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SELF-DUAL CONFORMAL STRUCTURES ON $l\mathbb{C}P^2$

ANDREAS FLOER

Abstract

We prove the existence of conformal structures with self-dual Weyl tensor on connected sums of arbitrarily many copies of two-dimensional complex projective space $\mathbb{C}P^2$. They are constructed from the standard conformal structures on $\mathbb{C}P^2$ by a gluing procedure.

1. Introduction

A conformal structure c on a smooth finite dimensional manifold M is an equivalence class $c = [g]$ of Riemannian metrics g on M, where $g_1 \sim g_2$ are (conformally) equivalent if $g_2 = f \cdot g_1$ for a smooth function $f\colon M \to \mathbb{R}_+$, the set of positive real numbers. We say that $(M, [g])$ is conformally flat if there exists a system of charts ψ, $M \supset U \to \mathbb{R}^n$ such that $\psi^* g \sim g_0$, where $\mathbb{R}^n$ is a Euclidean n-space, and g_0 is the Euclidean metric. The condition for the existence of such a restricted atlas can be stated as a nonlinear partial differential equation, called the integrability condition, on the conformal structure itself.

In two dimensions, a conformal structure is precisely specified by assigning an orthogonal direction to each direction in the tangent space T_xM of M at $x \in M$. If M is orientable, this yields a 1-1 correspondence with complex structures on M, so that a diffeomorphism ϕ of M is conformal (i.e., $\phi^* g \sim g$) if and only if it is holomorphic. It follows that every orientable conformal 2-dimensional manifold allows a conformal atlas, since it allows a holomorphic one. In dimensions higher than two, the set of conformal diffeomorphisms is much smaller. For the constant conformal structure on $\mathbb{R}^n$, $n > 2$, for example, it is a finite-dimensional group. Correspondingly, it is less likely to find a conformal atlas of M. In dimensions higher than three, the integrability condition for conformal structures is the Weyl tensor W, which is a component of the Riemannian curvature tensor R, i.e., of the integrability condition for the metric itself. (In dimension 3, the integrability condition is a first order differential

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equation in R.) A special feature of the Weyl tensor in dimension four is that it can be decomposed in an invariant way into two equally large parts $W_\pm$ (see [1]). Hence it makes sense to require a conformal structure to satisfy only "half" of the integrability condition. We say that c is self-dual if $W_-(c) = 0$.

One of the interesting facts about the self-duality equation is that its linearization is elliptic modulo the action of the diffeomorphism group of M. As a result, the set

$$\mathcal{S}\mathcal{C}(M) = \{c | W_-(c) = 0\}/\mathcal{D}iff(M) \tag{1.1}$$

of equivalence classes of self-dual conformal structures can be expected to be a smooth manifold whose dimension can be calculated by the Atiyah-Singer Index Theorem as the index of the corresponding elliptic system. In fact, this is the case under two additional conditions. The first is the vanishing of the cokernel of the linearization of W at $c \in \mathcal{S}\mathcal{C}(M)$. The second condition is that the group of diffeomorphisms acts freely on the space of self-dual connections. We can weaken this condition by including the case where the dimension of the group G_c of conformal diffeomorphisms of c is constant (locally) at c. Then the Atiyah-Singer Index Theorem yields

$$\dim_c \mathcal{S}\mathcal{C}(M) - \dim G_c = I(M) = \tfrac{1}{2}(29|\tau| - 15_\chi), \tag{1.2}$$

where τ is the signature of M, and χ is its Euler characteristic (see [3]). For example, $I(S^4) = -15$, and the self-dual (in fact, flat) conformal structure induced by the standard metric has the 15-dimensional conformal equivalence group $G = O(5, 1)$. Another simple example is $\mathbb{C}P^2$ with the standard (Fubini Study) metric, where $I(\mathbb{C}P^2) = \dim G(\mathbb{C}P^2) = 8$. We will in fact see (see Proposition 3.1 and Lemma 4.1) that DW is surjective in these cases. Examples where $\mathcal{S}\mathcal{C}$ has positive dimension were given by Poon [17], [18] on the connected sum of two and three copies of $\mathbb{C}P^2$, where the connected sum is performed in such a way that the intersection form is positive definite. Examples where (1.2) fails due to a nontrivial cokernel of DW_- are the Kummer surface K_3 (see [3]) and the flat 4-tori T^4 with $I(T^4) = 0$, $\dim \mathcal{S}\mathcal{C}(T^4) = 8$, and $\dim G_c = 4$.

By the above, one is drawn to compare the self-duality equation with the instanton equation in 4-dimensional manifolds. For example, one might expect $SC(M)$ to satisfy similar compactness properties as the moduli space of gauge equivalence classes of self-dual Yang-Mills connections. Here, the ends correspond to the "splitting" of an instanton family into "smaller" instantons (see [23], [24]). Conversely, under appropriate conditions, one can "glue" two or more existing instantons together (see [19]). In the case of self-dual conformal structures on compact 4-manifolds M and N, an analogous construction might yield self-dual conformal structures on the connected sum $M\#N$ (see [21]). In fact, Poon's constructions seem to "split up" in a certain limit into standard conformal structures on $\mathbb{C}P^2$. In this paper, we use Taubes' gluing method to prove

Theorem 1. *There exist self-dual conformal structures on any positive definite connected sum of finitely many copies of* $\mathbb{C}P^2$.

Most likely, the conformal structures can be represented by metrics of positive scalar curvature. Actually, we construct noncompact families of self-dual conformal structures, which have the dimension predicted by (1.2) and converge in some sense to the standard structures on $\mathbb{C}P^2$ (see Theorem 2 in the next section).

The method of the proof is the one suggested by [19]: First, by means of cutoff functions, one defines a family c_ρ of conformal structures on the connected sum which are self-dual outside some "small" subset. Then one inverts the linearization of W_- at these approximate solutions to obtain self-dual structures. The problem is to find Banach norms $\| \ \|_\rho$ on a chart of the space of conformal structures and $\| \ \|_{W_\rho}$ on the target space of W_- so that W_- is continuous and

(1) $\lim_{\rho\to\infty} \|W_-(c_\rho)\|_{W,\rho} = 0$;

(2) $DW_-(c_\rho)$ has a left inverse which is bounded independently of ρ.

Hence the problem is essentially reduced to finding the right "surgery" procedure. To prove (1), the connected sum must be performed on smaller and smaller balls in $\mathbb{C}P^2$, which seems to imply that the curvature of the metric obtained charges rapidly. It is then difficult to invert the linearizations in a controlled way. The new idea in the approach presented here is to consider the conformal structure c of $\mathbb{C}P^2$ on a (pointed) neighborhood of a point x as a conformal structure on a half cylinder which "approaches" the standard flat structure on $\mathbb{R}\times S^3$, where S^m is the standard unit m-sphere. This is just a matter of choosing an appropriate metric representing c. It turns out that one can choose a "cylindrical chart" $\omega\colon \mathbb{R}_+ \times S^3 \to \mathbb{C}P^2$ in a natural way so that the rate of approach is exponential. Now there is an obvious way to perform the connected sum, which is carried out in §2. To invert the linear operator, we use the Fredholm theory on asymptotically cylindrical manifolds recently developed by Lockhard and McOwen [13] (see §§3 and 4). This theory has been previously applied to instanton problems in [21].

The self-dual conformal structure on the cylindrical chart can be considered as a solution of an ordinary differential equation on the set of metrics on S^3 approaching a "hyperbolic" rest point. In fact, once a radial coordinate is fixed on a neighborhood U_x of a point $x \in M$, there exists a unique parametrization of $U_x - x$ by an open subset of $\mathbb{R} \times S^3$ mapping the radial coordinate into the time variable τ and the conformal structure into one represented by a metric of the form $(d\tau)^2 + g\tau$. Then the self-duality equation becomes a "dynamical" equation in the family g_τ of metrics on S^3. (In a similar way, the self-dual Yang-Mills equations on $\mathbb{R}\times S^3$ can be interpreted as the "flow equation" for a canonical vector field on the space of connections on S^3.) Of course, the dynamical system for g_τ lacks the continuity properties necessary to solve the initial value problem. However, the solutions that do exist should decay like trajectories of a finite-dimensional system. In this way, charts as constructed explicitly here may be found in more general situations. Moreover, it is likely that the gluing procedure described in §§2–4 can be carried out whenever such charts can be bound and DW_- is surjective for the self-dual manifolds involved. If the latter condition fails, one could try to reduce the problem to finite dimensions as in [22]. However, a more general gluing procedure based on complex analytic twistor method has meanwhile been established by Donaldson and Friedman [2]. Most recently, explicit formulas for certain families of self-dual conformal structures on $l\mathbb{C}P^2$ were obtained by C. LeBrun [12].

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2. The construction of self-dual structures

We consider $\mathbb{C}P^2$ as the quotient of $S^5 \subset \mathbb{R}^6 \simeq \mathbb{C}^3$ with respect to the phase operation of S^1 and $\mathbb{C}^3$. If S^5 is defined by means of Hermitian metric, then S^1 operators by isometries, and the quotient inherits a metric called the Fubini Study metric.

Let $x \in \mathbb{C}P^2$ be represented by the orbit of the S^1-action through $(1, 0, 0)$. Then the map

$$\omega\colon \mathbb{R} \times S^3 \to S^5/S^1 = \mathbb{C}P^2 \tag{2.1}$$
$$\omega(r, \theta) = [(1 + e^{2r})^{-1/2}(e^\tau, \theta)]$$

is injective. In fact, it is a diffeomorphism onto $\mathbb{C}P^2 - x - x^{\perp}$, where $x^{\perp} = S^3/S^1$ and $S^3 = \{(0, y, z) \mid |y|^2 + |z|^2 = 1\}$. Note that the map ω is uniquely determined given x and a frame of $\mathbb{C}P^2$ at x. We can reduce this parameter set to the U_2-bundle of unitary frames on $\mathbb{C}P^2$ using the complex structure of $\mathbb{C}P^2$. This also defines a complex structure on $\mathbb{R} \times S^3$.

Lemma 2.1. *With* $\gamma(\tau) = (1 + e^{2\tau})(1 + e^{-2\tau})$, *we have*

$$\gamma\omega^* g = \begin{pmatrix} 1 & 0 \\ 0 & (1 + e^{-2\tau})1 \end{pmatrix},$$

where the first factor is the complex plane generated by $\frac{\partial}{\partial \tau}$, *and the second factor is its orthogonal in* TS^3.

The proof of Lemma 2.1 proceeds by direct calculation. Given any chart ω as in (2.1), we fix a metric g_ω which is conformally equivalent to g and satisfies $\omega^* g_\omega = \gamma\omega^* g$ on $\mathbb{R}_+ \times S^3$. For any positive ρ, we also define g_ω^ρ by

$$\omega^* g_\omega^\rho = \beta_\rho \omega^* g_\omega, \tag{2.2}$$

where $\beta_\rho(\tau) = \beta(\tau - \rho)$, and $\beta\colon \mathbb{R} \to [0, 1]$ is a smooth function satisfying $\beta(\tau) = 0$ for $\tau \leq 0$ and $\beta(\tau) = 1$ for $\tau \geq 1$. Now let M denote the disjoint union of finitely many copies of $\mathbb{C}P^2$. For any pair $v = \omega_\pm$ of charts as in Lemma 2.1 with different centers and for any real number ρ, consider the equivalence relation generated by

$$\omega_+(\rho + \rho\theta) \sim \omega_-(-\rho - \tau, \overline{\theta}),$$

where $\theta \to \overline{\theta}$ is some fixed orientation reversing isometry of S^3. The metrics $g_{\omega_+}^\rho$ and $g_{\omega_-}^\rho$ then give rise to a smooth metric g_χ on $M/\sim$. This metric depends only on ρ, centers $v_\pm$ of $\omega_\pm$, and on the identification $T_{v_-} M \to T_{V_+} M$ defined by the two frames. More precisely, since the metric of Lemma 2.1 is invariant under rotations of $\mathbb{R} \times S^3$ around the "Hopf field" $i\frac{\partial}{\partial \tau}$, it only depends on ρ and an element of the bundle

$$SO_3(v_-, v_+) := U(T_{v_-} M, T_{v_+} M)/S^1 \tag{2.3}$$

over $M \times M - \Delta$. We may identify $SO_3(x, y)$ with the isometries of the spheres $x^{\perp}$ and $y^{\perp}$ induced by the two cylindrical charts $\omega_\pm$. We want to apply this procedure to finitely many point pairs simultaneously.

We denote therefore by V a finite set of unordered pairs $v = (v_+, v_-) \in M \times M - \Delta$ which do not intersect. It defines a graph Γ, whose vertices can be identified with the pairs in v in V, and whose edges can be identified with the components of M. We will always assume that Γ is connected. Then we denote by $\mathscr{V}_\Gamma$ the set of all V defining the same graph Γ.

Definition 2.1. Let $R \to \mathscr{V}_\Gamma$ denote the SO_3-bundle over $\mathscr{V}$ defined by (2.3). Then for every compact subset $K \subset \mathscr{V}_\Gamma$ and for ρ_K large enough, by the above procedure we define a manifold M_Γ and a map

$$\alpha: R|_K \times [\rho_K, \infty) \to S^2(TM_\Gamma),$$
$$\chi := (V, \boldsymbol{\sigma}, \boldsymbol{\rho}) \to g_\chi.$$

The restriction to large ρ is necessary to avoid overlaps . For a proof of Theorem 1, it suffices to consider a "simple" graph Γ which connects a given vertex with any number of vertices by a simple edge. However, the gluing procedure also works for a graph Γ. The reader may, however, prefer to check the construction first in the simple case.

Note that the unitary group $U(3)$ acts isometrically on $\mathbb{C}P^2$, and that the quotient

$$G = U(3)/S^1 = SU(3)/\mathbb{Z}_3$$

acts transitively. Therefore, the true parameter space is an open set in the quotient $R/G^E \times \mathbb{R}_+^\Gamma$. In fact, one verifies that if $\mathscr{C}(M_\Gamma)$ denotes the set of diffeomorphism classes of smooth conformal structures on M_Γ, then the maps

$$\alpha: R/G^E \times \mathbb{R}_+^\Gamma \supset R/G^E|_K \times [\rho_K, \infty)^V \to \mathscr{C}\mathscr{S}(M_\Gamma)$$

induced by Definition 2.1 are injective. They define smooth families wherever the dimension of the stabilizer group is locally constant. Here, a smooth family of conformal equivalence classes is defined as a smooth tensor field over a product space. In the presence of discrete (hence finite) stabilizer groups in R, smoothness can be defined locally in terms of a finite covering. From (1.2) we obtain

$$I(M_\Gamma) = 15|V| - 8|E| = \dim(R \times \mathbb{R}_\Gamma^v) - \dim G^E. \tag{2.4}$$

Hence the parameter space has the same dimension as the expected space of equivalence classes of self-dual conformal structures, provided that the dimension of the stabilizer group $G_V \subset G^E$ of $V \in R$ is either zero or that it at least does not jump at χ.

An example of a nontrivial stabilizer group is given by the graph connecting two different copies of $\mathbb{C}P^2$. Here, R is an SO_3-bundle over $\mathbb{C}P^2 \times \mathbb{C}P^2$ which is acted upon transitively by $(U_3/U_1)^2$ with stabilizer group $T^2 = S^1 \times S^1$. Hence $\dim R/G^E = 0$, and the image of α is 1-dimensional. The index formula (1.2) yields $I(\mathbb{C}P^2 \# \mathbb{C}P^2) = -1$, which predicts a 1-dimensional family of self-dual structures if their conformal equivalence groups are 2-dimensional. This is in fact the case for the self-dual conformal structures constructed by Poon [17].

One easily verifies that the above example is the only one exhibiting a symmetry group of constant nonzero dimension. An example for "exceptional" symmetry groups is given by the graph with three edges and two vortices v and w, which produces $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$. Here, the symmetry group contains S^1 if $v_+ \perp w_-$, and is discrete otherwise. The methods used in this paper do not apply to produce self-dual structures out of the exceptional initial conditions. In the remaining cases, we have the following result.

Theorem 2. *For $R \to \mathscr{V}_\Gamma$ as above and for each compact set $K \subset \mathscr{V}_\Gamma$ such that G_V is discrete for each $V \in K$, there exist ρ_K, $C \in \mathbb{R}_+$, and a local diffeomorphism*

$$\beta\colon R_K/G^E \times [\rho, \infty)^\Gamma \to \mathscr{S}\mathscr{C}(M_\Gamma),$$
$$\chi = (\Gamma, \rho, \theta) \mapsto [g_\chi + h_\chi]$$

for each $\rho > \rho_K$ with $\|h\|_\infty + \|\nabla h\|_\infty \le Ce^{-\rho/C}$.

In principle, our methods could be modified to apply to the case of $\mathbb{C}P^2 \# \mathbb{C}P^2$. However, this is omitted for the sake of brevity and in view of Poon's results. The proof of Theorem 2 is given in §4.

3. Fredholm theory for pointed conformal structures

A conformal structure on a smooth manifold M is usually defined as a section of the bundle Γ with fibers

$$\Gamma_x = (S_+^2 T_x P)/\mathbb{R}_+.$$

Here, $S_+^2 T_x M$ is the cone of positive symmetric bilinear forms on $T_x P$, and $\mathbb{R}_+$ operators by scalar multiplication. One therefore often represents a conformal structure by a Riemannian metric g. For 4-dimensional manifolds there exists another description of conformal structures, which does not make any reference to Riemannian geometry. Note that the Hodge duality isomorphism

$$*: \Omega^p M \to \Omega^{4-p} M$$

induced by a Riemannian metric depends for $p = 2$ only on the conformal structure. Since moreover $*$ is symmetric on Ω^2, it induces a splitting

$$\Omega^2 M = \Omega^- \oplus \Omega^+$$

of Ω^2 into the positive and negative eigenspace. These are called the self-dual and anti-self-dual 2-forms, respectively. Note that with respect to the bilinear form $\langle\alpha, \beta\rangle = (\alpha \cup \beta)[M]$ on Ω^2, Ω^- is negative definite. In fact, any smooth 3-dimensional subbundle of Ω^2 with this property defines a unique conformal structure for which it coincides with Ω^-. Infinitesimal deformations of a conformal structure are therefore described by sections of the bundle $\mathrm{Hom}(\Omega^-, \Omega^+)$. We use this second description whenever the invariance under smooth maps is important, and the first description for more practical calculations.

To describe the Weyl tensor, let so_4 denote the Lie algebra of the group of linear isometries (with respect to the conformal structure) of T_xP. Then there exists a unique splitting $so_4 = so_3^- \oplus so_3^+$ of Lie algebras. In the same way, we have a splitting of the bundle $so_4(M)$ of infinitesimal isometries of TM into $so_3^-(M) \oplus so_3^+(M)$. Since the Riemannian curvature tensor with respect to any metric representative g of γ takes values in $\Omega^2 \otimes so_4(M)$, we can consider its component in $\Omega^- \otimes so_3^-(M)$. Note that raising one index by the metric g defines an isomorphism $\Omega^2 \simeq so_4(M)$ mapping $\Omega^\pm$ into $so_3^\pm(M)$. It therefore induces a splitting of $\Omega^- \otimes so_3^-(M)$ into a trace part, an antisymmetric part, and a trace free symmetric part. We will denote the latter by $\Omega^- \otimes_5 so_3^-(M)$; the corresponding component of the curvature is the anti-self-dual Weyl tensor W_-.

Let us denote the linearization of W_- by

$$(3.1) \qquad D: C^\infty(\mathrm{Hom}(\Omega^-, \Omega^+)) \to C^\infty(\Omega^- \otimes_5 so_3^-(M).$$

It is a second order differential operator, which we shall examine more closely in §5. Moreover, the linearized operation of the diffeomorphism group on $\mathscr{C}$ is described by the operator

$$(3.2) \qquad L: C_0^\infty(TM) \to C_0^\infty(\mathrm{Hom}(\Omega^-, \Omega^+)); \; (LX)(\lambda) = \pi_+ L_X \lambda,$$

where L_X is the Lie derivative on the set of 2-forms, and π_+ is the projection onto Ω^+. Again, we show in §5 that L is a differential operator in the vector field X alone. By L^+ we denote the L^2-adjoint of L with respect to some fixed metric g. It is well known (see also §4 below) that (D, L^+) is an elliptic system of partial differential equations (of mixed order). For a compact manifold M, it therefore induces a Fredholm operator

$$(D, L^+)\colon U \to V \oplus W,$$

where $U = L^p_2(\mathrm{Hom}(\Omega^-, \Omega^+))$, $V = L^p_1(TP)$, $W = L^p(\Omega^- \otimes_5 so_3^-(M))$ are Sobolev spaces with respect to some metric on M. In fact, the following result is well known:

Lemma 3.1. *For $M = \mathbb{C}P^2$ with the standard conformal structure, the operator $(D, L^+)\colon U \to V \oplus W$ is injective.*

Proof. We follow a suggestion of C. LeBrun. It relies on the twistor construction (see [1]) which relates self-dual conformal structures on M to complex structures on a certain S^2-bundle Z over M. In the case of $M = \mathbb{C}P^2$, Z is a flag manifold whose complex structure does not admit any deformations. In fact, infinitesimal deformations of a complex structure on Z are described by $H^1(Z, \mathscr{O}(TZ))$, the first cohomology of the sheave of local holomorphic sections of the tangent bundle of Z, which is trivial in the case. We want to show in general that

$$H^1(Z, \mathscr{O}(TZ)) \cong \ker DW/\operatorname{Im} L. \tag{3.3}$$

To relate holomorphic sheaf cohomology on the 4-dimensional complex manifold Z to differential operators on the base M, we consider the 2-dimensional complex bundle $D \subset \Omega^{(0,1)}Z$ of $(0, 1)$-forms vanishing along the fibers. Consider the sheaves

$$\Omega^p_\pi(TZ) = \mathscr{O}_\pi(\Omega^p D \otimes T^{(1,0)}Z).$$

Here, $\mathscr{O}_\pi$ denotes the sheaf of sections which are holomorphic along the fibers of the twistor fibration. Then we have an exact sequence of sheaves

$$0 \longrightarrow \Omega^0(TZ) \longrightarrow \Omega^0_\pi(TZ) \xrightarrow{\bar\partial} \Omega^1_\pi(TZ) \xrightarrow{\bar\partial} \Omega^2_\pi(TZ) \longrightarrow 0$$

with $H^1(\Omega^0(TZ)) = H^1(TZ)$. The idea of the proof is now that the zeroth cohomologies of the above sheaves can be identified with spaces of sections of vector bundles over M, whereas the higher cohomologies tend to vanish because of the "softness" of the sheaves in the M-direction. For example, it follows from the LeRay-Singer spectral sequence that $H^1(\Omega^0_\pi(TZ)) = 0$. Hence splitting the above sequence into two short exact sequences with $K = \ker[\Omega^1_\pi(TZ) \xrightarrow{\bar\partial} \Omega^2_\pi(TZ)]$, we obtain long exact sequences

$$H^0(T^Z) \longrightarrow H^0(\Omega^1_\pi(TZ)) \longrightarrow H^0(K) \longrightarrow H^1(TZ) \longrightarrow 0,$$
$$0 \longrightarrow H^0(K) \longrightarrow H^0(\Omega^1_\pi(TZ)) \xrightarrow{\overline{\partial}_1} H^0(\Omega^2_\pi(TZ)) \longrightarrow H^1(K) \longrightarrow \cdots .$$

From the second sequence we learn that $H^0(K) = \ker \overline{\partial}_1$, so that

$$H^1(TZ) = \ker(\overline{\partial}_1)/\overline{\partial}_0 H^0(\Omega^0_\pi(TZ)).$$

Now let $S_\pm$ denote the spinor bundles over X. Then the zeroth cohomologies can be described as extensions

$$\begin{array}{lllll} 0 \to C^\infty(\Omega^2_-) & \to H^0(\mathscr{O}_\pi(TZ)) & \to C^\infty(T_{\mathbb{C}}M) & & \to 0 \\ 0 \to C^\infty(S_+ \otimes S^3_-) & \to H^0(Z, \Omega^1_\pi(TZ)) & \to C^\infty(\Omega^2_- \oplus (\Omega^2_- \otimes \Omega^2_+)) & & \to 0 \\ 0 \to C^\infty(S^4_+) & \to H^0(Z, \Omega^2_\pi(TZ)) & \to C^\infty(S_+ \otimes S^3_-) & & \to 0 \end{array}$$

such that the $\overline{\partial}$-operators induce the identities on $C^\infty(M, \Omega^2_-)$ and $C^\infty(S_+ \otimes S^3_-)$. Thus we have an induced isomorphism

$$\ker \overline{\partial}_1 = \ker(0 + \not{D}^2 \colon \Gamma(\Omega^2_-) \oplus \Gamma(\Omega^2_- \otimes \Omega^2_+) \to \Gamma(S^4_+)),$$

where $\not{D}^2 \colon \Gamma(\Omega^2_- \otimes \Omega^2_+) = \Gamma(S^2_+ \otimes S^2_-) \to \Gamma(S^4_+)$ is the operator

$$\not{D}^2 \varphi_{ABA'B'} \mapsto \nabla^{A'}_{(C} \nabla^{B'}_{D} \varphi_{AB)A'B'},$$

i.e., the linearization of W_-. Finally, ∂_0 induces the operator $L\colon \Gamma(T_{\mathbb{C}}M) = \Gamma(S_+ \otimes S_-) \to \Gamma(S^2_+ \otimes \mathscr{S}^2_-)$ given by $L\colon \gamma_{AB'} \mapsto \nabla^{(AB;)}_{(A_\gamma B')}$, i.e., the Lie derivatives of the conformal structure. This completes the proof of (3.3). q.e.d.

To describe pointed conformal structures, we will need Sobolev norms with special weights at distinguished points. Let us consider $\mathring{M} := M - \{x_1 \cdots x_n\}$ as a manifold with cylindrical ends; i.e., there exist cylindrical charts $\omega_i \colon \mathbb{R}_+ \times S^3 \to \mathring{M}$ covering $\mathring{M}$ up to a compact set M_0. On $\mathring{M}$, let us fix a metric g so that $\omega_i^* g$ coincides with the standard metric on $\mathbb{R}_+ \times S^3$. Let $\tau\colon \mathring{M} \to \mathbb{R}$ be a function coinciding with the τ-variable in the range of ω_i. Then consider the exponentially weighted norms

$$\|\xi\|_{:\,k} = \|e^{\varepsilon\tau}\xi\|_{k,p}, \tag{3.4}$$

where ξ is a section of a metric bundle over $\mathring{M}$, $\| \ \|_{k,p}$ is the usual Sobolev norm, and ε is a positive real number.

Definition 3.1. For a given conformal structure c on M, consider the bundles $\mathrm{Hom}(\Omega^-, \Omega^+)$, $\Omega^- \otimes so_3^-$, and $T\overset{\circ}{M}$ with metrics induced by the metric g on M. Then define for $p > 2$ and $\varepsilon > 0$

$$U_c = L^p_{2;\varepsilon}(\mathrm{Hom}(\Omega^-, \Omega^+)), \quad V_c = L^p_{1;\varepsilon}(TX), \quad W_c = L^p_{0;\varepsilon}(\Omega^- \otimes SO_3^-).$$

The reason for introducing exponentially weighted norms is the Fredholm theory developed in [8], [13], [14] for manifolds with cylindrical ends. In §5, we will exhibit (D, L^+) as an elliptic system of partial differential equations, which is asymptotically "constant" at the ends. The following result is then an application of Theorem 1.3 of [13]:

Proposition 3.1. *For every $c \in \mathscr{C}(M)$ and $\varepsilon \in \mathbb{R}$ we have a continuous operator*

$$(L^+, D) \colon U_c \to V_c \oplus W_c.$$

Moreover, there exists a discrete set $\sigma_\infty \subset \mathbb{R}$ (containing zero) so that if $\varepsilon \notin \sigma_\infty$, then (L^+, D) is Fredholm.

The discrete set σ_∞ is examined in §5. In particular, we will show that it contains zero. Note that the Fredholm index on (L^+, D) may depend on ε (for example, by Theorem 1.4 of [13] and Proposition 5.2 below, it jumps by $7n$ when ε passes through 0). From now on we will only consider positive values of ε which are smaller than the first positive element of σ_∞.

The set of pointed conformal self-dual structures, i.e., of self-dual conformal structures on $\mathbb{C}P^2$ up to diffeomorphisms which fix $x_1 \cdots x_n$, is a manifold of dimension $4n - 8$ whenever $n \geq 4$. If we assume this to be the case for all components of M, then one can show that for small positive ε, the kernel of (D, L^+) is isomorphic to the tangent space $\ker \overline{D}/\overline{L}V_0(M, \mathbf{x})$ of pointed conformal structures on $(M, \mathbf{x})$. Here, $\overline{D}$, and $\overline{L}$ denote the operators D and L on M, and $V_0(M)$ is the set of smooth vector fields X such that $X(x) = 0$ for all $x \in \mathbf{x}$. A priori, however, we will only need

Proposition 3.2. *We have an injection*

$$\pi e_* \colon \ker(D, L^+) \to \ker \overline{D}/\overline{L}V_0(M, \mathbf{x}).$$

Proof. Assume that $\pi e_* \xi = 0$ for some $\xi \in \ker(D, L^+)$. This means that $e_*\xi = \overline{LX}$ for some $\overline{X} \in \overline{V}_0$. Let X denote the vector field on $\overset{\circ}{M}$ satisfying $e_* X = \overline{X}$. Then $e_*(LX) = \overline{L}e_* X = L\overline{X} = e_*\xi$, so that $LX = \xi$. Hence $L^+ LX = 0$. But since $\xi \in V$, we can integrate by parts to obtain $\langle LX, LX \rangle = \langle X, L^+ LX \rangle = 0$.

4. Proof of theorems

To prove Theorems 1 and 2, we may restrict ourselves to a neighborhood of a given set V of points pairs $v = (v_+, v_-)$ in V_Γ (see §2). For technical reasons, we will modify the construction of Definition 2.1 as follows: On each $\mathbb{C}P^2_e$, $e \in E$, choose an additional point x_e not coinciding with and not perpendicular to any of the $v_\pm$, $v \in V$. Correspondingly, we consider $\mathring{M}_\Gamma$ as a closed manifold with $|E|$ points taken away. We also replace g_χ for $\chi = (V, \boldsymbol{\sigma}, \boldsymbol{\rho})$ by a metric on M_Γ which for τ large enough coincides with (2.1) under a cylindrical chart centered at x_e, $e \in E$. Outside a neighborhood of x_e, we leave g_χ as in Definition 2.1. On $\mathring{M}_\Gamma$, we now consider the Banach spaces U_χ, V_χ, and W_χ with the exponentially weighted norms of (3.4). The aim is to find $\xi \in U_x$ such that

$$W_-([g_\chi + \xi]) = 0, \qquad L^+_\chi \xi = 0 \tag{4.1}$$

on $\mathring{M}_\Gamma$. Since each such perturbation converges to zero in the end, it results in a continuous conformal structure on the compactification of $\mathring{M}_\Gamma$ (recall from §3 that $\mathrm{Hom}(\Omega^+, \Omega^-)$ has conformal weight zero). This will prove Theorem 1. A more careful analysis of ξ will also yield a proof of Theorem 2.

To obtain ξ, we will work with the "practical" description of conformal geometry in terms of metric representatives. Then U of Definition 3.1 has to be replaced by the space $U = L^p_{2;\varepsilon}(S^2_{\mathrm{trf}} TM_\Gamma)$, of tracefree symmetric tensors on TM_Γ. Since near any conformal structure represented by an asymptotically constant, the transformation $(S^2TP)/\mathbb{R} \to \mathrm{Hom}(\Omega^-, \Omega^+)$ has uniformly bounded derivatives, Propositions 3.1 and 3.2 remain true. Moreover, let us identify the target spaces $\Omega^- \otimes_5 so_3^-$ for different conformal structures near $[g_\chi]$. Consider therefore the metric on $\Omega^2 \otimes so_4$ induced by g_χ. Then the orthogonal projection

$$\pi \colon \Omega^2 \otimes so_4 \to (\Omega^- \otimes_5 so_3^-)_\gamma$$

define a linear isomorphism when restricted to any subspace $(\Omega^- \otimes_5 so_3^-)_{\gamma'}$, as long as γ' is close enough to γ. We can therefore define the function

$$\begin{aligned} W^\chi_- \colon U_\chi \supset \widetilde{U}_\chi &\to W_\chi \otimes V_\chi, \\ \xi &\mapsto (\pi W_-([g_\chi + \xi]), L^+_\chi \xi), \end{aligned} \tag{4.2}$$

where $\widetilde{U}_\chi$ is a suitable neighborhood of 0 in U_χ consisting of such ξ for which $g + \xi$ is a metric.

Lemma 4.1. *The map* W_-^χ *is smooth, with* $DW^\chi(0) = (DW_-([g_\chi]), L_\chi^+)$. *Moreover,* $W_-^\chi(\xi) = 0$ *is equivalent to* (4.1).

The proof of smoothness uses standard methods of [15]. For a treatment of the special problems arising at the ends, see for example the proof of Theorem 3 in [5]. The second statement of Lemma 4.1 is obvious if $\widetilde{U}_\chi$ is chosen small enough.

Now consider the first order expansion

$$\widetilde{W}_\chi(\xi) = \widetilde{W}_-(c) + (D_\chi, L_\chi)\xi + N_\chi(\xi). \tag{4.3}$$

The idea is to invert the linear operator (D_χ, L_χ) and to covert the equation $W_\chi(\xi) = 0$ into a fixed point problem for a contractive function in ξ. This method has been previously used in [19]. It can be formalized as follows:

Lemma 4.2. *Assume that a smooth map* $f\colon E \to F$ *between Banach spaces* E *and* F *has an expansion*

$$f(\xi) = f(0) + Df(0)\xi + N(\xi)$$

so that $Df(0)$ *has a right inverse* G, *and for* $\xi, \zeta \in E$

$$\|GN(\xi) - GN(\zeta)\|_E \le C_N(\|\xi\| + \|\zeta\|_E)\|\xi - \zeta\|_E$$

for some constant C_N. *If* $\|Gf(\xi)\|_E \le (8C_N)^{-1}$, *then the zero set of* f *in* $B_\varepsilon = \{\xi \in E \mid \|\xi\| < \varepsilon\}$ *with* $\varepsilon = (4C_N)^{-1}$ *is a smooth manifold of dimension equal to the dimension of* $\ker df$. *In fact, if we define* $K_\varepsilon = \{\xi \in \ker Df(0) \mid \|\xi\|_E < \varepsilon\}$, *then there exists a smooth function*

$$\phi\colon K_\varepsilon \to K^\perp := GF \subset E,$$

with $f(\xi + \phi(\xi)) = 0$ *so that all zeros of* f *in* B_ε *are of the form* $\xi + \phi(\xi)$. *Moreover, we have the estimate*

$$\|\phi(0)\|_E \le 2\|Gf(0)\|_E.$$

The proof of Lemma 4.2, like the proof of the implicit function theorem, is a simple application of the contraction principle. In order to obtain an inverse which is bounded independently of the parameters ρ_v, we have to modify the norms on U_χ. First let us define a function $\tau_\chi\colon M_\Gamma \to \mathbb{R}$ which coincides with the τ-variable defined by the cylindrical charts of §2 except on a neighborhood of the gluing set, where it is continued smoothly. For fixed $p > 2$ and $\varepsilon \in (0, 1)$, we then define norms $\|\ \|_{:\,k}$ as in (3.4) with τ replaced by τ_χ. We denote them by $\|\ \|_{U,\chi} = \|\ \|_{:\,2}$, $\|\ \|_{V,\chi} = \|\ \|_{:\,1}$, and $\|\ \|_{W,\chi} = \|\ \|_{:\,0}$, respectively. To find a suitable range for a right inverse G of $D(\chi, L_\chi^\pm)$, we have to factor out the "asymptotic kernels" in U_γ. By this we mean families $\xi_\chi \in U - \chi$ for which $\|\xi_\chi\|_{U,\chi} = 1$ but $\|(D_\chi, L_\chi^+)\xi_\chi\|_{V\oplus W,\chi} \to 0$, where $\rho(\chi) \to \infty$. Such families can be constructed, for example, from elements of the kernel of (D, L^+) on the components of M (see Proposition 3.2) by using appropriate cutoff functions

$$\beta_e\colon M - \gamma \to [0, 1], \qquad e \in E_\Gamma,$$

which are identically equal to 1 on the ith component of $M_{\rho-1} = \tau^{-1}[0, \rho - 1]$, and vanish on a neighborhood of $S_\alpha^\beta = \omega_v(0 \times S^3)$ for $\alpha \in V_\Gamma$. Here, $\omega_v[-\rho_v, \rho_v] \times S^3 \to M_\Gamma$ is the parametrization constructed from the cylindrical charts of §2. Additional asymptotic kernels arise due to these long cylinders in M_Γ. Consider the standard metric on $\mathbb{R} \times S^3$, and let (D_∞, L_∞^+) denote the corresponding linear operator. Then asymptotic kernels can be constructed from τ-independent kernel elements of (D_∞, L_∞^+), by using appropriate cutoff functions again. In Proposition 5.2 below we show that there exists a 7-dimensional space

$$\kappa \subset \mathscr{C}^\infty(S^2_{\mathrm{trf}}(T(\mathbb{R} \times S^3)))|_{0\times S^3} \tag{4.4}$$

such that for $\xi(\tau, \theta) = \xi_0(\theta)$ with $\xi_0 \in \kappa$, we have $(D_\infty, L_\infty^+)\xi = 0$. These considerations motivate the definition

$$\begin{aligned} U_\chi^\perp = \{\xi \in U_\chi | &(1)\ \xi|S_\alpha^3 \perp \text{ for } \alpha \in V_\Gamma, \\ &(2)\ \beta_e\xi \perp \ker_{M_e}(D, L^+) \text{ on } M_e \text{ for } e \in E_\Gamma\}. \end{aligned} \tag{4.5}$$

Lemma 4.3. *For each compact set K as in Definition 2.1 there exist positive constants ρ_0 and C, and continuous linear operators*

$$G_\chi\colon V_\chi \oplus W_\chi \to U_\chi^\perp,$$

for $\rho(\chi) \geq \rho_0$ with $G_\chi \circ (D_\chi, L_\chi) = \mathrm{id}$ and

$$\|G_\chi(X, h)\|_{U,\xi} \leq \mathscr{C}(\|X\|_{V,\chi} + \|h\|_{W,\chi}).$$

Proof. We first show that there exists a constant C such that for ρ large enough and $\xi \in U_\chi^\perp$,

$$\|\xi\|_U \leq C\{\|L_\chi^+\xi\|_{V,\chi} + \|D_\chi\xi\|_{W,\chi}\}. \tag{4.6}$$

Since by (2.4) and Proposition 3.2, the codimension of $U_\chi^\perp$ is less than or equal to the index of D_χ predicted by (1.2), it follows that (D_χ, L_χ)

is surjective on $U_\chi^\perp$. To prove (4.6) we proceed indirectly. For $\chi_n = (\Gamma_n, \sigma_n, \rho_n) \in Kx[\rho_n, \infty)^V$ with $\rho(\chi_n) \to \infty$, let c_n denote the conformal class of $g_{\chi n}$. If (4.6) is wrong, then there exist such a sequence and a sequence $\xi_n \in U_{c_n}^\perp$ such that with $(D_{\chi_n}, L_{\chi_n}) = (D_n, L_n)$

$$\lim_{\rho\to\infty}(\|L_n^+\xi_n\|_{V,\chi_n} + \|D_n\xi_n\|_{W,\chi_n}) = 0, \tag{4.7}$$

$$\|\xi_n\|_{U,\chi_n} = 1. \tag{4.8}$$

To derive a contradiction, consider for each $v \in V$ the parametrization ω_v and the sequence ζ_{vn} on $\mathbb{R}\times S^3$ defined by

$$\zeta_{vn} = \begin{cases} \exp(\varepsilon\rho_{vn}\omega_v^*\xi_n & \text{on } (-\rho_{vn}, \rho_{vn})\times S^3, \\ 0 & \text{otherwise}. \end{cases}$$

On $\mathbb{R}\times S^3$, define the weight $w(\tau, \theta) = e^{-\varepsilon|\tau|}$. Then it follows from (4.8) that $\|e^w\zeta_{vn}\|_p$ is bounded independently of n. Passing to a subsequence (which we still denote by ζ_{vn}), we can assume that $\zeta_{vn} \to \zeta_{v\infty}$ weakly in the Banach space $U_{-\varepsilon}$ defined by this norm. We want to show that $\zeta_{v\infty} = 0$. Therefore note that for each positive constant R, the restriction $\zeta_{vn;R}$ of ζ_{vn} to $[-R, R]\times S^3$ satisfies

$$\|\xi_{vn,R}\|_{2,p} \le \mathscr{C}_R, \tag{4.9}$$

independently of n. Thus $\zeta_{vn,R} \to \zeta_{v\infty}$ weakly with respect to this norm. In particular, by (1) of (4.5),

$$\zeta_{v\infty}|_{0\times S^3}\perp\kappa. \tag{4.10}$$

Moreover, it follows from (4.7) that for each fixed R,

$$\lim_{n\to\infty}(\|L_\infty^+\zeta_{vn,R}\|_{1,p} + \|D_\infty\zeta_{vn,R}\|_p) = 0. \tag{4.11}$$

Since both seminorms in (4.11) are continuous with respect to the norm in (4.9), they are also weakly lower semicontinuous, so that $(L^+, D)\zeta_{v\infty} = 0$. But this together with (4.10) implies that $\zeta_{v\infty} = 0$ by Proposition 5.3.

Now consider $\beta_e\xi_n\perp\ker(D, L^+)$ on $M - e$ (see (2) of (4.5)). Since $e^{\varepsilon\rho_{vn}}\|\xi_n\|_{1,p} \to 0$ on any bounded neighborhood of $S_n^3 \subset M_n$ by the above, we have

$$\lim_{n\to\infty}(\|D\beta_e\xi_n\|_W + \|L^+\beta_e\xi_n\|_V) = 0.$$

Hence $\beta_e\xi_n \to 0$ in $U(M_e)$ by Proposition 3.2. Combining these two facts, we obtain a contradiction to (4.8). This proves that (D, L^+) satisfies (4.6) on $U_\chi^\perp$ for $\rho(\chi)$ large enough. q.e.d.

The hypotheses of Lemma 4.2 are now satisfied for ρ large enough due to the following two estimates.

Lemma 4.4. *There exists a constant* C *such that for all* χ

$$\|W_-(\gamma_\chi)\|_{W,\chi} \le Ce^{-\rho(\chi)(2-\varepsilon)}$$

and

$$\|N_\chi(\xi) - N_\chi(\xi)\|_{W,\chi} \le C(\|\xi\|_{U,\chi} + \|\zeta\|_{U,\chi})\|\zeta - \xi\|_{U,\chi}.$$

Proof. The first estimate is obvious from the construction of g_χ. To prove the second estimate, note that the Riemannian curvature tensor of a metric g is of the form

$$R_g = L(g^{-1}\nabla\nabla g) + Q(g^{-1}\nabla g), \tag{4.12}$$

where L is linear and Q is quadratic. The linear part of $g^{-1}\nabla\nabla g$ is

$$D(g^{-1}\nabla\nabla g)h = g^{-1}hg^{-1}\nabla\nabla g + g^{-1}\nabla\nabla h,$$

from which it follows that the remainder of the first order expansion of $g^{-1}\nabla\nabla g$ is

$$\begin{aligned} N_1(h) &:= (g+h)^{-1}\nabla\nabla(g+h) - g^{-1}(\nabla\nabla_g + \nabla\nabla_h - hg^{-1}\nabla\nabla g) \\ &= \{(g+h)^{-1} - g^{-1}\}\nabla\nabla(g+h) - g^{-1}hg^{-1}\nabla\nabla g \\ &= \{(g+h)^{-1} - g^{-1} + g^{-1}hg^{-1}\}\nabla\nabla(g+h) - g^{-1}hg^{-1}\nabla\nabla h. \end{aligned}$$

Hence we have the pointwise estimate

$$|N_1(h)(x)| \le C_1|h(x)|^2|\nabla\nabla(g+h)(x) + | + |h(x)|\,|\nabla\nabla h(x)|.$$

Standard Sobolev embeddings now yield

$$\begin{aligned} \|N_1(h)\|_p &\le C_1\|h\|_\infty^2\|\nabla\nabla(g+h)\|_p + \|h(x)\|_\infty\|\nabla\nabla h\|_p \\ &\le C_2\|h\|_u^2. \end{aligned}$$

This gives the desired estimate for the first term in (4.12). The second term is treated in a similar way.

5. The elliptic complex

The aim of this section is to determine the linearization of the self-duality equation at the standard solution on the cylinder $\mathbb{R}\times S^3$. We will frequently work in local coordinates, where the index zero corresponds to the $\mathbb{R}$-direction and italic indices i, j, k, ... run from 1 to 3. We can describe a conformal structure by its unique representative g satisfying

$g_{00} = 1$ or, equivalently, by the induced metric g^3 on S^3 and a one-form $\alpha_i = g_{0i}$ measuring the angles between the τ-direction and TS^3. We will therefore make the identification

$$\mathrm{Hom}(\Omega^-, \Omega^+) \simeq \Omega_3 \oplus \Omega_6,$$

where $\Omega_3 = \Omega^1(S^3)$, and Ω_6 is the symmetric tensor product. We will also use the isomorphisms $T(\mathbb{R} \times S^3) = \Omega_1 \oplus \Omega_3$ and $\Omega^- \otimes_5 so_3^- = \Omega_5$, where Ω_5 is the traceless symmetric component of $\Omega_3 \otimes \Omega_3$. The irreducible decompositions

$$\Omega_3 \otimes \Omega_3 \to \Omega_1 \oplus \Omega_3 \oplus \Omega_5,$$
$$\Omega_5 \otimes \Omega_3 \to \Omega_3 \oplus \Omega_5 \oplus \Omega_7$$

define symbols of the following operators:

$$\begin{gathered}
\mathrm{div} = -d^* = -d^* : \Omega_3 \to \Omega_1 : \quad \mathrm{div}\,\lambda = \nabla_i \lambda_i, \\
{}^*d : \Omega_3 \to \Omega_3; \qquad ({}^*d\lambda)_i = \varepsilon_{ijk} \nabla_j \lambda_k, \\
d_s : \Omega_3 \to \Omega_6; \qquad (d_s \lambda)_{ij} = \tfrac{1}{2}(\nabla_i \lambda_j + \nabla_j \lambda_i), \\
\mathrm{div} = -d_s^* : \Omega_6 \to \Omega_3 (\mathrm{div}\, h)_j = \nabla_i h_{ij}, \\
\not{d} : \Omega_5 \to \Omega_5; \qquad (\not{d}h)_{ij} = e_{imn+} \nabla_m h_{nj} + \varepsilon_{jmn} \nabla_m h_{ni}, \\
d_7 : \Omega_5 \to \Omega_7.
\end{gathered}$$

The explicit formulas refer to a normal chart for g, and $D^* = {}^*d^*$ and D_s^* are the L^2-adjoints of d and d_s. It is easy to see that ${}^*d$ and $\not{d}$ are formally self-adjoint in L^2. Of course, d_s splits into its traceless component and

$$\mathrm{tr}\, d_s = \mathrm{div} \equiv -d^*.$$

Other relations are

$$\begin{aligned}
&\not{d}\, d_S = d_S * d, \\
(5.1) \qquad & d_S^* d_S = \tfrac{1}{2} * d * + d * d * - \mathrm{Ric} \\
&\qquad = \tfrac{1}{2} d^* d + d\, d^* - \mathrm{Ric}.
\end{aligned}$$

The first statement follows from

$$\begin{aligned}
2(d_S^* d_S X)_j &= -\nabla_j (\nabla_i X_j + \nabla_j X_i) \\
&= (-\nabla^2 X) - [\nabla_i, \nabla_j] X_i + \nabla_j d^* X \\
&= (-\nabla^2 X)_j - R_{ijia} X_\alpha + (d\, d^* X)_j \\
&= (-\nabla^2 X)_j - \mathrm{Ric}_{j\alpha} X_\alpha + (d\, d^* X)_j
\end{aligned}$$

and the "Weizenböck-formula"

$$\begin{aligned}-((d\,d^* - d^*\,d)X)_j &= \nabla_j\nabla_i X_i + \nabla_i(\nabla_i X_j - \nabla_j X_i)\\ &= (\nabla^2 X)_j + R_{jii\alpha}X_\alpha = (\nabla^2 X)_j - \mathrm{Ric}_{j\alpha}X_\alpha.\end{aligned}$$

To prove the second relation, note that we can ignore the curvature terms, since the curvature of S^3 has no component in $\mathrm{Hom}(\Omega_3, \Omega_5)$. We are now ready to calculate L and D in the following special case.

Proposition 5.1. *Let the derivative in the zero direction be denoted by a dot. Then the complex* (L, D) *for the standard conformal structure on* $\mathbb{R}\times S^3$ *has the form*

$$\begin{aligned}&L\colon C^\infty(\Omega_1\oplus\Omega_3)\to C^\infty(\Omega_3\oplus\Omega_6),\\ &L(f, X) = (df + \dot X, 2(d_S X - \dot f g)),\end{aligned}$$

and

$$\begin{aligned}&D\colon C^\infty(\Omega_3\oplus\Omega_6)\to C^\infty(\Omega_5),\\ &D(\alpha, h) = \left(\tfrac12 h - \tfrac12 \not{d}\dot h - Eh + d_S({}^*\,d\alpha - \dot\alpha)_5\right)_f.\end{aligned}$$

Here, E *is the linearization of the traceless Ricci tensor around the standard metric on* S^3.

Proof. Let ϕ_t be a smooth family of diffeomorphisms on $\mathbb{R}\times S^3$ with $(\frac{d}{dt}\phi_t)_t = X$. Then we have with $X_0 = f$:

$$\begin{aligned}(L_\chi\alpha)_i = \left.\frac{d}{dt}\right|_{t=0}(\phi_t^* g)_{00}^{-1}\phi_t^* g_{0i} &= (L_X g)_{0i} - (L_X g)_{00}g_{0i}\\ &= \nabla_0 X_i + \nabla_i X_0 = (\dot X + df)_i.\end{aligned}$$

Similarly,

$$\begin{aligned}(L_X h)_{ij} = \left.\frac{d}{dt}\right|_{t=0}(\phi_t^* g)_{00}^{-1}\phi_t^* g_{ij} &= (L_X g)_{ij} - (L_X g)_{00}g_{ij}\\ &= \nabla_i X_j + \nabla_j X_i - 2\nabla_0 X_0 g_{ij} = 2(d_S X - \dot f g)_{ij}.\end{aligned}$$

The operator D is calculated first for $\alpha = 0$; its full form follows then from $D^\circ L = 0$. Let g be a metric on $\mathbb{R}\times\mathbb{R}^3$ satisfying $g_{00} = 1$ and $g_{0i} = 0$. Then the Christoffel symbols $\Gamma_{ij,k} = \frac12(\nabla_i g_{ij} + \nabla_j g_{ik} - \nabla_k g_{ij})$ coincide with those of the induced metric on $\mathbb{R}^3$, whereas $\Gamma_{0,ij} = -\frac12$ and $\Gamma_{0,i0} = 0$. We only have to consider the highest order part of the curvature tensor:

$$\begin{aligned} R^L_{ijkl} &= \Gamma_{i,jk|l} - \Gamma_{i,jl|k} = R^L_{ijkl}(g^3), \\ R^L_{0jkl} &= \Gamma_{0,jk|l} - \Gamma_{0,jl|k} = \tfrac{1}{2}(\dot g_{jl|k} - \dot g_{jk|l}), \\ R^L_{0j0l} &= \Gamma_{0,j0|l} - \Gamma_{0,jl|0} = \tfrac{1}{2}\ddot g_{jl}. \end{aligned}$$

By means of the isomorphism

$$\begin{aligned} &\Omega_3 \oplus \Omega_3 \to \Omega^2(\mathbb{R}\times S^3), \\ (5.2)\qquad &\omega = dt\Lambda\pi^* E + \pi^*(*B), \\ &E_k = \omega_{ok}, \qquad B_i = \tfrac{1}{2}\varepsilon_{ijk}\omega_{jk}, \end{aligned}$$

we can write R^L as a bilinear form, $\langle (E, B), R^L(E', B')\rangle = \alpha_{ij}E_iE'_j + \beta_{ij}(E_iB'_j + E'_iB_j) + \gamma_{ij}B_iB_j$ on $\Omega_3 \oplus \Omega_3$ with

$$\begin{aligned} \alpha_{jl} &= R_{0j0l} = \tfrac{1}{2}\ddot g_{jl}, \\ \beta_{ij} &= \tfrac{1}{2}\varepsilon_{jkl}R_{0ikl} = \frac{1}{4}\varepsilon_{jkl}(\dot g_{il|k} - \dot g_{ik|l}) = \tfrac{1}{2}\varepsilon_{jkl}\dot g_{il|k}, \\ \gamma_{ij} &= \tfrac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}R_{klmn} = -(\mathrm{Ric}_{ij} - \tfrac{1}{2}\delta_{ij}\operatorname{tr}\mathrm{Ric}), = -\mathrm{Ein}_{ij}, \end{aligned}$$

the Einstein tensor of the 3-dimensional metric g. Since, by (5.2), the (anti) self-dual forms in $\Omega^2(\mathbb{R}\times S^3)$ correspond to the (anti) diagonal in $\Omega_3 \oplus \Omega_3$, W_- is up to nonlinear terms the traceless component of

$$(W_-)_{ij} = \tfrac{1}{2}\ddot g_{ij} - \tfrac{1}{2}\varepsilon_{jkl}\dot g_{il|k} - \mathrm{Ein}_{ij}, = (\tfrac{1}{2}\ddot g - \tfrac{1}{2}\not{d}\dot g - \mathrm{Ein})_{ij}.$$

We conclude that modulo trace,

$$D(0, h) = \tfrac{1}{2}\ddot h - \tfrac{1}{2}\not{d}\dot h - Eh.$$

To determine D in general, note that for each $\alpha = C^\infty_-(\Omega_3)$ there exists an $X \in \mathscr{C}^\infty(\Omega_3)$ such that $\dot X = \alpha$. Then by the modulus trace

$$\begin{aligned} D(\alpha, 0) &= D((\alpha, 0) - L(0, X)) = D((\alpha, 0) - (\dot X, 2d_SX)) \\ &= -2D(0, d_SX) = -2(\tfrac{1}{2}d_S\ddot X - \tfrac{1}{2}\not{d}d_S\dot X - E\,d_SX) \\ &= -d_S\dot\alpha + \not{d}d_S\alpha = d_S(*d\alpha - \dot\alpha). \end{aligned}$$

This completes the proof of Proposition 5.1. q.e.d.

The L^2-adjoint of L is

$$(5.3)\qquad L^+(\alpha, h) = (d^*\alpha + 2\operatorname{tr}\dot h, 2d_S^*h - \dot\alpha).$$

Hence $(L^+, D)\colon C^\infty(\Omega_6 \oplus \Omega_3) \to C^\infty(\Omega_1 \oplus \Omega_3 \oplus \Omega_5)$ is elliptic. To apply the Fredholm theory of [13], we have to consider the asymptotics of

(D, L^+) at the cylindrical ends, i.e., the translationally invariant system (D, L^+) on $\mathbb{R} \times S^3$ with respect to the standard product metric. It turns out that the space of solutions of the equation $(D, L^+)\xi = 0$ decomposes naturally into solutions of the form $\xi(\tau, \theta) = e^{\tau\lambda}\xi_\lambda(\theta)$. Here, ξ is a solution of $(D_\lambda, L_\lambda^+)\xi = 0$, and (D_λ, L_λ^+) is obtained from (D, L^+) by replacing the time derivative by multiplication with λ. With Proposition 5.1 and (5.3), this reads

$$0 = 2\lambda \operatorname{tr} h + d^*\alpha, \qquad 0 = \lambda\alpha - 2\, d_S^* h,$$
$$0 = (\tfrac{1}{2}\lambda^2 h - \tfrac{1}{2}\lambda \not{d} h - Eh + d_S(* \, d\alpha - \lambda\alpha))_5.$$

Now define the asymptotic spectrum of (D, L^+) as

$$\sigma_\infty = \{\lambda \in \mathbb{C} | \ker(D_\lambda, L_\lambda^+)_\mathbb{C} \neq 0\}.$$

(Our definition of the spectrum differs from the one in [13] by a factor i.) It follows from the above that purely imaginary $\lambda \in \sigma_\infty$ gives rise to bounded elements in the kernel of (D_∞, L_∞^+), and thereby destroys the Fredholm property of the operator (D, L^+) of Proposition 3.1 for the usual (unweighted) Sobolev norms. In general, by Theorem 1.3 of [13], the operator of Proposition 3.1 is Fredholm if and only if ε is not the real part of any $\lambda \in \sigma_\infty$. Note that directly from the ellipticity of (D_λ, L_λ^+) we see that the set $\operatorname{Re}\sigma_\infty = \{\operatorname{Re}\lambda | \lambda \in \theta_\infty\}$ is a discrete set in $\mathbb{R}$ (see e.g. [13]). Now Proposition 3.1 follows from

Proposition 5.2. *If* $\lambda \in \sigma_\infty$ *with* $\operatorname{Re}\lambda = 0$, *then* $\lambda = 0$. *Moreover, we have a seven-dimensional space*

$$\kappa := \ker(D_0, L_0^+) \simeq \mathbb{R} \oplus so_3.$$

For the Fredholm theory of (D_∞, L_∞^+), Proposition 5.2 has the following consequence: Define U_ε^∞, V_ε^∞, and W_ε^∞ as in Definition 3.1 with $P = \mathbb{R} \times S^3$ and weighted norms

$$\|\xi\|_{k,p;\varepsilon} = \|e^{\varepsilon\rho}\xi\|_{k,p},$$

where ρ is a smooth function satisfying $\rho(\tau, \theta) = |\tau|$ for $|\tau| \geq 1$. Then a Fourier transformation in τ yields the corollary (see [13]):

Proposition 5.3. *For small* $\varepsilon > 0$, (D, L^+): $U_\varepsilon^\infty \to V_\varepsilon^\infty \oplus W_\varepsilon^\infty$ *is surjective, and there is an isomorphism* $\kappa \to \ker(D, L^+)$; $\xi \mapsto \overline{\xi}(\tau, \theta) = \xi(\theta)$.

Proof of Proposition 5.2. We first calculate the kernel of

$$(L_0^+, D_0)C^\infty(\Omega_3 \oplus \Omega_6) \to C^\infty(\Omega_1 \oplus \Omega_3 \oplus \Omega_5),$$
$$(\alpha, h) \mapsto (d^*\alpha - 2\, d_s^* h, d_s * d\alpha - Eh).$$

Since, on S^3, $*d$ is an isomorphism on $\ker d^*$, and no nonisometric conformal vector field can be divergence-free, the Ω_3-component of the kernel is

$$\ker(d_5 * d) \cap \ker d^* \simeq \{*d\xi | d_5(*d\xi) = 0\}$$
$$\simeq \{\phi \in \Omega_3 | d_5\phi = 0 \text{ and } d^*\phi = 0\} \simeq so_3.$$

The Ω_6-component $\ker E \cap \ker d_s^* = \mathbb{R}$ is the tangent space to the space of diffeomorphism classes of traceless Einstein metrics on S^3, which is parametrized by the volume.

It remains to prove (L_∞^+, D_∞^+) has no purely imaginary spectrum. For $\mu \in \mathbb{R}$ and $\lambda = i\mu$, assume that $(\alpha, h) = 0$ in $C^\infty(\Omega_3 \oplus \Omega_5)$ solves $(L_\lambda^+, D_\lambda)(\alpha, h) = 0$. We have to distinguish the case where (α, h) is "pure gauge", i.e., $(\alpha, h) = L_\lambda(f, X)$ for $(f, X) \in C^\infty(\Omega_1 \oplus \Omega_3)$. In this case, $D_\lambda(\alpha, h) = 0$ holds trivially since $D_\lambda \circ L - \lambda = (D \circ L)_\lambda = 0$. Hence the only condition on (f, X) is

$$L_\lambda^+ L_\lambda(f, X) = (L^+L)_\lambda(f, X) = 0.$$

Note that in the case $\lambda = i\mu$, $\mu \in \mathbb{R}$, $L_\lambda^+ = (L_\lambda)^+$, so that $(L^+L)_\lambda = (L_\lambda)^+ L_\lambda$ is nonnegative. Since moreover L_λ itself is injective for $\lambda \neq 0$, we conclude that $L_\lambda^+ L_\lambda$ is an isomorphism. Hence the "pure gauge" part of the spectrum cannot have purely imaginary nonzero elements.

To examine the general spectrum of (D_∞, L_∞^+) we can now use this fact to replace the condition $L_\lambda^+(\alpha, h) = 0$ by a more convenient gauge. First, we can eliminate the angular component α by adding $L_\lambda(0, \frac{1}{\lambda}\alpha)$. Hence there exists a nonzero $h \in \Omega_6$ satisfying the tracefree part of

$$\lambda^2 h - \lambda \not{d} h - 2Eh = 0. \tag{5.4}$$

Note that the same is true for

$$h' = h + \tfrac{1}{2}L_\lambda(-\lambda f, df) = h + d_s df + \lambda^2 fg,$$

for any smooth function f. We want to use this remaining gauge freedom to find h' which also satisfies an extension of (5.4) to the trace component. To define an appropriate extension, we first calculate E:

Lemma 5.1.

$$E = -\tfrac{1}{8}\not{d}^2 + \tfrac{1}{4}(d_5 d_S^* + d_5 d \operatorname{tr} + \pi_5).$$

Proof. Note that $E = \alpha/\not{d}^2 + \beta d_S d_S^* + \gamma(d_S d \operatorname{tr} + g d^* d_S) + \delta g d^* d \operatorname{tr} + Z$ as a self-adjoint second order operator. The coefficients above are obtained by checking certain known identities for E. First, since E is invariant under diffeomorphisms, we know that $E\,d_S X$ vanishes for each vector field X on S^3. With (5.1), this yields

$$\begin{aligned}0 &= (\alpha \not{d}^2\, d_S + \beta\, d_S\, d_S^* + \gamma\, d_S\, d\,\mathrm{tr}\, d_S + Z)\\ &= d_5\{\alpha * d * d + \beta(\tfrac{1}{2} * d * d + d * d * - \mathrm{Ric}) - \gamma\phi * d * + Z\}\\ &= d_5\{(\alpha + \tfrac{1}{2}\beta) * d * d + (\beta + \gamma)\, d * d * - \beta\,\mathrm{Ric} + Z\};\end{aligned}$$

hence $\beta = \gamma = -2\alpha$ and $Z = -2\alpha\,\mathrm{Ric}$. The value $\alpha = -\frac{1}{8}$ is obtained by calculating $E(gf)$ explicitly for some function f, and Lemma 5.1 is proved. q.e.d.

We choose the extension

$$\widetilde{E} = -\tfrac{1}{8}\not{d}^2 + \tfrac{1}{4}\{d_S\, d_S^* + d_S\, d\,\mathrm{tr} + g\, d^*\, d_S^* + g\, d^*\, d\,\mathrm{tr} + 1\} \tag{5.5}$$

which is in addition a self-adjoint operator. Then the additional operator acting on h is

$$D_\lambda^{\mathrm{tr}} = \mathrm{tr}\,\lambda^2 - \lambda\not{d} - 2\widetilde{E} = \lambda^2\,\mathrm{tr} - d^*\, d_S^* - d^*\, d\,\mathrm{tr} - \tfrac{1}{2}.$$

Here, we have used $\mathrm{tr}\not{d} = 0$ and $\mathrm{tr}\, d_S = -d^*$. Now the equation $D_\lambda^0(h + (d_S\, d + \lambda^2 g)) = 0$ can be solved uniquely for f, since the operator

$$\begin{aligned}D_\lambda^0(d_S\, d + \lambda^2 g) &= ((\lambda^2 - 1)\,\mathrm{tr} - d^*\, d_S^* - d^*\, d\,\mathrm{tr})(\lambda^2 g + d_S\, d)\\ &= 3(\lambda^2 - 1)\lambda^2 + (\lambda^2 - 1)\,\mathrm{tr}\, d_S\, d - \lambda^2(d^*\, d_S^* g + d^*\, d\,\mathrm{tr}\, g)\\ &\quad - d^*\, d_S^*\, d_S\, d - d^*\, d\,\mathrm{tr}\, d_S\, d\\ &= 3(\lambda^2 - 1)\lambda^2 - (\lambda^2 - 1)\, d^*\, d - \lambda^2(d^*(-d) + 3\, d^*\, d)\\ &\quad - d^*(\tfrac{1}{2} d^*\, d + d\, d^* - 1)\, d + d^*\, d\, d^*\, d\\ &= 3(\lambda^2 - 1)\lambda^2 = (\lambda^2 - 1)\Delta - 2\lambda^2\Delta + \Delta\end{aligned}$$

with $\Delta = d\, d^* + d^*\, d$ is positive for $\lambda = i\mu$. Thus if there exists a nonzero (α, h) satisfying $(D_l, L_\lambda^+)(\alpha, h) = 0$ for $0 \neq \lambda \in i\mathbb{R}$, then there also exists a nonzero h satisfying the full equation (5.4) with E replaced by $\widetilde{E}$. This would imply that $(\mu^2 + 2\widetilde{E})h = 0$ and $\not{d}h = 0$, since $\widetilde{E}$ and $\not{d}$ are self-adjoint and real. But inserting the second condition in (5.5) we see that $\widetilde{E}$ is a positive semidefinite operator on $\ker\not{d}$. Hence the proof of Proposition 5.2 is complete.

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UNIVERSITY OF CALIFORNIA, BERKELEY