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SURFACES AND BRANCHED SURFACES TRANSVERSE TO PSEUDO-ANOSOV FLOWS ON 3-MANIFOLDS

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Abstract

Given a circular pseudo-Anosov flow φ on an irreducible, atoroidal 3-manifold M, we classify all closed surfaces in M which are transverse and "almost transverse" to φ, generalizing the Schwartzmann-Fried classification of cross-sections to φ. In particular, there exists an "almost transverse" surface representing any class in $H_2(M; \mathbf{Z})$ which is nonnegative on all homology directions of φ. As an application, if σ is a fibered face of the unit ball of Thurston's polyhedral norm on $H_2(M; \mathbf{R})$, we give conditions under which Oertel's conjecture can be verified, that there exists a single taut branched surface in M carrying norm-minimizing representatives of every class in $\mathrm{Cone}(\sigma)$, and in particular carrying fiber representatives of every class in $\mathrm{int}(\mathrm{Cone}(\sigma))$.

0. Introduction

The study of fibrations of 3-dimensional manifolds over the circle gained great impetus with the introduction in [12] of Thurston's norm on the homology and cohomology of a 3-manifold. The norm x on $H_2(M; \mathbf{R})$ is defined in the following manner. Given $\alpha \in H_2(M; \mathbf{Z}) \subset H_2(M; \mathbf{R})$, $x(\alpha)$ is defined as the infimum, over all embedded surfaces A representing α, of

$$\chi_-(A) = -\chi(A - \text{spherical components of } A).$$

x is then extended by homogeneity and continuity to all of $H_2(M; \mathbf{R})$. In general, x is only a seminorm, but if M has no nonseparating spheres or tori, and in particular when M is irreducible and atoroidal, then x is a norm. Thurston showed that the unit ball $B_x = B_x(M)$ of x is always a polyhedron with integrally defined faces. Moreover, there is a certain collection of top-dimensional faces of B_x, called the fibered faces, such that a class $\alpha \in H_2(M; \mathbf{Z}) \subset H_2(M; \mathbf{R})$ is represented by a fiber of some

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fibration $M \to S^1$ if and only if $\alpha \in \operatorname{int}(\operatorname{Cone}(\sigma))$ for some fibered face σ of B_x.

A parallel development in 3-manifolds was the study of branched surfaces, which are used to systematize and classify all incompressible surfaces in a 3-manifold (see [13] and [8]). In [9], Oertel considered the problem of applying branched surface theory to the study of *norm-minimizing surfaces*, i.e. surfaces which realize the Thurston norm in their homology class. A *taut* branched surface is a transversely oriented branched surface Σ which carries some norm-minimizing surface with positive weights on every sector. Oertel showed that every surface carried by a taut Σ is norm-minimizing. Oertel also showed that there exist finitely many taut branched surfaces $\Sigma_1, \cdots, \Sigma_n$ such that any norm-minimizing surface is carried by one of $\Sigma_1, \cdots, \Sigma_n$. Also, the homology classes carried by a single taut branched surface all lie in a single face of B_x. Along these lines, one question of particular interest asked by Oertel is whether, given a face σ of B_x, there exists a single taut branched surface Σ carrying all norm-minimizing representatives of every class in $\operatorname{Cone}(\sigma)$. A recent counterexample of Sterba-Boatwright [11] shows that this is false in general, and Sterba-Boatwright's counterexample can be modified to produce a fibered face σ.

A weaker question asked by Oertel is whether a taut branched surface exists carrying *some* norm-minimizing representative of every class in $\operatorname{Cone}(\sigma)$. We shall show how, under certain conditions, this conjecture can be verified. In order to state our theorem, we recall some results due to David Fried.

Fried showed in [3] that the fibered faces of Thurston's norm on an irreducible atoroidal 3-manifold M can be very neatly related to certain nonsingular flows on M. In particular, he showed that to each fibered face σ, there is a naturally associated nonsingular flow φ with the property that for every class $\alpha \in H_2(M; \mathbf{Z})$, $\alpha \in \operatorname{int}(\operatorname{Cone}(\sigma))$ if and only if α is represented by a cross-section to φ, in which case the first return map of φ to S is pseudo-Anosov. The flow φ is uniquely determined by this property, up to isotopy and reparametrization. φ is called the *pseudo-Anosov* flow on M corresponding to σ. Fried also showed that σ is dual to the collection of periodic orbits of φ, in the following sense. Let $\operatorname{Cone}(D_\varphi) \subset H_1(M; \mathbf{R})$ denote the smallest convex closed cone containing the homology class of every periodic orbit of φ. Fried showed that for each $\alpha \in H_2(M; \mathbf{R})$, $\alpha \in \operatorname{Cone}(\sigma)$ if and only if $\langle \alpha, c \rangle \geq 0$ for every $c \in \operatorname{Cone}(D_\varphi)$, where $\langle \cdot, \cdot \rangle$ denotes the intersection pairing. Equivalently, $\alpha \in \operatorname{Cone}(\sigma)$ if and only if $\langle \alpha, \gamma \rangle \geq 0$ for every periodic orbit γ of

φ. Thus, the cones $\mathrm{Cone}(\sigma)$ and $\mathrm{Cone}(D_\varphi)$ are dual cones under the intersection pairing. The set $\mathrm{Cone}(D_\varphi)$ occurs in Fried's work under the guise of homology directions of φ. Homology directions are used by Fried [4] to formulate a general theory of cross-sections to flows; this theory was a rediscovery of work due to Schwartzmann [10], who studied cross-sections using asymptotic cycles.

Since φ is the suspension flow of a pseudo-Anosov homeomorphism $f: S \to S$, by suspending the singular periodic orbits of f we obtain a finite collection of singular periodic orbits of φ. The singular orbits of φ can cause trouble in certain situations. For instance, we have evidence that Oertel's conjecture fails in general, because of the behavior of singular orbits (the construction of counter-examples is beyond the methods of this paper). However, by making a simple additional assumption about φ, we can prove Oertel's conjecture.

Branched Surface Theorem. *Let M be an irreducible, atoroidal 3-manifold, σ a fibered face of $B_x(M)$, and φ the pseudo-Anosov flow associated to σ. Suppose that for each singular periodic orbit γ of φ, $[\gamma] \in \mathrm{int}(\mathrm{Cone}(D_\varphi))$. Then there exists a taut branched surface $\Sigma \subset M$ carrying norm-minimizing representatives of every class in* $\mathrm{Cone}(\sigma)$.

The hypothesis of this theorem is equivalent to the statement that for every $\alpha \in \mathrm{Cone}(\alpha)$ and for each singular orbit γ of φ, $\langle \alpha, \gamma \rangle > 0$. Such examples of pseudo-Anosov flows can be easily constructed. For instance, suppose $f: S \to S$ is a pseudo-Anosov surface homeomorphism having a unique singular periodic orbit, and let φ be the suspension flow of f. Then φ has a unique singular periodic orbit γ, and the hypothesis can be easily verified. To see how, let $I(\gamma)$ denote the index of γ, given by $I(\gamma) = 1 - N(\gamma)/2$, where $N(\gamma)$ is the number of stable separatrices of each point in the singular orbit of f. If A is any cross-section to φ, an application of the Euler-Poincaré formula shows that $\chi(A) = \langle A, I(\gamma)\cdot\gamma \rangle$. Thus, the linear functional $\langle \cdot, -I(\gamma)\cdot\gamma \rangle$ restricted to $\mathrm{Cone}(\sigma)$ is equal to the restriction of the Thurston norm x to $\mathrm{Cone}(\sigma)$. Since x is positive on $\mathrm{Cone}(\sigma)$, and since $I(\gamma) < 0$, it follows that $\langle \alpha, \gamma \rangle > 0$ for all $\alpha \in \mathrm{Cone}(\sigma)$.

The method of our proof of the Branched Surface Theorem is to classify not only the cross-sections of the flow φ, which was already done by Fried, but to classify all surfaces transverse to the flow, up to isotopy along the flow. Here is a weakened version of our main transverse surface theorem, which is sufficient to prove the above branched surface theorem:

Weak transverse surface theorem. *Let M be an irreducible, atoroidal 3-manifold, σ a fibered face of $B_x(M)$, and φ the pseudo-Anosov flow* *associated to* σ. *Suppose that for each singular periodic orbit* γ *of* φ, $[\gamma] \in \mathrm{int}(\mathrm{Cone}(D_\varphi))$. *Then every integral class in* $\mathrm{Cone}(\sigma)$ *is represented by a surface which is transverse to* φ.

Using these transverse surfaces, it is then a simple matter to construct a branched surface Σ carrying representatives of each class in $\mathrm{Cone}(\sigma)$. Σ will also be transverse to the flow, and tautness easily follows, proving the Branched Surface Theorem. This is carried out in §1.

If transverse surfaces existed without the assumption on singular periodic orbits, then the branched surface theorem would also be true without this assumption. Unfortunately, singular orbits provide obstructions to the existence of transverse surfaces—there are counter-examples which show that the transverse surface theorem is false in this generality. However, after "perturbing" the singular periodic orbits of φ in a certain manner, transverse surfaces can be constructed.

In §1 we shall define a *dynamic blowup* of a singular periodic orbit γ. Roughly speaking, this means that γ is pulled apart into a union of annuli, to produce a new flow $\varphi^\#$ which is identical to φ except for the new annuli. Each new annulus R is preserved by $\varphi^\#$, and the flow lines on R spiral from one boundary component of R to the other. Given a singular periodic orbit γ, there are several ways to dynamically blow up γ, depending on how γ is pulled apart.

Given $\alpha \in H_2(M; \mathbf{R})$ and a periodic orbit γ of φ, we say that γ is α-*null* if $\langle \alpha, \gamma \rangle = 0$. Given a surface $A \subset M$ representing α, we say that A is *almost transverse* to φ if there is a way to dynamically blow up each α-null singular orbit of φ, so that if $\varphi^\#$ is the resulting flow, then up to isotopy A is transverse to $\varphi^\#$. Here is our main result:

Transverse Surface Theorem. *Let* M *be an irreducible, atoroidal* 3-*manifold,* σ *a fibered face of* $B_x(M)$, φ *the pseudo-Anosov flow associated to* σ. *Then every integral class in* $\mathrm{Cone}(\sigma)$ *is represented by a surface which is almost transverse to* φ.

Comment. Under the hypotheses of the weak branched surface theorem, it follows that there are no α-null singular orbits. In particular, the blown up flow $\varphi^\#$ is identical to φ. The weak transverse surface theorem follows immediately.

It seems likely that Sterba-Boatwright's counterexample can be souped up to disprove Oertel's weaker conjecture. Singular orbits on the boundary of $\mathrm{Cone}(D_\varphi)$ may prove to be an obstruction to this conjecture. However, the Transverse Surface Theorem suggests that φ itself can be viewed as a unifying object for understanding norm-minimizing surfaces whose classes are in $\mathrm{Cone}(\sigma)$. More recent results of the author show that, in fact, for every norm-minimizing surface A with $[A] \in \text{Cone}(\sigma)$, A is almost transverse to φ. If each singular orbit γ of φ satisfies $[\gamma] \in \text{int}(\text{Cone}(D_\varphi))$, then each such surface A is transverse to φ.

The Transverse Surface Theorem is really a result in dynamics. The proof, given in §2, relies first of all on a dynamical analysis of the lifted flow $\tilde{\varphi}^\#$ on a certain $\mathbf{Z}$-covering space $\tilde{M}$ of M. This analysis is the content of the $\mathbf{Z}$-Spectral Decomposition Theorem, the main result of the companion paper ([5] and [17]). The one place in the present paper where we must allow α-null singular orbits to be blown up is in the citation of the $\mathbf{Z}$-Spectral Decomposition Theorem. Next, using Conley's construction of Lyapounov functions [1],.we prove the existence of transverse surfaces to the lifted flow $\tilde{\varphi}^\#$. Finally, using a combinatorial argument, we show that a transverse surface to $\tilde{\varphi}^\#$ can be chosen so as to project to the desired transverse surface to $\tilde{\varphi}^\#$ in the class α. The combinatorial argument also yields Theorem 2.11, which classifies all transverse surfaces to $\varphi''^\#$, up to isotopy along flow lines. In addition, we use the combinatorial analysis to show in §3 the existence of a Lyapounov cocycle representing the cohomology class Poincaré dual to α.

The Schwartzmann-Fried theory of cross-sections actually applies to an arbitrary flow φ on a compact manifold M^n, and even to an arbitrary closed invariant set I of φ. In the language of Fried, a flow φ has a cross-section to I in a class $\alpha \in H_{n-1}(M; \mathbf{Z})$ if and only if α is positive on $\text{Cone}(D(\varphi; I))$, the cone of all homology directions of φ on I. As a consequence, if $\text{Cone}(D(\varphi; I))$ is contained in an open half-space of $H_1(M; \mathbf{R})$, then I is a *circular* invariant set, i.e., it possesses some cross-section. Our transverse surface theorem can be restated in this language: when $n = 3$ and φ is a circular pseudo-Anosov flow, φ has an almost transverse surface in a class $\alpha \in H_2(M; \mathbf{Z})$ if and only if α is nonnegative on $\text{Cone}(D_\varphi)$. In §4, we show how the Transverse Surface Theorem and the Branched Surface Theorem can sometimes be generalized along these lines when I is a basic set of an Axiom A flow. In particular, we shall show that these theorems, properly interpreted, apply when I is 1-dimensional. We shall also mention under what general conditions these theorems apply to an arbitrary basic set. Since Axiom A flows do not have singular orbits, the problems such orbits engender do not occur, but other problems do arise.

As our main interest is in application to the topology of 3-manifolds, we shall for the most part concentrate entirely on the case of pseudo-Anosov flows on 3-manifolds.

The paper is organized as follows.

In §1, we state the Transverse Surface Theorem, and use it to prove the Branched Surface Theorem.

In §2, the main section of the paper, we state the main result from the companion paper [5], the **Z**-Spectral Decomposition Theorem, and use it to prove the Transverse-Surface Theorem.

In §3, we prove a theorem about the existence of "Lyapounov cocycles", under the same hypotheses as the Transverse Surface Theorem.

In §4, we consider generalizations of the Transverse Surface Theorem and the Branched Surface Theorem to the setting of basic sets of Axiom A flows.

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1. Statement of the Transverse Surface Theorem, and application to the Branched Surface Theorem

Let M be an oriented, irreducible, atoroidal 3-manifold. We consider $H_2(M;\mathbf{Z})$ to be embedded in the natural way as a lattice in $H_2(M;\mathbf{R})$. The Thurston norm x on $H_2(M;\mathbf{R})$ is defined as follows. For each $\alpha \in H_2(M;\mathbf{Z})$, $x(\alpha)$ is the minimum, over all embedded oriented surfaces $A \subset M$ representing α, of $\chi_-(A) = -\chi(A - \text{spherical components})$. An oriented surface $A \subset M$ is said to be *norm-minimizing* if A contains no spheres or tori and $x([A]) = \chi_-(A)$. Thurston showed in [12] that x extends to a norm on $H_2(M;\mathbf{R})$. Moreover, he proved the following theorem describing the structure of the unit ball B_x of x:

1.1 Theorem (*Thurston*). *The unit ball B_x of x is a polyhedron in $H_2(M;\mathbf{R})$ with integrally defined faces; more specifically, x is the supremum of finitely many integrally defined linear functionals on $H_2(M;\mathbf{R})$. Moreover, there is a specific set of top-dimensional faces of B_x, called fibered faces, with the property that a class $\alpha \in H_2(M;\mathbf{Z})$ is represented by a fiber of some fibration $M \to S^1$ if and only if $\alpha \in \operatorname{int}(\operatorname{Cone}(\sigma))$ for some fibered face σ of B_x. Also, if α is represented by a fiber A, then every norm-minimizing surface representing α is isotopic to A.*

Fried has shown that there is a connection between the fibered faces of B_x and certain nonsingular flows on M. Recall that a *cross-section* to a nonsingular flow φ on M is a surface A transverse to φ, which intersects every flow line. Every flow line which leaves A must return to A, and so there is a well-defined *first return map* on A.

1.2 Theorem (*Fried*). *There is a natural way to associate, to each fibered face σ of B_x, a nonsingular flow φ on M with the following property. For each class $\alpha \in H_2(M;\mathbf{R})$, α is represented by a cross-section A to φ if and only if $\alpha \in \operatorname{int}(\operatorname{Cone}(\sigma))$, in which case the first return map on A is pseudo-Anosov. φ is uniquely characterized by this property, up to isotopy and reparametrization. φ is called a pseudo-Anosov flow associated to σ. Moreover, if $\alpha \in \operatorname{int}(\operatorname{Cone}(\sigma))$, and A is any surface representing α and transverse to φ, then A is a cross-section to φ, and A is unique up to isotopy along flow lines.*

Consider a fibered face σ of B_x, and let φ be the pseudo-Anosov flow associated to σ. Let $T\varphi$ be the oriented tangent line bundle to φ, let E be the normal plane bundle to $T\varphi$, and let $\chi_E \in H^2(M;\mathbf{Z})$ be the Euler class of E. By the universal coefficients theorem, we can regard χ_E as a linear functional on $H_2(M;\mathbf{Z})$. If A is any cross-section to φ, the restriction of E to A is isomorphic to the tangent bundle of A. It follows that $\langle -\chi_E, A\rangle = -\chi(A) = x(A)$. Since there is a cross-section representing every lattice point in $\mathrm{int}(\mathrm{Cone}(\sigma))$, it follows that $-\chi_E$ and x agree on $\mathrm{Cone}(\sigma)$.

Consider an oriented surface $A \subset M$. Suppose that there is an oriented tangent line bundle V on M homotopic to $T\varphi$, such that up to isotopy A is transverse to V, and $TA \oplus V$ is positively oriented in M. Then we say that A is *weakly transverse* to φ. If this is so, let N be the normal plane bundle to V, and notice that $E \approx TM/T\varphi \approx TM/V \approx N$, so $\chi_E = \chi_N$. Since M is irreducible and V is nonsingular, it follows that A has no sphere components. And by the transversality condition on A and V, it follows that $N \mid A$ is isomorphic as an oriented plane bundle to TA, so $\langle \chi_N, A\rangle = \chi(A)$. Letting $[A]$ denote the homology class of A, we therefore have $x[A] \leq -\chi(A) = \langle -\chi_E, A\rangle \leq x[A]$, where the first inequality follows by definition of x, and the second inequality follows by convexity of x. Thus, both inequalities are equalities, and we have proved:

1.3 Corollary. *Let $A \subset M$ be an oriented surface which is weakly transverse to φ. Then A is norm-minimizing, and $[A] \in \mathrm{Cone}(\sigma)$.*

To obtain the Transverse Surface Theorem, we shall have to allow weakly transverse surfaces, i.e., surfaces transverse to a flow $\varphi^{\#}$ whose tangent line bundle is homotopic to $T\varphi$. In general $\varphi^{\#}$ can be a very different flow than φ, from the point of view of dynamical behavior—in particular, $\varphi^{\#}$ need not be isotopic to φ. However, for purposes of the Transverse Surface Theorem only special deformations of φ will be allowed, hence $\varphi^{\#}$ will be an easily understood object. The deformed flow $\varphi^{\#}$ will be obtained by dynamically blowing up certain singular orbits of φ. The definition of dynamic blowups is taken from [7].

First we define dynamic blowups in the context of pseudo-Anosov maps. Let s be a singular fixed point of a pesudo-Anosov map f, and consider first the case where f does not rotate the separatrices. To obtain a dynamic blowup of s, replace s by a finite set of pseudo-Anosov fixed points which are connected in a tree pattern by invariant paths. Here is a more precise description. Let D be a coordinate disc centered on s. List the stable and unstable separatrices in circular order as $\{\ell_n \mid n \in \mathbf{Z}/2N\}$, where $N \geq 3$. Let $p_n = \ell_n \cap \partial D$. Choose an embedded tree $T = T_s \subset D$, such that T intersects ∂D transversely in the set $\{p_n\}$, and every interior vertex of T is of even valance ≥ 4. Let $\ell_n^{\#}$ be the edge of T incident on p_n, and let $T^\circ = \mathrm{cl}(T - \{\ell_n^{\#}\})$. With these conditions on T, the map f can be replaced by a map $f^{\#}$ which is semiconjugate to f, and by a semiconjugacy which collapses T° to the point s, so that $f^\#$ has a prong singularity at each interior vertex of T, f leaves T° invariant, and $f^\#$ acts as a translation on $\mathrm{int}(E)$ for each edge E of T°. We say that $f^\#$ is obtained by *dynamically blowing up* s. Notice that there is more than one way to dynamically blow up s—the dynamic blowup is determined by choosing the tree T, which can be done in finitely many ways up to isotopy.

When f rotates the separatrices at s through a fraction K/N of a complete rotation, a dynamic blowup is similarly defined with the additional proviso that T is invariant under a K/N rotation of D.

If γ is a singular periodic orbit of a pseudo-Anosov flow φ, a dynamic blowup of γ is defined as follows. Choose a local cross-section near γ, having a pseudo-Anosov singular fixed point s, and choose a dynamic blowup of s by picking a tree T as above. This can be suspended, to obtain a dynamic blowup of γ. The result is determined up to isotopy by the choice of T. The effect is to introduce several annuli, each of which is invariant under the blown-up flow $\varphi^\#$, one annulus for each orbit of edges of T° under the rotation action. There is a semiconjugacy taking $\varphi^\#$ to φ, which collapses each invariant annulus to the orbit γ. Notice that $T\varphi^\#$ is homotopic to $T\varphi$.

To state our theorem, consider a class $\alpha \in H_2(M; \mathbf{Z})$. Given an oriented surface A representing α, we say that A is *almost transverse* to φ if there is a way to dynamically blow up each α-null singular orbit of φ, so that if $\varphi^\#$ is the resulting flow, then up to isotopy A is transverse to $\varphi^\#$, and $TA \oplus T\varphi^\#$ is positively oriented in M. Notice that almost transversality implies weak transversality.

1.4 Transverse Surface Theorem. *Let M^3 be irreducible and atoroidal, σ a fibered face of $B_x(M)$, φ a pseudo-Anosov flow associated to σ. Given a class $\alpha \in H_2(M; \mathbf{Z})$, α is represented by a surface A almost transverse to φ if and only if $\alpha \in \mathrm{Cone}(\sigma)$.*

One direction of this theorem follows immediately from Corollary 1.3. The proof of the other direction will be given in the following section. For the moment, we make several comments on the Transverse Surface Theorem, and then we shall use it to prove the Branched Surface Theorem.

The theorem can be restated in the following manner. Given α, there exists a way to dynamically blow up each α-null singular orbit, so that if $\varphi^\#$ is the resulting flow, then α is represented by a surface A transverse to $\varphi^\#$. Since there are finitely many α-null singular orbits, and since each orbit can be blown up in finitely many ways, then there are finitely many choices for the blown up flow $\varphi^{\#}$. The choice of $\varphi^{\#}$ depends on α, in a way which is made explicit in the proof of the **Z**-spectral decomposition theorem (see [7]).

Since Fried's theorem covers the case when $\alpha \in \operatorname{int}(\operatorname{Cone}(\sigma))$, we shall only be interested in the case that $\alpha \in \partial(\operatorname{Cone}(\sigma))$. Because the faces of B_x are integrally defined, then for every subface $\sigma' \subset \sigma$, there exists some integral class $\alpha \in \operatorname{int}(\operatorname{Cone}(\sigma'))$. In particular, the hypotheses of the Transverse Surface Theorem are not vacuous.

Notice that by the easy half of Fried's theorem, if A is a transverse surface representing the class α, then A cannot be a section of φ when $\alpha \in \partial(\operatorname{Cone}(\sigma))$. For instance, Fried's theory in [4] guarantees the existence of a closed orbit c of φ such that $\langle \alpha, c \rangle = 0$, and A must miss any such c. The existence of such orbits is the primary difficulty in the construction of transverse surfaces. To deal with these orbits in an organized manner, we shall utilize a result from the companion paper ([5] and [7]), the **Z**-Spectral Decomposition Theorem, which is stated in the beginning of §2.

An interesting feature of the theory will be that the surfaces transverse to $\varphi^{\#}$ in a given class $\alpha \in \partial(\operatorname{Cone}(\sigma))$ are not unique up to isotopy. However, the collection of all transverse surfaces representing α can be precisely described, and classified up to isotopy along flow lines of $\varphi^{\#}$. For instance, we shall show that there are finitely many transverse surfaces representing α, up to isotopy along flow lines. This description is given in Theorem 2.11, Classification of Transverse Surfaces.

The motivation for the Transverse Surface Theorem comes from the following question, which was originally asked in a more general form by Oertel: Given a fibered face σ of $B_x(M)$, does there exist a single taut branched surface $\Sigma \subset M$ carrying every fiber whose class is in $\operatorname{Cone}(\sigma)$? For Σ to be taut means that it is transversely orientable, and it carries some norm-minimizing surface having positive weights on every branch. When this happens, Oertel proves that every surface carried by Σ is norm-minimizing. (See [8] or [9] for definitions of *branched surfaces* and *carrying*, and [9] for the proofs of the results mentioned here.)

Given a face σ of $B_x(M)$, suppose a taut branched surface Σ does exist carrying representatives of all integral classes in $\operatorname{int}(\operatorname{Cone}(\sigma))$. Since the set of rational homology classes carried by Σ is necessarily closed, Σ must carry norm-minimizing representatives of every integral point in $\partial(\operatorname{Cone}(\sigma))$ as well. Thus, one idea for the construction of Σ is to start with the vertices $\{p_1, \cdots, p_N\}$ of σ, take transversely oriented norm-

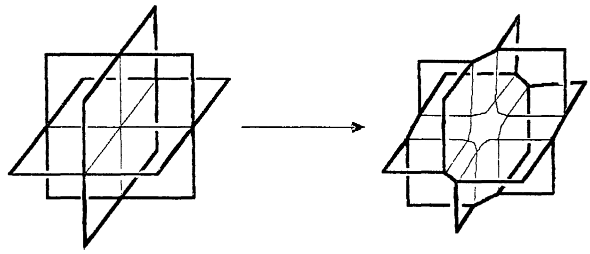

DIAGRAM 1. Smoothing the intersection locus of $\bigcup A_i$ to form a branched surface

minimizing surfaces $\{A_1, \cdots, A_n\}$ with $[A_i] \in \text{Cone}(p_i)$, perform isotopies so that the surfaces A_i are in general position, and then smooth the intersection locus to form a transversely oriented branched surface Σ carrying each A_i. Diagram 1 shows the local model for smoothing the intersection locus, in such a way that the transverse orientation on Σ agrees with the transverse orientations on each of the A_i. The diagram indicates three sheets of surfaces, intersecting in a triple point; the transverse orientations on the sheets are chosen so as to point into the octant of the observer. Using Proposition 3 of [9], together with the fact that the classes $[A_i]$ form a positive spanning set for $\text{Cone}(\sigma)$, it follows that Σ carries a representative of any integral homology class $\alpha \in \text{Cone}(\sigma)$.

However, this argument is incomplete. There is no guarantee that Σ is taut, so the surfaces carried by Σ need not be norm-minimizing. Moreover, the argument does not use anything special about the face σ, but Sterba-Boatwright [11] has constructed a manifold M and a face σ of $B_x(M)$ for which there is no taut branched surface carrying representatives of all classes in $\text{Cone}(\sigma)$.

Here is where flows come into play. Let σ be a fibered face of $B_x(M)$, and let φ be a pseudo-Anosov flow on M associated to σ, as given in Fried's Theorem 1.2. Suppose φ satisfies the hypothesis that for each singular periodic orbit γ of φ, $[\gamma] \in \text{int}(\text{Cone}(D_\varphi))$. Then for $\alpha \in \text{Cone}(\sigma)$, it follows that no singular periodic orbit is α-null. The Transverse Surface Theorem produces a surface A representing α which is almost transverse to φ, but since no singular orbit is α-null, it follows that A is transverse to φ, proving the Weak Transverse Surface Theorem. Let $\{p_1, \cdots, p_N\}$ be the vertices of σ, which are necessarily rational. By the Weak Transverse Surface Theorem, surfaces $\{A_1, \cdots, A_N\}$ transverse to φ can be chosen so that $[A_n] \in \text{Cone}(p_n)$. The surfaces A_n have a natural transverse orientation compatible with the direction of the flow. Now perform isotopies along flow lines of φ so that the surfaces A_n are in general position, and smooth the intersection locus of the collection $\{A_n\}$ as in the diagram. This creates a branched surface Σ transverse to the flow. It follows that any surface carried by Σ is transverse to the flow, and so is norm-minimizing by Corollary 1.3. Notice that Σ carries a surface in the class $[A_1] + \cdots + [A_N]$ with positive weights on every branch. Thus, Σ is taut. As noted in the previous paragraph, Σ carries a representative A of each homology class $\alpha \in \mathrm{int}(\mathrm{Cone}(\sigma))$, with A transverse to φ; since A is norm-minimizing, the final clause of Fried's Theorem 1.2 shows that A is a cross-section to φ.

This proves the Branched Surface Theorem stated in the introduction, which we restate here in more detail:

1.5 Branched Surface Theorem. *Let σ be a fibered face of $B_x(M)$ for an oriented, irreducible, atoroidal 3-manifold M. Suppose that for each singular periodic orbit of φ, $[\gamma] \in \mathrm{int}(\mathrm{Cone}(D_\varphi))$. Then there exists a taut branched surface $\Sigma \subset M$ carrying representatives of every integral point in $\mathrm{Cone}(\sigma)$. In particular, Σ carries fiber representatives of each integral point in $\mathrm{int}(\mathrm{Cone}(\sigma))$. Moreover, Σ is transverse to the pseudo-Anosov flow φ associated to σ.*

If one tries to generalize this theorem by eliminating the hypothesis on singular orbits, the following difficulty is encountered. For each vertex V of σ, the Transverse Surface Theorem produces a surface A transverse to a flow $\varphi^\#$ obtained by blowing up each V-null singular orbit in a certain way. Thus, different vertices yield surfaces transverse to different flows, and so the construction given above for Σ breaks down. If it is true that each singular orbit γ is in a face of $\mathrm{Cone}(D_\varphi)$ of codimension at most 1, then the proof can be patched up. In that case, for each singular orbit γ, there is at most one vertex V of σ such that γ is V-null, and we can blow up all of the singular orbits simultaneously to obtain a flow $\varphi^\#$ such that each vertex V is represented by a surface transverse to $\varphi^\#$. The real trouble comes when some singular orbit γ is contained in face of $\mathrm{Cone}(D_\varphi)$ of codimension ≥ 2, in which case γ is V-null for two or more vertices V of σ. In this case, the dynamic blowups of γ required by the Transverse Surface Theorem might be different for different vertices, and the proof breaks down completely.

Our current suspicion is that Sterba-Boatwright's construction can be souped up to produce a fibered face σ for which even the weak version of Oertel's conjecture is false.

2. Proof of the Transverse Surface Theorem

The proof of this theorem can be motivated by examining the methods developed by Schwartzmann [10] and, later, Fried [4], in which a general classification theory for cross-sections to a flow φ is developed. Fried's version of the theory is stated in terms of *homology directions* of φ; we shall adopt this method. Homology directions are elements of the projective homology space $H_1(M; \mathbf{R})/\mathbf{R}^+$, topologized as the disjoint union of a sphere and a point. The set of homology directions for φ, denoted D_φ, is defined to be those elements of $H_1(M; \mathbf{R})/\mathbf{R}^+$ which are approximated by long, almost closed flow segments of φ. For example, every closed orbit defines a homology direction. When φ is a transitive Markov flow, e.g., when φ is a circular pseudo-Anosov flow, then the inverse image of D_φ under the projection $H_1(M; \mathbf{R}) \to H_1(M; \mathbf{R})/\mathbf{R}^+$, denoted $\mathrm{Cone}(D_\varphi)$, is the smallest convex closed cone containing the homology class of every periodic orbit of φ, minus the origin if there are no homologically trivial periodic orbits. This is more or less consistent with the notation $\mathrm{Cone}(D_\varphi)$ used in the introduction. Given a class $\alpha \in H_2(M; \mathbf{Z})$ and a homology direction d, although the intersection number of α and d cannot be defined, the intersection sign $\alpha(d) \in \{-1, 0, +1\}$ is well defined. Fried proves that a class $\alpha \in H_2(M; \mathbf{Z})$ is represented by a cross-section to φ if and only if $\alpha(d) > 0$ for every $d \in D_\varphi$. To do this, he lifts φ to a flow $\tilde{\varphi}$ on the $\mathbf{Z}$-cover $\tilde{M} \to M$ associated to the Poincaré dual of α in $H^1(M; \mathbf{Z})$. $\tilde{M}$ is a noncompact manifold with two ends $-\infty$ and $+\infty$, and Fried uses the fact that α is positive on homology directions to prove that every orbit of $\tilde{\varphi}$ goes from $-\infty$ in negative time to $+\infty$ in positive time. From this, it follows by elementary topological considerations that $\tilde{M}$ is homeomorphic to $S \times \mathbf{R}$ for some compact surface S, and the flow $\tilde{\varphi}$ is equivalent to the flow in the $\mathbf{R}$ direction. Then one carefully chooses a cross-section to $\tilde{\varphi}$ which is disjoint from all its covering translates, and which therefore projects down to a cross-section to φ in the class α.

In order to prove the Transverse Surface Theorem, we shall start by trying to mimic as much of Fried's proof as possible. Given an integral homology class $\alpha \in \partial(\mathrm{Cone}(\sigma))$, let $\tilde{M} \to M$ be the $\mathbf{Z}$-cover associated to the Poincaré dual of α, and let $\tilde{\varphi}$ be the lifted flow of φ. $\tilde{M}$ is still a noncompact manifold with two ends $-\infty$ and $+\infty$. Unfortunately, flow lines of $\tilde{\varphi}$ no longer go from $-\infty$ in negative time to $+\infty$ in positive time. The trouble is that there exist homology directions having zero intersection number with α; any closed orbit of φ representing such a homology direction lifts to a closed orbit of $\tilde{\varphi}$.

So to prove the Transverse Surface Theorem, we shall require a precise analysis of the qualitative dynamics of the lifted flow $\tilde{\varphi}$ on $\tilde{M}$, i.e., an analysis of the asymptotic behavior of orbits of $\tilde{\varphi}$. This analysis is contained in the **Z**-Spectral Decomposition Theorem stated below, which is taken from the companion paper [5] and from [7]. To state the theorem, first we need to recall some generalities about flows, which are taken from Conley's book [1].

Given any flow φ, we shall use the notation $x \cdot t$ as a shorthand for $\varphi_t(x)$. Let a fixed flow φ on a metric space N be given, with metric d. For our main application, N will be the space $\tilde{M}$ and d will be any **Z**-equivariant metric. Given a set $X \subset N$, the set $L_+(X)$, known in the literature as the ω-limit set of X, is the set of all limit points of sequences of the form $x_i \cdot t_i$, where $x_i \in X$ and $t_i \to +\infty$ as $i \to +\infty$. The set $L_-(X)$, known as the α-limit set, is similarly defined by letting $t_i \to -\infty$ as $i \to +\infty$. An *attractor* is any closed invariant set $A \subset N$ such that, for some neighborhood U of A, $L_+(U) = A$. A *repeller* is similarly defined by the condition that $L_-(U) = A$. Given $\varepsilon > 0$ and $T > 0$, an ε, T *chain* from x to x' is a pair of sequences of the form $(x = x_0, x_1, \cdots, x_n = x'; t_1, \cdots, t_n)$ such that for $1 \leq i \leq n$, $t_i > T$, and $d(x_{i-1} \cdot t_i, x_i) < \varepsilon$. If $x = x'$, this is called an ε, T *cycle* through x. Given $X \subset N$, $R_+(X)$, the forward chain limit set of X, is defined as the set of all points y such that for all ε, $T > 0$ there exists $x \in X$ and an ε, T chain from x to y. $R_-(X)$ is similarly defined by taking chains ending at points of X. The chain recurrent set R of φ is the set of all points $x \in N$ such that $x \in R_+(x)$, i.e., there exists an ε, T cycle through x for all ε, $T > 0$. It is a fact that the restriction of φ to R is a chain recurrent flow, i.e. the chain recurrent set of $\varphi|R$ is all of R. A closed, invariant set $C \subset R$ is *chain connected* or *chain transitive* if for any x, $y \in C$ and any ε, $T > 0$, there exists an ε, T chain from x to y. C is called a *chain component* of R if it is a maximal chain connected set. It is elementary to prove that the chain components form a partition of R into closed sets.

Here is the main theorem about the orbit structure of $\tilde{\varphi}$, taken from the companion papers [5] and [7]:

2.1 Z-Spectral Decomposition Theorem. *Let M be an oriented, irreducible, atoroidal 3-manifold, σ a fibered face of $B_x(M)$, φ a pseudo-Anosov flow on M associated to σ, $\alpha \in H_2(M; \mathbf{Z}) \cap \partial(\mathrm{Cone}(\sigma))$ a primitive element of $H_2(M; \mathbf{Z})$. There is a way to dynamically blow up each α-null singular orbit of φ, so that if $\varphi^\#$ is the resulting flow, then the* *following conditions hold. Let* $\tilde{M} \to M$ *be the* **Z***-cover associated to* α, *and* $\tilde{\varphi}^{\#}$ *the flow on* $\tilde{M}$ *which lifts* $\varphi^{\#}$. *Let* $R = R(\tilde{\varphi}^{\#})$ *be the chain recurrent set of* $\tilde{\varphi}^{\#}$. *Then the following hold:*

(A) *Each chain component of* R *is compact.*

(B) *There are finitely many orbits of chain components of* R *under the action of* **Z**.

(C) *For any point* $x \in \tilde{M} - R$, *either* $L_+(x) = \{+\infty\}$ *or* $L_+(x)$ *is contained in some chain component of* R.

(D) *Similarly, for any* $x \in \tilde{M} - R$, *either* $L_-(x) = \{-\infty\}$ *or* $L_-(x)$ *is contained in a chain component of* R.

(E) *If* $L_+(x) \neq \{+\infty\}$, *then there exists a neighborhood* U *of* $+\infty$ *such that for any chain component* C *of* R, *if* $C \subset U$ *then* $C \subset R_+(x)$; *a similar statement holds if* $L_-(x) \neq \{-\infty\}$.

(F) *Let* $T\colon \tilde{M} \to \tilde{M}$ *generate the* **Z***-action on* $\tilde{M}$, *so that* T *moves points towards* $+\infty$. *Let* $p\colon \tilde{M} \to \mathbf{R}$ *be a continuous map such that* $p(T(x)) = p(x) + 1$. *Then there is a constant* K *such that for any* $x \in \tilde{M}$ *and any* $t \geq 0$, $p(x \cdot t) \geq p(x) - K$.

The name of the theorem suggests an analogy with Smale's Spectral Decomposition Theorem, which says in part that the chain recurrent set of an Axiom A flow with no cycles consists of finitely many compact, chain transitive components. Properties (A) and (B) of the **Z**-Spectral Decomposition Theorem say that in the presence of a free, properly discontinuous, cocompact **Z**-action, if the flow is equivariant with respect to **Z**, then the same conclusions hold as in Smale's theorem, up to the action of **Z**.

Theorem 2.1 as stated above differs in two respects from the original **Z**-spectral decomposition theorem of [5]. First of all, the result stated in [5] contains an error, in that it does not require blowing up α-null singular orbits; there are counterexamples which show that the theorem fails without this requirement. This error was corrected in [7].

Second of all, property (F) in the conclusion of Theorem 2.1 does not appear in either [5] or [7]. But the proof can be easily supplied by mimicking an argument of Fried, taken from Theorem H of [4]. Suppose there are sequences $x_i \in \tilde{M}$ and $t_i \geq 0$, such that if we set $y_i = x_i \cdot t_i$, then $p(y_i) - p(x_i) \to -\infty$ as $i \to \infty$. Consider the flow segment $[x_i, y_i]$ of $\tilde{\varphi}^{\#}$. This projects to a flow segment X_i of $\varphi^{\#}$ in M. We produce a closed curve $\overline{X}_i$ by concatenating X_i with any curve Y_i connecting the endpoints of X_i in M, such that length(Y_i) is uniformly bounded; the diameter of M can be used to bound length(Y_i). It follows that $\langle \alpha, \overline{X}_i \rangle \to -\infty$ as $i \to \infty$. On the other hand, by employing the symbolic dynamics of the pseudo-Anosov flow φ together with the semiconjugacy from $\varphi^{\#}$ to φ, one can show that for each $\overline{X}_i$ there is an equation in homology of the form $[\overline{X}_i] = [\gamma_1] + \cdots + [\gamma_N] + \varepsilon$, where each γ_n is a periodic orbit of φ, and ε is contained in some predetermined bounded subset of $H_1(M)$; this argument is contained in Theorem H of [4]. Since each periodic orbit defines an element of D_φ, we have $\langle \alpha, \gamma_i \rangle \geq 0$, and thus $\langle \alpha, \overline{X}_i \rangle$ is bounded away from $-\infty$, a contradiction.

Notational convention. For the rest of §2, we shall drop the superscript $^{\#}$ from $\varphi^{\#}$. Thus, the symbol φ will be used to denote the flow which satisfies the conclusions of the **Z**-Spectral Decomposition Theorem.

Besides the finiteness statements (A) and (B), property (E) is most interesting. It says that $\tilde{\varphi}$ is well behaved near the ends of $\tilde{M}$. To be specific, suppose that we are given $x \in \tilde{M}$ such that $L_+(x) \neq +\infty$ and $L_-(x) \neq -\infty$; for example, this is true if $x \in R(\tilde{\varphi})$. By property (E), every chain component sufficiently close to $+\infty$ is in $R_+(x)$, and every chain component sufficiently close to $-\infty$ is in $R_-(x)$. By property (B), together with the fact that **Z** acts properly discontinuously on $\tilde{M}$, we have:

2.2 Proposition. *Given $x \in \tilde{M}$, if $L_+(x) \neq \{+\infty\}$ and $L_-(x) \neq \{-\infty\}$, then $R_-(x) \cup R_+(x)$ contains all but finitely many chain components of R.*

As the chain recurrent set of $\tilde{\varphi}$ is equivariant, it projects to a closed invariant set of the flow φ. For later uses, we shall need a characterization of this set, taken from the companion paper [5] and [7]. Given M, σ, φ, and α as in the statement of the above theorem, we define the *chain kernel* of α to be the set $R(\alpha)$ consisting of all points $x \in M$ with the following property: for each ε, $T > 0$, there exists an ε, T cycle $(x_i; t_i)_{i \in \mathbf{Z}/I}$ such that if X is the closed curve $x_0 \cdot [0, t_1] * p_1 * x_1 \cdot [0, t_2] * p_2 * \cdots * x_{I-1} \cdot [0, t_I] * p_I$, where p_i is a curve from $x_{i-1} \cdot t_i$ to x_i staying in the ε ball around x_i, then $\langle \alpha, X \rangle = 0$. If $q \colon \tilde{M} \to M$ is the **Z**-covering associated to α, then in Proposition 7.1 of the companion paper [5] it is proven that $q^{-1}(R(\alpha)) = R(\tilde{\varphi})$. Theorem 3.8 and §4 of [5] also gives a method for describing $R(\alpha)$ in terms of the symbolic dynamics of φ. (See also [7]).

Now we turn to the proof of the Transverse Surface Theorem. Let $\mathscr{C}(R)$ denote the set of chain components of R. We define a partial order on $\mathscr{C}(R)$ as follows. Given $C, C' \in \mathscr{C}(R)$, we say that $C < C'$ if $C' \subset R_+(C)$ but $C' \neq C$. It is trivial to prove:

2.3 No Cycles Lemma. *The relation $<$ on $\mathscr{C}(R)$ is transitive and nonreflexive, and hence is a strict partial order.*

Given two disjoint subsets $\mathscr{C}_-, \mathscr{C}_+ \subset \mathscr{C}(R)$, we say that the ordered pair $(\mathscr{C}_-, \mathscr{C}_+)$ is a *partial cut* of $\mathscr{C}(R)$ if, for each $C_- \in \mathscr{C}_-$ and $C_+ \in \mathscr{C}_+$, it is not true that $C_+ < C_-$. When $\mathscr{C}(R) - (\mathscr{C}_- \cup \mathscr{C}_+)$ is finite, then we say that $\mathscr{C}_-, \mathscr{C}_+$ form a *cofinite* partial cut. When $\mathscr{C}(R) = \mathscr{C}_- \cup \mathscr{C}_+$, then they form a *cut*. We will often use $\mathscr{C}_\pm$ as shorthand notation for an ordered pair $(\mathscr{C}_-, \mathscr{C}_+)$. We also say that $\mathscr{C} \subset \mathscr{C}(R)$ is a *half cut* if $\mathscr{C}$ and $\mathscr{C}(R) - \mathscr{C}$ form a cut, in some ordering.

One property of cuts that we shall need is that they are well situated with respect to the ends of $\tilde{M}$.

2.4 Lemma: Cuts separate $-\infty$ from $+\infty$. *Given a cut $\mathscr{C}_\pm$, the closure of the set $\bigcup\{C \in \mathscr{C}_-\}$ does not contain $+\infty$, and the closure of the set $\bigcup\{C \in \mathscr{C}_+\}$ does not contain $-\infty$.*

Proof. Fix $C \in \mathscr{C}_+$ and $x \in C$. Since $L_+(x) \neq \{+\infty\}$, it follows from property (E) of the **Z**-Spectral Decomposition Theorem that there is a neighborhood U of $+\infty$ such that for each $C' \in \mathscr{C}(R)$, if $C' \subset U$ then $C' \in R_+(x)$. If $C' \in R_+(x)$ then clearly $C < C'$, so by definition of a cut, $C' \in \mathscr{C}_+$. Since $C' \in \mathscr{C}_+$ whenever $C' \subset U$, then the closure of the set $\bigcup\{C \in \mathscr{C}_-\}$ misses U. The proof for $\bigcup\{C \in \mathscr{C}_+\}$ is similar. q.e.d.

In order to verify the existence of transverse surfaces, we shall need to have our hands on some cuts.

2.5 Lemma: Existence of cuts. *There exist cuts of $\mathscr{C}(R)$.*

Proof. Pick some $C_1 \in \mathscr{C}(R)$. From Proposition 2.2, it follows that there exist sets $\mathscr{C}_-^1$, $\mathscr{C}_+^1 \subset \mathscr{C}(R)$ such that $\mathscr{C}_-^1 < C_1 < \mathscr{C}_+^1$, and $\mathscr{C}(R) - (\mathscr{C}_-^1 \cup \mathscr{C}_+^1)$ is finite. An application of the No Cycles Lemma shows that $\mathscr{C}_\pm^1$ is a cofinite partial cut of $\mathscr{C}(R)$.

The rest is a formal consequence of the definitions of partial orders and cuts. We inductively define partial cuts $\mathscr{C}_\pm^n$ of $\mathscr{C}(R)$ so that for all n, $\mathscr{C}_-^{n-1} \subset \mathscr{C}_-^n$ and $\mathscr{C}_+^{n-1} \subset \mathscr{C}_+^n$, and at least one of these two containments is strict if $\mathscr{C}_\pm^{n-1}$ is not a cut. Since $\mathscr{C}_\pm^n$ is cofinite, and since the set $\mathscr{C}(R) - (\mathscr{C}_-^n \cup \mathscr{C}_+^n)$ decreases with increasing n as long as it is nonempty, it follows by induction that for some n, $\mathscr{C}_\pm^n$ is a cut of $\mathscr{C}(R)$.

We will show the inductive step for $n = 2$, assuming that $\mathscr{C}(R) \neq \mathscr{C}_-^1 \cup \mathscr{C}_+^1$. Pick some $C_2 \in \mathscr{C}(R) - (\mathscr{C}_-^1 \cup \mathscr{C}_+^1)$. We define $\mathscr{C}_\pm^2$ in two cases:

Case (1). If there exists $C_+ \in \mathscr{C}_+^1$ such that $C_+ < C_2$, then put $C_2 \in \mathscr{C}_+^2$, and for any $C' \in \mathscr{C}(R) - (\mathscr{C}_-^1 \cup \mathscr{C}_+^1)$ such that $C_2 < C'$, put C' in C_+^2; also, put all of $\mathscr{C}_+^1$ in $\mathscr{C}_+^2$, and set $\mathscr{C}_-^2 = \mathscr{C}_-^1$. To see that $\mathscr{C}_\pm^2$ is a partial cut, note that if there existed $C_- \in \mathscr{C}_-^1$ and $C' \in \mathscr{C}_+^2 - \mathscr{C}_+^1$

such that $C' < C_-$, then we would have $C_+ < C_2 \le C' < C_-$, which is impossible.

Case (2). If there does not exist $C_+ \in \mathscr{C}^1_+$ such that $C_+ < C_2$, then put $C_2 \in \mathscr{C}^2_-$, and for any $C' \in \mathscr{C}(R) - (\mathscr{C}^1_- \cup \mathscr{C}^1_+)$ such that $C' < C_2$, put $C' \in \mathscr{C}^2_-$; also, put all of $\mathscr{C}^1_-$ in $\mathscr{C}^2_-$, and set $\mathscr{C}^2_+ = \mathscr{C}^1_+$. If there existed $C' \in \mathscr{C}^2_- - \mathscr{C}^1_-$ and $C_+ \in \mathscr{C}^1_+$ such that $C_+ < C'$, then it would follow that $C_+ < C' \le C_2$, which violates the hypothesis of case 2. q.e.d.

Here is a more precise version of the existence of cuts that we shall have need of:

2.6 Lemma. *Let $\mathscr{C}'_\pm$ be a partial cut such that $\mathscr{C}'_-$ and $\mathscr{C}'_+$ are both half cuts, and let $C_- \in \mathscr{C}(R) - \mathscr{C}'_+$, $C_+ \in \mathscr{C}(R) - \mathscr{C}'_-$ be given so that $C_- < C_+$. Then there exists a cut $\mathscr{C}_\pm$ such that $\mathscr{C}'_- \cup \{C_-\} \subset \mathscr{C}_-$ and $\mathscr{C}'_+ \cup \{C_+\} \subset \mathscr{C}_+$.*

Proof. Define $\mathscr{C}^1_-$ to include $\mathscr{C}'_-$, C_-, and any C' such that $C' < C_-$. Define $\mathscr{C}^1_+$ to include $\mathscr{C}'_+$, C_+, and any C' such that $C_+ < C'$. The hypotheses show that $\mathscr{C}^1_\pm$ is a partial cut. Now proceed as in Existence of Cuts. q.e.d.

The set $\mathrm{Cuts}(\mathscr{C}(R))$ consisting of all cuts of $\mathscr{C}(R)$ itself has a partial order defined on it. Given distinct $\mathscr{C}^i_\pm \in \mathrm{Cuts}(\mathscr{C}(R))$ for $i = 1, 2$, we say that $\mathscr{C}^1_\pm < \mathscr{C}^2_\pm$ if $\mathscr{C}^1_-, \mathscr{C}^2_+$ forms a partial cut of $\mathscr{C}(R)$.

We shall show that the elements of the set $\mathrm{Cuts}(\mathscr{C}(R))$ correspond in a natural way to certain surfaces transverse to the flow $\tilde\varphi$. A transverse surface S to $\tilde\varphi$ is said to *simply separate the ends of* $\tilde M$ if $\tilde M - S$ consists of two components, $\tilde M_-(S)$ limiting on $-\infty$ and $\tilde M_+(S)$ limiting on $+\infty$. Two transverse surfaces S, S' to $\tilde\varphi$ are said to be *flow isotopic* if there is an isotopy from S to S' which moves points along flow lines; this is equivalent to the existence of a continuous function $r\colon S \to \mathbf{R}$ such that the map $x \to x \cdot r(x)$ is a homeomorphism from S onto S'. We will use $[S]$ to denote the flow isotopy class of a transverse surface to $\tilde\varphi$. The set of flow isotopy classes of transverse surfaces which simply separate the ends of $\tilde M$ has a natural partial order defined on it: given nonflow isotopic S, S' which simply separate the ends of $\tilde M$, $[S] < [S']$ if there are representative S_0, S'_0 such that $S'_0 \subset \tilde M_+(S_0)$.

The following proposition is the key technical result in the proof of the Transverse Surface Theorem:

2.7 Proposition: Cuts and transverse surfaces correspond. *There exists a natural, order preserving bijection between* $\mathrm{Cuts}(\mathscr{C}(R))$ *and the set of* *flow isotopy classes of transverse surfaces which simply separate the ends of* $\tilde{M}$.

Proof. Given a transverse surface S which simply separates the ends of $\tilde{M}$, note that $S \cap R = \varnothing$. To see why, given $T > 0$, let S_T be the surface obtained by flowing S forward for time T, and let ε be the minimal distance from S to S_T. If $X = (x_0, x_1, \cdots, x_n; t_1, \cdots, t_n)$ is any ε, T chain starting from $x_0 \in S$, then $x_0 \cdot t_1 \in \tilde{M}_+(S_T)$, so $x_1 \in \tilde{M}_+(S)$. Continuing by induction, we see that $x_n \in \tilde{M}_+(S)$. Since $S \cap \tilde{M}_+(S) = \varnothing$, it is impossible for X to be a cycle. In other words, no point on S is contained in an ε, T cycle. Thus, $S \cap R = \varnothing$.

Thus, we can define a partition $\mathscr{C}_-, \mathscr{C}_+$ of $\mathscr{C}(R)$ by saying that for $C \in \mathscr{C}(R)$, $C \in \mathscr{C}_-$ if $C \subset \tilde{M}_-(S)$, and $C \in \mathscr{C}_+$ if $C \subset \tilde{M}_+(S)$. To see that this is a cut of $\mathscr{C}(R)$, suppose that $C_+ < C_-$ for some $C_- \in \mathscr{C}_-$, $C_+ \in \mathscr{C}_+$. Pick an ε, T chain $X = (x_0, \cdots, x_n; t_1, \cdots, t_n)$ from $x_0 \in C_+$ to $x_n \in C_-$ for some small ε and large T. Let $\overline{X}$ be the path from x_0 to x_n obtained from X by interpolating a short path from $x_{i-1} \cdot t_i$ to x_i for each $1 \leq i \leq n$. Since x_0, x_n are in opposite components of $\tilde{M} - S$, $\overline{X}$ must intersect S. If ε is sufficiently small and T is sufficiently large, it is easy to see by examining a neighborhood of each intersection point that the intersection number of S and $\overline{X}$ must be positive. But since $x_0 \in \tilde{M}_+(S)$ and $x_n \in \tilde{M}_-(S)$, we arrive at a contradiction.

This shows that every transverse surface S which simply separates the ends of $\tilde{M}$ defines a cut $\mathscr{C}_\pm(S)$ by the condition that $\mathscr{C}_-(S) = \{C \in \mathscr{C}(R) | C \subset \tilde{M}_-(S)\}$, $\mathscr{C}_+(S) = \{C \in \mathscr{C}(R) | C \subset \tilde{M}_+(S)\}$. Moreover, it is clear that flow isotopic surfaces define the same cut.

Now we must show the other direction: that every cut $\mathscr{C}_\pm$ determines a transverse surface $S(\mathscr{C}_\pm)$ which simply separates the ends of $\tilde{M}$, well defined up to flow isotopy. We must do this in such a way that for each cut $\mathscr{C}_\pm$, $\mathscr{C}_\pm(S(\mathscr{C}_\pm)) = \mathscr{C}_\pm$, and for each transverse surface S simply separating the ends of $\tilde{M}$, $S(\mathscr{C}_\pm(S))$ is flow isotopic to S.

The idea is as follows: throw away every $C \in \mathscr{C}_+$ together with its unstable set, and throw away every $C \in \mathscr{C}_-$ together with its stable set, and show that what remains is a flow on a surface cross $\mathbf{R}$. Then choose S to be any section to the remaining flow. The argument which makes this idea rigorous goes back to an idea of A. J. Schwartz. In modern terminology, one employs the notion of a Lyapounov function. This approach follows the material contained in Conley's book [1], in particular, in §5 of Chapter II.

Here are the definitions and results that we need. Given a flow on a compact metric space Γ, an *attractor repeller pair* (A, A^*) consists of an attractor $A \subset \Gamma$ and a repeller $A^* \subset \Gamma$ such that $A \cap A^* = \emptyset$, and for any $x \in \Gamma - (A \cup A^*)$, $L_+(x) \subset A$ and $L_-(x) \subset A^*$. Proposition B of the above-mentioned section of Conley's book shows that every attractor repeller pair has a *Lyapounov function*, which is a continuous function $g: \Gamma \to [0, 1]$ such that $g^{-1}(1) = A^*$, $g^{-1}(0) = A$, and g is strictly decreasing on orbits in $\Gamma - (A \cup A^*)$. Note that if $\Gamma - (A \cup A^*)$ is a manifold, then for any $r \in (0, 1)$, $g^{-1}(r)$ is a codimension 1 submanifold transverse to the flow, which separates Γ into two components, one containing A^* and the other containing A.

For our space Γ, we take the end compactification $\tilde{M}^c = \tilde{M} \cup \{-\infty, +\infty\}$ of $\tilde{M}$, and for the flow on $\tilde{M}^c$ we extend $\tilde{\varphi}$ to the flow $\tilde{\varphi}^c$ for which $-\infty$ and $+\infty$ are stationary points. For the set A, we take

$$\bigcup\{\text{Unstable}(C) \mid C \in \mathscr{C}_+\} \cup \{+\infty\},$$

and for the set A^*, we take $\bigcup\{\text{Stable}(C) \mid C \in \mathscr{C}_-\} \cup \{-\infty\}$; by definition, $\text{Stable}(C) = \{x \mid L_+(x) \subset C\}$, and $\text{Unstable}(C) = \{x \mid L_-(x) \subset C\}$. In order to apply the results of the previous paragraph, we need only show that (A, A^*) is an attractor repeller pair in Γ.

To show that A is closed, first we show that for any $C \in \mathscr{C}_+$, $\text{cl}(C \cup \text{Unstable}(C)) \subset A$. Consider a convergent sequence $y_i \to z$ in $\tilde{M} \cup \{-\infty, +\infty\}$, with $y_i \in C \cup \text{Unstable}(C)$. If there is an infinite subsequence $y_{i(n)} \in C$, then $z \in C$, so we can assume that $y_i \in \text{Unstable}(C)$. We can choose a sequence of real numbers $r_i < 0$ and pass to a subsequence so that $x_i = y_i \cdot r_i$ converges to some point in C.

Now we distinguish five cases:

(i) $z \in C'$ for some $C' \in \mathscr{C}(R)$;
(ii) $z \in \text{Unstable}(C')$ for some $C' \in \mathscr{C}(R)$;
(iii) $z = +\infty$.
(iv) $z = -\infty$.
(v) $L_-(z) = -\infty$.

In case (i), for each $\varepsilon, T > 0$, we can choose i so that $y_i \in B(z, \varepsilon)$. Using the flow segment $[x_i, y_i]$, we can construct an ε, T chain from a point in C to z. It follows that $C \le C'$, so $C' \subset A$ and thus $z \in A$. In case (ii), we use the fact that $\text{cl}(C \cup \text{Unstable}(C))$ is a closed invariant set of $\tilde{\varphi}^c$ to conclude that $C' \subset \text{cl}(C \cup \text{Unstable}(C))$, hence $C < C'$ by case (i), so $C' \cup \text{Unstable}(C') \subset A$ and thus $z \in A$. In case (iii), we immediately have $z \in A$.

In case (iv), we apply property (F) of the **Z**-Spectral Decomposition Theorem to obtain a contradiction. Let $p: \tilde{M} \to R$ be as in property (F).

Since x_i converges to a point in C, the values $p(x_i)$ stay in a bounded subset of R. But since $y_i \to -\infty$, then $p(y_i) \to -\infty$. This implies that $p(y_i) - p(x_i) \to -\infty$, contradicting property (F).

In case (v), use the fact that $\mathrm{cl}(C \cup \mathrm{Unstable}(C))$ is a closed invariant set of $\tilde{\varphi}^c$ to conclude that $-\infty \in \mathrm{cl}(C \cup \mathrm{Unstable}(C))$, which has already been ruled out in case (iv).

This finishes the proof that $\mathrm{cl}(C \cup \mathrm{Unstable}(C)) \subset A$. Incidentally, we have also proven that $\mathrm{cl}(C \cup \mathrm{Unstable}(C))$ is bounded away from $-\infty$.

To complete the proof that A is closed, it remains to consider a sequence $y_i \in A$, such that $y_i \in C_i \cup \mathrm{Unstable}(C_i)$ for some 1-1 sequence $C_i \in \mathscr{C}_+$. Let T be the generator of the $\mathbf{Z}$-action on $\tilde{M}$. By property (B) of the $\mathbf{Z}$-spectral decomposition theorem, we can pass to a subsequence so that there exist $C \in \mathscr{C}_+$ and a sequence $b_i \in \mathbf{Z}$ such that $n_i \to +\infty$ and $C_i = T^{n_i}(C)$. Since $C \cup \mathrm{Unstable}(C)$ is bounded away from $-\infty$, it follows easily that $y_i \to +\infty$.

Appealing to symmetry we see that A^* is also closed. Clearly A, A^* are invariant sets of $\tilde{\varphi}^c$, and evidently $A \cap A^* = \emptyset$. Given $x \in \tilde{M}^c - A \cup A^*$, the $\mathbf{Z}$-Spectral Decomposition Theorem shows that $L_-(x) \subset A^*$ and $L_+(x) \subset A$. It follows immediately that (A, A^*) is an attractor repeller pair.

Thus, there exists a Lyapounov function g for the attractor repeller pair (A, A^*), and we construct a transverse surface $S = S(\mathscr{C}_\pm)$ to $\tilde{\varphi}$ by setting $S = g^{-1}(1/2)$. Note that S separates Γ into two components, one containing A^* and the other containing A. Thus, S simply separates the ends of $\tilde{M}$, and since $A^* - \{-\infty\} \subset \tilde{M}_-(S)$ and $A - \{+\infty\} \subset \tilde{M}_+(S)$, we see that $\mathscr{C}_\pm(S(\mathscr{C}_\pm)) = \mathscr{C}_\pm$.

Also, let S' be any other transverse surface which simply separates the ends of $\tilde{M}$, such that $\mathscr{C}_\pm(S') = \mathscr{C}_\pm$; we must show that S and S' are flow isotopic. Since $\bigcup \mathscr{C}_- \subset \tilde{M}_-(S')$, it follows that $A^* \subset \tilde{M}_-(S') \cup \{-\infty\}$; similarly, $A \subset \tilde{M}_+(S') \cup \{+\infty\}$. Thus, $S' \subset \Gamma - (A \cup A^*)$. Now the existence of the Lyapounov function g shows that $\Gamma - (A \cup A^*)$ is homeomorphic to $S \times (0, 1)$ under the map $(x, t) \to y$, where y is defined by the conditions that $g(y) = t$ and $y \in x \cdot \mathbf{R}$; moreover, the flow on $S \times (0, 1)$ is in the $(0, 1)$ direction. Since S' is a cross-section to this flow, it is clear that S' and S are flow isotopic.

We have shown that the correspondence $S \to \mathscr{C}_\pm(S)$ induces a bijection between the set $\mathrm{Cuts}(\mathscr{C}(R))$, and the set of flow isotopy classes of transverse surfaces which simply separate the ends of $\tilde{M}$. To finish the proof of Proposition 2.7, we must show that the partial orders correspond.

To do this, let $\mathscr{C}_\pm$, $\mathscr{C}'_\pm$ be cuts such that $\mathscr{C}_-$, $\mathscr{C}'_+$ is a partial cut; we must show that $S = S(\mathscr{C}_\pm)$ and $S' = S(\mathscr{C}'_\pm)$ can be chosen within their flow isotopy classes so that $S' \subset \tilde{M}_+(S)$.

Consider the collection $\{\mathscr{C}^i_\pm | i = 1, \cdots, I\}$ of all cuts such that $\mathscr{C}_- \subset \mathscr{C}^i_-$ and $\mathscr{C}^i_+ \subset \mathscr{C}'_+$. Let g^i be a Lyapounov function for the cut $\mathscr{C}^i_\pm$, as defined above. Let $g = \sum g^i$. Notice that image(g) is the interval $[0, I]$. Notice also that for any $C \in \mathscr{C}(R) - (\mathscr{C}_- \cup \mathscr{C}'_-)$, g is constant on C; moreover, $g(C) \in (0, I)$, for by Lemma 2.6, the refined version of Lemma 2.5, Existence of Cuts, there exist two cuts $\mathscr{C}^i_\pm$ and $\mathscr{C}^{i'}_\pm$ with $i, i' \in \{1, \cdots, I\}$ such that $C \subset \mathscr{C}^i_+$ and $C \subset \mathscr{C}^{i'}_-$. Since $\mathscr{C}(R) - (\mathscr{C}_- \cup \mathscr{C}'_+)$ is a finite set, then there exists some $\varepsilon > 0$ such that $g^{-1}(0, \varepsilon)$ and $g^{-1}(I-\varepsilon, I)$ are disjoint from elements of $\mathscr{C}(R)$. In particular, it follows that $S = g^{-1}(\varepsilon/2)$ and $S' = g^{-1}(|I| - \varepsilon/2)$ are disjoint surfaces such that $S = S(\mathscr{C}_\pm)$ and $S' = S(\mathscr{C}'_\pm)$, and clearly $S' \subset \tilde{M}_+(S)$. q.e.d.

Now we shall employ Proposition 2.7 to construct transverse surfaces to the downstairs flow φ in the homology class α. The idea is to take a cut $\mathscr{C}_\pm$ of $\mathscr{C}(R)$, and project the surface $S(\mathscr{C}_\pm)$ down to M. The problem is that the projection map might not be an embedding on $S(\mathscr{C}_\pm)$. One might try to isotope $S(\mathscr{C}_\pm)$ along flow lines so that it is disjoint from all its covering translates, but there is an obstruction to performing such an isotopy. To see the obstruction, suppose we start with a transverse surface S to φ in the homology class α. Let $\tilde{S}$ be a particular lift of S. Let T be the generator of the $\mathbf{Z}$-action on $\tilde{M}$. Then it is clear that $T(\tilde{S}) \subset \tilde{M}_+(\tilde{S})$. T also acts on Cuts$(\mathscr{C}(R))$, preserving the partial order, in such a way that $T(\mathscr{C}_\pm(\tilde{S})) = \mathscr{C}_\pm(T(\tilde{S}))$. Thus, $\mathscr{C}_\pm(\tilde{S}) < T(\mathscr{C}_\pm(\tilde{S}))$. In other words, in order for $\mathscr{C}_\pm \in$ Cuts$(\mathscr{C}(R))$ to produce a transverse surface to φ, it is necessary that $\mathscr{C}_\pm < T(\mathscr{C}_\pm)$. When this happens, we say that $\mathscr{C}_\pm$ is *moved strictly forward* by $\mathbf{Z}$. First we shall show that this is also a sufficient condition for $S(\mathscr{C}_\pm)$ to embed into M, and then we shall show how to construct such cuts.

2.8 Proposition. *The operation which assigns to* $\mathscr{C}_\pm \in$ Cuts$(\mathscr{C}(R))$ *the flow isotopy class of* $S(\mathscr{C}_\pm)$ *restricts to a bijection between the collection of cuts which are moved strictly forward by* $\mathbf{Z}$, *and the collection of flow isotopy classes of transverse surfaces* S *simply separating the ends of* $\tilde{M}$ *such that* $T(S) \subset \tilde{M}_+(S)$.

Proof. We know that if $\mathscr{C}_\pm$ is moved strictly forward by $\mathbf{Z}$, then $S(\mathscr{C}_\pm)$ and $S(T(\mathscr{C}_\pm))$ can be chosen so that $S(T(\mathscr{C}_\pm)) \subset \tilde{M}_+(S(\mathscr{C}_\pm))$. We must choose these surfaces so that in addition, $T(S(\mathscr{C}_\pm)) = S(T(\mathscr{C}_\pm))$. In other words, setting $S = S(\mathscr{C}_\pm)$, we must choose a flow isotopy from S to some S' so that $T(S') \subset \tilde{M}_+(S')$. To show this, we adapt an argument of Fried, found in [4, 355–356].

Clearly there exists some $n > 0$ so that, setting τ equal to the 2^n power of T, $\tau(S) \subset \tilde{M}_+(\tau^{-1}(S))$. We shall find a flow isotopy from S to some S' so that $\tau(S') \subset \tilde{M}_+(S')$; applying induction on n completes the proof. Let $A = \{x \in S | x \in \tilde{M}_-(\tau^{-1}(S))\}$, and let $B = \{x \in S | x \in \tilde{M}_+(\tau(S))\}$. Note that $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \varnothing$, since $\tau(S) \subset \tilde{M}_+(\tau^{-1}(S))$. Notice also that $\tau(\mathrm{cl}(A))$ can be flowed forward to $\mathrm{cl}(B)$, that is, for some continuous function $s\colon \mathrm{cl}(A) \to [0, +\infty)$, the map $x \to s(x) \cdot \tau(x)$ from $\tau(\mathrm{cl}(A))$ to $\mathrm{cl}(B)$ is a homeomorphism; the existence of s follows from the fact that $\tau(S)$ can be flowed forward past S. Now choose a continuous function $t\colon S \to [0, +\infty)$ such that $t(x) > s(x)$ for all $x \in \mathrm{cl}(A)$, $t(x) = 0$ for all x in a neighborhood of $\mathrm{cl}(B)$, and $x \cdot t(x) \in \tilde{M}_-(\tau(S))$ for all $x \in S - \mathrm{cl}(B)$. Setting $S' = \{x \cdot t(x) | x \in S\}$, it is clear that S' is the desired transverse surface.

2.9 Corollary. *There exists a natural bijection between* **Z***-orbits of cuts of* $\mathscr{C}(R)$ *which are moved strictly forward by* **Z***, and flow isotopy classes of surfaces in* M*, transverse to* φ*, whose homology class is* α.

Now we give a finitistic description of the collection of all cuts of $\mathscr{C}(R)$ which are moved strictly forward by $\mathbf{Z}$. Utilizing this description, we will show that such cuts do actually exist. The remainder of this section is highly combinatorial in nature, centering on the notion of a directed graph. First we review the main definitions and elementary properties concerning directed graphs.

Recall that a *directed graph* or *digraph* is a finite 1-dimensional CW complex Γ each of whose edges has a preferred orientation. Nodes(Γ) denotes the collection of all *nodes* or 0-cells of Γ. An oriented 1-cell e is called a *directed edge* of Γ, and we use Head(e) and Tail(e) to denote those nodes such that $\partial e = -\mathrm{Tail}(e) + \mathrm{Head}(e)$.

A *directed path* in Γ is a path of the form $\gamma = e_1 * e_2 * \cdots * e_K$, where e_k $(k = 1, \cdots, K)$ are directed edges, and $\mathrm{Head}(e_k) = \mathrm{Tail}(e_{k+1})$ for $k = 1, \cdots, K-1$, and we write $\mathrm{Tail}(\gamma) = \mathrm{Tail}(e_1)$ and $\mathrm{Head}(\gamma) = \mathrm{Head}(e_K)$; if the e_k are not necessarily directed, γ is simply a *nondirected path* in Γ, or when the context is clear, simply a *path.* A *directed loop* is a directed path γ such that $\mathrm{Tail}(\gamma) = \mathrm{Head}(\gamma)$; *nondirected loops*, or simply *loops*, are similarly defined. A finite directed graph Γ is *strongly connected* if, for any two distinct nodes $x, y \in \Gamma$, there exists a directed path from x to y; equivalently, either Γ is the trivial digraph consisting of a single node and no edges, or Γ is a connected digraph with a directed loop through each edge. Given an arbitrary digraph Γ, a *strong component* of Γ is a maximal strongly connected subgraph; evidently each node of Γ is contained in a unique strong component of Γ, and two distinct strong components are disjoint. Note that not every edge in Γ is necessarily contained in a strong component.

A finite directed graph Γ is *transitive* if there exists some N such that for each $n \geq N$ and any two nodes $x, y \in \Gamma$, there exists a directed path of length n from x to y; if one builds a square matrix M whose rows and columns are indexed by nodes of Γ, where $M(x, y)$ counts the number of directed edges from x to y, then transitivity of Γ is equivalent to the existence of a power of M having all positive entries.

We shall occasionally have need to discuss the homology and cohomology of a directed graph Γ; by this, we shall always mean cellular homology and cohomology, using the CW-structure on Γ and the given orientations on edges, which specifies the chain and cochain groups. A *nonnegative cohomology class* of Γ is an element $U \in H^1(\Gamma; \mathbf{Z})$ such that for each directed loop γ, $U(\gamma) \geq 0$; if the inequality is always strict, then U is called a *positive cohomology class.* A *nonnegative cocycle* of Γ is an element $u \in C^1(\Gamma; \mathbf{Z})$ such that for each directed edge e, $u(e) \geq 0$.

Here is the digraph we shall be interested in. First construct a digraph $\tilde{\Gamma} = \tilde{\Gamma}(\alpha)$, whose nodes are the elements of $\mathscr{C}(R)$, with a directed edge pointing from C to C' when $C < C'$, and there is no $C'' \in \mathscr{C}(R)$ such that $C < C'' < C'$. Clearly the defining property of directed edges is invariant under the action of $\mathbf{Z}$ on $\mathscr{C}(R)$. Thus, $\mathbf{Z}$ acts on $\tilde{\Gamma}$, and it is easily seen that the action is free. The quotient graph $\tilde{\Gamma}/\mathbf{Z}$ is denoted $\Gamma = \Gamma(\alpha)$.

We assert that $\tilde{\Gamma}$, and therefore also Γ, is connected. That is, for $C, C' \in \mathscr{C}(R)$, there is an edge path in $\tilde{\Gamma}$ from C to C'. By property (E) of the $\mathbf{Z}$-Spectral Decomposition Theorem, there exists $C_0 \in \mathscr{C}(R)$ such that $C < C_0$ and $C' < C_0$. Thus, it suffices to show that if $C < C'$, then there is a directed path from C to C'. To do this, it clearly suffices to prove that there are only finitely many $C_1 \in \mathscr{C}(R)$ such that $C < C_1 < C'$. If there were infinitely many possibilities for C_1, then by property (B) of the $\mathbf{Z}$-Spectral Decomposition Theorem, they would have to accumulate on either $-\infty$ or $+\infty$, and it would follow that either $-\infty \in R_+(C)$, or $+\infty \in R_-(C')$. But this contradicts the $\mathbf{Z}$-Spectral Decomposition Theorem by a now-familiar argument.

Associated to the $\mathbf{Z}$-covering $\tilde{\Gamma} \to \Gamma$ is a cohomology class $U = U(\alpha) \in H^1(\Gamma; \mathbf{Z})$: for any closed loop γ of Γ, if $\tilde{\gamma}$ is a lift of γ with $\operatorname{Tail}(\tilde{\gamma}) = x$

and $\text{Head}(\tilde{\gamma}) = y$, then $y = T^{U(\gamma)}(x)$. Note that U is positive. To see why, notice that $\tilde{\Gamma}$, as a topological space, has a natural compactification by adding points $\pm\infty$, where for each $x \in \text{Nodes}(\Gamma)$, $T^i(x) \to +\infty$ as $i \to +\infty$, and $T^i(x) \to -\infty$ as $i \to -\infty$. This compactification has the property that for any infinite directed path $e_1 * e_2 * e_3 * \cdots$, $\text{Tail}(e_k) \to +\infty$ as $k \to +\infty$; this is a consequence of the **Z**-Spectral Decomposition Theorem and the definition of $\tilde{\Gamma}$. Since there is a directed path from $x_n = T^{n\cdot U(\gamma)}(x_0)$ to $x_{n+1} = T^{(n+1)\cdot U(\gamma)}(x_0)$ for each n, where $x_0 = \text{Tail}(\tilde{\gamma})$, it follows that $x_n \to +\infty$ as $n \to \infty$, and so $U(\gamma) > 0$.

The digraph $\Gamma(\alpha)$ is strongly connected. To see why, given nodes x, y of $\Gamma(\alpha)$, choose $\tilde{x} \in \tilde{\Gamma}(\alpha)$ lying over x. As a consequence of property (E) of the **Z**-Spectral Decomposition Theorem and the argument given above, there is a directed path from $\tilde{x}$ to any node of $\tilde{\Gamma}(\alpha)$ sufficiently close to $+\infty$. Thus, choosing $\tilde{y}$ over y close to $+\infty$, there is a directed path from $\tilde{x}$ to $\tilde{y}$, which projects down to a directed path from x to y.

Note. In §8 of the companion paper [5], a direct proof of strong connectivity of $\Gamma(\alpha)$ and positivity of $U(\alpha)$ is given, without using property (E); in fact, property (E) is obtained as a consequence. So what we have in effect done here is to show that, in the presence of properties (A)–(D) of the **Z**-Spectral Decomposition Theorem, property (E) is equivalent to strong connectivity of $\Gamma(\alpha)$ and positivity of $U(\alpha)$.

Here is the proposition which gives a finitistic understanding of the set of **Z**-orbits of cuts of $\mathscr{C}(R)$ which are moved strictly forward by $\mathbf{Z}$.

2.10 Proposition. *There exists a natural bijection between the set* {*nonnegative cocycles on* $\Gamma(\alpha)$ *in the cohomology class* $U(\alpha)$} *and the set* {**Z**-*orbits of cuts of* $\mathscr{C}(R)$ *which are moved strictly forward by* $\mathbf{Z}$}.

Proof. Let u be a nonnegative cocycle on $\Gamma(\alpha)$ representing $U(\alpha)$. Let $\tilde{u}$ be the lifted cocycle on $\tilde{\Gamma}(\alpha)$. Since the covering map $p\colon \tilde{\Gamma}(\alpha) \to \Gamma(\alpha)$ corresponds to $U(\alpha)$, then the lifted cohomology class $\tilde{U}(\alpha)$ is trivial in $H^1(\tilde{\Gamma}(\alpha); \mathbf{Z})$ since $\tilde{u}$ represents $\tilde{U}(\alpha)$, it follows that $\tilde{u}$ is a coboundary. Choose a 0-dimensional cocycle $\nu \in C^0(\tilde{\Gamma}(\alpha); \mathbf{Z})$ such that $\delta\nu = \tilde{u}$; thus, $\nu\colon \text{Nodes}(\tilde{\Gamma}(\alpha)) \to \mathbf{Z}$. Since $\tilde{\Gamma}(\alpha)$ is connected, ν is unique up to an additive integer constant.

The nonuniqueness of ν can be related to the **Z**-action on $\tilde{\Gamma}(\alpha)$ as follows. Let T denote the standard generator of the **Z**-action on $\tilde{\Gamma}(\alpha)$, so that $T^n(c) \to +\infty$ as $n \to +\infty$ for each $c \in \text{Nodes}(\tilde{\Gamma}(\alpha)) = \mathscr{C}(R)$. Since $\alpha \in H^1(M; \mathbf{Z})$ was chosen to be primitive, then $U(\alpha) \in H^1(\Gamma(\alpha); \mathbf{Z})$ is primitive. It follows that for any $c \in \text{Nodes}(\tilde{\Gamma}(\alpha))$, if γ is a path from c to $T(c)$, then $p \circ \gamma$ is a closed loop in $\tilde{\Gamma}(\alpha)$ such that $U(\alpha)(p \circ \gamma) = +1$, so $\tilde{u}(\gamma) = +1$. Thus, $\nu(T(c)) - \nu(c) = \tilde{u}(\gamma) = +1$, so $\nu(T(c)) = \nu(c) + 1$. This is true for any $c \in \text{Nodes}(\tilde{\Gamma}(\alpha))$, and it follows that addition of a constant integer n to ν corresponds to replacing ν by $\nu \circ T^n$. Thus, the 0-cochain ν satisfying $\delta\nu = \tilde{u}$ is unique up to the action on $\mathbf{Z}$ on $\tilde{\Gamma}(\alpha)$.

Fixing the choice of ν, we define a cut $C_\pm$ as follows: $C_- = \{c \in \text{Nodes}(\tilde{\Gamma}(\alpha)) | \nu(c) \leq 0\}$, and $C_+ = \{c \in \text{Nodes}(\tilde{\Gamma}(\alpha)) | \nu(c) > 0\}$. Evidently $C_\pm$ is a partition on $\mathscr{C}(R) = \text{Nodes}(\tilde{\Gamma}(\alpha))$. To prove the cut property, suppose that $c_+ < c_-$ for $c_- \in C_-$ and $c_+ \in C_+$; thus, $\nu(c_-) \leq 0$ and $\nu(c_+) > 0$, and in particular, $\nu(c_-) < \nu(c_+)$. Since $c_+ < c_-$, this means that there is a directed path γ with $\text{Tail}(\gamma) = c_+$ and $\text{Head}(\gamma) = c_-$. Since $\tilde{u}$ is nonnegative, $\tilde{u}(\gamma) \geq 0$. Thus, $\nu(c_-) - \nu(c_+) = \nu(\partial\gamma) = \delta\nu(\gamma) = \tilde{u}(\gamma) \geq 0$, implying $\nu(c_-) \geq \nu(c_+)$, a contradiction.

Also, the cut $C_\pm$ is moved strictly forward by $\mathbf{Z}$: this follows from the fact that $T(C_+) = \{c \in \text{Nodes}(\tilde{\Gamma}(\alpha)) | \nu(c) > 1\} \subset C_+$. Finally, the fact that ν is uniquely defined up to the action of $\mathbf{Z}$ shows that $C_\pm$ is uniquely defined up to the action of $\mathbf{Z}$. Thus, we have defined a map {nonnegative cocycles on $\Gamma(\alpha)$ in the cohomology class $U(\alpha)$} $\to$ {$\mathbf{Z}$-orbits of cuts of $\mathscr{C}(R)$ which are moved strictly forward by $\mathbf{Z}$}.

We show that this is a 1-1 correspondence by defining an inverse map. Let $C_\pm$ be a cut moved strictly forward by $\mathbf{Z}$. Define a 0-cochain ν as follows. For each $c \in \mathscr{C}(R)$, let $n \in \mathbf{Z}$ be such that $T^n(c) \in C_-$ and $T^{n+1}(c) \in C_+$ and define $\nu(C) = n$; existence of n follows from the fact that C_- contains all c sufficiently close to $-\infty$, and C_+ contains all c sufficiently close to $+\infty$; uniqueness of n follows from the fact that $C_\pm$ is moved strictly forward by $\mathbf{Z}$. Evidently $\nu \circ T = \nu + 1$, so $\tilde{u} = \delta\nu$ is a 1-cocycle on $\tilde{\Gamma}(\alpha)$ which is equivariant with respect to $\mathbf{Z}$, and this is the lift of a 1-cocycle u on $\Gamma(\alpha)$. Moreover, if γ is a closed loop in $\Gamma(\alpha)$, then

$$\begin{aligned} u(\gamma) &= \tilde{u}(\tilde{\gamma}) = \nu(\text{Head}(\tilde{\gamma})) - \nu(\text{Tail}(\tilde{\gamma})) \\ &= \nu(T^{U(\alpha)(\gamma)}(\text{Tail}(\tilde{\gamma})) - \nu(\text{Tail}(\tilde{\gamma}))) = U(\alpha)(\gamma), \end{aligned}$$

where $\tilde{\gamma}$ is any path lifting of γ; it follows that u represents $U(\alpha)$.

To see that u is nonnegative, it suffices to show that $\tilde{u}$ is nonnegative. For any directed path γ in $\tilde{\Gamma}(\alpha)$, we must show that $\tilde{u}(\gamma) = \nu(\text{Head}(\gamma)) - \nu(\text{Tail}(\gamma)) \geq 0$. Since $\tilde{u}$ is equivariant, it suffices to assume that $\nu(\text{Tail}(\gamma)) = 1$, and we must show that $\nu(\text{Head}(\gamma)) \geq 1$. If this were not the case, then we would have $\text{Tail}(\gamma) \in C_+$ and $\text{Head}(\gamma) \in C_-$, and since γ is directed, we would have $\text{Tail}(\gamma) < \text{Head}(\gamma)$ in the partial order on $\mathscr{C}(R)$, contradicting the fact that $C_\pm$ is a cut.

This defines a map $\{\mathbf{Z}$-orbits of cuts of $\mathscr{C}(R)$ which are moved strictly forward by $\mathbf{Z}\} \to \{$ nonnegative cocycles on $\Gamma(\alpha)$ in the cohomology class $U(\alpha)\}$, which is easily seen to be the inverse of the map previously defined. q.e.d.

As a direct corollary, we have the following theorem. In order to state this result precisely, we shall revert to the notation φ for the original circular pseudo-Anosov flow, and $\varphi^{\#}$ for the flow obtained by blowing up singular orbits of φ.

2.11 Theorem: Classification of Almost Transverse Surfaces. *Let M^3 be irreducible and atoroidal, σ a fibered face of $B_x(M)$, and φ a pseudo-Anosov flow associated to σ. Given $\alpha \in \partial(\mathrm{Cone}(\sigma)) \cap H_2(M; \mathbf{Z})$, let $\varphi^{\#}$ be any flow obtained by dynamically blowing up α-null singular orbits of φ, such that $\varphi^{\#}$ satisfies the conclusion of the $\mathbf{Z}$-Spectral Decomposition Theorem. Let $\Gamma(\alpha)$, $U(\alpha)$ be respectively the digraph and the cohomology class constructed from $\varphi^{\#}$ by the above process. Then there exists a natural bijection between the set $\{$nonnegative cocycles of $\Gamma(\alpha)$ in the cohomology class $U(\alpha)\}$ and the set $\{$flow isotopy classes of transverse surfaces to $\varphi^{\#}$ in the homology class $\alpha\}$.*

Before proceeding with the proof of the Transverse Surface Theorem, a few comments about Theorem 2.11 are in order.

In §8 of the companion paper [5] (see also [7]), we show how the digraph $\Gamma(\alpha)$ and the cohomology class $U(\alpha)$ can be constructed from the symbolic dynamics of the circular pseudo-Anosov flow φ. In particular, given a cross-section A to φ with pseudo-Anosov first return map f, $\Gamma(\alpha)$ and $U(\alpha)$ can be constructed in terms of a Markov partition for f. Thus, the classification of Theorem 2.11 is in a certain sense an effective classification, once one has a Markov partition.

The statement of Theorem 2.11 leaves open the following question: suppose that $\varphi^{\#}$ is obtained from φ by blowing up α-null singular orbits in some bad way, so that $\varphi^{\#}$ does not satisfy the conclusions of the $\mathbf{Z}$-Spectral Decomposition Theorem. How does one classify the transverse surfaces to $\varphi^{\#}$ in the class α? The answer is that there are none. To see why, one needs to look at the proof of the $\mathbf{Z}$-Spectral Decomposition Theorem. If $\varphi^{\#}$ has a transverse surface A in the class α, then it can be easily verified that, in the language of [5] and [7], $\langle \alpha, \zeta \rangle \geq 0$ for every periodic quasi-orbit ζ of $\varphi^{\#}$. As shown in [7], this is enough to verify the conclusions of the $\mathbf{Z}$-Spectral Decomposition Theorem.

So in order to finish off the proof of the Transverse Surface Theorem, we need only exhibit a nonnegative cocycle in the class $U(\alpha)$. This is a purely combinatorial problem in graph theory, and it probably exists in the literature somewhere, but for completeness here is a proof.

2.12 Proposition: Existence of nonnegative cocycles. *Given a strongly connected digraph* Γ *and a positive cohomology class* U *of* Γ, *there exists a nonnegative cocycle of* Γ *in the class* U.

Proof. For each cocycle c representing U, we shall define a measure of how far c is from being nonnegative, and then we shall set up an induction argument by proving that if c is not nonnegative, then c differs by a coboundary from some c' which is closer to being nonnegative.

Choose a cocycle c representing U. Let $T_0(c)$ be the subgraph of directed edges on which c is zero, $T_-(c)$ the subgraph of edges on which c is negative, and $T_\leq(c) = T_0(c) \cup T_-(c)$; when the context is clear, we drop the argument c. Notice that $T_\leq$ and T_- are acyclic directed graphs, i.e., they contain no directed loops, for U would take on a nonpositive value on such a loop. As a simple consequence, setting $T = T_\leq$ or T_-, there exist nodes $x \in T$ such that no directed edge in T starts at x; such a node is called a *local maximum* of T. Moreover, for any node $x \in T$, there exists a directed path γ in T with $\mathrm{Tail}(\gamma) = x$ and $\mathrm{Head}(\gamma)$ some local maximum of T; such paths are called *maximal paths* in T starting at x.

For each local maximum x of T_-, let T_x be the union of all maximal paths in $T_\leq$ starting at x. If x itself is a local maximum of $T_\leq$, then T_x consists of the node x alone. Otherwise, T_x is an acyclic directed graph rooted at x (though not necessarily a tree). Now define the *trough* of c to be the subgraph $\mathrm{trough}(c) = \bigcup\{T_x \mid x \text{ is a local maximum of } T_-\}$. Notice that there may be other edges of $T_\leq - T_-$ not contained in $\mathrm{trough}(c)$. Such edges cannot be reached from T_- via a directed path contained completely in $T_\leq$; we think of the union of these edges as forming the "plateau" of c.

Intuitively, c tells how much each directed edge of Γ rises or falls; T_- is the subgraph where c is falling; and $\mathrm{trough}(c)$ contains the subgraph where c bottoms out, although $\mathrm{trough}(c)$ may also contain portions of T_-. One measure of how far c is from being nonnegative is the total absolute value of c on directed edges of T_-. In order to shrink this, one might try taking a local maximum $x \in T_-(c)$, and replacing c by $c' + \delta x$, which for each directed edge $e \in T_-(c)$ with $\mathrm{Head}(e) = x$, has the effect that $|c'(e)| = |c(e)| - 1 < |c(e)|$. Unfortunately, if there is an edge $e' \in T_0$ with $\mathrm{Tail}(e') = x$, then $e' \in T_-(c')$, and we have not reduced the size of

T_- . So the trough has to be taken into account to get a good measure of how far a cocycle is from being nonnegative.

Now we define the complexity of c to be the ordered pair of nonnegative integers $(m_1(c), m_2(c))$, where $m_1(c) = \sum\{-c(e) | e$ is a directed edge of $T_-(c)\}$, and $m_2(c)$ is the number of edges in trough(c). Let complexities be given the dictionary order. Notice that if $m_1(c) = 0$, then $T_-(c) = \varnothing$, so c is nonnegative. When $m_1(c) \neq 0$, we shall show that c can be altered by a coboundary so that the complexity is decreased.

First a word on coboundaries. If x is a node of Γ, we also use x to denote the 0-dimensional cochain with value 1 on x and value 0 on all remaining nodes of Γ. The coboundary δx has the following effect on a directed edge e of Γ: if Head(e) = Tail$(e) = x$, then $\delta x(e) = 0$; if Head$(e) = x \neq$ Tail(e), then $\delta x(e) = +1$; if Tail$(e) = x \neq$ Head(e), then $\delta x(e) = -1$; and otherwise, $\delta x(e) = 0$.

Let γ be a directed path of minimal length in the trough such that $x =$ Tail(γ) is a local maximum of $T_-(c)$ and $y =$ Head(γ) is a local maximum of $T_{\leq}(c)$; if there happens to be a local maximum of $T_-(c)$ which is also a local maximum on $T_{\leq}(c)$, then this path has length zero, and $x = y$ is a local maximum of $T_-(c)$. Let $c' = c + \delta y$.

Since y is a local maximum for $T_{\leq}(c)$, then for any edge e with Tail$(e) = y$, we have $e \not\subset T_{\leq}(c)$, so $c(e) > 0$. If in addition, Head$(e) = y$, then $c'(e) = c(e)$; whereas if Head$(e) \neq y$, then $c'(e) = c(e) - 1$; in either case, we see that Tail$(e) = y$ implies $c'(e) \geq 0$. For any edge e with Tail$(e) \neq y$, evidently $c'(e) \geq c(e)$. It follows that $T_-(c') \subset T_-(c)$, and for every $e \subset T_-(c')$, $c'(e) \geq c(e)$. Therefore, $m_1(c') \leq m_1(c)$.

If $x = y$, then there exists an edge $e \subset T_-$ with Head$(e) = y$, and Tail$(e) \neq y$; thus, $\delta y(e) \geq 0$, so $c'(e) > c(e)$, and it follows that $m_1(c') < m_1(c)$.

If $x \neq y$, we will show that $m_2(c') < m_2(c)$. Since γ is the shortest path in trough(c) from a local maximum of $T_-(c)$ to a local maximum of $T_{\leq}(c)$, then no edge of $T_-(c)$ ends at y; it follows that $T_-(c') = T_-(c)$.

To show that Trough$(c') \subset$ Trough(c), let e be a directed edge in Trough(c'), which we can assume is not in $T_-(c') = T_-(c)$, and let $\gamma' = e'_1 * \cdots * e'_N$ be a shortest directed path in $T_{\leq}(c')$ with $e = e'_N$ and Tail(e'_1) = Tail$(\gamma') = x'_0$ a local maximum for $T_-(c')$. We write $x'_n =$ Head(e'_n) for $1 \leq n \leq N$. Since $T_-(c) = T_-(c')$, then $x'_0 \in T_-(c)$, and moreover x'_0 is a local maximum for $T_-(c)$. Now we prove the following two properties by double induction:

$A(n)$: $\quad x'_n \neq y \qquad (0 \leq n < N)$;

$B(n)$: $\quad e'_1 * \cdots * e'_n \subset \text{trough}(c) \quad (1 \leq n \leq N)$; vacuously true for $n = 0$.

To start, notice that $x'_0 \neq y$, because the shortest path γ from a local maximum of $T_-(c)$ to a local maximum of $T_{\leq}(c)$ has positive length.

Assuming $A(n)$ and $B(n)$ by induction for some $0 \leq n < N$, it follows that $\delta y(e'_{n+1}) \geq 0$, so $c(e'_{n+1}) - c'(e'_{n+1}) = -\delta y(e'_{n+1}) \leq 0$, so $c(e'_{n+1}) \leq c'(e'_{n+1}) = 0$, implying that $e'_{n+1} \subset T_{\leq 0}(c)$; thus, $e'_1 * \cdots * e'_{n+1} \subset \text{trough}(c)$, proving $B(n+1)$.

Assuming $A(n-1)$ and $B(n)$ by induction for some $1 \leq n \leq N$, we claim that $x'_n \neq y$. For if $x'_n = y$, then $\text{Head}(e'_n) = y$ and $\text{Tail}(e'_n) = x'_{n-1} \neq y$, so $\delta y(e'_n) = +1$. However, $e'_n \subset \text{trough}(c) \cap \text{trough}(c')$; and by minimality of γ', $e'_n \not\subset T_-(c') = T_-(c)$. Thus, $e'_n \in T_0(c') \cap T_0(c)$, i.e. $c'(e'_n) = c(e'_n) = 0$, contradicting the equation $c' = c + \delta x'_n$. Thus, $x'_n \neq y$, proving $A(n)$.

This finishes the proof that $\text{trough}(c') \subset \text{trough}(c)$.

Finally, in order to conclude that $m_2(c') < m_2(c)$, we must show that $\text{trough}(c') \neq \text{trough}(c)$. Let e be the final edge in γ, so $\text{Head}(e) = y$. By minimality of γ, $\text{Tail}(e) \neq y$, so $\delta y(e) = +1$, implying that $c'(e) > c(e) = 0$. Thus, $e \subset \text{trough}(c) - \text{trough}(c')$. It follows that $m_2(c') < m_2(c)$. This proves Proposition 2.12 and, therefore, the Transverse Surface Theorem.

3. Lyapounov cocycles

Now we show that, under the hypotheses of the Transverse Surface Theorem, the cohomology class dual to α contains a cocycle which is well behaved along the orbits of the flow $\varphi^\#$. Such a cocycle is called a *Lyapounov cocycle*, and is defined as follows.

A map $f: \mathbf{R} \to S^1$ is said to be *strictly increasing* if it lifts, through the universal covering map $r \to e^{2\pi i \cdot r}$, to a strictly increasing map $\tilde{f}: \mathbf{R} \to \mathbf{R}$. Given a manifold M and a flow $\varphi^\#$ on M, a *Lyapounov cocycle* for $\varphi^\#$ is a continuous function $\omega: M \to S^1$ with the property that for every $x \in M$, the map $r \to \omega(x \cdot r)$ is either strictly increasing or constant. ω can be thought of as a 1-dimensional cocycle in the sense of singular cohomology, via the correspondence $\omega \to \omega^*(d\theta)$; this correspondence is 1-1, up to rigid rotations of S^1. If $\tilde{M} \to M$ is the $\mathbf{Z}$-covering map corresponding to the cohomology class of $\omega^*(d\theta)$, then ω lifts to a map

$\tilde{\omega}: \tilde{M} \to \mathbf{R}$ which is a Lyapounov function in the generalized sense of [1] for the lifted flow $\tilde{\varphi}^{\#}$, i.e., $\tilde{\omega}$ is either constant or increasing along an orbit of $\tilde{\varphi}^{\#}$, depending on whether or not the orbit is contained in the chain recurrent set. Moreover, $\tilde{\omega}$ is $\mathbf{Z}$-equivariant. Conversely, a $\mathbf{Z}$-equivariant Lyapounov function for $\tilde{\varphi}^{\#}$ gives rise to a Lyapounov cocycle for $\varphi^{\#}$.

Given $x \in M$, when the map $r \to \omega(x \cdot r)$ is strictly increasing, then x is said to be a *regular point* of ω; otherwise x is a *critical point* of ω. Given $z \in S^1$, if there exists a critical point $x \in \omega^{-1}(z)$, then z is a *critical value* of ω; otherwise z is a *regular value.*

3.1 Theorem. *Let M^3 be irreducible and atoroidal, σ a fibered face of $B_x(M)$, φ a pseudo-Anosov flow associated to σ and $\alpha \in \partial(\mathrm{Cone}(\sigma))$ an integral homology class. Let $\varphi^{\#}$ be any flow obtained by blowing up the α-null singular orbits of φ, so that $\varphi^{\#}$ satisfies the $\mathbf{Z}$-Spectral Decomposition Theorem. Then there exists a Lyapounov cocycle ω for $\varphi^{\#}$ such that the cohomology class of ω is Poincaré dual to α, and ω has only finitely many critical values. Moreover, the set of critical points of ω is exactly the chain kernel of α, $R(\alpha)$.*

This theorem can be thought of as a generalization of the Transverse Surface Theorem, for if θ is any regular value of ω, then $\omega^{-1}(\theta)$ is a transverse surface to $\varphi^{\#}$ in the class α.

Proof. For notational convenience, we shall as before drop the superscript $^{\#}$, and just write φ for $\varphi^{\#}$.

Let $\tilde{M} \to M$ be the $\mathbf{Z}$-covering map associated to α, T a generator of the action of $\mathbf{Z}$ on $\tilde{M}$, $\tilde{\varphi}$ the lifted flow of φ on $\tilde{M}$ and $R = R(\tilde{\varphi})$ the chain recurrent set of $\tilde{\varphi}$. Choose a cut $C_{\pm} \in \mathrm{Cuts}(R)$ which is moved strictly forward by $\mathbf{Z}$. Choose a transverse surface S to $\tilde{\varphi}$ simply separating the ends of $\tilde{M}$, corresponding to $C_{\pm}$, so that $T(S) \subset \tilde{M}_{+}(S)$, and write $S' = T(S)$.

Enumerate the elements of $\mathscr{C}(R)-(C_{-} \cup T(C_{+}))$ as $\{c_n | n=1, \cdots, N\}$, in such a way that $c_n < c_{n'}$ implies that $n < n'$. To see that this can be done, choose c_1 to be a minimal element of $\mathscr{C}(R)-(C_{-} \cup T(C_{+}))$, which means that for any $c \in \mathscr{C}(R)-(C_{-} \cup T(C_{+}))$, it is false that $c < c_1$. Then choose c_2 to be a minimal element of $\mathscr{C}(R)-(C_{-} \cup T(C_{+}) \cup \{c_1\})$, c_3 a minimal element of $\mathscr{C}(R)-(C_{-} \cup T(C_{+}) \cup \{c_1, c_2\})$, etc.

Notice that if we define $C_{-}^{n} = C_{-} \cup \{c_1, \cdots, c_n\}$ and $C_{+}^{n} = \mathscr{C}(R)-C_{+}^{n}$, then $C_{\pm}^{n}$ is a cut, and $C_{\pm} = C_{\pm}^{0} < C_{\pm}^{1} < \cdots < C_{\pm}^{N} = T(C_{\pm})$. Thus, we can find transverse surfaces simply separating the ends of $\tilde{M}$, $S = S_0, S_1, S_2, \cdots, S_N = S'$, representing the cuts $C_{\pm}^{0}, C_{\pm}^{1}, \cdots, C_{\pm}^{N}$, so that $S_n \subset \tilde{M}_+(S_{n-1})$ for all $n = 1, \cdots, N$. Consider the manifold $M_n \subset \tilde{M}$ bounded by $S_{n-1} \cup S_n$, and the restricted semiflow $\tilde{\varphi} \mid M_n$. Evidently the chain recurrent set of the semiflow $\tilde{\varphi} \mid M_n$ is precisely the set c_n. Thus, according to the theorem of Conley [1], there exists a complete Lyapounov function for $\tilde{\varphi} \mid M_n$, i.e., a continuous function $g_n \colon M_n \to [0, 1]$ such that $S_{n-1} = g_n^{-1}(0)$, $S_n = g_n^{-1}(1)$, g_n is constant on c_n, and g_n is strictly increasing on orbits in $M_n - c_n$.

Now let $M(S, S') \subset \tilde{M}$ be the manifold bounded by S and S', and let $g \colon M(S, S') \to [0, 1]$ be the function defined by

$$g(x) = (g_n(x) + n - 1)/N$$

whenever $x \in M_n$; g is evidently a well-defined continuous function on $M(S, S')$. Moreover, g is a complete Lyapounov function for $\tilde{\varphi} \mid M(S, S')$, i.e., $g^{-1}(0) = S$, $g^{-1}(1) = S'$, g is constant on orbits in the chain recurrent set $c_1 \cup \cdots \cup c_N$ of $\tilde{\varphi} \mid M(S, S')$, and g is strictly increasing off orbits of the chain recurrent set. Since $T(S) = S'$, g can be extended to a $\mathbf{Z}$-equivariant Lyapounov function for $\tilde{\varphi}$, still denoted g. Thus, g is the lift of a Lyapounov cocycle $\omega \colon M \to S^1$ for φ whose cohomology class is Poincaré dual to α.

To see that the critical set of ω is precisely the chain kernel $R(\alpha)$, recall from the comments after the statement of the $\mathbf{Z}$-Spectral Decomposition Theorem that $R(\tilde{\varphi})$ is the full inverse image of $R(\alpha)$. It is evident that each $\mathbf{Z}$-orbit of components of $R(\tilde{\varphi})$ is represented exactly once in the list $\{c_1, \cdots, c_N\}$. Thus, $R(\alpha)$ consists precisely of the image of $c_1 \cup \cdots \cup c_N$ downstairs in M, which is exactly the critical set of ω. Since ω is constant on the image downstairs of each c_n, then ω has only finitely many critical values.

4. Generalizations

The reader may have noticed that most of §2 is independent of the assumption that φ is a pseudo-Anosov flow on a 3-manifold M. In fact, all that is needed for the arguments of §2 to work is a version of the $\mathbf{Z}$-Spectral Decomposition Theorem. In the companion paper [5], generalizations of the $\mathbf{Z}$-Spectral Decomposition Theorem are discussed. We shall review those results, and see what consequences they have for analogues of the Transverse Surface Theorem.

Consider a manifold M, a flow φ on M, and a closed invariant set I of M. Given a codimension-1 submanifold $N \subset M$, not necessarily properly embedded, if N is transverse to I and $I \cap N \subset \operatorname{int}(N)$ we say that N is a *transverse surface* to I. N determines a Cech cohomology class $\alpha_N \in H^1(I; \mathbf{Z})$. If $\alpha \in H^1(M; \mathbf{Z})$ restricts to α_N, we say that N is compatible with α. Following Fried [4], we say that N is a *cross-section* to I if, in addition, every orbit of I intersects N. The analogue of the cone on a fibered face of Thurston's norm is the set $\mathrm{CS}(\varphi; I) \subset H^1(M; \mathbf{R})$ consisting of the closure of all rays through classes in $H^1(M; \mathbf{Z})$ which are compatible with cross-sections to I. This cone can be described by utilizing Fried's homology directions, as follows.

Let the projective homology space $D_M = H_1(M; \mathbf{R})/\mathbf{R}_+$ be topologized as a sphere disjoint union a point. The collection of *homology directions* of $\varphi|I$, denoted $D(\varphi; I)$, is defined as follows. Given $x \in I$, a *closing sequence* based at x is a sequence of the form $(x_i, t_i | i = 1, 2, \cdots)$, where $x_i \to x$, $T_i \to \infty$, and $x_i \cdot t_i \to x$. Let γ_i be the path $x_i \cdot [0, t_i] * p_i$, where p_i is any path from $x_i \cdot t_i$ to x_i staying within a small neighborhood of x. Then any limit point d of the projective homology classes of the γ_i is defined to be an element of $D(\varphi; I)$. It is an elementary consequence of the definitions that $D(\varphi; I)$ is a closed subset of D_M, so $\mathrm{Cone}(D(\varphi; I))$ is a closed cone in $H_1(M; \mathbf{R})$, possibly with the origin deleted. Fried proves that for each $\alpha \in H^1(M; \mathbf{Z})$, α is compatible with some cross-section to I if and only if $\alpha(d) > 0$ for every $d \in D(\varphi; I)$. Thus, $\mathrm{CS}(\varphi; I)$ is the dual cone to $\mathrm{Cone}(D(\varphi; I))$. In particular, if $\mathrm{CS}(\varphi; I) \neq \varnothing$, then $\mathrm{CS}(\varphi; I)$ is a convex, closed cone with nonempty interior.

Now we describe the general notion of $\mathbf{Z}$-spectral decomposition. Let φ be a flow on a manifold M, I a closed, invariant set of φ, and $\alpha \in H^1(M; \mathbf{Z})$ a primitive class. Let $\tilde{M} \to M$ be the $\mathbf{Z}$-covering map associated to α, $\tilde{\varphi}$ the lifted flow of φ on $\tilde{M}$, and $\tilde{I}$ the total lift of I. Thus, $\tilde{I}$ is a closed, invariant set of the flow $\tilde{\varphi}$. We say that $\tilde{I}$ has a *$\mathbf{Z}$-spectral decomposition* if the following conditions hold:

(A) Each chain component of $R = R(\tilde{\varphi} \mid \tilde{I})$ is compact.

(B) There are finitely many orbits of chain components of R under the action of $\mathbf{Z}$.

(C) For any $x \in \tilde{I} - R$, either $L_+(x) = \{+\infty\}$ or $L_+(x)$ is contained in some chain component of R.

(D) Similarly, for any $x \in \tilde{I} - R$, either $L_-(x) = \{-\infty\}$ or $L_-(x)$ is contained in some chain component of R.

(E) If $L_+(x) \neq \{+\infty\}$, then there exists a neighborhood U of $+\infty$ such that for any chain component C of R, if $C \subset U$ then $C \subset R_+(x)$; a similar statement holds when $L_-(x) \neq \{-\infty\}$.

(F) Let $T: \tilde{M} \to \tilde{M}$ generate the $\mathbf{Z}$-action on $\tilde{M}$, so that T moves points towards $+\infty$. Let $p: \tilde{M} \to R$ be a continuous map such that $p(T(x)) = p(x) + 1$. Then there is a constant K such that for any $x \in \tilde{I}$ and any $t \geq 0$, $p(x \cdot t) \geq p(x) - K$.

In [5], the existence of a $\mathbf{Z}$-spectral decomposition is reduced to a property of the *chain kernel* of α. This is the set $R(\alpha; I)$ consisting of all points $x \in I$ such that for all ε, $T > 0$, there exists an ε, T cycle X in I through x, such that $\alpha(\overline{X}) = 0$, where $\overline{X}$ is a closed path in M obtained by interpolating short paths in the gaps of X. We say that a closed invariant set J of φ is *homologically taut* if there exists a neighborhood U of J such that the image of $H_1(U; \mathbf{R}) \to H_1(M; \mathbf{R})$ is spanned by $\mathrm{Cone}(D(\varphi; J))$.

The following proposition is proven in [5], except that condition (F) is not included. But that condition can be verified exactly as in §2.

Proposition. *Let I be a closed, hyperbolic invariant set of φ. Suppose that $\alpha(d) \geq 0$ for all $d \in D(\varphi; I)$, but α is neither identically positive nor identically zero. If $R(\alpha; I)$ is homologically taut, then $\tilde{I}$ has a $\mathbf{Z}$-spectral decomposition. In particular, if $R(\alpha; I)$ is a 1-dimensional hyperbolic invariant set, then it is homologically taut.*

Corollary. *If I is a 1-dimensional hyperbolic invariant set, and α is nonnegative but not everywhere zero or everywhere positive on $D(\varphi; I)$, then $\tilde{I}$ has a $\mathbf{Z}$-spectral decomposition.*

Assuming that $\tilde{I}$ has a $\mathbf{Z}$-spectral decomposition, the proof of the Transverse Surface Theorem goes through practically unchanged. Most of the changes needed are to deal with the fact that we are concentrating on the invariant set $\tilde{I}$, rather than the whole manifold $\tilde{M}$. $\tilde{I}$ can be compactified by adding two points $-\infty$ and $+\infty$. A transverse surface N to $\tilde{I}$ *simply separates the ends* if it separates $\tilde{I}$ into two components, $\tilde{I}_-(N)$ limiting on $-\infty$ and $\tilde{I}_+(N)$ limiting on $+\infty$. The definition of flow isotopy needs to be changed: two transverse surfaces N_1, N_2 are flow isotopic if there are subsurfaces $N_i' \subset N_i$, such that $I \cap \mathrm{int}(N_i) = I \cap \mathrm{int}(N_i')$, and N_1', N_2' are isotopic along flow lines. Also, the partial order on flow isotopy classes is defined as $[N] < [N']$ if N, N' can be chosen so that $N' \cap \tilde{I} \subset \tilde{I}_+(N)$. With these definitional changes, one proves in the same manner that $\mathrm{Cuts}(\mathscr{C}(R))$ is in order preserving 1-1 correspondence with the set of flow isotopy classes of transverse surfaces to $\tilde{I}$ which simply separate the ends. The remainder of the proof of the Transverse Surface Theorem goes through unchanged. Thus, we have:

General Transverse Surface Theorem. *Given a flow φ on a manifold M, an invariant set I, and an integral cohomology class α, let $\tilde{M} \to M$*

be the **Z***-covering map associated to* α, $\tilde{\varphi}$ *the lifted flow and* $\tilde{I}$ *the total lift of* I. *If* $\tilde{I}$ *has a* **Z***-spectral decomposition, then there exists a transverse surface to* φ *compatible with* α. *In particular, if* I *is a* 1*-dimensional basic set of an Axiom A flow, and if* α *is nonnegative but not identically zero on* $D(\varphi; I)$, *then there exists a transverse surface to* φ *compatible with* α.

We say that Σ is a *transverse branched surface* to I if Σ is a codimension-1 branched submanifold of M such that $\Sigma \cap I \subset \text{int}(\Sigma)$, and Σ is transverse to I. A cross-section N to I is *carried* by Σ if there exists a smaller cross-section $N' \subset N$ such that N' is isotopic along the flow into a regular neighborhood of Σ. In order to apply the arguments of §1, we need only that $\text{CS}(\varphi; I)$ be positively spanned by finitely many integral classes. In [4], this is proven when I is a basic set of an Axiom A flow. Thus, we have:

General Branched Surface Theorem. *Suppose* I *is a closed invariant set for a flow* φ *on a manifold* M. *Let* $\alpha_1, \cdots, \alpha_N \in H^1(M; \mathbf{Z})$ *be a positive spanning set for* $\text{CS}(\varphi; I)$. *If each* α_n *is represented by a transverse surface to* φ, *then there exists a transverse branched surface to* I *carrying all cross-sections compatible with cohomology classes of* M. *In particular, this holds if any of the following conditions hold*:

(i) *the total lift* $\tilde{I}$ *in the* **Z***-covering of* $\tilde{M}$ *associated to any* α_n *has a* **Z***-spectral decomposition for* $n = 1, \cdots, N$;

(ii) φ *is Axiom* A, I *is a basic set of* φ, *and* $R(\alpha_n; I)$ *is homologically taut for* $n = 1, \cdots, N$;

(iii) φ *is Axiom* A *and* I *is a* 1*-dimensional basic set of* φ.

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