J. DIFFERENTIAL GEOMETRY
34 (1991) 143-164

THE LOCAL BEHAVIOUR OF HOLOMORPHIC CURVES IN ALMOST COMPLEX 4-MANIFOLDS

DUSA McDUFF

Abstract

In this paper we prove various results about the positivity of intersections of holomorphic curves in almost complex 4-manifolds which were stated by Gromov. We also show that the virtual genus of any closed holomorphic curve in an almost complex 4-manifold is nonnegative. These technical results form the basis of the classification of rational and ruled symplectic 4-manifolds given in [5].

1. Introduction

Let (V, J) be an almost complex manifold, and let (Σ, J_0) be a Riemann surface. A (parametrized) *J-holomorphic curve* in (V, J) is a map $f: \Sigma \to V$ which preserves the almost complex structures, i.e., $df \circ J_0 = J \circ df$. The unparametrized curve $\operatorname{Im} f$ is denoted by C. We will consider both local and global questions. In the former case, C will be the image of a disc D centered at the origin in the complex plane $\mathbb{C}$, and in the latter it will be closed, i.e., the image of a compact Riemann surface Σ without boundary. Throughout, we will assume that (V, J) and all maps are C^∞-smooth, unless there is explicit mention to the contrary. Further, we will assume that f is not a multiple covering, i.e., that f does not factor as $f' \circ \gamma$, where $\gamma: \Sigma \to \Sigma$ is a J_0-holomorphic self-map of degree > 1.

Our first aim is to prove the following result, which was stated by Gromov in [3, 2.1.C_2].

Theorem 1.1. *Two closed distinct J-holomorphic curves C and C' in an almost complex 4-manifold (V, J) have only a finite number of intersection points. Each such point x contributes a number $k_x \geq 1$ to the algebraic intersection number $C \cdot C'$. Moreover, $k_x = 1$ only if the curves C and C' intersect transversally at x.*

Received December 1, 1989 and, in revised form, July 16, 1990. The author was partially supported by National Science Foundation grant #DMS 8803056.

Thus, if C and C' are distinct closed holomorphic curves, $C \cdot C' = 0$ if and only if C and C' are disjoint. Further $C \cdot C' = 1$ if and only if C and C' meet exactly once transversally, and so at a point which is nonsingular on both curves. (A point $x \in C$ is called *nonsingular* if $f^{-1}(x)$ consists of a single point z such that $df_z \neq 0$. Otherwise, it is *singular*. Further, x is said to be *critical* if it is the image of a point z such that $df_z = 0$.) In fact, one does not need C and C' to be closed here: it is enough that they be compact with only interior points of intersection. Further, when $\dim V > 4$, there is an analogous result for intersections of a J-holomorphic curve C with an immersed J-holomorphic submanifold X of (V, J) of codimension 2 (see [3, 2.1. C_2']).

In the course of proving the above theorem, we establish the following result.

Proposition 1.2. *Every J-holomorphic map $f \colon \Sigma \to V$ may be C^1-approximated by a J'-holomorphic immersion $f' \colon \Sigma \to V$, where J' is C^0-close to J and equals J except in small spherical shells $B_x(\varepsilon_2) - B_x(\varepsilon_1)$ about each critical point x of* $\operatorname{Im} f$.

Our second result is a homological version of the adjunction formula for holomorphic curves in almost complex 4-manifolds. Let $c \in H^2(V; \mathbb{Z})$ be the first Chern class of the complex vector bundle (TV, J). Then the virtual genus $g(C)$ of a closed curve C is defined to be the number

$$g(C) = 1 + \tfrac{1}{2}(C \cdot C - c(C)).$$

If C is an embedded copy of a Riemann surface Σ of genus g_Σ, the equalities

$$c(C) = c_1(TC) + c_1(\nu_C) = 2 - 2g + C \cdot C$$

show that the virtual genus $g(C)$ equals the genus g_Σ of Σ. We claim that the converse holds, namely,

Theorem 1.3. *Let C be a J-holomorphic image of the closed Riemann surface Σ of genus g. Then $g(C)$ is an integer which is greater than or equal to g_Σ, with equality if and only if C is embedded.*

This follows easily if C is immersed, and so the crucial step is to show that each critical point of C increases $C \cdot C$, and hence also $g(C)$. To do this, we define the local self-intersection number $\text{L.Int}(f, x)$ of a critical point and then prove:

Theorem 1.4. *For every critical point x of C,* $\text{L.Int}(f, x) > 0$.

The proofs involve extending the results of Nijenhuis and Woolf about the existence of perturbations of holomorphic curves in almost complex manifolds. Their results apply to nonsingular curves. In order to deal with singularities, we follow Gromov [3, 2.4. B_1] and lift to the branched cover.

Unfortunately, when J is not integrable, its lift $\tilde{J}$ may only be Lipschitz smooth. However we have enough control of its Lipschitz constant to be able to push through Nijenhuis and Woolf's arguments.

One very interesting question which is not touched upon here concerns the type of the knot formed by the intersection of a J-holomorphic curve $\operatorname{Im} f$ in $\mathbb{C}^2$ with a small 3-sphere centered at a singular point x on $\operatorname{Im} f$, where $\mathbb{C}^k$ denotes a complex k-space. If J is integrable, such a knot is an iterated torus knot. It is not known if more general knots can occur in the nonintegrable case.

This paper is organized as follows. In §2, we recall some general results about J-holomorphic curves, and then establish a local normal form for a singular holomorphic curve in a 4-dimensional manifold V. The hard work occurs in §3 where we construct desingularizations of singular curves. We define the local self-intersection number in §4, and prove the main theorems in §5. I wish to thank R. Piene, D. Arapura, and G. Mess who helped me understand the classical adjunction formula, Gromov for discussing the nonintegrable case with me at length, and Oh for making some useful comments about an earlier version of the paper.

2. General facts about J-holomorphic curves

Let $D = D(R_0)$ be the disc of radius R_0 about the point $0 \in \mathbb{C}$ with its usual complex structure, and consider a map f from D to an almost complex manifold (V, J) which takes the origin $0 \in D$ to the point $x \in V$. Since all questions considered here are local, we may identify the target space V with $\mathbb{C}^n$. We will always do this in such a way that x corresponds to the origin $(0, \cdots, 0)$ and J pushes forward to an almost complex structure (also called J) on $\mathbb{C}^n$ which equals the standard structure J_0 at $(0, \cdots, 0)$. Such an identification will be called *standard*. The points of D will be denoted by z, and the operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ by ∂ and $\bar{\partial}$ as usual. We begin by reformulating the equation satisfied by a J-holomorphic curve.

Lemma 2.1 [6, (3.3)]. *The map* $f\colon (D, 0) \to (\mathbb{C}^n, (0, \cdots, 0))$ *is* J-*holomorphic near* $\{0\}$ *if and only if it satisfies an equation of the form*

$$\bar{\partial} f^i + a^i_{\bar{m}}(f(z)) \bar{\partial} \bar{f}^m = 0 \tag{2.1.1}$$

on a possibly smaller disc D', *where* $a^i_{\bar{m}}$ *is an* $n \times n$ *matrix of complex valued functions on* $\mathbb{C}^n$ *which vanishes at* $(0, \cdots, 0)$ *and* $f^i\colon \mathbb{C} \to \mathbb{C}$, $i = 1, \cdots n$, *are the component functions of* f.

Proof. Consider the complexified tangent bundle $TV \otimes \mathbb{C}$ and cotangent bundle $T^*V \otimes \mathbb{C}$, where, to avoid confusion, we have written V instead of $\mathbb{C}^n$. The former space has basis $\partial/\partial w^1, \cdots, \partial/\partial w^n, \partial/\partial \overline{w}^1, \cdots, \partial/\partial \overline{w}^n$ over $\mathbb{C}$, and the latter has basis $dw^1, \cdots, dw^n, d\overline{w}^1, \cdots, d\overline{w}^n$, where $w^1, \cdots, w^n$ are the coordinates on $V = \mathbb{C}^n$. Let P be the projection of $TV \otimes \mathbb{C}$ onto the subspace consisting of vectors which have type $(1, 0)$ with respect to J. Then $P = (\mathrm{Id} - iJ)/2$, and we write P^i_m, $P^{\bar{i}}_m$, etc., for its components with respect to the above basis. These are complex valued functions on V which all vanish at the point $(0, \cdots, 0)$ except for the functions P^i_i which take the value 1.

The 1-forms on (V, J) of J-type $(1, 0)$ consist of the elements in $\operatorname{Im} P^t$, where P^t is the transpose of P. Hence this space is spanned by the forms

$$P^i_m\, dw^m + P^i_{\overline{m}}\, d\overline{w}^m \quad \text{and} \quad P^{\bar{i}}_m\, dw^m + P^{\bar{i}}_{\overline{m}}\, d\overline{w}^m,$$

where $i, m = 1, \cdots, n$. It is well known that f is J-holomorphic if and only if it pulls the 1-forms of J-type $(1, 0)$ back to 1-forms of type $(1, 0)$ on $D \subset \mathbb{C}$. Hence we require that the 1-forms

$$f^*(P^i_m\, dw^m + P^i_{\overline{m}}\, d\overline{w}^m) = P^i_m(f(z))\, df^m + P^i_{\overline{m}}(f(z))\, d\overline{f}^m$$

and

$$f^*(P^{\bar{i}}_m\, dw^m + P^{\bar{i}}_{\overline{m}}\, d\overline{w}^m) = P^{\bar{i}}_{\overline{m}}(f(z))\, df^m + P^{\bar{i}}_{\overline{m}}(f(z))\, d\overline{f}^m$$

have type $(1, 0)$. Since the type $(0, 1)$ part of the 1-form dg on D is $\overline{\partial} g$, this is equivalent to the equations

$$P^i_m(f(z))\overline{\partial} f^m + P^i_{\overline{m}}(f(z))\overline{\partial}\, \overline{f}^m = 0$$

and

$$P^{\bar{i}}_m(f(z))\overline{\partial} f^m + P^{\bar{i}}_{\overline{m}}(f(z))\overline{\partial}\, \overline{f}^m = 0.$$

Now let P_1 be the $n \times n$ complex matrix (P^i_m), P_2 be $(P^i_{\overline{m}})$, P_3 be $(P^{\bar{i}}_m)$, and P_4 be $(P^{\bar{i}}_{\overline{m}})$, and let $J_1, \cdots, J_4$ (resp. $Q_1, \cdots, Q_4$) be similar submatrices for J (resp. $Q = (\mathrm{Id} + iJ)/2$). Then it is easy to check that at the point $(0, \cdots, 0)$, we have $P_1 = \mathrm{Id}$, and $P_2 = P_3 = P_4 = 0$. In particular, there is a disc $D' \subset D$ such that P_1 is invertible at all points of $f(D')$. Observe also that, because J is real, $\overline{J}_1 = J_4$ and $\overline{J}_2 = J_3$. It follows easily that

$$\begin{aligned} \overline{P}_1 &= Q_4 = \mathrm{Id} - P_4, & \overline{P}_2 &= Q_3 = -P_3, \\ \overline{P}_3 &= Q_2 = -P_2, & \overline{P}_4 &= Q_1 = \mathrm{Id} - P_1. \end{aligned}$$

Using the fact that $PQ = 0$, it is now not hard to show that $P_3 = -\overline{a}P_1$ and $P_4 = -\overline{a}P_2$, where $a = P_1^{-1}P_2$ (see [6, 3.3]). Thus the second set of equations above follows from the first, and the first is equivalent to

$$\overline{\partial} f^i + a^i_{\overline{m}}(f(z))\overline{\partial}\,\overline{f}^m = 0.$$

Note that the $a^i_{\overline{m}}$ do vanish at $(0, \cdots, 0)$ as claimed. Nijenhuis and Woolf show by a simple calculation using the above identities that the matrix $(1 - a\overline{a})$ is invertible on $\operatorname{Im} f$, and that $P_1 = (1 - a\overline{a})^{-1}$. Conversely, given any a such that $(1 - a\overline{a})$ is invertible, one can use the above identities to construct P and hence J.

Corollary 2.2 [2, §1]. *Any map f which satisfies (2.1.1) is a solution of a system of equations of the form*

$$\partial\overline{\partial} f^i = \psi_i(f^m, \partial f^m, \overline{\partial} f^m, \partial \overline{f}^m, \overline{\partial}\,\overline{f}^m) \quad \textit{for } i = 1, \cdots, n,$$

where $\psi_i = 0$ when all its arguments equal zero.

Proof. In shorthand notation we may write (2.1.1) as

$$\overline{\partial} f + a\overline{\partial}\,\overline{f} = 0.$$

Thus $\partial\overline{\partial} f + a\partial\overline{\partial}\,\overline{f}$ is a function of f and its first derivatives. By taking conjugates, we see that $\partial\overline{\partial}\,\overline{f} + \overline{a}\partial\overline{\partial} f$ is too. Therefore, substituting for $\partial\overline{\partial}\,\overline{f}$ in the first equation, we find $(1 - a\overline{a})\partial\overline{\partial} f$ also has this property. The result now follows by multiplying by $(1 - a\overline{a})^{-1}$.

Lemma 2.3. *If two J-holomorphic curves $f, f' : \Sigma \to (V, J)$ have the same ∞-jet at a point z of a connected Riemann surface Σ, then $f = f'$.*

Proof. Since Σ is connected, it is enough to prove this locally. Hence, by Corollary 2.2, we may assume that f and f' are solutions of the following system of equations on D:

$$\partial\overline{\partial} h^i = \psi_i(h^m, \partial h^m, \overline{\partial} h^m, \partial \overline{h}^m, \overline{\partial}\,\overline{h}^m) \quad \text{for } i = 1, \cdots, n,$$

and that $g = f - f'$ vanishes to infinite order at $0 \in D$. Because g and its derivatives are bounded on D, it is easy to check that g satisfies differential inequalities of the form

$$|\partial\overline{\partial} g^i(z)|^2 \leq M \sum_m (|g^m|^2 + |\partial g^m|^2 + |\overline{\partial} g^m|^2), \tag{2.3.1}$$

for $i = 1, \cdots, n$ and all $z \in D$. In this situation, Aronszajn's strong unique continuation theorem [1, Remark 3] implies that g is identically zero in D. q.e.d.

We will need the following generalization of this result.

Lemma 2.4. *Suppose that the functions $a^i_{\overline{m}}$ of Lemma 2.1 vanish at all points of the axis $(u, 0, \cdots, 0)$ and that f is a solution of (2.1.1) whose* *component functions* f^i, $i \geq 2$, *are infinitely tangent to* 0 *at* $z = 0$. *Then each* f^i, $i \geq 2$, *is identically zero.*

Proof. It suffices to show that $g = (0, f^2, \cdots, f^n)$ satisfies the differential inequalities (2.3.1), since then the result follows from Aronszajn's theorem. To establish (2.3.1) for g^i when $i \geq 2$, observe that in the equation $\partial\bar{\partial} f^i = \psi_i(f^m, \partial f^m, \bar{\partial} f^m, \partial \bar{f}^m, \bar{\partial}\bar{f}^m)$ of Corollary 2.2 the only terms in ψ_i, which involve the function f^1, also contain a factor ∂b^i or $\partial \bar{b}^i$, where we have written $b^i(z) = a^i_{\bar{1}}(f^1(z), f^2(z), \cdots, f^n(z))$. But

$$\partial b^i = \partial b^i/\partial w^1 \cdot \partial f^1/\partial z + \sum_{m\geq 2} \partial b^i/\partial w^m \cdot \partial f^m/\partial z.$$

Since $\partial b^i/\partial w^1 = 0$ on $(u, 0, \cdots, 0)$, its value at $(f^1, f^2, \cdots, f^n)$ is bounded by $\mathrm{const}(\sum_{m\geq 2}|f^m|)$. Therefore, ∂b^i satisfies an inequality

$$|\partial b^i(z)|^2 \leq M \sum_{m\geq 2}(|f^m|^2 + |\partial f^m|^2 + |\bar{\partial} f^m|^2),$$

and the result follows. q.e.d.

We will now specialize to the case $n = 2$. For convenience, we will use the coordinates (u, v) on $\mathbb{C}^2$, where $u = u_1 + iu_2$, $v = v_1 + iv_2$.

Lemma 2.5. *Any standard identification of* V *with* $\mathbb{C}^2$ *may be changed by a diffeomorphism* Ψ *of* $(\mathbb{C}^2, (0, 0))$ *to a standard identification such that* $J = J_0$ *at all points on the coordinate axes* $\{u = 0\}$ *and* $\{v = 0\}$ *which are sufficiently close to* $(0, 0)$. *Moreover, we may choose* Ψ *to be* 1-*tangent to the identity at* $(0, 0)$.

Proof. By [6, Theorem III], there are local J-holomorphic curves, A_u and A_v say, through $(0, 0)$ which are tangent to the coordinate axes $\{v = 0\}$ and $\{u = 0\}$. Since every almost complex structure on a manifold of real dimension 2 is diffeomorphic to the usual integrable structure, there is a diffeomorphism Ψ' of $\mathbb{C}^2$ which takes (A_u, J) and (A_v, J) into the coordinate axes with the standard structure J_0. Thus we may choose Ψ' so that the restriction of $\Psi'_*(J)$ to the tangent space of each coordinate axis equals J_0 near $(0, 0)$. Clearly, we may assume that $\Psi' = \mathrm{Id} + O(2)$.

Next we adjust Ψ' near these axes to a map Ψ such that $\Psi_*(J) = J_0$ in the normal directions too. For the axis $\{v = 0\}$, for example, this may be done by composing Ψ' with a map L of the form

$$L(u_1, u_2, v_1, v_2) = (u_1, u_2, v_1, v_2) + (a, b, c, d)v_2,$$

where a, b, c, d are functions of $u = (u_1, u_2)$. In fact, if $(A, B, C, 1+D)$ are the components of the vector $\Psi'_* J(\partial/\partial v_1)$ at $(u, 0)$, then we may take $a = -A/(1+D)$, $b = -B/(1+D)$, $c = -C/(1+D)$, and $1+d = 1/(1+D)$. Observe that because A, B, C, and D vanish at $(0, 0)$, $L = \mathrm{Id} + O(2)$, so that $\Psi = \mathrm{Id} + O(2)$ also.

Proposition 2.6. *Let f be a nonconstant J-holomorphic map of $(D, 0)$ to an almost complex 4-manifold (V, x) which has a critical point at x, and assume, as always, that f is not a multiple covering. Then there is a standard identification of V with $\mathbb{C}^2$, such that the axis $\{v = 0\}$ is J-holomorphic and f has the form*

$$f(z) = (z^k, z^m) + O(m+1),$$

where $m > k$, k does not divide m, and $O(m+1)$ denotes a function of z and $\bar{z}$ which vanishes to order m at $z = 0$.

Proof. Choose any standard identification of V with $\mathbb{C}^2$, and consider the Taylor expansion $T(f)$ of $f = (f^1, f^2)$ in terms of z and $\bar{z}$. Since f is nonconstant, it follows from Lemma 2.3 that this expansion is not identically zero. Let k be the order of its first nonzero term. Observe that $k > 1$ by hypothesis. Since $J = J_0$ at $(0, 0)$, the functions $a^i_{\bar{m}}$ vanish at $(0, 0)$, and so the coefficients $a^i_{\bar{m}}(f(z))$ in equation (2.1.1) start with terms of order k. Hence, by (2.2.1), the term in $T(f)$ of order k is annihilated by $\bar{\partial}$ and so is a function of z alone. Therefore, we may change coordinates by a J_0-holomorphic linear transformation of $\mathbb{C}^2$ in such a way that

$$f(z) = (z^k + O(k+1), O(r)), \quad \text{where } r \geq k+1.$$

Now, change coordinates as in Lemma 2.5 so that $J = J_0$ along the axis $\{v = 0\}$. (Since $\Psi = \mathrm{Id} + O(2)$, this does not change the above form of f.) Then the functions $P^i_{\bar{m}}$ and $a^i_{\bar{m}}$ vanish at all points $(u, 0)$, and so the functions $a^i_{\bar{m}}(f(z))$ vanish to order $r - 1$ at $z = 0$. Hence (2.1.1) implies that all terms in $T(f)$ of order $\leq r$ are functions of z alone. Thus

$$f(z) = (z^k + q(z), bz^r) + O(r+1),$$

where $q(z)$ is a polynomial in z and $b \in \mathbb{C}$. If $b \neq 0$, we can put f into the required form (i.e., absorb q and b) by a J_0-holomorphic change of coordinates on $\mathbb{C}^2$ which does not move the axis $\{v = 0\}$. Note also that if r happens to equal pk we can eliminate the term z^r in f^2 by the J_0-holomorphic change of coordinates $(u, v) \mapsto (u, v - u^p)$, and then repeat the above argument.

We claim that there always is a finite number r for which $b \neq 0$.[1] For if not, the above process either terminates after a finite number of steps with $T(f) = (z^k, 0)$, or it continues indefinitely. In the latter case, one can choose the sequence of coordinate changes so that it converges on the level of jets. Hence there is a coordinate system in which $T(f) = (z^k, 0)$. We may also arrange that the axis $\{v = 0\}$ is J-holomorphic. Thus, in either case, f is infinitely tangent to the J-holomorphic map f' defined by $f'(z) = (z^k, 0)$. But then $f = f'$ by Lemma 2.3, which contradicts the hypothesis that f is not a multiple covering. q.e.d.

It follows immediately from Proposition 2.6 that a map $f : (D, 0) \to (\mathbb{C}^2, 0)$, which has an expansion in which k and m are mutually prime, is an embedding on a deleted neighbourhood $D' - \{0\}$ of $\{0\}$, where $D' \subset D$. We will see in Lemma 5.3 below that this holds for all f, but for now will recall a partial result from [4]. Note that it, as well as its corollary, holds in all dimensions.

Lemma 2.7 [4, 4.4]. *All critical points of a J-holomorphic map $f : D \to V$ are isolated. Further, if C and C' are distinct connected J-holomorphic curves, then every accumulation point in the intersection $C \cap C'$ is critical on both curves.*

Proof. When $n = 2$ the first statement follows immediately from Proposition 2.6. Since all that we used is the fact that the first nonzero term in the Taylor expansion for f involves z only (and not $\bar{z}$), the proof works in all dimensions.

To prove the second statement, assume that $x \in C \cap C'$ is a nonsingular point of C', and let $f = (f^1, f^2, \cdots, f^n)$ (resp. f') parametrize C (resp. C'). Choose coordinates so that $f'(z) = (z, 0, \cdots, 0)$ and $J = J_0$ on the axis $(z, 0, \cdots, 0)$. Since the first nonzero terms in the Taylor expansions of the f^i involve only z, it is easy to see that if one of the f^i, for $2 \leq i \leq n$, has a nonzero Taylor expansion at $\{0\}$, x must be an isolated point of $\operatorname{Im} f \cap \operatorname{Im} f'$. On the other hand, if these Taylor expansions all vanish, it follows immediately from Lemma 2.4 that the f^i, $i \geq 2$, all vanish. But then $C = C'$ locally (and hence globally) contrary to hypothesis.

Corollary 2.8. *Given any J-holomorphic map $f : (D, 0) \to (\mathbb{C}^n, 0)$, there is a neighbourhood $D' \subset D$ of $\{0\}$ and a sequence of points*

[1]The author wishes to thank H. Hofer for pointing out a mistake in her original proof of this fact. This mistake also occurred in Lemma 4.4(iii) of [4]: a correct version of the latter argument appears in Lemma 2.7.

$X = \{x_j : j \geq 1\}$ converging to $\{0\}$ such that $f|D' - \{0\}$ is an immersion, and $f|D' - X \cup \{0\}$ is an embedding.

3. Desingularizing holomorphic curves

We begin by showing that any J-holomorphic map can be C^1-approximated by a J-holomorphic immersion. Recall that $D(R)$ denotes the disc of radius R.

Proposition 3.1. *Let $f: D(R_0) \to \mathbb{C}^2$ be a J-holomorphic map of the form*

$$f(z) = (z^k, z^m) + O(m+1).$$

Then if $R > 0$ is sufficiently small, there is a constant $C > 0$ such that for every $\varepsilon \in \mathbb{C} - \{0\}$ with $|\varepsilon| \leq 0.1$ there is a J-holomorphic immersion $f_\varepsilon: D(R) \to \mathbb{C}^2$ of the form

$$f_\varepsilon(z) = (z^k + O(k+1), \varepsilon z + O(2))$$

such that $\|f_\varepsilon - f\|_{C^1} \leq C|\varepsilon|$ on $D(R)$.

Nijenhuis and Woolf proved a similar result when $k = 1$. In order to reduce to their case, we lift f over the branched covering map $\psi: \mathbb{C}^2 \to \mathbb{C}^2$ given by $(u, v) \mapsto (u^k, v)$. The lifted curve will be $\widetilde{J}$-holomorphic, where $\widetilde{J}$ is the pullback of J by ψ, and our first task is to study $\widetilde{J}$. In order that this have nice properties, we must choose our coordinates on $\mathbb{C}^2$ so that $J = J_0$ at all points of the branching axis $\{u = 0\}$, which is possible by Lemma 2.5. As in Lemma 2.5 we will identify $\mathbb{C}^2$ with $\mathbb{R}^4$ using coordinates (u_1, u_2, v_1, v_2), where $u = u_1 + iu_2$ and $v = v_1 + iv_2$, and we will write $B^4(R)$ for the ball with center $(0, 0)$ in $\mathbb{C}^2$ of radius R.

Lemma 3.2. *Suppose that $J = J_0$ at all points of the axes $\{u = 0\}$ and $\{v = 0\}$. Then the pullback $\widetilde{J} = d\psi^{-1} \circ J \circ d\psi$ of J by ψ is Lipschitz continuous, and also equals J_0 at all points of the axes $\{u = 0\}$ and $\{v = 0\}$. Further, the partial derivatives $\partial \widetilde{J}/\partial v_q$ are continuous everywhere, and, for all $R > 0$, the mixed partials $\partial^2 \widetilde{J}/\partial u_p \, \partial v_q$ are uniformly bounded in $B^4(R) - \{u = 0\}$.*

Proof. Denote ku^{k-1} by $a + ib$ so that a and b are homogeneous polynomials of degree $k - 1$ in u_1, u_2. Then

$$d\psi|_{(u,v)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$d\psi^{-1}|_{\psi(u,v)} = \begin{pmatrix} a/k^2|u|^{2k-2} & -b/k^2|u|^{2k-2} & 0 & 0 \\ b/k^2|u|^{2k-2} & a/k^2|u|^{2k-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us write $J = J_0 + A$, where A is a 4×4 matrix whose entries A_{ij} are functions in the u_p, v_q. Since $A = 0$ on the coordinate axes $\{u = 0\}$, $\{v = 0\}$, each term in the Taylor expansion of A_{ij} about $(0, 0)$ is divisible by some u_p and some v_q. Hence, if $\widetilde{A}_{ij}(u, v) = A_{ij}(u^k, v)$, the $\widetilde{A}_{ij}$ are smooth functions and each term in their Taylor expansions has order $\geq k$ in the u_p and order ≥ 1 in the v_q.

Now write $\widetilde{J} = J_0 + B$, so that $B = d\psi^{-1} \circ A \circ d\psi$. Then, for example,

$$B_{13}(u, v) = a\widetilde{A}_{13}/k^2|u|^{2k-2} - b\widetilde{A}_{23}/k^2|u|^{2k-2},$$

if $u \neq 0$. Because a and b are homogeneous polynomials in u_1, u_2 of degree $k - 1$, and because of the above remarks about the $\widetilde{A}_{ij}$, we may extend B_{13} to a Lipschitz continuous function by setting it equal to 0 when $u = 0$. Further, the partials $\partial B_{13}/\partial v_q$ and $\partial^2 B_{13}/\partial u_p \partial v_q$ clearly have the desired properties. One can treat the other terms B_{ij} similarly. (In fact, the terms with $i = 1$ or 2 and $j = 3$ or 4 are the worst.) q.e.d.

Now consider the lift $\tilde{f}$ of f over ψ which has the formula

$$\tilde{f}(z) = (\tilde{f}^1(z), \tilde{f}^2(z)) = (z(1 + h(z)), z^m + O(m+1)).$$

Here $1 + h(z)$ is the kth root of a function of the form $1 + O(m+1)/z^k$ and so, because $m > k$, is at least C^1-smooth. (Recall that $O(m+1)$ denotes a function of z and $\bar{z}$ of order $\geq m + 1$, and so its quotient by z^k need not be C^∞.)

Lemma 3.3. *There is a C^2-smooth change of coordinates Φ on $\mathbb{C}^2$ which is the identity on the axis $\{u = 0\}$ and is such that $\Phi \circ \tilde{f}(z) = (z, 0)$. Further, we may suppose that $\Phi_*(\widetilde{J}) = J_0$ along the axis $\{u = 0\}$ and $\{v = 0\}$ and that the partial derivatives of $\Phi_*(\widetilde{J})$ have the same properties as those of $\widetilde{J}$.*

Proof. Take $\Phi'(u, v) = ((\tilde{f}^1)^{-1}(u), v - \tilde{f}^2 \circ (\tilde{f}^1)^{-1}(u))$. Then Φ' satisfies all conditions in the first sentence of the lemma. Let $J' = \Phi'_*(\widetilde{J})$. Since Φ' is 1-tangent to the identity map Id on the axis $\{u = 0\}$, it is clear that $J' = J_0$ along this axis. Further, because the map $z \mapsto (z, 0)$ is J'-holomorphic, $J' = J_0$ on the tangent bundle of $\{v = 0\}$, and so we only need to alter Φ' in the directions normal to this axis. Since

$m > k \geq 2$, the function $\tilde{f}^2 \circ (\tilde{f}^1)^{-1}(u)$ is $O(3)$. Therefore, if we define L as in Lemma 2.5, the functions A, B, C, and D are $O(2)$ so that $L = \mathrm{Id} + O(3)$. Hence $\Phi = \mathrm{Id} + O(3)$ which implies that the partials of $\Phi_*(\widetilde{J})$ have the required properties. q.e.d.

Let $\widehat{J} = \Phi_*(\widetilde{J})$. We are interested in solutions f of the equations

$$\bar{\partial} f^i + a^i_{\overline{m}}(f(z))\bar{\partial}\overline{f}^m = 0, \qquad i = 1, 2, \tag{3.3.1}$$

where $a^i_{\overline{m}}$ are the appropriate functions for $\widehat{J}$ defined as in Lemma 2.1. The crucial step in the proof of Proposition 3.1 is the following estimate. Consider the norms $\|\cdot\|$ and $\|\cdot\|'$ defined on functions $g: D(R) \to \mathbb{C}^2$ by

$$\|g\| = |g| + L(g) \quad \text{and} \quad \|g\|' = \max\{\|\partial g\|, \|\bar{\partial} g\|\}, \tag{3.3.2}$$

where $|g| = \sup\{|g^1(z)|, |g^2(z)|: z \in D(R)\}$, and $L(g)$ is the Lipschitz constant of g. (Note: $\|\cdot\|$ is slightly different from the corresponding norm in [6].) Further, given $R > 0$ and $\varepsilon \in \mathbb{C}$, let L_ε denote the space of maps $g: (D(R), 0) \to (\mathbb{C}^2, (0, 0))$ of the form

$$g(z) = z(1, \varepsilon) + \tilde{g}(z),$$

where $\tilde{g}$ is Lipschitz with Lipschitz constant $\leq |\varepsilon|$. We will suppose that $R \leq 1$ and that $|\varepsilon| \leq 0.1$ so that $\operatorname{Im} g \subset B^4(2)$ for all $g \in L_\varepsilon$.

Lemma 3.4. *There is a constant K which is independent of ε and R such that*

$$\|a^i_{\overline{m}}(g(z))\| \leq K|\varepsilon|R$$

for i, $m = 1, 2$, all $g \in L_\varepsilon$, and all $\varepsilon \in \mathbb{C}$ for which $|\varepsilon| \leq 0.1$.

Proof. Let $a(z)$ denote one of the functions $a^i_{\overline{m}}(z)$. By Lemma 3.3 and the definition of the $a^i_{\overline{m}}$ given in Lemma 2.1, a vanishes when $u = 0$ or $v = 0$ and its partial derivatives $|\partial a/\partial v_q|$ and $|\partial^2 a/\partial u_p \partial v_q|$ are uniformly bounded in $B^4(2) - \{u = 0\}$, say by c_1. Since $\partial a/\partial u_p = 0$ when $v = 0$ (and $u \neq 0$), we find that

$$|\partial a/\partial u_p(u, v)| \leq c_1|v|, \quad \text{if } u \neq 0. \tag{3.4.1}$$

Hence $|a(u, v)| \leq c_1|u||v|$. Similarly, because the functions $|\partial a/\partial v_q|$ are continuous in $B^4(2)$ and equal 0 when $u = 0$, and because $|\partial^2 a/\partial u_p \partial v_q| \leq c_1$, we have

$$|\partial a/\partial v_q(u, v)| \leq c_1|u|. \tag{3.4.2}$$

Since $|\varepsilon| < 0.1$, the image $\operatorname{Im} g$ of an element of L_ε clearly lies in the cone $\Lambda = \{(u, v) : |v| \le 4|\varepsilon|\,|u|\}$ and so, by the above, $|a| \le 4|\varepsilon|c_1 R$ on $\operatorname{Im} g$. Further,

$$\begin{aligned} L(a(g)) &= \sup\{|a(g(z)) - a(g(z'))|/|z - z'| : z, z' \in D(R)\} \\ &\le \sup\{|a(g^1(z), g^2(z)) - a(g^1(z'), g^2(z))|/|z - z'|\} \\ &\quad + \sup\{|a(g^1(z'), g^2(z))| - |a(g^1(z'), g^2(z'))|/|z - z'|\}. \end{aligned}$$

Without loss of generality, we may suppose that $|g^2(z)| \le |g^2(z')|$, so that the point $(g^1(z'), g^2(z))$ does belong to the cone Λ. Then we may apply (3.4.1) and (3.4.2) to deduce that

$$\begin{aligned} L(a(g)) &\le c_1 \cdot L(g^1) \cdot \sup|g^2(z)| + c_1 \cdot L(g^2) \cdot \sup|g^1(z)| \\ &\le c_1(1 + |\varepsilon|) \cdot 4|\varepsilon|R + c_1 \cdot 2|\varepsilon|R. \end{aligned}$$

Hence $\|a(g)\| = |a| + L(g) \le K|\varepsilon|R$ provided that $K \ge 7c_1$. q.e.d.

Following [6, (5.2)], we now replace (3.3.1) by the equivalent system of integral equations

$$f^i = Sf^i - T[a^i_{\overline{m}}(f(z))\overline{\partial}\overline{f}^m], \tag{3.4.3}$$

where S and T are the integral operators of [6, §6]; these are defined for functions $h : \mathbb{C} \to \mathbb{C}$ and have the property that $\overline{\partial}Sh = 0$ and $\overline{\partial}Th = h$. We are looking for solutions which have a specified tangent at 0. To find them, let

$$\omega^i(f, g) = -T[a^i_{\overline{m}}(g(z))\overline{\partial}\overline{f}^m],$$

where $g : D \to \mathbb{C}^2$ is a Lipschitz function such that $g(0) = (0, 0)$; and set

(3.4.4)

$$\Theta^i(f, g)(z) = \omega^i(f, g)(z) - \omega^i(f, g)(0) - z \cdot \partial\omega^i(f, g)(0), \qquad i = 0, 1.$$

Observe that $\Theta^i(f, g)(0) = 0$. Then we look for solutions of the equation

$$f = z(1, \varepsilon) + \Theta(f, f) \quad \text{with } f(0) = (0, 0), \tag{3.4.5}$$

where $\Theta = (\Theta^1, \Theta^2)$. By construction, any solution of (3.4.5) is $\widehat{J}$-holomorphic and is tangent to $z(1, \varepsilon)$ at $z = 0$. If $\varepsilon = 0$, we know from Lemma 3.3 that the map $z \mapsto (z, 0)$ is a solution, and our aim is to perturb it. If $\widehat{J}$ were C^1, one could do this using the results of [6], but because $\widehat{J}$ is merely Lipschitz one has to work a little harder.

Proposition 3.5. *Let $\widehat{J}$ be any Lipschitz continuous almost complex structure on $\mathbb{C}^2$ which is such that the estimate of Lemma* 3.4 *holds. If*

$R > 0$ *is sufficiently small, then there is a solution* $\hat{f}_\varepsilon$ *of* (3.4.5) *in* L_ε *for every* ε *with* $|\varepsilon| \le 0.1$.

Granted this, it is easy to prove Proposition 3.1.

(3.6) *Proof of Proposition* 3.1. Define $f_\varepsilon = \psi \circ \Phi^{-1} \circ \hat{f}_\varepsilon$, where Φ is as in Lemma 3.3. Then f_ε is J-holomorphic and therefore C^∞. Its Taylor expansion clearly has the form $(z^k, \varepsilon z)+$ higher order terms. Further, the existence of the constant C (which depends on Φ) is immediate from the definition of L_ε. It remains to show that f_ε is an immersion. Observe first that $\hat{f}_\varepsilon$ is C^1 since $\widehat{J}$ is Lipschitz. Further the derivative of its first component never vanishes, so that it is an embedding. Clearly the only intersection of $\Phi^{-1} \circ \hat{f}_\varepsilon$ with the branching axis $u = 0$ occurs when $z = 0$, and hence at a point where the second component of $\Phi^{-1} \circ \hat{f}_\varepsilon$ has nonzero derivative. It follows easily that $\psi \circ \Phi^{-1} \circ \hat{f}_\varepsilon$ is an immersion.

(3.7) *Proof of Proposition* 3.5. As in [6, (5.3)], Proposition 3.5 is proved in two steps. We first show that if R is sufficiently small, for every $g \in L_\varepsilon$ there is a unique solution $f = \psi(g)$ in L_ε of the equation

$$f = z(1, \varepsilon) + \Theta(f, g).$$

Then we show that the map ψ is a continuous mapping from L_ε to itself. But L_ε is a compact convex subset of the locally convex topological linear space Λ of all Lipschitz maps from $D(R)$ to $\mathbb{C}^2$. Hence by the Schauder-Tychonoff fixed point theorem, ψ has a fixed point, which is the desired solution of (3.4.5). Although many of the details of this argument are the same as those in [6], we repeat them here for the sake of clarity.

Let $\mathbf{B} = \mathbf{B}_R$ be the space of maps $(D(R), 0) \to (\mathbb{C}^2, (0, 0))$ whose first derivatives are Lipschitz, with norm $\|\cdot\|'$ as defined in (3.3.2),and let $\mathbf{A} = \{f \in \mathbf{B} : \|f\|' \le 3\}$. By [6, (6.1.2), (6.1.4)], the integral operator T of (3.4.3) satisfies the inequality

$$\|Th\|' \le c_1 \|h\|$$

for some constant c_1 and any Lipschitz function h defined on $D(R)$. Further, we clearly have $\|ah\| \le \|a\|\, \|h\|$. Hence, by definition (3.4.4) and Lemma 3.4,

$$\begin{aligned} \|\Theta(f, g)\|' &\le 4c_1 \|a^i_{\overline{m}}(g(z))\|\, \|\overline{\partial}\, \overline{f}\| \\ &\le 4c_1 K|\varepsilon| R \|\overline{\partial}\, \overline{f}\| \le c_2 |\varepsilon| R \|f\|' \end{aligned} \tag{3.7.1}$$

for $g \in L_\varepsilon$, where $c_2 = 4c_1 K$. (The 4 appears here as $2\cdot 2$, one factor of 2 coming from the fact that m has two possible values, and the other coming from the two terms of Θ.) Thus, because Θ is linear in its first factor, for each $g \in L_\varepsilon$, the operator $\theta_g \colon \mathbf{A} \to \mathbf{B}$ defined by $\theta_g(f) = \Theta(f, g)$ satisfies the conditions of the following version of the contraction mapping principle.

Lemma 3.8 (see [6, Lemma 2.5a]). *Let $\psi \in \mathbf{B}$ be such that $\|\psi\|' \le 3/2$, and suppose that $\theta \colon \mathbf{A} \to \mathbf{B}$ is a map such that*

$$\|\theta(\Omega') - \theta(\Omega)\|' \le K_1 R \|\Omega' - \Omega\|'$$

and

$$\|\theta(\Omega)\|' \le K_2 R \quad \textit{for all } \Omega, \Omega' \in \mathbf{A},$$

where the constants K_1, K_2 are independent of R. Choose $\Omega_1 \in \mathbf{A}$, and define Ω_N recursively by $\Omega_{N+1} = \psi + \theta(\Omega_N)$. Then, for sufficiently small R, the following holds: $\Omega_N \in \mathbf{A}$ for all N, and the sequence $\{\Omega_N\}$ converges to a limit $\Omega \in \mathbf{A}$ which satisfies

$$(*) \qquad \Omega = \psi + \theta(\Omega).$$

Further, this is the only solution of $(*)$ in $\mathbf{A}$.

It follows, provided R is sufficiently small, that the equation $f = z(1, \varepsilon) + \theta_g(f)$ has a unique solution $\psi(g)$ in $\mathbf{A}$ for each $g \in L_\varepsilon$. Further, because $L(\theta_g(f)) \le \|\theta_g(f)\|' \le 3c_2|\varepsilon|R$, the solution $\psi(g)$ will be in L_ε provided $3c_2 R \le 1$.

It remains to prove that ψ is continuous as a map of L_ε to itself with the supremum norm $|\cdot|$. To this end, suppose that $f = \psi(g)$ and that $\{g_n\}$ is a sequence in L_ε such that $|g - g_n| \to 0$. Let $f_n = \psi(g_n)$. Because $\{f_n\}$ belongs to the set L_ε which is compact with respect to $|\cdot|$, by passing to a subsequence we may suppose that $f_n \to f' \in L_\varepsilon$. Clearly, it will suffice to show that $f' = f$. Using the remarks preceding (3.7.1) and the fact that Θ is linear in its first argument, one easily sees that

$$\begin{aligned} \|f_n - f_m\|' &\le \|\Theta(f_n - f_m, g_n)\|' + \|\Theta(f_m, g_n) - \Theta(f_m, g_m)\|' \\ &\le c_2|\varepsilon|R\|f_n - f_m\|' + c_3|g_n - g_m|\,\|f_m\|' \\ &\le |\varepsilon|/3\|f_n - f_m\|' + 3c_3|g_n - g_m|, \end{aligned}$$

since $f \in \mathbf{A}$. Hence the sequence $\{f_n\}$ is Cauchy with respect to the norm $\|\cdot\|'$. Since the space $C^{1,1}(D(R))$ of functions whose first derivatives are Lipschitz is complete with respect to $\|\cdot\|'$, the sequence $\{f_n\}$ has a limit in $C^{1,1}(D(R))$ which must be f'. Further $f' = z(1, \varepsilon) + \theta_g(f')$. Thus because of the uniqueness proved above, $f' = f$ as required. q.e.d.

Proposition 3.5 is enough to prove Theorem 1.1 and Proposition 1.2. However, in order to prove Theorems 1.3 and 1.4, we need to find a partial desingularization of our singular curve f.

Proposition 3.9. *Let $f: D(R_0) \to \mathbb{C}^2$ be a J-holomorphic map of the form*

$$f(z) = (z^k, z^r) + O(r+1).$$

If $R > 0$ is sufficiently small, then there is a constant $C > 0$ such that for every $\varepsilon \in \mathbf{C}-\{0\}$ with $|\varepsilon| \le 0.1$ there is a J-holomorphic map $f_\varepsilon: D(R) \to \mathbb{C}^2$ of the form

$$f_\varepsilon(z) = (z^k + O(k+1), \varepsilon^2 z^2 + O(3))$$

such that $\|f_\varepsilon - f\|_{C^1} \le C|\varepsilon|$ on $D(R)$. Further, f_ε is an immersion on $D(R) - \{0\}$.

Proof. This is essentially the same as the proof of Proposition 3.1 except that we use two branched covering maps. As before, we first lift f over the branched covering $\psi_u: \mathbb{C}^2 \to \mathbb{C}^2$ given by $(u, v) \mapsto (u^k, v)$ and change coordinates by the map Φ of Lemma 3.3 so that the lifted map $\hat{f}$ is just $z \mapsto (z, 0)$. We then lift over the branched covering $\psi_v: \mathbb{C}^2 \to \mathbb{C}^2$ given by $(u, v) \mapsto (u, v^2)$. If we denote the lift of $\hat{J}$ over ψ_v by $\overline{J}$, it is easy to check that $\overline{J} = J_0 + C$, where C is a matrix of functions which have the form

$$\rho(u_1, u_2)\sigma(v_1, v_2)/|u|^{2k-2}|v|^2,$$

where ρ and σ are sums of homogeneous polynomials of degrees at least $2k-1$ and 3 respectively. Hence the mixed partials $\partial^2\overline{J}/\partial u_p \partial v_q$ are uniformly bounded and continuous in $B^4(R) - \{u \text{ or } v = 0\}$. Further, the derivatives $\partial\overline{J}/\partial u_p$ are bounded and continuous in $B^4(R) - \{u = 0\}$ and vanish when $v = 0$, and similarly for $\partial\overline{J}/\partial v_q$. Thus the estimate of Lemma 3.4 holds for $\overline{J}$. Therefore, by Proposition 3.5, there is an embedded $\overline{J}$-holomorphic curve $\overline{f}_\varepsilon$ tangent to $z \mapsto (z, \varepsilon z)$. Composing this with $\psi_u \circ \Phi^{-1} \circ \psi_v$ gives us the desired curve f_ε.

Note 3.10. Clearly, by replacing ψ_v by the map $(u, v) \mapsto (u, v^p)$, one can construct for any $p < r$ a perturbation of f of the form

$$f_\varepsilon(z) = (z^k + O(k+1), \varepsilon^p z^p + O(p+1)).$$

4. The local self-intersection number

In this section we define the local self-intersection number and explain its elementary properties. Roughly speaking, if $f: D \to V$ is an embedding except at 0, this invariant counts the algebraic number of self-intersection points of an immersion f' which is a C^1-small perturbation of f. In order to make this definition precise, one must specify how the perturbation behaves near ∂D. We will do this by extending everything over S^2, but other approaches are possible.

Let $E \to D$ be a rank-2 complex bundle whose restriction to ∂D contains a line bundle L_0 with a given trivialization τ_0. Then, for each trivialization $\tau: L \to \partial D \times \mathbb{C}$ of the quotient line bundle $L = E/L_0$ over ∂D, let $E(\tau)$ be the rank-2 complex bundle over S^2 obtained by gluing $D^2 \times \mathbb{C}^2$ to E using the trivializations τ_0 and τ. Let $\chi(\tau)$ be the integral of the first Chern class $c_1(E(\tau))$ over S^2. Clearly, $\chi(\tau)$ depends on τ. In fact, if τ and τ' are two such trivializations, then $\tau' = \psi \circ \tau$ for some map $\psi: \partial D \to \mathbb{C} - \{0\}$, and, if n is the degree of ψ where ∂D is oriented as the boundary of D, then it is easy to check that $\chi(\tau') = \chi(\tau) + \deg \psi$.

Consider the special case when $E = f^*(TV)$, where the map $f: D \to V$ is a J-holomorphic immersion near ∂D. Then we may take L_0 to be the pullback of the complex line bundle tangent to $C = \operatorname{Im} f$ along ∂C. Thus L_0 has a natural identification with the restriction to ∂D of the tangent bundle TD, and we choose τ_0 so that it extends to a trivialization of TS^2 over $S^2 - \operatorname{Int}(D)$. Hence a trivialization τ of the normal bundle ν_C of C along ∂C determines the integer $\chi(\tau)$ as above.

Suppose further that all the singular points of f lie in the interior of D, i.e., that there is a neighbourhood A of ∂D such that f embeds $f^{-1}(f(A))$ onto $f(A)$. Then the trivialization τ also allows us to define the τ-self-intersection number $\operatorname{Int}(C, \tau)$ as the number of intersections of C with $\operatorname{Im} f_t$, for small t, where f_t is a generic perturbation of f which moves ∂C in the direction of τ. Alternatively, glue $D^2 \times \mathbb{C}$ to a neighbourhood of ∂C in V, so that the section $D^2 \times \{0\}$ extends C and so that the fibers $x \times \mathbb{C}$ for $x \in \partial D^2$ match up with the trivialized bundle ν_C over ∂C. This gives an almost complex manifold N_τ which contains a holomorphic image $\Gamma = C \cup D^2 \times \{0\}$ of S^2, and clearly $\operatorname{Int}(C, \tau)$ is just the self-intersection number $\Gamma \cdot \Gamma$ of Γ in N_Γ. Again, given two trivializations τ and τ', one can check that $\operatorname{Int}(C, \tau') = \operatorname{Int}(C, \tau) + n$.

Definition 4.1. Let $f: D \to V$ be a map which is J-holomorphic near ∂D and is such that all its singular points lie in the interior of D. We define the *self-intersection number* $\operatorname{Int}(f)$ of f to be $\frac{1}{2}[\operatorname{Int}(C, \tau) - \chi(\tau)] - 1$. By the above, this does not depend on the choice of τ. Further, if f is a J-holomorphic map on D such that $f|D' - \{0\}$ is an embedding for some disc D' such that $\{0\} \in D' \subset D$, we define the *local self-intersection number* $\operatorname{L.Int}(f, 0)$ to be equal to $\operatorname{Int}(f|D')$. Clearly, this is independent of the choice of D'. (We will see in Lemma 5.3 below that every

J-holomorphic f satisfies this hypothesis, but for the moment all we know is stated in Corollary 2.8.)

Lemma 4.2. (i) *If f is a J-holomorphic immersion with only interior transverse double points, then* $\mathrm{Int}(f)$ *is defined and equals the number of double points of f.*

(ii) *Let J_t be a continuous family of almost complex structures on V and let $f_t\colon D \to V$ be a continuous family of maps all of whose singular points lie inside D and which are J_t-holomorphic near ∂D. Then* $\mathrm{Int}(f_0) = \mathrm{Int}(f_1)$.

Proof. (i) Note that $\chi(\tau)$ may also be defined in terms of the manifold N_τ: in fact $\chi(\tau)$ is just the integral of $c_1(TN)$ over Γ. The result now follows because, for any holomorphically immersed curve Γ,

$$c_1(TN)(\Gamma) = c_1(T\Gamma)(\Gamma) + c_1(\nu_\Gamma)(\Gamma) = 2 + \Gamma \cdot \Gamma - 2m,$$

where m is the number of self-intersection points of Γ. (Recall that every transverse intersection of two J-holomorphic curves is positively oriented, and so has multiplicity $+1$.)

(ii) This is obvious.

Lemma 4.3. *Consider a J-holomorphic map $f\colon D \to V$ whose singularities all lie in the interior of D, and a J-holomorphic immersion $f'\colon D \to V$. If f' is sufficiently C^1-close to f, then the singularities of f' all lie in the interior of D and*

$$\mathrm{Int}(f) = \mathrm{Int}(f').$$

In particular, $\mathrm{Int}(f)$ *is a nonnegative integer.*

Proof. Let f be an embedding on the annulus $A_\lambda = \{z : \lambda \le |z| \le 1\}$ in $D = \{z : |z| \le 1\}$, and put any metric on V. If $x, y \in V$ are sufficiently close, let $\rho(x, y)$ be the unique shortest geodesic such that $\rho(x, y)(0) = x$ and $\rho(x, y)(1) = y$. If f' is sufficiently C^1-close to f, then the formula

$$g_t(z) = \rho(f'(z), f(z), t)$$

defines a family of embeddings g_t of A into V such that $g_0 = f'$ and $g_1 = f$. We now patch $f'|D - A$ to $f|\partial D$ to get a map $F\colon D \to V$ as follows. Let β be an increasing function of $[\lambda, 1]$ onto $[0, 1]$, which is constant near each endpoint. Then define

$$F(z) = \begin{cases} f'(z) & \text{if } |z| \le \lambda, \\ g_{\beta(|z|)}(z) & \text{if } |z| \ge \lambda. \end{cases}$$

Then $\|F - f\|_{C^1} \le c\|f' - f\|_{C^1}$, so that $F|A$ will be an embedding provided that $\|f' - f\|_{C^1}$ is sufficiently small. Thus F is an immersion of D.

We claim further that we may assume that the only double points of F are those of f' (which must therefore involve only the points of $D-A$). To see this, note that by choice of β, $F = f'$ outside some annulus $A_1 = \{z : \mu \le |z| \le 1\}$, where $\mu > 1/2$. Choose $\nu > 0$ so that the sets $f(D-A)$ and $f(A_1)$ are at least a distance ν apart, and then assume that $\|f' - f\|_{C^1}$ is so small that $d(f'(z), f(z)) \le c.\nu/3$, for all z. Then, by construction, $d(F(z), f(z)) \le c.\nu/3$, for all z, so that the sets $f'(D-A) = F(D-A)$ and $F(A_1)$ are disjoint.

Now, consider the homotopy F_t from f to F defined by

$$F_t(z) = \rho(f(z), F(z), t).$$

We may clearly assume that each F_t restricts to an embedding of A. Moreover, this homotopy is constant along ∂D and since it moves no point more than a distance $\nu/3$, it follows as above that all the singular points of F_t lie inside D. Finally, observe that because $F_t|A$ is an embedding, there is a family of almost complex structures J_t such that $F_t|A$ is J_t-holomorphic. Hence $\mathrm{Int}(f) = \mathrm{Int}(F_1)$ by Lemma 4.2. But, clearly

$$\mathrm{Int}(F_1) = \mathrm{Int}(F_1|D-A) = \mathrm{Int}(f'|D-A) = \mathrm{Int}(f').$$

Corollary 4.4. *If* $: D \to V$ *is a* J*-holomorphic map which is an embedding except at* 0, *then, provided that* f' *is sufficiently* C^1*-close to* f,

$$\mathrm{L.Int}(f, 0) = \mathrm{Int}(f') \ge 0.$$

5. Proof of the theorems of §1

(5.1) *Proof of Theorem* 1.1. Let C and C' be distinct J-holomorphic curves which intersect at the point $x \in V$. If x is an isolated point of intersection, then it makes sense to talk about the contribution of x to the intersection number $C \cdot C'$; for this is just the number of intersection points of $\overline{N}$ with N' (counted with multiplicities), where N and N' are little compact neighbourhoods of x in C and C' respectively which intersect only at x, and where $\overline{N}$ is a small perturbation of N which meets N' transversally.

We will first show that, if x is isolated, its contribution k_x to $C \cdot C'$ is positive, and is > 1 unless it is a transverse intersection point. In fact, it is clear that if the curves do intersect transversally at x, then $k_x = +1$ since the orientation provided by J is compatible with that on V. Therefore, we only have to consider what happens at singular points or points of tangency. Since the question is local, the only singularities which concern us are the critical points, i.e., the points where $df = 0$. Observe also that, by Proposition 2.5, every J-holomorphic curve does have a tangent space even at a critical point; in fact, in the coordinates of Proposition 2.6, it is given by $\{v = 0\}$. Therefore we may consider cases as follows.

Case (i). *C' is nonsingular at x, and C is not tangent to C'.* In this case, we may identify (V, x) with $(\mathbb{C}^2, (0, 0))$ as in Lemma 2.5 in such a way that $J = J_0$ at $(0, 0)$, C' is the curve $z \mapsto (0, z)$, and C is tangent to the axis $\{v = 0\}$. Thus, C may be parametrized near x by a map $f: D \to \mathbb{C}^2$ of the form

$$f(z) = (z^k + O(k+1), O(r)),$$

where $r > k$. Let $f_t(z) = t(z^k, 0) + (1-t)f(z)$ for $0 \le t \le 1$. By restricting D, we may assume that, for each $t \in [0, 1]$, the only intersection point of $\operatorname{Im} f_t$ with C' occurs when $z = 0$. Thus the multiplicity of x in $C \cdot C'$ equals that of the intersection of $\operatorname{Im} f_1$ with C', which is k.

Case (ii). *C' is nonsingular at x, and C is tangent to C'.* As in Proposition 2.6, we may choose coordinates so that C' is the axis $\{u = 0\}$, and C is parametrized by a map $z \mapsto (z^r, z^k) + O(r+1)$, where $r > k$. Thus the multiplicity is r. Note that, even if C is nonsingular, $r > 1$.

Case (iii). *Both curves are singular, but they do not have the same tangent.* In this case, we may choose them to be tangent to the coordinate axes so that they have parametrizations

$$z \mapsto (O(r), z^k + O(k+1)) \quad \text{and} \quad z \mapsto (z^p + O(p+1), O(q)),$$

where $r > k$ and $q > p$. As before, these maps may be homotoped to the maps $z \mapsto (0, z^k)$ and $z \mapsto (z^p, 0)$ without creating any new intersections. Thus the multiplicity is kp.

Case (iv). *The curves are singular and have the same tangent.* Parametrize a neighbourhood of x in C and C' by maps f and $f' : (D, 0) \to (V, x)$ which are embeddings on $D - \{0\}$, and choose $R > 0$ so that f may be approximated by the immersion $f_\varepsilon: D(R) \subset D \to V$ as in Proposition 3.1. Since x is an isolated point of intersection, we may assume that the curves $\operatorname{Im}(f|D(R))$ and $\operatorname{Im}(f'|D(R))$ meet only at x. If $|\varepsilon| > 0$ is sufficiently small, k_x is clearly given by the intersection number $C_\varepsilon \cdot C'$, where $C_\varepsilon = \operatorname{Im} f_\varepsilon$. Since C_ε is immersed and goes through the singular point x of C' it follows from (i) and (ii) above that the point x contributes ≥ 2 to $C_\varepsilon \cdot C'$. Further, by (i) and (ii), any other points of intersection of C_ε with C' contribute a positive number to $C_\varepsilon \cdot C'$. Hence $C_\varepsilon \cdot C' \ge 2$ as required.

Now suppose that there is an intersection point x which is not isolated. By Lemma 2.7, the curves must be singular there, and clearly they must also be tangent to each other. Suppose that $C \cdot C' = n$. Choose the number R in (iv) above to be so small that C and C' have $> n$ intersections outside the neighbourhoods $f(D(R))$ and $f'(D(R))$. Then let $\overline{C}$ be the closed immersed curve obtained by joining $C - f(D(R))$ to $f_\varepsilon(D(R'))$ as in Lemma 4.3, where R' is slightly smaller than R. Clearly, we may assume that the only intersection points of $\overline{C}$ with C' occur at points of $\overline{C}$ which are J-holomorphic, i.e., either in $C - f(D(R))$ or in $f_\varepsilon(D(R'))$. Then, by Lemma 2.7, there are only finitely many such points, and by the results above

$$\overline{C} \cdot C' \geq [C - f(D(R))] \cdot C' > n = C \cdot C'.$$

But $\overline{C} \cdot C' = C \cdot C'$ because $\overline{C}$ is homotopic to C. Therefore, this situation cannot occur.

(5.2) *Proof of Proposition* 1.2. Perturb f in a neighbourhood $B_x(\varepsilon)$ of each of its critical points x to a J-holomorphic immersion $f_{\varepsilon, x}$. Then obtain f' by patching these maps $f_{\varepsilon, x}$ to f by the technique of Lemma 4.3. Clearly f' is J-holomorphic except in the spherical shells $B_x(\varepsilon_2) - B_x(\varepsilon_1)$ where the patching takes place. It also is an immersion by construction. q.e.d.

Let us now consider self-intersections. First we must control the singularities of a single curve.

Lemma 5.3. *A J-holomorphic curve $f: D \to V$ has only finitely many points of self-intersection. In particular, if f has a critical point at $\{0\}$, f restricts to an embedding on some deleted neighbourhood $D' - \{0\}$ of $\{0\}$.*

Proof. By Theorem 1.1, it suffices to show that there cannot be distinct sequences $\{z_i\}$, $\{z_i'\}$ in D which converge to 0 and are such that $f(z_i) = f(z_i')$ for all i. If these exist, let D'' be a neighbourhood of $\{0\}$ which does not contain z_i or z_i' for $i = 1$ to $k+1$, where $k = \mathrm{Int}(f) < \infty$. By Proposition 3.1, there is $R > 0$ such that we may approximate f on $D(R)$ as closely as we want in the C^1-topology. Therefore, if we choose $R_1 < R_2$ so that $D(R_2) \subset D''$ and so that none of the z_j, z_j' lie in the annulus $D(R_2) - D(R_1)$, we may patch $f|D - D(R_2)$ to an immersion f_ε of $D(R_1)$. Then, by arguing as in the last paragraph of (5.1), we find:

$$k = \mathrm{Int}(f) \geq k + 1 + \mathrm{Int}(f_\varepsilon | D(R_1)) \geq k + 1,$$

since $\mathrm{Int}(f_\varepsilon | D(R_1)) \geq 0$. A contradiction. q.e.d.

It follows that the local self-intersection number $\mathrm{L.Int}(f, 0)$ can be defined as in Definition 4.1 for all f. We now prove Theorem 1.4 which states that, if f has a critical point at $\{0\}$, L.Int $(f, 0) > 0$. As in (5.1), we can deal with some cases without using the results of §3.

Lemma 5.4. *Suppose that* f *has the form*

$$f(z) = (z^k, z^r) + O(r+1),$$

where $r > k > 1$, *and* k *and* r *are mutually prime. Then* $\mathrm{L.Int}(f, 0) = (k-1)(r-1)$. *In particular, for any* f *which has a singularity of prime order* k, $\mathrm{L.Int}(f, 0) > 1$.

Proof. If k and r are mutually prime, the map f' defined by $f''(z) = (z^k, z^r)$ is an embedding except at $z = 0$. Further, by restricting f and f'' to a suitably small disc $D(R)$, one can apply the arguments of (4.3) and (4.4). Hence

$$\mathrm{L.Int}(f, 0) = \mathrm{Int}(f''|D(R)) = \mathrm{L.Int}(f'', 0).$$

Clearly, $\mathrm{L.Int}(f'', 0) = \mathrm{Int}(g_\varepsilon)$, where $g_\varepsilon(z) = (z^k + \varepsilon z, z^r)$ for small ε. The result now follows because g_ε has $(k-1)(r-1)$ transverse self-intersections when $\varepsilon \neq 0$.

(5.5) *Proof of Theorem* 1.4. Put f in the normal form of Proposition 2.6:

$$f(z) = (z^k, z^r) + O(r+1),$$

where $k < r$, and k does not divide r. If k and r are mutually prime, we may apply Lemma 5.4. Otherwise, for every $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \leq 0.1$, let $f_\varepsilon: D(R) \to V$ be the partial desingularization of f whose existence is asserted by Proposition 3.9. By Corollary 4.4, we have

$$\mathrm{L.Int}(f, 0) = \mathrm{Int}(f_\varepsilon|D(R)),$$

where $|\varepsilon|$ is sufficiently small. Further, Lemma 5.5 applies to f_ε and implies that $\mathrm{L.Int}(f_\varepsilon, 0) > 1$. (In fact, $\mathrm{L.Int}(f_\varepsilon, 0) = k - 1$ if k is odd and is $> k - 1$ otherwise. Note that our hypotheses on the pair k, r imply that k itself is at least 4.) But because f_ε is an immersion on $D(R) - \{0\}$, it follows from Lemma 4.2(i) that

$$\mathrm{Int}(f_\varepsilon|D(R)) \geq \mathrm{L.Int}(f_\varepsilon, 0).$$

Hence $\mathrm{L.Int}(f, 0) > 0$ as required. (In fact, we have shown that $\mathrm{L.Int}(f, 0) > 1$.)

(5.6) *Proof of Theorem* 1.3. If C is immersed, this follows as in Lemma 4.2(i). Otherwise, we may replace a neighbourhood of each singular point x of C by the image $\mathrm{Im}\, f_{\varepsilon,x}$ of an immersion $f_{\varepsilon,x}: D \to V$ as in (5.2). Then $\mathrm{Int}(f_{\varepsilon,x}) = \mathrm{L.Int}(f, 0)$ so that $f_{\varepsilon,x}$ is *not* an embedding.

Thus, C is homotopic to an immersed J'-holomorphic curve C' which is not embedded. Hence $g(C) > g_\Sigma$.

References

[1] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of the second order*, J. Math. Pure Appl. **36** (1957) 235–249.

[2] W. M. Boothby, S. Kobayashi & H. C. Wang, *A note on mappings and automorphisms of almost complex manifolds*, Ann. of Math. (2) **77** (1963) 329–334.

[3] M. Gromov, *Pseudo-holomorphic curves on almost complex manifolds*, Invent. Math. **82** (1985) 307–347.

[4] D. McDuff, *Examples of symplectic structures*, Invent. Math. **89** (1987) 13–36.

[5] ——, *Rational and ruled symplectic 4-manifolds*, J. Amer. Math. Soc. **3** (1990) 679–712.

[6] A. Nijenhuis & W. Woolf, *Some integration problems in almost-complex and complex manifolds*, Ann. of Math. (2) **77** (1963) 424–489.

State University of New York, Stony Brook