial\widetilde{M} - \gamma_v(-\infty)$.*

$\eta_{v,R}$ depends on the choice of $R > 0$ in the following way (Corollary 2 of [9]).

Lemma 1.2. *If $0 < r < R$, then $\eta_{v,R} \leq \eta_{v,r} \leq (R/r)\eta_{v,R}$.*

Write $\eta_v = \eta_{v,1}$. We want to show that the family of η_v-rectifiable curves in $\partial\widetilde{M}$ does not depend on v. For this define $B_{\eta_v}(x, \varepsilon) = \{y \in \partial\widetilde{M} - \gamma_v(-\infty) \mid \eta_v(x, y) < \varepsilon\}$ for $x \in \partial\widetilde{M} - \gamma_v(-\infty)$ and $\varepsilon > 0$. Thus $B_{\eta_v}(x, \varepsilon)$ is the projection in $\partial\widetilde{M}$ of the open ball of radius 1 about $\pi_{v,\log 1/\varepsilon}x$ in $(\theta_v^{-1}(\log 1/\varepsilon), d_{v,\log 1/\varepsilon})$. Define a function $\sigma: \mathbf{R} \to \mathbf{R}$ by $\sigma(s) = e^{bs} + (4\sinh s)/b$. σ is continuous and $\sigma(0) = 1$.

Lemma 1.3. *Let $v, w \in T^1\widetilde{M}$, and let $A \subset \partial\widetilde{M} - \{\gamma_v(-\infty), \gamma_w(-\infty)\}$ be compact. Assume that there is $\delta > 0$ and $\tau \in \mathbf{R}$ such that*

$$d(\pi_{v,\tau}(y), \pi_{w,\tau}(y)) < \delta \quad \textit{for all } y \in A.$$

If $B_{\eta_v}(x, r) \subset A$ for some $x \in A$ and $r \in (0, e^{-\tau})$ then $\eta_w(y, z) \leq \sigma(\delta)\eta_v(y, z)$ for all $y, z \in B_{\eta_v}(x, r/3)$.

Proof. Let $y, z \in B_{\eta_v}(x, r/3)$ and $\eta_v(y, z) = \varepsilon$. Since η_v is a distance, and $\eta_v(x, y) < r/3$, and $\eta_v(x, z) < r/3$, we have $z \in B_{\eta_v}(y, \frac{2}{3}r) \subset A$ and $t = \log(1/\varepsilon) > \tau$. By the definition of η_v there is a smooth curve $\phi: I \to \pi_{v,t}(B_{\eta_v}(y, \frac{2}{3}r))$ such that $\phi(0) = \pi_{v,t}(y)$, $\phi(1) = \pi_{v,t}(z)$, and that the length of ϕ equals 1. Since for every $u \in \partial\widetilde{M} = \{\gamma_v(-\infty), \gamma_w(-\infty)\}$ the function $t \to d(\pi_{v,t}(u), \pi_{w,t}(u))$ is decreasing, $\phi(I)$ is contained in the δ-neighborhood of $\theta_w^{-1}(t)$. Now the operator norm of the restriction of the projection $\pi_{t,w}$ to the δ-neighborhood of $\theta_w^{-1}(t)$ is not larger than $e^{b\delta}$; hence the length of $\pi_{w,t} \circ \phi$ does not exceed $e^{b\delta}$.

On the other hand, $d(\pi_{v,t}(y), \pi_{w,t}(y)) < \delta$ implies

$$d(\pi_{w,t} \circ \pi_{v,t}(y), \pi_{w,t}(y)) < 2\delta$$

(Lemma 9 of [9]). By the estimates in [11] this yields

$$d_{w,t}(\pi_{w,t} \circ \pi_{v,t}(y), \pi_{w,t}(t)) < \tfrac{2}{b} \sinh\delta,$$

i.e., $d_{w,t}(\pi_{w,t}(y), \pi_{w,t}(z)) < e^{b\delta} + \frac{4}{b}\sinh\delta = \sigma(\delta)$ and $\eta_{w,\sigma(\delta)}(y, z) < e^{-t} = \eta_v(y, z)$. The lemma now follows from Lemma 1.2.

Corollary 1.4. *Let $v, w \in T^1\widetilde{M}$, $x \in \partial\widetilde{M} - \{\gamma_v(-\infty), \gamma_w(-\infty)\}$, and θ be a Busemann function at x. If $s = \theta(\pi_{v,0}(x)) - \theta(\pi_{w,0}(x))$ then there is for every $\varepsilon > 0$ a neighborhood A of x in $\partial\widetilde{M}$ such that $(1-\varepsilon)\eta_v(y, z) \le e^s\eta_w(y, z) \le (1+\varepsilon)\eta_v(y, z)$ for all $y, z \in A$.*

Proof. Since $\eta_v = \eta_{\overline{v}}$ for all $\overline{v} \in W^{\mathrm{su}}(v)$, we may assume that $\gamma_v(\infty) = \gamma_w(\infty) = x$. Moreover $\eta_{\Phi^t w} = e^t\eta_w$ for all $t \in \mathbf{R}$ shows that it suffices to consider the case $\theta_{-v}(Pw) = 0$ which means $d(P\Phi^t v, P\Phi^t w) \to 0$ $(t \to \infty)$. For $\varepsilon > 0$ choose $\delta > 0$ sufficiently small such that $\sigma(\delta) < \min\{1+\varepsilon, 1/(1-\varepsilon)\}$. There is a number $\tau \in \mathbf{R}$ such that $d(P\Phi^t v, P\Phi^t w) < \delta/3$ for all $t \ge \tau$. Define

$$\widetilde{A} = \{y \in \partial\widetilde{M} \mid \max\{d(\pi_{v,\tau}(y), P\Phi^\tau v), d(\pi_{w,\tau}(y), P\Phi^\tau w)\} < \delta/3\}.$$

$\widetilde{A}$ is an open neighborhood of x in $\partial\widetilde{M}$ and if $y \in \widetilde{A}$ then

$$d(\pi_{v,\tau}(y), \pi_{w,\tau}(y)) < \delta.$$

Choose $r \in (0, e^{-\tau})$ such that $B_{\eta_v}(x, r) \cup B_{\eta_w}(x, r) \subset \widetilde{A}$ and let $A = B_{\eta_v}(x, r/3) \cap B_{\eta_w}(x, r/3)$. By 1.3, A has the required property.

Corollary 1.5. *For $v, w \in T^1\widetilde{M}$ every curve*

$$\varphi: I \to \partial\widetilde{M} \backslash \{\gamma_v(-\infty), \gamma_w(-\infty)\}$$

is rectifiable with respect to η_v *if and only if it is rectifiable with respect to* η_w.

By 1.5 we can call a curve $\varphi: I \to \partial\widetilde{M}$ *η-rectifiable* if φ is rectifiable with respect to some and hence any distance η_v with $\gamma_v(-\infty) \notin \varphi(I)$.

Corollary 1.6. $\eta_v \geq d_{v,0} \circ \pi_{v,0}$; *in particular, the projection into* $\theta_v^{-1}(0)$ *of a* η_v*-rectifiable curve* $\varphi: I \to \partial\widetilde{M} \backslash \{\gamma_v(-\infty)\}$ *is rectifiable with respect to* $d_{v,0}$.

The corollary follows from the comparison with the hyperbolic plane of constant curvature -1 (see [9]).

2. The distinguished distribution

The differential of the projection $P: T^1\widetilde{M} \to \widetilde{M}$ maps the tangent space at $v \in T^1\widetilde{M}$ of the strong unstable manifold $W^{su}(v)$ containing v isomorphically onto the orthogonal complement $v^\perp$ of v in $T_{P_v}\widetilde{M}$. Let R be the curvature tensor of $\widetilde{M}$; then the restriction to $v^\perp$ of the operator $R_v: w \to R(v, w)v$ is a self-adjoint automorphism of $v^\perp$ whose eigenvalues are not bigger than -1. For $\rho \geq 0$ define $E_\rho(v) = \{dPX \in v^\perp \mid X \in T_v W^{su}(v), R_{\Phi^t v} dPd\Phi^t X = -dPd\Phi^t X$ for all $t \in [0, \rho]\}$. $E_0(v)$ is the eigenspace of R_v with respect to the eigenvalue -1 and $E(v) = \bigcap_{\rho \geq 0} E_\rho(v)$ is a linear subspace of $v^\perp$. Let $Q: \widetilde{M} \to M$ be the canonical projection and let $E_\rho(dQv) = dQE_\rho(v)$ $(v \in T^1\widetilde{M},\ \rho \geq 0)$, resp. $E(dQv) = dQE(v)$.

For $v \in T^1M$ and $w \in v^\perp$ let $t \to \Lambda(v, t)w$ be the Jacobi field along the geodesic γ_v through $\Lambda(v, 0)w = w$ which vanishes asymptotically at $-\infty$, i.e., satisfies $\Lambda(v, t)w \to 0$ $(t \to -\infty)$. Then $w \to \lambda(v, t)w$ is an isomorphism of $v^\perp$ onto $(\Phi^t v)^\perp$ which maps $E(v)$ into $E(\Phi^t v)$ for $t \geq 0$. If $p: T^1M \to M$ denotes the canonical projection, then $\Lambda(v, t)\, dpX = dpd\Phi^t X$ for every $X \in T_v W^{su}(v)$.

Let $\|_t: T_{pv}M \to T_{p\Phi^t v}M$ be the parallel transport along the geodesic γ_v and let $k = \min\{\dim E(w) \mid w \in T^1M\}$. Then we have

Lemma 2.1. $k = h\text{-}\mathrm{rank}(M)$; *in particular,* $\Lambda(v, t)w = e^t \|_t w$ *for every* $v \in T^1M$ *with* $\dim E(v) = k$ *and every* $w \in E(v)$.

Proof. Let $v \in T^1M$; then there is an $h\text{-}\mathrm{rank}(v)$-dimensional subspace A of $v^\perp$ such that for every $w \in A$ we have $R_{\Phi^t v}(\|_t w) = -\|_t w$ for all $t \in \mathbf{R}$. Define $J_w(t) = e^t(\|_t w)$; clearly J_w is a Jacobi field along γ_v which satisfies $J_w(t) \in E(\Phi^t v)$ for all $t \in \mathbf{R}$. This means $\dim E(v) \geq h\text{-}\mathrm{rank}(v)$ and consequently $k \geq h\text{-}\mathrm{rank}(M)$.

To show $k \le h$-rank(M) let $v \in T^1M$ be such that $\dim E(v) = k$. For $w \in E(v)$ the assignment $t \to \tilde{w}(t) = \Lambda(v, t)w$ is a Jacobi field along γ_w with $R_{\Phi^t v}\tilde{w}(t) = -\tilde{w}(t)$ for all $t \in [0, \infty)$. The differential equation for Jacobi fields shows that for $t \ge 0$, $\tilde{w}$ is a solution of the differential equation

$$(*) \qquad \tilde{w}''(t) - \tilde{w}(t) = 0,$$

where $\tilde{w}'$ means covariant derivative. Every solution of this equation is determined by its initial conditions $w_0 = \tilde{w}(0)$ and $w_1 = \tilde{w}'(0)$, and can be written as $\tilde{w}(t) = \cosh t \|_t w_0 + \sinh t \|_t w_1$.

Now $\Lambda(\Phi^{-t}v, t)E(\Phi^{-t}v)$ is contained in $E(v)$ for every $t > 0$. This implies

$$\Lambda(\Phi^{-t}v, t)E(\Phi^{-t}v) = E(v),$$

since $\dim E(v) = k = \min\{\dim E(w) \mid w \in T^1\widetilde{M}\}$. In particular, the Jacobi field $t \to \Lambda(v, t)w$ is a solution of $(*)$ for every $t \in \mathbf{R}$.

Suppose $w_1 = \lambda w_0 + \overline{w}$ for some $\lambda \in \mathbf{R}$ and $\overline{w} \in \overline{w}_0^{\perp}$. Then

$$\|\tilde{w}(t)\|^2 = (\cosh t + \lambda \sinh t)^2 \|w_0\|^2 + \sinh^2 t \|\overline{w}\|^2.$$

Since $\|\tilde{w}(t)\| \to 0$ $(t \to -\infty)$, this shows $\overline{w} = 0$ and $\lambda = 1$, i.e.,

$$(**) \qquad \tilde{w}(t) = \cosh t \|_t w_0 + \sinh t \|_t w_0.$$

Thus h-rank$(v) \ge k$ which is the claim.

Remark 2.2. If $\dim E(v) = k$, then $\Lambda(v, t)$ maps the orthogonal complement of $E(v)$ in $v^{\perp}$ isomorphically onto $E(\Phi^t v)^{\perp} \subset (\Phi^t v)^{\perp}$: For let $t \to e(t)$ be a parallel vector field along the geodesic $\gamma_v : t \to P\Phi^t v$ such that $e(0) \in E(v)$, and let $\tilde{w}$ be any Jacobi field along γ_v. Then $d^2/dt^2 \langle e(t), \tilde{w}(t)\rangle = -\langle e(t), R_{\Phi^t v}\tilde{w}(t)\rangle$. Since $R_{\Phi^t v}$ is self-adjoint and $e(t)$ is an eigenvector of $R_{\Phi^t v}$ with respect to the eigenvalue -1, the map $R_{\Phi^t v}$ preserves the orthogonal complement of $E(\Phi^t v)$. Thus the function $t \to \langle e(t), \tilde{w}(t)\rangle$ satisfies the differential equation $(*)$.

For $w \in E(v)^{\perp}$, $\tilde{w} : t \to \Lambda(v, t)w$ is a Jacobi field along γ_v which satisfies $\langle e(0), \tilde{w}(0)\rangle = 0$. Then $(*)$ shows

$$\langle e(t), \tilde{w}(t)\rangle = \langle e(0), \tilde{w}'(0)\rangle \sinh t.$$

But $\|\tilde{w}(t)\| \to 0$ $(t \to -\infty)$ which is only possible if $\langle \tilde{w}'(0), e(0)\rangle = 0$, i.e., if $\langle e(t), \tilde{w}(t)\rangle = 0$ for all $t \in \mathbf{R}$.

Since for $v \in T^1M$ the restriction of dp to the tangent space $TW^{\mathrm{su}}(v) \subset TT^1M$ of $W^{\mathrm{su}}(v)$ at v is an isomorphism onto $v^{\perp}$, the spaces $E_p(v)$

($\rho > 0$) (resp. $E(v)$), have a unique lift to subspaces $E'_\rho(v)$ (resp. $E'(v)$) of $TW^{\mathrm{su}}(v)$. The definition of $\Lambda(v, t)$ shows $d\Phi^t E'(v) \subset E'(\Phi^t v)$ for all $t > 0$.

Lemma 2.3. *$D = \{w \in T^1M \mid \dim E(w) = k\}$ is an open Φ^t-invariant subset of T^1M.*

Proof. Choose $v \in T^1M$ such that $\dim E(v) = k$. Now $E_\rho(v)$ is a linear subspace of the finite-dimensional vector space $v^\perp$ and $E_\rho(v) \supset E_\tau(v)$ if $\rho \leq \tau$. This implies that there is $\rho > 0$ such that $E(v) = E_\rho(v)$.

The assignment $v \to TW^{\mathrm{u}}(v)$ is a continuous vector bundle over T^1M, and for every $t \geq 0$ the map $w \in TW^u(v) \to R_{\Phi^t v}\, dpd\Phi^t w \in (\Phi^t v)^\perp$ is continuous. Since by definition

$$E'_\rho(v) = \bigcap_{t\in[0,\rho]} \{w \in TW^{\mathrm{su}}(v) \mid R_{\Phi^t v}\, dpd\Phi^t w = -dpd\Phi^t w\},$$

this yields that the set $\{E'_\rho(v) \mid v \in T^1M\}$ is closed in TT^1M. In particular, there is an open neighborhood U of v in T^1M such that $\dim E'_\rho(w) \leq k = \dim E'_\rho(v)$ for all $w \in U$, i.e., $U \subset D$, and D is open. The Φ^t-invariance of D is clear from the definition of E.

Corollary 2.4. *There is an open Φ^t-invariant subset Ω of T^1M such that $v \to E'(v)$ is a smooth k-dimensional distribution on Ω.*

Proof. Recall (see [12]) that the tangent bundle TT^1M of T^1M admits a decomposition $TT^1M = T^1 \oplus T^{\mathrm{h}} \oplus T^{\mathrm{v}}$ into smooth subbundles as follows: T^1 is tangent to the flow lines of the geodesic flow on T^1M. The *vertical bundle* T^{v} is tangent to the fiber of $T^1M \to M$. T^{h} is the subbundle of the *horizontal bundle* with the property that for every $w \in T^1M$ the fiber T^{h}_w of T^{h} over w is mapped by dp isomorphically onto $w^\perp \subset TM$. The decomposition $TT^1M = T^1 \oplus (T^{\mathrm{h}} \oplus T^{\mathrm{v}})$ is invariant under the geodesic flow, i.e., $(T^{\mathrm{h}} \oplus T^{\mathrm{v}})_{\Phi^t w} = d\Phi^t(T^{\mathrm{h}} \oplus T^{\mathrm{v}})_w$ for all $w \in T^1M$.

For $u \in T^{\mathrm{h}} \oplus T^{\mathrm{v}}$ write $u = u^{\mathrm{h}} + u^{\mathrm{v}}$, where $u^{\mathrm{h}} \in T^{\mathrm{h}}$, $u^{\mathrm{v}} \in T^{\mathrm{v}}$. Now for every $w \in T^1M$ there is a canonical isomorphism A of the fiber T^{v}_w of the bundle T^{v} at w with $w^\perp \subset T_{pw}M$. For $t \in \mathbf{R}$ define $A_t(w) = \{u \in T^{\mathrm{h}} \oplus T^{\mathrm{v}} \mid dp(d\Phi^t u)^{\mathrm{h}} = A(d\Phi^t u)^{\mathrm{v}}\}$. $A_t = \bigcup_{w\in T^1M} A_t(w)$ is a smooth m-dimensional vector bundle over T^1M.

We showed in 2.1 that $E'(w) = \bigcap_{t\geq 0} A_t(w)$ for all $w \in D$. Assume that the curvature of M is not constant, i.e., $k < m = \dim A_0$. Then there is $\tau > 0$ such that $A_\tau(v) \neq A_0(v)$ for some $v \in D$.

Let $q = \min\{\dim(A_0(w) \cap A_\tau(w)) \mid w \in D\} < m$. Since D is open and $A_0 \cap A_\tau$ is closed in TT^1M, there is an open subset $\widetilde{U}$ of D such that $\dim(A_0(w) \cap A_\tau(w)) = q$ for all $w \in \widetilde{U}$. But this implies that the restriction of $A_0 \cap A_\tau$ to $\widetilde{U}$ is a smooth vector bundle on $\widetilde{U}$. Repeating this argument at most $m-k-1$ times we find an open subset U of D such that the restriction of E' to U is a smooth k-dimensional vector bundle. Since E' is invariant under the geodesic flow, E' is then a smooth bundle over an open Φ^t-invariant subset Ω of D. q.e.d.

Let $\widetilde{\Omega} \subset T^1\widetilde{M}$ be the preimage of Ω under the restriction of dQ to $T^1\widetilde{M}$. E' lifts to a smooth distribution $\widetilde{E}$ on $\widetilde{\Omega}$ which is tangent to the strong unstable foliation.

Let $v \in \widetilde{\Omega}$ and let $\varphi: I \to \widetilde{\Omega}$ be a curve through $\varphi(0) = v$ which is tangent to $\widetilde{E}$. Then $P\varphi$ is a curve on the horosphere $\theta_v^{-1}(0)$ such that $(P\varphi)'(t) \in E(\varphi(t))$ for all $t \in I$.

Lemma 2.5. *$\pi_{v,\infty} \circ P \circ \varphi$ is an η_v-rectifiable curve in $\partial\widetilde{M}$.*

Proof. Assume without loss of generality that $P \circ \varphi \subset \theta_v^{-1}(0)$ is parametrized proportional to arc length with respect to $d_{v,0}$. Then there is $\alpha > 0$ such that $d_{v,0}(P\varphi(t), P\varphi(t+\varepsilon)) \le \alpha\varepsilon$ for all $\varepsilon > 0$, $t \in [0, 1-\varepsilon]$. Let $s > 0$; by the definition of E the length of $\pi_{v,s} \circ P \circ \varphi$ on $\theta_v^{-1}(s)$ equals $e^s\alpha$, i.e., $P\varphi(I)$ can be covered by $\sim e^s\alpha$ balls of radius $e^{-s}/2$ with respect to the distance η_v. This means that the η_v-length of $\pi_{v,\infty} \circ P \circ \varphi$ does not exceed α.

3. η-rectifiable curves in $\partial\widetilde{M}$

In this section we investigate the η-rectifiable curves in $\partial\widetilde{M}$. It follows from 2.5 that the space of these curves is not empty.

Let $\varphi: I \to \partial\widetilde{M}$ be an η-rectifiable curve. For $v \in T^1\widetilde{M}$ with $\gamma_v(-\infty) \notin \varphi(I)$ there is a unique curve $\overline{\varphi}: I \to W^{su}(v)$ such that $\varphi(s) = \pi_{v,\infty} \circ P \circ \overline{\varphi}(s)$ for all $s \in I$. Since by 1.6 $\pi_{v,0} \circ \eta_v \ge d_{v,0}$, $\overline{\varphi}$ is rectifiable as a curve in the Riemannian manifold $W^{su}(v)$, is hence differentiable almost everywhere.

Recall that a point $w \in T^1M$ is called *recurrent* if for every neighborhood U of w the orbit $t \to \Phi^t w$ $(t \in [0, \infty))$ meets U infinitely often. The Birkhoff ergodic theorem implies that with respect to the Lebesgue measure almost every $w \in T^1M$ is recurrent.

The purpose of this section is to show

Proposition 3.1. *If $dQ\overline{\varphi}(s)$ is recurrent for almost every $s \in I$ then $(P\overline{\varphi})'(s) \in E(\overline{\varphi}(s))$ for almost every $s \in I$.*

For the proof of 3.1 we need the following preparations.

Lemma 3.2. *If $\dim E_\rho(v) < q$ for some $v \in T^1M$, and $\rho, q > 0$, then there is a neighborhood U of v in T^1M and $\varepsilon > 0$ such that for all $w \in U$ and every q-dimensional subspace A of $w^\perp$ the determinant of the restriction to A of the map $\Lambda(w, \rho)$ is not smaller than $e^{q\rho}(1+\varepsilon)$.*

Proof. For $w \in v^\perp$ write $\tilde{w}(t) = \Lambda(v, t)w$. Since the curvature of M does not exceed -1, it follows that $\|\tilde{w}(t)\| \geq e^{t-s}\|\tilde{w}(s)\|$ for all $s \geq 0$, $t \geq s$, where $\|\;\|$ is the norm associated to the Riemannian metric of M (see [11]). In particular,

$$\langle \tilde{w}(t), \tilde{w}'(t)\rangle / \|\tilde{w}(t)\| = \frac{d}{dt}\|\tilde{w}(t)\| \geq \|\tilde{w}(0)\| e^t,$$

and the determinant of the restriction of $\Lambda(v, t)$ to any q-dimensional subspace A of $v^\perp$ is not smaller than e^{qt}. Moreover, $\Lambda(v, t+s) = \Lambda(\Phi^s v, t)\Lambda(v, s)$ shows

$$\frac{d}{dt}\det(\Lambda(v, t) \mid A)_{s=0} \geq q e^{qs}.$$

Choose a q-dimensional subspace A of $v^\perp$ with the property that $\det(\Lambda(v, \rho) \mid A)$ is less than or equal to the determinant of $\Lambda(v, \rho)$ restricted to any other q-dimensional subspace of $v^\perp$. Since $\dim E_\rho(v) < q$, there is $\tau < \rho$ such that $\Lambda(v, \tau)A = \overline{A} \not\subset E_0(\Phi^\tau)$.

Let $w_1, \cdots, w_q$ be an orthonormal basis of $\overline{A}$, and let $e_1(t), \cdots, e_q(t)$ be parallel vector fields along the geodesic γ_v such that $e_i(0) = w_i$. Then for all $s \geq 0$

$$\det(\Lambda(\Phi^r v, s) \mid \overline{A}) \geq \prod \langle e_i(\tau + s), \overline{w}_i(s)\rangle,$$

where $\overline{w}_i(s) = \Lambda(\Phi^\tau v, s)w_i$. Hence,

$$\begin{aligned}&\frac{d}{ds}\det(\Lambda(\Phi^\tau v, s) \mid \overline{A})_{s=0}\\ &\qquad \geq \sum_{i=1}^{q} \langle e_1(s), \overline{w}_1(s)\rangle \cdots \langle e_i(s), \overline{w}_i'(s)\rangle \cdots \langle e_q(s), \overline{w}_q(s)\rangle_{s=0},\end{aligned}$$

and this sum is not smaller than q. The differential equation for Jacobi fields shows

$$\frac{d^2}{ds^2}\det(\Lambda(\Phi^\tau v, s) \mid \overline{A})_{s=0} \geq q(q-1) + \sum_{i=1}^{q} \langle w_i, -R(\Phi^\tau v, w_i)\Phi^\tau v\rangle.$$

Since $\overline{A} \not\subset E_0(\Phi^\tau v)$ and since the eigenvalues of the map $R_{\Phi^\tau v}$ are not larger than -1, it follows

$$\frac{d^2}{ds^2}\det(\Lambda(\Phi^\tau v\,, s) \mid \overline{A})_{s=0} > q^2\,,$$

which together with $\Lambda(v\,, \tau + s) = \Lambda(\Phi^\tau v\,, s)\Lambda(v\,, \tau)$ and the above considerations implies that $\det(\Lambda(v\,, \rho) \mid A) = e^{q\rho}(1 + 2\varepsilon)$ for some $\varepsilon > 0$. By continuity there is then a compact neighborhood U of v in T^1M as claimed in the lemma. q.e.d.

Assume $\varphi\colon I \to \partial\widetilde{M}$ satisfies the assumptions of 3.1. For $s \in I$ let $\mathrm{Lip}(s)$ be the local dilation at s of the map φ with respect to the distance η_v. Then almost every $s \in I$ has the following properties:

(i) $\mathrm{Lip}(s) < \infty$.
(ii) The differential $\overline{\varphi}'(s)$ of $\overline{\varphi}$ at s exists.
(iii) $dQ(\overline{\varphi}(s))$ is recurrent.

Thus 3.1 follows from

Lemma 3.3. *If* $s \in I$ *satisfies* (i)–(iii), *then* $\overline{\varphi}'(s) \in \widetilde{E}(\varphi(s))$.

Proof. Assume $s \in I$ satisfies (i)–(iii), let $w = dQ\overline{\varphi}(s)$, and choose $\rho > 0$ such that $E(w) = E_\rho(w)$. By 3.2 there is a compact neighborhood U of w in T^1M and $\varepsilon > 0$ such that for all $u \in U$ and every $(k+1)$-dimensional subspace A of $u^\perp$, the determinant of the restriction to A of the map $\Lambda(u\,, \rho)$ is not smaller than $e^{(k+1)\rho}(1 + \varepsilon)$.

Assume $(P\overline{\varphi})'(s) \notin E(\overline{\varphi}(s))$; in particular, $\overline{\varphi}'(s) \neq 0$, and let $A \subset w^\perp$ be the linear hull of $E(w)$ and the projection $dQ(P\overline{\varphi})'(s)$ of $(P\overline{\varphi})'(s)$ into TM. Let $d_0 = d_{v,0}$; since $\overline{\varphi}'(s) \neq 0$ and $\mathrm{Lip}(s) < \infty$, there are numbers $\sigma > 0$, $\nu \in (0, 1)$ such that $d_0(P\overline{\varphi}(s), P\overline{\varphi}(s+t)) \geq \nu t$ and $\eta_v(\varphi(s), \varphi(s+t)) \leq t/\nu$ for all $t \leq \sigma$, i.e., $d_0(P\overline{\varphi}(s), P\overline{\varphi}(s+t)) \geq \nu^2\eta_v(\varphi(s), \varphi(s+t))$.

Denote again by $\|_t\colon (\Phi^t w)^\perp \to w^\perp$ the parallel transport along the geodesic $s \to \Phi^{-s}w$ in M. Since w is recurrent, there are numbers $\{t_j\} \in \mathbf{R}$ such that $t_{j+1} \geq t_j + \rho$ and $\Phi^{t_j}w \in U$. The choice of U then implies that the determinant of the map $\|_{-t_j}\Lambda(w\,, t_j) \mid A$ is not smaller than $e^{(k+1)t_j}(1 + \varepsilon)^j$.

However $\|_{-t}\Lambda(w\,, t)u = e^t u$ for every $u \in E(w)$ and $t > 0$ by 2.1, and $\|_{-t}\Lambda(w\,, t)$ leaves the orthogonal complement $E(w)^\perp$ of $E(w)$ invariant. Thus if $e \in A \cap E(w)^\perp$, then $e^{-kt}\det\|_{-t}\Lambda(w\,, t) \mid A = \lambda_t$ is the norm of the vector $\|_{-t}\Lambda(w\,, t)e$ and $\lambda_{t_j} \geq e^{t_j}(1 + \varepsilon)^j$ by the choice of t_j.

Now the projection of $P\overline{\varphi}'(s)$ into TM equals $\alpha e + \tilde{e}$ for some $\tilde{e} \in E(w)$ and some $\alpha \neq 0$. Choose $j \in \mathbf{Z}$ such that $t_j > 1$ and $(1+\varepsilon)^{-j} < \nu^2|\alpha|/4\|ae + \tilde{e}\|$, and denote by exp the exponential map at $\pi_{t_j}\varphi(s)$ on the horosphere $\theta_v^{-1}(t_j)$ with respect to the induced Riemannian structure. The above arguments show that the norm of $(\pi_{t_j}\varphi)'(s)$ is not smaller than $|\alpha|\lambda_{t_j} > 4e^{t_j}\|P\overline{\varphi}'(s)\|/\nu^2 > 1$; thus if $\beta \in (0, 1]$ is such that $u = \beta(\pi_{t_j}\varphi)'(s)$ is a unit vector in the tangent space of $\theta_v^{-1}(t_j)$ at $\pi_{t_j}\varphi(s)$, then $\|d\pi_o u\| < e^{-t_j}\nu^2/4$, and there is $\tau_0 \in (0, \sigma)$ such that $\|d\pi_0 \frac{d}{d\tau}\exp \tau u\| < e^{-t_j}\nu^2/2$ and

$$d_{v,t_j}(\exp \tau u, \pi_{t_j}\varphi(s + \tau\beta)) < \nu^2\tau/4$$

for all $\tau < \tau_0$. Thus

$$d_{v,t_j}(\pi_{t_j}\varphi(s + \tau_0\beta), \ \varphi\pi_{t_j}(s)) > \tau_0(1 - \nu^2/4),$$

which implies that

$$\eta_v(\varphi(s + \tau_0\beta), \varphi(s)) > e^{-t_j}\tau_0(1 - \nu^2/4).$$

On the other hand,

$$d_0(\pi_0\varphi(s), \exp \tau_0 u) + d_0(\pi_0 \exp \tau_0 u, \pi_0\varphi(s + \tau_0\beta)) < e^{-t_j}3\nu^2\tau_0/4.$$

Since $\tau_0 < \sigma$, $|\beta| \leq 1$, and $\nu^2 < 1$, this contradicts the fact that $d_0(\pi_0\varphi(s), \pi_0\varphi(s + t)) \geq \nu^2\eta_v(\varphi(s), \varphi(s + t))$ for all $t < \sigma$ and finishes the proof of the lemma.

4. The foliation on $\partial\widetilde{M}$

By 2.5 the assignment $E' \colon w \to E'(w)$ is a smooth k-dimensional distribution on $\Omega \subset T^1M$. Thus for $w \in \Omega$ and $i \geq 1$ we can consider the vector space $E'_i(w) \subset TW^{\mathrm{su}}(w)$ which is spanned by $E'(w)$ and the values at w of the commutators up to order i of the vector fields which are tangent to E'. Since the dimension of $E'_i(w)$ is locally nondecreasing, there is an open subset U of Ω such that the dimension of E'_i is constant on U for every $i \in \{1, \cdots, m\}$. In particular, E'_m is an integrable distribution on U (recall $m = \dim W^{\mathrm{su}}$). Now E' is invariant under the geodesic flow on T^1M, hence the same is true for E'_i. Thus, by the ergodicity of the geodesic flow, E'_i are smooth distributions on an open subset of Ω of full measure which we may identify (by abuse of notation) with Ω. Let $\widetilde{\Omega}$ be the lift of Ω to $T^1\widetilde{M}$. E_i' lifts to a distribution on $\widetilde{\Omega}$ which we denote by $\widetilde{E}_i$.

Lemma 4.1. *Let D be an open Φ^t-invariant subset of T^1M. If $v \in D$, and $-v$ is recurrent, then the whole strong unstable manifold $W^{\mathrm{su}}(v)$ is contained in D.*

Proof. Given $\tilde{v} \in T^1\widetilde{M}$, the distance $d_{\tilde{v},0}$ on the horosphere $PW^{\mathrm{su}}(\tilde{v})$ lifts to a distance $d_{\tilde{v}}$ on $W^{\mathrm{su}}(\tilde{v})$. These distances are clearly invariant under the action of the isometry group of $\widetilde{M}$ on $T^1\widetilde{M}$, hence they project to distances d_v on $W^{\mathrm{su}}(v)$ $(v \in T^1M)$.

For $v \in D$ there is a neighborhood U of v in D and $\varepsilon > 0$ such that for all $w \in U$ the ε-neighborhood of w in $(W^{\mathrm{su}}(w), d_w)$ is contained in D. If $-v$ is recurrent, there are numbers $t_j \in \mathbf{R}$ such that $t_j \to -\infty$ and $\Phi^{t_j}v \in U$. This implies that the ε-neighborhood of $\Phi^{t_j}v$ in $W^u(\Phi^{t_j}v, d_{\Phi^{t_j}v})$ is contained in D.

By the choice of the distances d_w the image under Φ^{-t_j} of $W^{\mathrm{su}}(\Phi^{t_j}v)\cap D$ contains the $e^{-t_j}\varepsilon$-neighborhood of v in $(W^{\mathrm{su}}(v), d_v)$. Since D is invariant under the geodesic flow, this shows $W^{\mathrm{su}}(v) \subset D$. q.e.d.

Define $\overline{\Omega} = \{v \in \widetilde{\Omega} \mid dQ(w)$ and $dQ(-w)$ are recurrent for almost every $w \in W^{\mathrm{su}}(v)\}$ (with respect to the Lebesgue measure on $W^{\mathrm{su}}(v)$). Since the strong unstable foliation is absolutely continuous with respect to the Lebesgue measure (see [14]), $\overline{\Omega}$ is a subset of $\widetilde{\Omega}$ of full measure which is invariant under the action of Γ and the geodesic flow. Lemma 4.1 shows $W^{\mathrm{su}}(v) \subset \widetilde{\Omega}$ for every $v \in \overline{\Omega}$; in particular, the distribution $\widetilde{E}_m$ is defined on all of $W^{\mathrm{su}}(v)$. Thus the maximal integral manifolds of $\widetilde{E}_m$ induce a smooth foliation of $W^{\mathrm{su}}(v)$ which projects to a foliation $\mathfrak{F}_v$ of $\theta_v^{-1}(0)$.

Lemma 4.2. *Let $v, w \in \overline{\Omega}$ and let $\varphi: I \to \theta_v^{-1}(0)$ be a smooth curve which is tangent to E_v and such that $\gamma_w(-\infty) \notin \pi_{v,\infty}\varphi(I)$. Then $\pi_{w,0} \circ \pi_{v,\infty}\varphi$ is contained in a leaf of $\mathfrak{F}_w$.*

Proof. Without loss of generality we may assume that the map $\varphi: I \to \theta_v^{-1}(0)$ is an embedding. Then there is an open neighborhood U of $\varphi(I)$ in $\theta_v^{-1}(0)$ and local coordinates $(x^1, \cdots, x^m): U \to (-2, 2)^m$ on U with the following properties:

(i) $\pi_{v,\infty}U \not\ni \gamma_w(-\infty)$.
(ii) $x^i(\varphi(0)) = 0$ for $i \in \{1, \cdots, m\}$.
(iii) The local vector field $\partial/\partial x_1$ is tangent to E_v.
(iv) φ is an integral curve of $\partial/\partial x_1$.

Let λ_m be the Lebesgue measure on the Riemannian manifold $\theta_v^{-1}(0)$. Recall that the *m-dimensional modulus* $M_m(\Psi)$ of a family Ψ of rectifiable nontrivial curves in $\theta_w^{-1}(0)$ is the infimum of all integrals $\int \rho^m \, d\lambda^m$, where ρ runs through the family of all nonnegative Borel functions on $\theta_w^{-1}(0)$ with the property that $\int_\psi \rho \geq 1$ for every $\psi \in \Psi$.

Identify U with $(-2, 2)^m$ via the coordinates $(x^1, \cdots, x^m)$. Let $I^{m-1} = [-1, 1]^{m-1} \subset \mathbf{R}^{m-1}$, and for $x \in I^{m-1}$ denote by $\varphi(x)$ the curve $t \to (t, x)$ on $\theta_v^{-1}(0)$. Then for every open subset B of I^{m-1} the m-dimensional modulus of the path family $\Psi_B = \{\varphi(x) \mid x \in B\}$ does not vanish [19].

Let $\overline{\varphi}(x) = \pi_{w,0} \circ \pi_{v,\infty}\varphi(x)$; then $\overline{\Psi}_B = \{\overline{\varphi}(x) \mid x \in B\}$ is a family of rectifiable curves in $\overline{U} = \pi_{w,0} \circ \pi_{v,\infty}(U)$. We claim $M_m(\overline{\Psi}_B) > 0$. To see this observe first that by 2.5, 1.4, 1.6, and the absolute continuity of $\pi_{w,0} \circ \pi_{v,\infty}$ with respect to the Lebesgue measure (see [14]) there is a number $L > 0$ such that for every $x \in I^{m-1}$ and $t \in I$ we have

(a) The local dilation of

$$\pi_{w,0} \circ \pi_{v,\infty}\colon (\varphi(x)(I), d_{v,0}) \to (\overline{\varphi}(x)(I), d_{w,0})$$

at $\varphi(x)(t)$ does not exceed L.

(b) The Jacobian at $\overline{\varphi}(x)(t)$ of the map $(\pi_{w,0} \circ \pi_{v,\infty})^{-1}\colon \overline{U} \to U$ with respect to the Lebesgue measure does not exceed L.

Let $\overline{\rho}\colon \overline{U} \to [0, \infty)$ be a Borel function such that $\int_{\overline{\varphi}(x)} \overline{\rho} \geq 1$ for every $x \in B$. Define $\rho = \overline{\rho} \circ \pi_{w,0} \circ \pi_{v,\infty}$; then (a) shows $\int_{\varphi(x)} \rho \geq 1/L$ for every $x \in B$ and consequently $\int_U \rho^m \geq L^{-m} M_m(\Psi_B)$. On the other hand, (b) implies $\int_U \rho^m \leq L \int_{\overline{U}} \overline{\rho}^m$, i.e., $\int_{\overline{U}} \overline{\rho}^m \geq L^{-m-1} M_m(\Psi_B)$. Since $\overline{\rho}$ was arbitrary, this means $M_m(\overline{\Psi}_B) > 0$ as claimed.

Let λ^1 be the 1-dimensional Hausdorff-measure in $(\overline{U}, d_{w,0})$ and let $A = \{\overline{u} \in \overline{U} \mid dQ(\overline{u}) \text{ is recurrent}\}$. Then $\overline{U} \backslash A$ is a set of measure zero, and hence the m-dimensional modulus of the path family $\Psi = \{\varphi(x) \mid x \in I^{m-1}, \lambda^1(\overline{\varphi}(x)(I) \backslash A) > 0\}$ vanishes (see [19]). Since $M_m(\overline{\Psi}_B) > 0$ for every open subset B of I^{m-1}, this means that $\overline{B} = \{x \in I^{m-1} \mid \overline{\varphi}(x) \notin \Psi\}$ is dense in I^{m-1}.

Now 3.3 shows that $\overline{\varphi}(x)$ is tangent almost everywhere to E_w for every $x \in \overline{B}$, hence by continuity and absolute continuity it is contained in a leaf of $\mathfrak{F}_w$. Since $\overline{B}$ is dense in I^{m-1} and the leaves of $\mathfrak{F}_w$ are locally closed, this implies that every curve $\overline{\varphi}(x)$ $(x \in B)$ is contained in a leaf of $\mathfrak{F}_w$; in particular, this is true for $\overline{\varphi}(0) = \pi_{w,0} \circ \pi_{v,\infty}\varphi$. q.e.d.

Assume $\dim \widetilde{E}_m = p$. Since $\overline{\Omega}$ and the distribution $\widetilde{E}_m$ are Γ-invariant, 4.2 shows that the foliations $\mathfrak{F}_v$ $(v \in \Omega)$ induce a Γ-invariant p-dimensional foliation $\mathfrak{F}$ on $\partial\widetilde{M}$. Here the leaf of $\mathfrak{F}$ containing ξ consists of all $\zeta \in \partial\widetilde{M}$ for which there is $v \in \overline{\Omega}$ and a curve φ on $\theta_v^{-1}(0)$ which is contained in a leaf of $\mathfrak{F}_v$ and such that the projection of φ into $\partial\widetilde{M}$ joins ξ to ζ. For $l \in \mathbf{N}$ denote by C^l the cube $\{x = (x_1, \cdots, x_l) \in \mathbf{R}^l \mid |x_i| < 1\}$ in $\mathbf{R}^l$.

Corollary 4.3. *Every leaf of $\mathfrak{F}$ is everywhere dense in $\partial\widetilde{M}$.*

Proof. Assume that there is an open subset U of $\partial\widetilde{M}$ and a leaf F of $\mathfrak{F}$ which does not meet U. Choose $\xi \in F$; by the definition of $\mathfrak{F}$ there is a homeomorphism α of an open neighborhood A of ξ in $\partial\widetilde{M}\backslash U$ onto C^m such that $\alpha^{-1}(C^p \times \{y\})$ is contained in a leaf of $\mathfrak{F}$ for every $y \in C^{m-p}$. In particular, every leaf of $\mathfrak{F}$ through a point of A meets the topological boundary ∂A of A.

Let $v \in \overline{\Omega}$ be such that γ_v is the axis of an isometry $\Psi \in \Gamma$ and that, moreover, $\gamma_v(\infty) \in A$, and $\gamma_v(-\infty) \in U$. Now Ψ acts as a homeomorphism on $\partial\widetilde{M}$ leaving $\mathfrak{F}$ invariant; moreover, there is $k \in \mathbf{Z}$ such that $\Psi^k \partial A \subset U$. But this means that every leaf of $\mathfrak{F}$ through a point of $\Psi^k A \supset A$ meets U; in particular, this is true for F, a contradiction which shows the claim.

Corollary 4.4. $\dim \widetilde{E}_m = m$, *i.e., $\mathfrak{F}$ is the trivial foliation.*

Proof. Assume to the contrary that $\dim \widetilde{E}_m = p < m$. Let $v \in \overline{\Omega}$ be such that γ_v is the axis of an isometry $\Psi \in \Gamma$. There is an open neighborhood A of $\xi = \gamma_v(\infty)$ in ∂M and a homeomorphism α of A onto C^m such that for every $y \in C^{m-p}$ the set $\alpha^{-1}(C^p \times \{y\})$ is contained in a leaf of $\mathfrak{F}$. Similarly we can find a homeomorphism β of an open neighborhood B of $\zeta = \gamma_v(-\infty)$ in $\partial\widetilde{M}$ with according properties. α and β may be chosen in such a way that $\alpha(\xi) = \beta(\zeta) = 0$. Moreover, by the arguments in the proof of 4.3 we may assume $A \cup B = \partial\widetilde{M}$ and $\Psi B \in B$.

Since $\alpha^{-1}(C^p \times \{0\})$ is homeomorphic to C^p $(p \geq 1)$, the complement of ξ in $\alpha^{-1}(C^p \times \{0\})$ has at most two components F_1, F_2. F_1, F_2 are subsets of the leaf F of $\mathfrak{F}$ through ξ which is invariant under Ψ. Let ∂F_i $(i = 1, 2)$ be the intersection of the closure of F_i with $\partial\widetilde{M}\backslash A$. ∂F_i is a connected subset of $B \cap F$ and hence there is $y_i \in C^{m-p}$ such that $G_i = F_i \cup \beta^{-1}(C^p \times \{y_i\})$ is a connected subset of F.

Since the image of F_1 under Ψ^{-1} is a connected component of $F\backslash\xi \cap \Psi^{-1}A \subset A$, we have $\Psi^{-1}F_1 \subset F_j$ for $j = 1$ or $j = 2$; thus replacing Ψ by Ψ^2 if necessary we may assume $\Psi^{-1}F_i \subset F_i$.

Suppose $y_i \neq 0$; then there is $k > 0$ such that $\beta^{-1}(C^p \times \{y_i\}) \cap \Psi^k B = \emptyset$. Now $\widetilde{G}_i = G_i \backslash \Psi^{-k} F_i$ is a connected subset of F, and hence $\Psi^k \widetilde{G}_i$ is a connected subset of $F \cap B$ containing ∂F_i. But this means $\Psi^k \widetilde{G}_i \subset \beta^{-1}(C^p \times \{y_i\})$, in contradiction to the choice of k. Thus $y_i = 0$ for $i = 1, 2$, and consequently $\alpha^{-1}(C^p \times \{0\}) \cup \beta^{-1}(C^p \times \{0\})$ is a leaf of $\mathfrak{F}$ which is closed in $\partial \widetilde{M}$. This contradicts 4.3 and shows the corollary. q.e.d.

Since by 4.4, $\widetilde{E}_m$ is a smooth distribution on $\widetilde{\Omega}$ which equals the tangent bundle of the strong unstable foliation, we have

Corollary 4.5. *The strong unstable foliation is smooth on* $\widetilde{\Omega}$.

5. The Carnot-Carathéodory metric induced by E

In §4 we showed that for $v \in \overline{\Omega}$ the distribution E_v on $\theta_v^{-1}(0)$ generates the whole tangent space of the horosphere. Thus we can consider the Carnot-Carathéodory metric δ_v on $\theta_v^{-1}(0)$ which is induced by E_v. Let $\bar{l}_v$ be the length-pseudo metric on $\partial \widetilde{M} - \gamma_v(-\infty)$ induced by η_v.

Lemma 5.1. *If* $W^{\mathrm{su}}(v) \subset \widetilde{\Omega}$, *then* $\bar{l}_v \leq \delta_v \circ \pi_{v,0}$; *in particular,* $\bar{l}_v$ *is a distance on* $\partial \widetilde{M} - \gamma_v(-\infty)$.

Proof. Since δ_v is a complete length metric on $PW^{\mathrm{su}}(v)$ (see [7]), the distance between any two points $x, y \in PW^{\mathrm{su}}(v)$ can be realized by a minimizing geodesic. Let $\phi: [0, \rho] \to PW^{\mathrm{su}}(v)$ be such a geodesic parametrized by arc length. Then $\delta_v(\phi(s), \phi(s+\varepsilon)) = \varepsilon$ for all $\varepsilon > 0$, $s \in [0, \rho - \varepsilon]$, hence $e^{-t} d_{v,t}(\phi(s), \phi(s+\varepsilon)) \leq \varepsilon$ for all $t \in \mathbf{R}$. This shows $\eta_v(\pi_{v,\infty}\phi(s), \pi_{v,\infty}(\phi(s+\varepsilon)) \leq \varepsilon$ by the definition of η_v, hence $\bar{l}_v \leq \delta_v \circ \pi_{v,0}$ as claimed. q.e.d.

Combining 4.5 and 1.4 we obtain

Lemma 5.2. $\bar{l}_v$ *is a distance on* $\partial \widetilde{M} - \gamma_v(-\infty)$ *for every* $v \in T^1 \widetilde{M}$.

Corollary 5.3. *Let* $v \in T^1 \widetilde{M}$ *be such that* $W^{\mathrm{su}}(v) \subset \widetilde{\Omega}$, *and let* $\varphi: I \to \theta_v^{-1}(0)$ *be a rectifiable curve which is tangent almost everywhere to* E_v. *Then the* δ_v*-length of* φ *coincides with the* η_v*-length of* $\pi_{v,\infty} \circ \varphi$.

Proof. If $\varphi: I \to \theta_v^{-1}(0)$ is tangent almost everywhere to E_v, then the δ_v-length of φ coincides with the length of φ with respect to the distance $d_{v,0}$ (see [17]). Since $\eta_v \geq d_{v,0} \circ \pi_{v,0}$, the η_v-length of φ is not smaller than its δ_v-length. The reverse inequality follows from 5.1 and the fact that $\bar{l}_v \geq \eta_v$.

By ergodicity of the geodesic flow on T^1M and the Birkhoff ergodic theorem there is $v \in T^1M$ and a Borel set $\overline{A} \subset W^{su}(v)$ of vanishing Lebesgue measure such that for every $w \in W^{su}(v) - \overline{A}$ the orbit $\{\Phi^t dQ(w) \mid t \in [0, \infty)\}$ of $dQ(w)$ is dense in T^1M. Let $A = \pi_{v,\infty}\overline{A} \cup \{\gamma_v(-\infty)\}$. Then A is a measure zero set with respect to the Lebesgue measure class on $\partial\widetilde{M}$, and for every $w \in T^1\widetilde{M}$ with $\pi(w) \in \partial\widetilde{M} - A$ the orbit $\{\Phi^t dQ(w) \mid t \in [0, \infty)\}$ is dense in T^1M. Clearly we may assume that A is Γ-invariant. Let χ_A be the characteristic function of A.

For $v \in T^1\widetilde{M}$ and a curve $\varphi: I \to \partial\widetilde{M} - \{\gamma_v(-\infty)\}$ let $\tilde{l}_v(\varphi)$ be the η_v-length of φ. For $x, y \in \partial\widetilde{M} - \{\gamma_v(-\infty)\}$ and $\varepsilon > 0$ define $l_\varepsilon(x, y) = \inf\{\tilde{l}_v(\varphi) \mid \eta_v(x, \varphi(0)) < \varepsilon,\ \eta_v(y, \varphi(1)) < \varepsilon,\ \int_\varphi(1 - \chi_A) = 0\}$ and $l_v(x, y) = \limsup_{\varepsilon\to 0} l_v(x, y)$. Clearly l_v is a pseudo-metric on $\partial\widetilde{M} - \{\gamma_v(-\infty)\}$.

Lemma 5.4. *If $W^{su}(v) \subset \widetilde{\Omega}$, then $l_v = \delta_v \circ \pi_{v,0}$.*

Proof. We show first $l_v \le \delta_v \circ \pi_{v,0}$. For this let $x, y \in \theta_v^{-1}(0)$, $x \ne y$, and let $\varphi: I \to \theta_v^{-1}(0)$ be a minimizing geodesic with respect to δ_v joining $\varphi(0) = x$ to $\varphi(1) = y$. Then φ is a smooth curve which is tangent to E_v and parametrized proportional to arc length ([10]). Thus there is a smooth section of X of E_v on a neighborhood of $\varphi(I)$ in $\theta_v^{-1}(0)$ of constant norm $\delta = \delta_v(x, y)$ whose restriction of $\varphi(I)$ equals the tangent of φ. Let $\varepsilon > 0$ and let $\mathscr{U}$ be an open neighborhood of x in $\theta_v^{-1}(0)$ with the following properties:

(i) $\pi_{v,\infty}\mathscr{U} \subset B_{\eta_v}(\pi_{v,\infty}x, \varepsilon)$.

(ii) For every $z \in \mathscr{U}$ the integral curve φ_z of X through $\varphi_z(0) = z$ exists on $[0, 1]$ and satisfies $\eta_v(\pi_{v,\infty}\varphi_z(1), \pi_{v,\infty}(y)) < \varepsilon$.

The considerations in the proof of 4.2 show that the m-dimensional modulus of the path family $\{\varphi_z \mid z \in \mathscr{U}\}$ does not vanish. Let χ be the characteristic function of $\pi_{v,0}(A)$ and let Ψ be the family of all locally rectifiable curves ψ in $\theta_v^{-1}(0)$ such that $\int_\psi(1 - \chi) > 0$. Since $\pi_{v,0}(A)$ is a set of vanishing Lebesgue measure, the m-dimensional modulus of Ψ vanishes. But this means that there is $z \in \mathscr{U}$ such that $\varphi_z \notin \Psi$. By 5.3 the η_v-length of $\pi_{v,\infty} \circ \varphi_z$ equals $\delta = \delta_v(x, y)$, and consequently $l_\varepsilon(\pi_{v,\infty}x, \pi_{v,\infty}y) \le \delta$ by the choice of $\mathscr{U}$. Since $\varepsilon > 0$ was arbitrary, this shows $l_v \le \delta_v \circ \pi_{v,0}$, in particular l_v is finite on $\partial\widetilde{M} - \{\gamma_v(-\infty)\}$.

To show the reverse inequality let $x, y \in \partial\widetilde{M} - \{\gamma_v(-\infty)\}$ and let $\varepsilon > 0$. Then there is a curve $\varphi: I \to \partial\widetilde{M} - \{\gamma_v(-\infty)\}$ with $\eta_v(x, \varphi(0)) < \varepsilon$,

$\eta_v(y, \varphi(1)) < \varepsilon$, $\int_\varphi(1-\chi) = 0$ and $\tilde{l}_v(\varphi) \le l_v(x, y) + \varepsilon$. Let $\overline{\varphi} = \pi_{v,0} \circ \varphi$, then $\overline{\varphi}$ is rectifiable with respect to $d_{v,0}$, hence differentiable almost everywhere. Moreover by the choice of A and 3.1, $\overline{\varphi}$ is tangent almost everywhere to E_v. Thus $\tilde{l}_v(\varphi)$ equals the δ_v-length of $\overline{\varphi}$ (Lemma 5.3) which shows $\delta_v(\overline{\varphi}(0), \overline{\varphi}(1)) \le \tilde{l}_v(\varphi) \le l_v(x, y) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this yields the lemma.

Remark. 5.4 shows in particular that l_v is a distance on $\partial\widetilde{M} - \{\gamma_v(-\infty)\}$ inducing the standard topology for all $v \in T^1\widetilde{M}$ such that $W^{su} \subset \widetilde{\Omega}$.

Next we investigate the relation between the distances l_v $(v \in T^1\widetilde{M})$.

Lemma 5.5. *Let $v, w \in T^1\widetilde{M}$ and $x \in \partial\widetilde{M} - \{\gamma_v(-\infty), \gamma_w(-\infty)\}$. Assume that there is $\lambda > 0$, $\sigma > 0$ such that $\eta_w(y, z) \le \lambda\eta_v(y, z)$ for all $y, z \in B_{\eta_v}(x, \sigma)$. Then $l_w(y, z) \le \lambda l_v(y, z)$ for all $y, z \in B_{l_v}(x, \sigma/3)$.*

Proof. Let $y, z \in B_{l_v}(x, \sigma/3)$ and let $\varepsilon > 0$. Since $\eta_v \le l_v$, there is a curve $\varphi: I \to \partial\widetilde{M} - \{\gamma_v(-\infty)\}$ with $\varphi(0) \in B_{\eta_v}(x, \sigma/3) \cap B_{\eta_v}(y, \varepsilon)$, $\varphi(1) \in B_{\eta_v}(x, \sigma/3) \cap B_{\eta_v}(z, \varepsilon)$, $\tilde{l}_v(\varphi) < \min\{2\sigma/3, l_v(y, z) + \varepsilon\}$ and such that $\int_\varphi(1-\chi) = 0$. Then necessarily $\varphi(I) \subset B_{\eta_v}(x, \sigma)$, hence the η_w-length of φ does not exceed $\lambda\tilde{l}_v(\varphi)$. Since $\varepsilon > 0$ was arbitrary, the lemma follows.

Corollary 5.6. *Let $v, w \in T^1\widetilde{M}$, and $x \in \partial\widetilde{M} - \{\gamma_v(-\infty), \gamma_w(-\infty)\}$, and let θ be a Busemann function at x. If $\tau = \theta(\pi_{v,0}(x)) - \theta(\pi_{w,0}(x))$, then for every $\varepsilon > 0$ there is a neighborhood A of x in $\partial\widetilde{M}$ such that $(1-\varepsilon)l_v(y, z) \le e^\tau l_w(y, z) \le (1+\varepsilon)l_v(y, z)$.*

Proof. By 1.4 there is a neighborhood $\widetilde{A}$ of x in $\partial\widetilde{M}$ such that $(1-\varepsilon)\eta_v(y, z) \le e^\tau\eta_w(y, z) \le (1+\varepsilon)\eta_w(y, z)$ for all $y, z \in \widetilde{A}$. Choose $\sigma > 0$ such that $B_{\eta_v}(x, \sigma) \cup B_{\eta_w}(x, \sigma) \subset \widetilde{A}$. By 5.4, $A = B_{l_v}(x, \sigma/3) \cap B_{l_w}(x, \sigma/3)$ satisfies the claim.

Lemma 5.7. *Let $v, w \in T^1\widetilde{M}$ and let $A \subset \partial\widetilde{M} - \gamma_v(-\infty)$ be compact. Assume that there is $\tau \in \mathbf{R}$ and $x \in A$ such that $\pi_{v,\infty}B_d(\pi_{v,\tau}x, 3) \subset A$ and $d(\pi_{v,\tau}(y), \pi_{w,\tau}(y)) < 1$ for all $y \in A$. Then $B_{\eta_w}(x, e^{-\tau}) \subset A$.*

Proof. For $y \in B_{\eta_w}(x, e^{-\tau})$ there is a curve $\phi: I \to \theta_w^{-1}(\tau)$ of length smaller than 1 such that $\phi(0) = \pi_{w,\tau}(x)$ and $\phi(1) = \pi_{w,\tau}(y)$. If $\pi_{w,\infty}\phi(I) \not\subset A$, then there is a first $s \in I$ such that $d(\pi_{v,\tau}(x), \pi_{v,\tau}\phi(s)) = 3$. But for this s we have $\phi(s) \in A$, hence

$$d(\pi_{w,\tau}(x), \pi_{w,\tau}(\phi(s)) > d(\pi_{v,\tau}(x), \pi_{v,\tau}\phi(s)) - 2 = 1,$$

in contradiction to the choice of ϕ.

Lemma 5.8. *Let* $v \in T^1\widetilde{M}$ *and let* $A \subset \partial\widetilde{M} - \gamma_v(-\infty)$ *be compact. Then for every* $\varepsilon > 0$ *there is a neighborhood* U *of* v *in* $T^1\widetilde{M}$ *such that*

(i) $(1-\varepsilon)\eta_w(y, z) \le \eta_v(y, z) \le (1+\varepsilon)\eta_w(y, z)$,

(ii) $(1-\varepsilon)l_w(y, z) \le l_v(y, z) \le (1+\varepsilon)l_v(y, z)$

for all $y, z \in A$ *and* $w \in U$.

Proof. Let $x = \gamma_v(\infty)$ and choose $r > 0$ such that $A \subset B_{l_v}(x, r)$. Define $\tau = \log 1/(24r)$ and let B be the closure of $\pi_{v,\infty}B_d(\pi_{v,\tau}(x), 3)$ in $\partial\widetilde{M}$. Given $\varepsilon \in (0, 1)$ let $\rho > 0$ be such that

$$\sigma(\rho) < \min\{1/(1-\varepsilon), 1+\varepsilon\},$$

where σ is as in 1.3. There is an open neighborhood U of v in $T^1\widetilde{M}$ such that $d(\pi_{w,\tau}(y), \pi_{v,\tau}(y)) < \rho$ for all $y \in B$ and $w \in U$ (compare the proof of Lemma 7 in [9]). Then $B_{\eta_w}(x, 24r) \subset B$ for all $w \in U$ by 5.5. Now 1.3 shows $\eta_w(y, z) \le \sigma(\rho)\eta_v(y, z)$ for $y, z \in B_{\eta_v}(x, 6r)$; in particular, $B_{\eta_v}(x, 3r) \subset B_{\eta_w}(x, 6r)$, and $\eta_v(\bar{y}, \bar{z}) \le \sigma(\rho)\eta_w(\bar{y}, \bar{z})$ for all $\bar{y}, \bar{z} \in B_{\eta_w}(x, 6r)$ $(w \in U)$. Since $A \subset B_{\eta_w}(x, r)$, this is (i).

It follows from 5.5 that $l_w(y, z) \le \sigma(\rho)l_v(y, z)$ for all $y, z \in B_{l_v}(x, 2r)$, and that $l_v(\bar{y}, \bar{z}) \le \sigma(\rho)l_w(\bar{y}, \bar{z})$ for all $\bar{y}, \bar{z} \in B_{l_w}(x, 2r)$. Thus $B_{l_v}(x, r) \subset B_{l_w}(x, 2r)$, and this is (ii) since $A \subset B_{l_v}(x, r)$.

Corollary 5.9. *There is a number* $\nu > 0$ *such that* $\eta_v \ge \nu l_v$ *for all* $v \in T^1\widetilde{M}$.

Proof. Let $D \subset M$ be a compact fundamental domain for the action of the isometry group on $\widetilde{M}$. For $w \in T^1\widetilde{M}|_D$ define $u(w) = \sup\{r > 0 | B_{\eta_w}(\gamma_w(\infty), r) \subset B_{l_w}(\gamma_w(\infty), 1)\}$. We claim $\nu = \inf\{u(w) | w \in T^1\widetilde{M}|_D\} > 0$.

To show this choose a sequence $\{v_i\} \subset T^1\widetilde{M}|_D$ such that $u(v_i) \to \nu$. By compactness of $T^1\widetilde{M}|_D$ we may assume that $\{v_i\}$ converges to $v \in T^1\widetilde{M}|_D$. Let $x_i = \gamma_{v_i}(\infty)$, $x = \gamma_v(\infty)$ and write $l_i = l_{v_i}$, $l = l_v$, $\eta_i = \eta_{v_i}$, and $\eta = \eta_v$. Define $\rho = \sup\{r > 0 \mid B_\eta(x, r) \subset B_l(x, 1/4)\} > 0$. Let $\overline{B} = \overline{B}_\eta(x, 2)$. By 5.8 there is a number $i_0 > 0$ such that for all $i \ge i_0$

(i) $x_i \in B_l(x, \rho/4)$,

(ii) $\eta(y, z)/2 \le \eta_i(y, z) \le 2\eta(y, z)$ for all $y, z \in \overline{B}$, and

(iii) $l(y, z)/2 \le l_i(y, z) \le 2l(y, z)$ for all $y, z \in \overline{B}$.

Let $i \ge i_0$. Then $\eta(x, y) = 2$ implies $\eta_i(x, y) \ge 1$, hence by the connectedness of the η_i-balls this means $B_{\eta_i}(x, 1) \subset \overline{B}$. Since $l_i(x, x_i) < \rho/2 < 1/4$, by (i) and (iii) we have $B_{l_i}(x_i, 1) \supset B_{l_i}(x, 1/2) \supset B_l(x, 1/4)$

and in the same vein $B_{\eta_i}(x_i, \rho/4) \subset B_{\eta_i}(x, \rho/2) \subset B_\eta(x, \rho)$. This shows $\nu \geq \rho/4$.

Now ν is the constant which we are looking for. Let $v \in T^1\widetilde{M}$ be arbitrary, $x \in \partial\widetilde{M} - \gamma_v(-\infty)$, $\varepsilon > 0$, $\tau = \log 1/\varepsilon$. Let $w \in W^{\mathrm{su}}(\Phi^\tau v)$ be such that $\gamma_w(\infty) = x$. Clearly $\eta_w = \eta_v/\varepsilon$, $l_w = l_v/\varepsilon$. By the choice of D there is an isometry Ψ of $\widetilde{M}$ such that $\Psi(Pw) \in D$. Let $\tilde{w} = d\Psi(w)$; Ψ then induces an isometry of $(\partial\widetilde{M} - \gamma_u(-\infty), \eta_u/\varepsilon)$ onto $(\partial\widetilde{M} - \gamma_{\tilde{w}}(-\infty), \eta_{\tilde{w}})$. By the definition of ν we have $B_{l_{\tilde{w}}}(\Psi(x), 1) \supset B_{\eta_{\tilde{w}}}(\Psi(x), \nu)$, hence $B_{l_v}(x, \varepsilon) \supset B_{\eta_v}(x, \nu\varepsilon)$, which finishes the proof. q.e.d.

Again let $v \in \widetilde{\Omega}$ be such that $W^{\mathrm{su}}(v) \subset \widetilde{\Omega}$. Then δ_v is a Carnot-Carathéodory metric on $PW^{\mathrm{su}}(v)$ induced by a smooth *generic* distribution (see [15]). Hence δ_v admits at Pv a *tangent cone* (see [15]) which consists of a nilpotent homogeneous Lie group $\mathfrak{N}_v$ equipped with a left-invariant Carnot-Carathédory metric $\delta_{\infty,v}$. $(\mathfrak{N}_v, \delta_{\infty,v})$ is determined by the property that for every $r > 0$ the compact balls $\overline{B}_{\lambda\delta_v}(Pv, r)$ converge as $\lambda \to \infty$ in the Hausdorff-sense to the closed ball of radius r in $(\mathfrak{N}_v, \delta_{\infty,v})$. $\mathfrak{N}_v$ admits a one-parameter group $\{\Delta_t | t > 0\}$ of automorphisms which act as a group of homotheties with respect to $\delta_{\infty,v}$. The Lie algebra automorphism associated to Δ_t is diagonalizable over $\mathbf{R}$; its eigenvalues are $t, t^2, \cdots, t^p$ where $p \geq 1$ is the least integer such that $\dim \widetilde{E}_p(v) = m$. In the sequel we will mean by a *homothety* always an automorphism of a nilpotent homogeneous Lie group of the above kind.

Let $\Upsilon \subset \widetilde{\Omega}$ be the set of points which project onto a periodic point of the geodesic flow in T^1M. For $v \in \Upsilon$ there is an isometry $\Psi \in \Gamma$ which acts as a translation on γ_v, i.e., $d\Psi\Phi^s v = \Phi^{s+\tau}v$ for some $\tau > 0$ and all $s \in \mathbf{R}$. By 4.1 we have $W^{\mathrm{su}}(v) \subset \widetilde{\Omega}$; thus for every integer j $d\Psi^j$ induces an isometry of $(PW^{\mathrm{su}}(v), \delta_v)$ onto $(PW^{\mathrm{su}}(\Phi^{j\tau}v), \delta_{\Phi^{j\tau}v})$ and the transformation $\pi_{v,0} \circ \Psi^j$ of $PW^{\mathrm{su}}(v)$ is an isometry of $(PW^{\mathrm{su}}(v), \delta_v)$ onto $(PW^{\mathrm{su}}(v), e^{j\tau}\delta_v)$ with fixed point Pv. This means that for every $r > 0$ the balls $\overline{B}_{\delta_v}(Pv, r)$ and $\overline{B}_{e^{j\tau}\delta_v}(Pv, r)$ are isometric; hence $(PW^{\mathrm{su}}(v), \delta_v)$ is isometric to $(\mathfrak{N}_v, \delta_{\infty,v})$.

Fix an arbitrary $\overline{v} \in \Upsilon$ and define $(\mathfrak{N}, \delta_\infty) = (\mathfrak{N}_{\overline{v}}, \delta_{\infty,\overline{v}})$.

Lemma 5.10. *$(\partial\widetilde{M} - \gamma_w(-\infty), l_w)$ is isometric to $(\mathfrak{N}, \delta_\infty)$ for every $w \in T^1\widetilde{M}$.*

Proof. We show first the claim for $w \in \Upsilon$. In this case $(\partial\widetilde{M} - \gamma_w(-\infty), l_w)$ is isometric to its tangent cone at any of its points. Let $x \in \partial\widetilde{M} - \{\gamma_{\overline{v}}(-\infty), \gamma_w(-\infty)\}$; it suffices to show that for every $r > 0$ the

Hausdorff-distance of the compact balls of radius R in the tangent cone at x of the metric spaces $(\partial\widetilde{M} - \gamma_{\overline{v}}(-\infty), l_{\overline{v}})$ and $(\partial\widetilde{M} - \gamma_w(-\infty), l_w)$ vanishes (see [7]).

Let $\varepsilon > 0$; by 5.4 there is a number $\beta > 0$ and a neighborhood A of x in $\partial\widetilde{M}$ such that

$$(\diamond) \qquad (1-\varepsilon)l_{\overline{v}}(y, z) \leq \beta l_w(y, z) \leq (1+\varepsilon)l_{\overline{v}}(y, z)$$

for all $y, z \in A$. Choose $\lambda_0 > 0$ sufficiently large such that $B_{l_v}(x, r/\lambda_0) \cup B_{\beta l_w}(x, r/\lambda_0) \subset A$. Since $(\diamond)$ is invariant under rescaling of $l_{\overline{v}}$ and l_w with the same factor, it follows $\sup\{|\lambda l_{\overline{v}}(y, z) - \lambda\beta l_w(y, z)| \mid y, z \in B_{\lambda l_v}(x, r) \cup B_{\lambda\beta l_w}(x, r)\} < \varepsilon r$ for all $\lambda \geq \lambda_0$. But this means that for $\lambda \geq \lambda_0$ the Hausdorff-distance of $B_{\lambda l_v}(x, r)$ and $B_{\lambda\beta l_w}(x, r)$ does not exceed εr. Since $\varepsilon > 0$ was arbitrary, the claim now follows from the definition of the tangent cone.

Now let $w \in T^1\widetilde{M}$ be arbitrary, $x = \gamma_w(\infty)$ and $r > 0$. Since $\widetilde{\Omega} \subset T^1\widetilde{M}$ is open and dense, and the periodic points of the geodesic flow in T^1M are dense, Υ is dense in $T^1\widetilde{M}$. Thus for $r, \varepsilon > 0$ by 5.6 there is a $v \in \Upsilon$ such that $(1-\varepsilon)l_v(y, z) \leq l_w(y, z) \leq (1+\varepsilon)l_v(y, z)$ for all $y, z \in \overline{B}_{l_v}(x, r) \cap \overline{B}_{l_w}(x, r)$. But this means as above that the Hausdorff-distance of $\overline{B}_{l_w}(x, r)$ and the compact ball of radius r in $(\mathfrak{N}, \delta_\infty)$ is not larger than εr. This is true for every $\varepsilon > 0$; hence these balls are isometric. But $r > 0$ was arbitrary, so this is the claim.

6. The group G of 1-quasiconformal transformations in $\partial\widetilde{M}$

Let (X, d) be a metric space. We call a distance δ on X *quasiconformally equivalent* to d if $\limsup_{\varepsilon\to 0} q(x, \varepsilon) = 1$ for every $x \in X$, where $q(x, \varepsilon) = \inf\{\beta \geq 1 \mid B_d(x, r) \subset B_\eta(x, \varepsilon) \subset B_d(x, \beta r)$ and $B_\eta(x, \overline{r}) \subset B_d(x, \varepsilon) \subset B_\eta(x, \beta\overline{r})$ for some $r, \overline{r} > 0\}$. A class of conformally equivalent metrics on X is called a *generalized conformal structure* on X. A homeomorphism f of X is called 1-*quasiconformal* if it preserves the generalized conformal structure. The set of 1-quasiconformal transformations of X has a natural group structure.

By 5.5 the distances l_v $(v \in T^1\widetilde{M})$ define a generalized conformal structure on $\partial\widetilde{M}$. Thus we can consider the group G of 1-quasiconformal transformations on $\partial\widetilde{M}$ with respect to this structure. For example, the action of an isometry of $\widetilde{M}$ on $\partial\widetilde{M}$ defines an element of G. In this section we investigate the structure of G.

Every $\zeta \in \partial\widetilde{M}$ determines a subgroup G_ζ of the isotropy group of G at ζ as follows: If we identify $\partial\widetilde{M} - \zeta$ with $(\mathfrak{N}, \delta_\infty)$, then G_ζ is the semidirect product of the 1-parameter group of homotheties of δ_∞ and the isometry group of $(\mathfrak{N}, \delta_\infty)$. This does not depend on the identification, i.e., on the choice of the distance l_v $(v \in W^{\mathrm{u}}(\zeta))$ on $\partial\widetilde{M} - \zeta$.

Since the distance δ_∞ on $\mathfrak{N}$ is left-invariant, G_ζ acts transitively on $\partial\widetilde{M} - \zeta$. More precisely, the following is true: δ_∞ is defined by a subspace L of the Lie algebra of $\mathfrak{N}$ and a scalar product $\langle\ ,\ \rangle$ on L. Then every $\psi \in G_\zeta$ admits a decomposition $\psi = \psi_1 \circ \psi_2 \circ \psi_3$, where ψ_1 is a left translation in $\mathfrak{N}$, ψ_2 is an automorphism of $\mathfrak{N}$ leaving L and $\langle\ ,\ \rangle$ invariant, and ψ_3 is a homothety (see [8]). In particular, the isotropy subgroup $G_{\zeta,x}$ of G_ζ at any point $x \in \partial\widetilde{M} - \zeta$ is the direct product of a compact group $Z_{\zeta,x}$ and an abelian group $T_{\zeta,x}$ which is naturally isomorphic to the multiplicative group of positive reals.

We want to show that G_ζ coincides with the isotropy group of G at ζ. For this we need the following preparation:

Let $\psi \in G$, $x \in \partial\widetilde{M}$, and let U be an open neighborhood of x in $\partial\widetilde{M}$ such that $\partial\widetilde{M} - (U \cup \psi U) \neq \emptyset$, and $\zeta \in \partial\widetilde{M} - (U \cup \psi U)$. Then $\partial\widetilde{M} - \zeta$ can be identified with $(\mathfrak{N}, \delta_\infty)$, and the restriction of ψ to U is a 1-quasiconformal homeomorphism of $U \subset (\mathfrak{N}, \delta_\infty)$ onto $\psi U \subset (\mathfrak{N}, \delta_\infty)$. For $y \in U$ let $\mathrm{Lip}_\psi(y)$ be the local dilation of ψ at y with respect to δ_∞.

Choose a compact neighborhood A of x in U, and define for $R < \inf\{\delta_\infty(y, z) \mid y \in \psi(A), z \in \partial\widetilde{M} - \psi(U)\}$ and $y \in A$, $D_R(y) = \delta_\infty(y, \psi^{-1}\partial B_{\delta_\infty}(\psi(y), R))$. The map $D_R : A \to \mathbf{R}^+$ is continuous.

The following lemma shows that ψ is locally Lipschitz.

Lemma 6.1 (Pansu [17, 18.4]). *$\mathrm{Lip}_\psi(y) \leq R/D_R(y)$ for all $y \in A$.*

Compare the proof of the following lemma with [17, 18.5].

Lemma 6.2. *For every $\zeta \in \partial\widetilde{M}$, G_ζ is the isotropy subgroup of G at ζ.*

Proof. Since G acts as a topological transformation group transitively on $\partial\widetilde{M}$, the isotropy groups of G at different points are mutually isomorphic; thus it suffices to show the lemma for any particular point of $\partial\widetilde{M}$.

For this let $\mathrm{Id} \neq \Lambda \in \Gamma$ be an isometry with axis γ which is oriented in such a way that $\Lambda\gamma(t) = \gamma(t + \log\tau)$ for some $\tau > 1$ and all $t \in R$. Then the points $\zeta = \gamma(-\infty)$ and $\xi = \gamma(\infty)$ are fixed by the restriction L of Λ to $\partial\widetilde{M}$.

Identify as before $\partial\widetilde{M}-\zeta$ (resp. $\partial\widetilde{M}-\xi$) with $(\mathscr{N}, \delta_\infty)$, with identity at ξ (resp. ζ). Denote the resulting space by $(\mathscr{N}_\zeta, \delta_\zeta)$ (resp. $(\mathscr{N}_\xi, \delta_\xi)$) and let $\{\Delta_t^\zeta \mid t > 0\}$ (resp. $\{\Delta_t^\xi \mid t > 0\}$) be its 1-parameter group of homotheties. Then there is an automorphism $A_\zeta \in Z_{\zeta,\xi}$ (resp. $A_\xi \in Z_{\xi,\zeta}$) such that $L = \Delta_{1/\tau}^\xi \circ A_\xi = \Delta_\tau^\zeta \circ A_\zeta$.

Let now $\Psi \in G$ be such that $\Psi(\zeta) = \zeta$. By 6.1 the restriction of Ψ to (N_ζ, δ_ζ) is locally Lipschitz, hence Δ-differentiable almost everywhere [17]. Thus Ψ can be composed with suitable translations of N_ζ in such a way that the resulting map fixes ξ and is Δ-differentiable at ξ.

This map can be written as a product of an element of $G_{\zeta,\xi}$ and a map $\varphi \in \overline{G}_{\zeta,\xi} = \{\overline{\psi} \in G \mid \overline{\psi}(\zeta) = \zeta, \overline{\psi}(\xi) = \xi\}$ whose Δ-differential exists at ξ and equals the identity.

For $y \in \partial\widetilde{M} - \{\xi, \zeta\}$ and $R > 0$ let $B_\zeta(y, R)$ (resp. $B_\xi(y, R)$) be the ball of radius R about y in (N_ζ, δ_ζ) (resp. (N_ξ, δ_ξ)). As above let $D_R(y) = \delta_\xi(y, \varphi^{-1}\partial B_\xi(\varphi(y), R))$; for $k > 0$ we then have

$$\begin{aligned} D_{k\tau}(y)/(k\tau) &= \delta_\xi(y, \varphi^{-1}L^k\partial B_\xi(L^{-k}\varphi(y), 1))/(k\tau) \\ &= \delta_\xi(L^{-k}y, L^{-k}\varphi^{-1}L^k\partial B_\xi(L^{-k}\varphi(y), 1)). \end{aligned}$$

Since $L^{-k}\varphi(y) \to \zeta (k \to \infty)$, we can find numbers $k_0 > 0$, $R_0 > 0$ in such a way that $\partial B_\xi(L^{-k}\varphi(y), 1) \subset B_\zeta(\xi, R_0)$ for all $k \geq k_0$. Now φ is Δ-differentiable at ξ, and its Δ-differential equals the identity; thus $\Delta_{k\tau}^\zeta \circ \varphi^{-1} \circ \Delta_{1/k\tau}^\zeta \to \mathrm{Id}$ uniformly on $B_\zeta(\xi, R_0)$ and hence also $L^{-k} \circ \varphi^{-1} \circ L^k = (A_\zeta^k)(\Delta_{k\tau}^\zeta \circ \varphi^{-1} \circ \Delta_{1/k\tau}^\zeta)(A_\zeta^{-k}) \to \mathrm{Id}$ uniformly on $B_\zeta(\xi, R_0)$ (recall that $A_\zeta \in Z_{\zeta,\xi}$). But $L^{-k}y \to \zeta (k \to \infty)$ and consequently $D_{k\tau}(y)/(k\tau) \to 1 (k \to \infty)$ or $\delta_\xi(\varphi(z), \varphi(y)) \leq \delta_\xi(z, y)$ for all $y, z \in \partial\widetilde{M} - \xi$ by 6.1. The same argument applied to φ^{-1} then shows that φ is an isometry of (N_ξ, δ_ξ), i.e., $\varphi \in Z_{\xi,\zeta}$. In particular the restriction of φ to $\partial\widetilde{M} - \{\xi, \zeta\}$ is smooth, and the restriction of ψ to $\partial\widetilde{M} - \zeta$ is Δ-differentiable everywhere. Thus we obtain a group homomorphism $F_\zeta \colon \overline{G}_{\zeta,\xi} \to G_{\zeta,\xi}$ by mapping ψ to its Δ-differential at ζ. The kernel $\ker F_\zeta$ of F_ζ is a normal closed subgroup of $Z_{\xi,\zeta}$.

We have to show that $\ker F_\zeta = \mathrm{Id}$. Let $\varphi \in \ker F_\zeta$; it suffices to prove that φ fixes pointwise the distance sphere $\partial B_\zeta(\xi, 1)$ of radius 1 about ξ in (N_ζ, δ_ζ).

Since $Z_{\xi,\zeta}$ and hence $\ker F_\zeta$ are compact, for every $\varepsilon > 0$ there is a number $k(\varepsilon) > 0$ such that $\delta_\zeta(x, \alpha x) \leq \varepsilon/(k\tau)$ for all $\alpha \in \ker F_\zeta$ and all

$k \geq k(\varepsilon)$, all $x \in B_\zeta(\xi, 1/k\tau)$. For $x \in B_\zeta(\xi, 1)$ we then have $L^k x \in B_\zeta(\xi, 1/(k\tau))$ and consequently $\delta_\zeta(x, \varphi(x)) = (k\tau)\delta_\zeta(L^k x, L^k\varphi(x)) = (k\tau)\delta_\zeta(L^k x, A_\xi^k \circ \Delta_{1/(k\tau)}^\xi \circ \varphi(x)) = (k\tau)\delta_\zeta(L^k x, (A_\xi^k \circ \varphi \circ A_\xi^{-1})L^k x) \leq \varepsilon$ (recall that φ commutes with $\{\Delta_t^\xi \mid t > 0\}$). But $\varepsilon > 0$ was arbitrary, which shows the required property of φ and finishes the proof of the lemma.

7. The action of G on $T^1\widetilde{M}$

In this section we show that G acts naturally on $T^1\widetilde{M}$. We start with an examination of the geometry of the hyperbolic plane H of constant curvature -1. Choose $u \in \partial H$, a Busemann function θ at u, and denote by π_t the projection onto $\theta^{-1}(t)$ along the geodesics which are asymptotic to u. For $t \in \mathbf{R}$ let d_t be the induced distance on $\theta^{-1}(t)$. If $x \in \theta^{-1}(t)$ and $y \in H$, then the hyperbolic distance $d_H(x, y)$ of x and y only depends on $\varepsilon = d_t(x, \pi_t y)$ and $\rho = \theta y - \theta x$; this number will be denoted by $r(\varepsilon, \rho)$. The function $r\colon [0, \infty) \times \mathbf{R} \to [0, \infty)$ has the following properties:

(a) r is continuous and increasing in the first variable.

(b) $r(0, \rho) = |\rho|$ for all $\rho \in \mathbf{R}$.

(c) $r(\varepsilon, 0) < \varepsilon$ for all $\varepsilon > 0$.

(d) $r(\varepsilon, \rho) = r(e^\rho\varepsilon, -\rho)$ for all $\varepsilon > 0$ and $\rho \in \mathbf{R}$.

(e) $r(\varepsilon, \rho) \leq r(\overline{\varepsilon}, \overline{\rho}) + r(e^{\overline{\rho}}(\varepsilon - \overline{\varepsilon}), \rho - \overline{\rho})$ for $\overline{\varepsilon} \leq \varepsilon$ and $\rho, \overline{\rho} \in \mathbf{R}$.

(a)–(c) are obvious. To show (d) observe that d_H is symmetric, and for $x, y \in H$ with $\theta(x) = t$ and $\theta(y) = t + \rho$ we have $d_{t+\rho}(y, \pi_{t+\rho}x) = e^\rho\, d_t(x, \pi_t(y))$. (e) is an immediate consequence of the triangle inequality for d_H.

For every $\zeta \in \partial\widetilde{M}$ we can now define a distance d_ζ on $\widetilde{M}$ as follows: Let $v \in W^{\mathrm{u}}(\zeta)$ and $x, y \in \widetilde{M}$. If $\theta_v(x) = t$ and $\theta_v(y) - t = \rho$, define $d_\zeta(x, y) = r(l_{\Phi^t v}\pi_{v,\infty}(x), \pi_{v,\infty}(y)), \rho)$. Since $l_{\Phi^{t+r} v} = e^\rho l_{\Phi^t v}$, by (d) above d_ζ is symmetric. The triangle inequality of d_ζ follows from the triangle inequality for $l_{\Phi^t v}$ and properties (a) and (e) of the function r.

Remark. If $W^{\mathrm{u}}(\zeta) \subset \widetilde{\Omega}$, then it is easy to see that d_ζ is the Carnot-Carathéodory metric on $\widetilde{M}$, which is induced by the distribution $E_\zeta\colon Pw \to \operatorname{span}\{w, E_w(Pw)\}$.

Lemma 7.1. *There is $\chi > 0$ such that $d_\zeta(p, q) \leq \chi d(p, q)$ whenever $d(p, q) \geq 1$. χ does not depend on ζ.*

Proof. Let $v, w \in W^{\mathrm{u}}(\zeta)$ be such that $d(Pv, Pw) = 1$, and assume $\theta_v(Pw) \geq 0$. Then $d(Pv, \pi_{v,0}Pw) \leq 1$, hence $d_{v,0}(Pv, \pi_{v,0}Pw) \leq (2\sinh b/2)/b$ by [11], which by 1.2 means

$$\eta_v(\gamma_v(\infty), \pi_{v,\infty}(Pw)) \leq (2\sinh b/2)/b.$$

It thus follows from 5.8 that $l_v(\gamma_v(\infty), \pi_{v,\infty}(Pw)) \leq (2\sinh b/2)/(b\nu)$, i.e., $d_\zeta(Pv, \pi_{v,0}(Pw)) \leq (2\sinh b/2)/(b\nu)$; here $\nu > 0$ is the constant from 5.8. But $\theta_v(Pw) \leq 1$, hence $d_\zeta(\pi_{v,0}(Pw), Pw) \leq 1$ and $d_\zeta(Pv, Pw) \leq (2\sinh b/2)/(b\nu) + 1$.

For $v, w \in W^{\mathrm{u}}(\zeta)$ with $d(Pv, Pw) = r \geq 1$, choose a minimizing geodesic $\phi: [0, r] \to \widetilde{M}$ which is parametrized by arc length and joins $\phi(0) = Pv$ to $\phi(r) = Pw$. The above argument yields

$$d_\zeta(\phi(s), \phi(s+\varepsilon)) \leq \chi/2$$

for all $\varepsilon \in (0, 1]$ and $s \in [0, r-\varepsilon]$, hence $d_\zeta(Pv, Pw) \leq \chi([r]+1)/2 \leq \chi r$, which is the claim. q.e.d.

Since $d_\zeta \geq d$, 7.1 shows in particular that every isometry of d_ζ is a *pseudo-isometry* of $\widetilde{M}$ (for the definition and basic properties of pseudo-isometries see [16] or [17]), hence admits a unique extension to a transformation of $\partial\widetilde{M}$.

Now every $\psi \in G_\zeta$ can be extended to a transformation $\Theta\psi$ of $\widetilde{M}$ as follows: Choose $v \in W^{\mathrm{u}}(\zeta)$. Then there is a number $\lambda(\psi) > 0$ such that $l_v(\psi x, \psi y) = \lambda(\psi) l_v(x, y)$; $\lambda(\psi)$ does not depend on the choice of v. For $w \in W^{\mathrm{u}}(\zeta)$ define $\Theta\psi(Pw) = \pi_{w,\log\lambda(\psi)}\psi\pi_{w,\infty}(Pw)$. Then Θ is clearly a homomorphism of G_ζ into the group of transformations of $\widetilde{M}$.

Lemma 7.2. *Θ is an isomorphism of G_ζ onto the subgroup* Iso_ζ *of the isometry group of d_ζ, which fixes ζ, and the inverse of Θ is the restriction mapping.*

Proof. Let $\psi \in G_\zeta$ and $t = \log\lambda(\psi)$. Since for every $w \in W^{\mathrm{u}}(\zeta)$ the map $\pi_{w,t}\pi_{w,\infty}$ is an isometry of $(\theta_w^{-1}(0), l_w\pi_{w,\infty})$ onto $(\theta_w^{-1}(t), e^{-t}l_{\Phi^t w}\pi_{w,\infty})$, $\Theta\psi$ maps $(\theta_w^{-1}(0), l_w\pi_{w,\infty})$ isometrically into $(\theta_w^{-1}(t), l_{\Phi^t w}\pi_{w,\infty})$. By the definition of $\Theta\psi$ and d_ζ, this means that $\Theta\psi$ is an isometry of d_ζ. Consequently $\Theta\psi$ is a pseudo-isometry of $\widetilde{M}$ and admits a unique extension to a transformation of $\partial\widetilde{M}$. But the image under $\Theta\psi$ of a geodesic γ in $\widetilde{M}$ with $\gamma(-\infty) = \zeta$ is a geodesic $\overline{\gamma}$ in $\widetilde{M}$ with $\overline{\gamma}(-\infty) = \zeta$ and $\overline{\gamma}(\infty) = \psi\gamma(\infty)$; hence this extension is just ψ.

It remains to show that $\Theta G_\zeta = \mathrm{Iso}_\zeta$, i.e., that Θ is surjective. Let $\Psi \in \mathrm{Iso}_\zeta$ and $w \in W^{\mathrm{u}}(\zeta)$. Since ΘG_ζ is transitive on $\widetilde{M}$, we may assume that $\Psi(Pw) = Pw$. Now Ψ is an isometry of d_ζ, which fixes ζ. Thus the image under Ψ of the geodesic ray $\gamma\colon [0, \infty) \to \widetilde{M}$, $s \to \pi_{w,-s}(Pw)$ is a globally minimizing geodesic with respect to d_ζ, which is contained in the β-tubular neighborhood of $\gamma[0, \infty)$ for some $\beta > 0$; in particular, $\theta_w \Psi\gamma(s) \in [-s, -s+\beta)$ for all $s \in [0, \infty)$.

Let again H be the hyperbolic plane of constant curvature -1, $u \in \partial H$, and θ be a Busemann function at u. If $\phi_H\colon [0, r] \to H$ is a geodesic parametrized by arc length with $\theta\phi_H(r) - \theta\phi_H(0) = \rho$, then the function $\sigma(\rho, r)\colon s \to \theta\phi_H(s) = \theta\phi_H(0)$ only depends on ρ and r. If $\phi\colon [0, r] \to H$ is any curve parametrized by arc length with $\theta\phi(r) - \theta\phi(0) = \rho$ and $\theta\phi(s) - \theta\phi(0) \neq \sigma(\rho, r)(s)$ for some $s \in [0, r]$, then $d_H(\phi(0), \phi(r)) < r$. Moreover for every $\tau_0 > 0$ and $s \in [0, \infty)$ there is a number $r_0 > 0$, such that $|s + \sigma(\tau - r, r)(s)| < \varepsilon$ for all $\tau < \tau_0$ and $r > r_0$.

This consideration and the definition of d_ζ imply the following: If $\psi\colon [0, r] \to \widetilde{M}$ is a minimizing geodesic with respect to d_ζ, which is parametrized by arc length and satisfies $\theta_w\phi(r) - \theta_w\phi(0) = \rho$, then $\theta_w\phi(s) - \theta_w\phi(0) = \sigma(\rho, r)(s)$. This applies in particular to the geodesic $\Psi\gamma$. Let $\varepsilon > 0$, $s \in [0, \infty)$, and choose $r > 0$ such that $|s + \sigma(\tau - r, r)(s)| < \varepsilon$ for all $\tau < \beta$. Since $\theta_w\Psi\gamma(r) + r < \beta$ and $\theta_w\Psi\gamma(s) = \sigma(\theta_w\Psi\gamma(r), r)(s)$, this implies $\theta_w\Psi\gamma(s) < \varepsilon - s$, and consequently Ψ leaves γ pointwise fixed.

For $y \in \widetilde{M}$ let $\rho(y) = \limsup_{s\to\infty} d_\zeta(y, \gamma(s)) - s$. Then $\rho \geq \theta_w$; but if $\theta_w(y) = t$, then $d_\zeta(y, \gamma(s)) - s \leq t + l_{\Phi^{-s}w}(\pi_{w,\infty}(y), \pi_{w,\infty}(Pw)) = t + e^{-s} l_w(\pi_{w,\infty}(y), \pi_{w,\infty}(Pw)) \to t$, which means $\rho = \theta_w$. Since ρ is clearly invariant under Ψ, Ψ leaves the horospheres at ζ invariant.

Now for every $y \in \partial\widetilde{M} - \zeta$ the curve $s \to \pi_{w,s}(y)$ is a minimizing geodesic with respect to d_ζ (recall $d_\zeta \geq d$) which realizes the d_ζ-distance between the horospheres at ζ, and every geodesic with this property is (up to reparametrization) of this form. Thus Ψ permutes the geodesics which are asymptotic to ζ, i.e., Ψ commutes with the projections $\pi_{w,t}$ $(t \in \mathbf{R})$. Moreover, the definition of d_ζ shows that Ψ induces an isometry of $(\theta_w^{-1}(0), l_w\pi_{w,\infty})$, i.e., $\Psi \in \Theta G_\zeta$ is claimed.

Remark. 7.2 shows in particular that every element of Iso_ζ is uniquely determined by its restriction to $\partial\widetilde{M}$. Thus if $\Psi\colon \widetilde{M} \to \widetilde{M}$ is an isometry of $(\widetilde{M}, d_\zeta)$ onto $(\widetilde{M}, d_{\overline{\zeta}})$ for some $\zeta, \overline{\zeta} \in \partial\widetilde{M}$ with $\Psi\zeta = \overline{\zeta}$, then Ψ is uniquely determined by its restriction to $\partial\widetilde{M}$.

Corollary 7.3. *For $g \in G$ and $\zeta \in \partial\widetilde{M}$ there is a unique isometry $\overline{\Theta}(\zeta, g)$ of $(\widetilde{M}, d_\zeta)$ onto $(\widetilde{M}, d_{g\zeta})$, whose restriction to $\partial\widetilde{M}$ equals g.*

Proof. Let J be an isometry of $(\widetilde{M}, d_\zeta)$ onto $(\widetilde{M}, d_{g\zeta})$. By 7.1, J has an extension $\overline{J}$ to a transformation of $\partial\widetilde{M}$, and this extension is an element of G which maps ζ to $g(\zeta)$. Thus $g\overline{J}^1 \in G_\zeta$ and hence $J\Theta(g\overline{J}^1)$ is an isometry of $(\widetilde{M}, d_\zeta)$ onto $(\widetilde{M}, d_{g\zeta})$, whose restriction to $\partial\widetilde{M}$ equals g. By the above remark, $\Theta(g\overline{J}^1)$ is unique.

Corollary 7.4. *There is a natural homomorphism J of G into the group of topological transformations of $T^1\widetilde{M}$ with the following properties*:

(i) *J is continuous, i.e., the map $G \times T^1\widetilde{M} \to T^1\widetilde{M}$, $(g, v) \to (Jg)(v)$ is continuous.*

(ii) *The action of JG on $T^1\widetilde{M}$ is transitive.*

(iii) *For every $g \in G$, $J(g)$ preserves the weak unstable foliation and commutes with the geodesic flow.*

(iv) *$J\Psi = d\Psi$ for all $\Psi \in \Gamma$.*

(v) *For every $g \in G$ the restriction of $J(g)$ to $\widetilde{\Omega} \cap J(g)^{-1}\widetilde{\Omega}$ is smooth.*

Proof. For $\zeta \in \partial\widetilde{M}$ let $W^{\mathrm{u}}(\zeta) = \{v \mid \gamma_v(-\infty) = \zeta\}$ be the leaf of the weak unstable foliation defined by ζ. The canonical projection maps $W^{\mathrm{u}}(\zeta)$ diffeomorphically onto $\widetilde{M}$; hence for $g \in G$ we can define $J(g)$ by $J(g)|_{W^{\mathrm{u}}(\zeta)} = (P|_{W^{\mathrm{u}}(g\zeta)})^{-1} \circ \overline{\Theta}(\zeta, g) \circ P|_{W^{\mathrm{u}}(\zeta)}$. Since $\overline{\Theta}(\zeta, gh) = \overline{\Theta}(h\zeta, g)\overline{\Theta}(\zeta, h)$, J is a homomorphism of G into the group of transformations of $T^1\widetilde{M}$. The properties (i)–(v) follow directly from the definition of J.

8. Proof of the theorem

We continue to use the notation of §§1–7. Consider the homomorphism J of 7.4; we want to show that $J(g)$ preserves the fibers of $T^1\widetilde{M}$ for every $g \in G$, hence decends to a transformation of $\widetilde{M}$. For this we need the following preparation:

Let $v \in \Upsilon$; then $\theta_v^{-1}(0)$ can naturally be identified with the nilpotent Lie group $\mathfrak{N}$ in such a way that the identity of $\mathfrak{N}$ equals Pv. Denote by $\mathfrak{n}$ the Lie algebra of $\mathfrak{N}$; with respect to this identification the derived algebra $[\mathfrak{n}, \mathfrak{n}]$ is a distribution on $\theta_v^{-1}(0)$ which is complementary to E_v.

Lemma 8.1. *$[\mathfrak{n}, \mathfrak{n}]$ is equal to the orthogonal complement of E_v in $\theta_v^{-1}(0)$.*

Proof. By the choice of v there is $\Psi \in \Gamma$ and $\tau > 0$ such that $d\Psi v = \phi^\tau v$. Then $\pi_{v,0} \circ \Psi$ is the composition of an isometry of $(\mathfrak{N}, d_\infty)$ fixing Pv and the dilation $\Delta_{e^{-\tau}}$; in particular, $\pi_{v,0} \circ \Psi$ is an automorphism of $\mathfrak{N}$ and hence preserves the distribution $[\mathfrak{n}, \mathfrak{n}]$ (7.6 of [8]). Moreover, $[\mathfrak{n}, \mathfrak{n}]$ is the only $\pi_{v,0} \circ \Psi$-invariant distribution on $\theta_v^{-1}(0)$, which is complementary to E_v. On the other hand, by Remark 2.2 the orthogonal complement $E_v^{\perp}$ of E_v in $\theta_v^{-1}(0)$ is invariant under $\pi_{v,0} \circ \Psi$ as well; thus $E_v^{\perp} = [\mathfrak{n}, \mathfrak{n}]$. q.e.d.

Since Υ is dense in $T^1\widetilde{M}$, by continuity we obtain that the statement of 8.1 holds for every horosphere $\theta_v^{-1}(0)$ in $\widetilde{M}$. Let again $v \in \overline{\Omega}$, and identify $\theta_v^{-1}(0)$ with the Lie group $\mathfrak{N}$ as before.

Lemma 8.2. *Let X be a left-invariant unit vector field on $\mathfrak{N} \sim \theta_v^{-1}(0)$ tangent to E_v and let $\psi: \mathbf{R} \to \theta_v^{-1}(0)$ be a maximal integral curve of X. Then $A = \{\pi_{v,t}\psi(s) \mid s, t \in \mathbf{R}\}$ is a totally geodesic embedded plane of constant curvature -1 in $\widetilde{M}$.*

Proof. It suffices to show that for every $\varepsilon > 0$ the geodesic γ in $\widetilde{M}$ joining $\psi(0)$ to $\psi(\varepsilon)$ is contained in A. Since A is a smooth embedded plane in $\widetilde{M}$, and $\psi(0)$ can be joined in A to $\psi(\varepsilon)$ by a curve of length $r(\varepsilon, 0)$, we have to show that $d(\psi(0), \psi(\varepsilon))$ is not smaller than $r(\varepsilon, 0)$.

Thus let $\varphi: I \to \widetilde{M}$ be a minimizing geodesic joining $\varphi(0) = \psi(0)$ to $\varphi(1) = \psi(\varepsilon)$ and let $\overline{\varphi}(t) = \pi_{v,0}\varphi(t)$. Then $\overline{\varphi}$ is a smooth curve in $\mathfrak{N}$ with $\overline{\varphi}(0) = \psi(0)$, $\overline{\varphi}(1) = \psi(\varepsilon)$, and can be decomposed as $\overline{\varphi}(t) = \overline{\varphi}_1(t)\overline{\varphi}_2(t)$, where $\overline{\varphi}_1'(t)$ is the orthogonal projection of $\overline{\varphi}'(t)$ into the left-invariant distribution E_v, and $\overline{\varphi}_2$ is tangent to $[\mathfrak{n}, \mathfrak{n}]$. Then $\overline{\varphi}_1(1) = \psi(\varepsilon) \cdot \overline{\varphi}_2(1)^{-1}$, and the length of $\overline{\varphi}_1$ is not smaller than

$$d_{\infty}(\psi(0), \psi(\varepsilon) \cdot \overline{\varphi}_2(1)^{-1}) \geq \varepsilon.$$

Let $s(t) = \theta_v(\varphi(t))$. Since $\overline{\varphi}'(t) = \overline{\varphi}_1'(t) + \overline{\varphi}_2'(t)$ is an orthogonal decomposition, the length of φ is not smaller than the integral

$$\int_0^1 (s'(t)^2 + e^{s(t)}\|\overline{\varphi}_1'(t)\|^2)^{1/2}\, dt,$$

which is not smaller than $r(\varepsilon, 0)$. This is the claim. q.e.d.

By continuity we obtain

Corollary 8.3. *For every $v \in \widetilde{\Omega}$ and $X \in E(v)$ there is a unique totally geodesic embedded plane of constant curvature -1 in $\widetilde{M}$ whose tangent space at Pv is spanned by v and X.*

Now let $g \in G$ and let $H \subset \widetilde{M}$ be a totally geodesic embedded plane of constant curvature -1 as in 8.3. The definition of $J(g)$ then implies that the restriction of $J(g)$ to the unit tangent bundle T^1H of H is a fiber preserving isometry onto the unit tangent bundle of a totally geodesic embedded plane $\tilde{H}$ of constant curvature -1.

Now for $v \in T^1\widetilde{M}$ the tangent space at v of the fiber of the bundle $T^1\widetilde{M}$ over $\widetilde{M}$ has a natural identification with the orthogonal complement $v^\perp$ of v in the tangent space of $\widetilde{M}$ at the foot point Pv. Under this identification $E(v)$ can be viewed as a subspace of the tangent space of the fiber. Consequently the assignment $v \to E(v)$ is a smooth distribution on $\tilde{\Omega}$ tangent to the fibers with the following properties:

(i) For every $u \in \widetilde{M}$ the canonical homeomorphism of $P^{-1}(u)$ onto $\partial\widetilde{M}$ maps the distribution E on $P^{-1}(u) \cap \tilde{\Omega}$ onto the distribution on $\partial\widetilde{M}$, which is the projection of the distributions $\tilde{E}$. (Recall that the restriction to $P^{-1}(u) \cap \tilde{\Omega}$ of the canonical homeomorphism of $P^{-1}(u)$ to $\partial\widetilde{M}$ is smooth.)

(ii) For every $u \in \widetilde{M}$ and $v \in P^{-1}(u) \cap \tilde{\Omega}$, every $0 \neq X \in E(v)$ is tangent to the unit tangent bundle of a unique totally geodesic embedded plane of constant curvature -1 in $\widetilde{M}$.

Now by (i) for every $v \in \tilde{\Omega}$ the distribution E generates the whole tangent space at v of the fiber of $T^1\widetilde{M}$. Thus every point of the connected component of v in $P^{-1}(Pv) \cap \tilde{\Omega}$ can be joined to v by a curve φ which is tangent to E. Let $g \in G$; if, moreover, φ is contained in $J(g)^{-1}\tilde{\Omega}$, then by (ii) and 8.3 the image under $J(g)$ of φ is a differentiable curve which is tangent to a fiber of $T^1\widetilde{M}$. Since $\tilde{\Omega} \cap J(g)^{-1}\tilde{\Omega}$ is open and dense in $T^1\widetilde{M}$, this implies that $J(g)$ is a bundle map for every $g \in G$, i.e., $J(g)$ projects to a homeomorphism $\bar{J}(g)$ of $\widetilde{M}$. Since $J(g)$ commutes with the geodesic flow on T^1M, $\bar{J}(g)$ maps each geodesic in $\widetilde{M}$ isometrically onto a geodesic, i.e., $\bar{J}(g)$ is an isometry of $\widetilde{M}$. Thus by 7.4(ii), $\widetilde{M}$ is homogeneous. Since $\widetilde{M}$ admits a compact quotient, this is only possible if $\widetilde{M}$ is symmetric, i.e., if $\widetilde{M}$ is locally symmetric (see [10]). This finishes the proof of the theorem. q.e.d.

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