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SOME NONDIFFEOMORPHIC HOMEOMORPHIC HOMOGENEOUS 7-MANIFOLDS WITH POSITIVE SECTIONAL CURVATURE

MATTHIAS KRECK & STEPHAN STOLZ

1. Introduction

One believes that the existence of a metric with positive sectional curvature imposes strong restrictions on the topology of a manifold. If a manifold admits such a metric, one can ask whether there is a second smooth structure on this manifold admitting again a metric with positive sectional curvature. Up to now no such examples were known.

In the case of the 7-sphere Gromoll and Meyer showed that there is a metric on an exotic 7-sphere with nonnegative sectional curvature, but it is still open whether there is a metric with positive sectional curvature [7].

If the manifolds are homogeneous spaces and the metrics are also homogeneous, a classification of a certain class of such manifolds up to isometry is known (compare [2]). In particular there is a family of such homogeneous spaces, called Wallach spaces, $N_{k,l} = \mathrm{SU}(3)/i_{k,l}(S^1)$, parametrized by integers k, l with $kl(k+l) \neq 0$. Here $i_{k,l}\colon S^1 \to S^1 \times S^1$ is the homomorphism defined by $z \to (z^k, z^l)$, and we identify $S^1 \times S^1$ with a fixed maximal torus of $\mathrm{SU}(3)$. Replacing k by rk and l by rl does not change the subgroup $i_{k,l}(S^1)$, and we will assume throughout the paper that k and l are coprime. These spaces admit a metric of positive sectional curvature which is $\mathrm{SU}(3)$-invariant [2, Theorem 3.2]. If in addition $k \not\equiv 1 \bmod 3$, they also admit an Einstein metric [14] (this restriction seems unnecessary; compare [4, Theorem 9.101, p. 264]); the metrics constructed in [2], however, are not Einstein.

Another motivation for our interest in the Wallach spaces is the following. In [13] we gave a homeomorphism and diffeomorphism classification

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of a similar family of 7-dimensional homogeneous spaces $M_{k,l}$ which are quotients of $SU(3) \times SU(2) \times U(1)$ by a subgroup with Lie algebra isomorphic to $SU(2) \times U(1) \times U(1)$. The most surprising discovery was that these manifolds are sometimes homeomorphic, but not diffeomorphic. There are speculations that this phenomenon cannot occur for homogeneous spaces G/H with G simple and simply connected. The Wallach spaces provide a test case for this.

The manifold $N_{k,l}$ is simply connected, and its cohomology ring looks as follows: $H^2(N_{k,l}; \mathbf{Z}) \cong \mathbf{Z}$, $H^3(N_{k,l}; \mathbf{Z}) = 0$, and $H^4(N_{k,l}; \mathbf{Z})$ is a cyclic group of order $N(k, l) = k^2 + kl + l^2$ generated by u^2, where u is a generator of $H^2(N_{k,l}, \mathbf{Z})$. Thus the cohomology of the Wallach spaces $N_{k,l}$ has the same structure as the cohomology of the manifolds $M_{k,l}$, and the theory developed in [13, §3] can be used to classify the Wallach spaces, too.

We work in the oriented category, i.e., the manifolds under consideration are oriented manifolds, and diffeomorphisms (resp. homeomorphisms) are orientation-preserving. We orient $N_{k,l}$ by the convention described in (4.3) below.

Theorem. *Assume* $(k, l) = 1 = (\tilde{k}, \tilde{l})$.

(i) $N_{k,l}$ *is homeomorphic to* $N_{\tilde{k},\tilde{l}}$ *if and only if* $N(k, l) = N(\tilde{k}, \tilde{l})$ *and* $kl(k+l) \equiv \tilde{k}\tilde{l}(\tilde{k}+\tilde{l}) \bmod 2^3 \cdot 3 \cdot N$, *where* $N = N(k, l) = k^2+kl+l^2$.

(ii) $N_{k,l}$ *is diffeomorphic to* $N_{\tilde{k},\tilde{l}}$ *if and only if* $N(k, l) = N(\tilde{k}, \tilde{l})$ *and* $kl(k+l) \equiv \tilde{k}\tilde{l}(\tilde{k}+\tilde{l}) \bmod 2^5 \cdot 7^{\lambda(N)} \cdot 3 \cdot N$, *where* $N = N(k, l)$ *and*

$$\lambda(N) = \begin{cases} 0 & \text{if } N \equiv 0 \bmod 7, \\ 1 & \text{otherwise.} \end{cases}$$

It turns out that the question whether there are two pairs k, l and $\tilde{k}, \tilde{l}$ of coprime integers such that the condition for homeomorphism holds, but the condition for diffeomorphism does not hold, is a difficult one. A computer calculation by Peter Gilkey revealed that for $N < 10^6$ no such pairs exist. An attempt to prove that this holds in general failed. It turns out that the above question can be expressed as a question about the arithmetic of the ring of algebraic integers in the cyclotomic field $\mathbf{Q}[\omega]$, where $\omega = (1 + \sqrt{-3})/2$ is a primitive sixth root of unity, since $k^2 + kl + l^2 = \mathrm{N}(\alpha)$ and $kl(k + l) = \mathrm{Tr}(\alpha^3/3\sqrt{-3})$ for $\alpha = k + l\omega$. Here $\mathrm{N}(\alpha) = \alpha\overline{\alpha}$ is the norm and $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha}$ is the trace of an element $\alpha \in \mathbf{Q}[\omega]$. Using the fact that $\mathbf{Z}[\omega]$ has unique factorization one obtains:

Corollary. *There are Wallach manifolds which are homeomorphic, but not diffeomorphic whose fourth cohomology group has order* N *if and only if there are primitive elements* $\gamma, \delta \in \mathbf{Z}[\omega]$ *such that* $N = \mathrm{N}(\gamma)N(\delta)$ *and* $2^3 \cdot 3 \cdot N$ *divides* $\mathrm{Tr}(\gamma^3/3\sqrt{-3})\,\mathrm{Tr}(\delta^3)$, *but* $2^5 \cdot 3 \cdot 7^\lambda \cdot N$ *does not, where* $\lambda = 0$ *if* N *is divisible by* 7 *and* $\lambda = 1$ *otherwise.*

A computer search for such elements γ, δ by Don Zagier showed:

Corollary. *If* M *and* $\widetilde{M}$ *are homeomorphic Wallach spaces whose fourth cohomology group has order* $N < 2955367597$, *then* M *is diffeomorphic to* $\widetilde{M}$. *On the other hand, the Wallach spaces* $N_{-56788,5227}$ *and* $N_{-42652,61213}$ *are homeomorphic, but not diffeomorphic. The order of their fourth cohomology is* 2955367597.

This is the first example of homeomorphic but not diffeomorphic manifolds admitting metrics with positive sectional curvature as well as the first example of such manifolds which are homogeneous spaces of the form G/H with G simple and simply connected. Furthermore both manifolds admit Einstein metrics.

If M and $\widetilde{M}$ are homeomorphic but not diffeomorphic manifolds of the type under consideration, then there exists an exotic 7-sphere Σ such that the connected sum $M\#\Sigma$ is diffeomorphic to $\widetilde{M}$. This follows from Theorem 3.1 below (note that in our situation $\bar{s}_i = s_i$ for $i = 2, 3$). Moreover the exotic sphere Σ is uniquely determined by M and $\widetilde{M}$, since it is detected by the s_1-invariant. The group of exotic 7-spheres is isomorphic to $\mathbf{Z}/28$ generated by $\Sigma(E_8)$, the boundary of the Milnor manifold $M(E_8)$ [10].

In the example mentioned above this exotic sphere is $7 \cdot \Sigma(E_8)$. Zagier and later Odlyzko have used large computers to find 23 other examples. The corresponding exotic spheres in these cases are always divisible by 7 (and all possibilities occur) except in one case where it represents a generator. The range of N in these examples is between the number above and approximately $2 \cdot 10^{20}$. It is open whether there are only finitely many such pairs. A table containing the first fourteen examples can be found at the end of §5.

Two manifolds of type $N_{k,l} = \mathrm{SU}(3)/i_{k,l}(S^1)$ are obviously diffeomorphic if the corresponding subgroups $i_{k,l}(S^1)$ are conjugate in $\mathrm{SU}(3)$. The example below shows that the converse is not true. We note that the subgroups $i_{k,l}(S^1)$ and $i_{\tilde{k},\tilde{l}}(S^1)$ are conjugate in $\mathrm{SU}(3)$ if and only if the complex numbers $\tilde{\alpha} = \tilde{k} + \tilde{l}\omega$ and $\alpha = k + l\omega$ are in the same orbit of the action of the Weyl group W of $\mathrm{SU}(3)$. Here we identify the Lie algebra of the maximal torus $\mathrm{SU}(3)$ with the complex numbers. The generators of W act on $\mathbf{C}$ by complex conjugation resp. multiplication by a third root of unity.

Example. The subgroups $i_{-4638661, 582656}(S^1)$ and $i_{-2594149, 5052965}(S^1)$ are not conjugate in $SU(3)$, but the corresponding quotient manifolds are diffeomorphic.

The interest in this example comes from the following problem.

Problem (Wu-yi Hsiang [9, Problem 1, p. 160]). Let G be a given simple, compact connected Lie group and H a nontrivial subgroup of G, $M = G/H$. Is it true that the only nontrivial differentiable G-action on the manifold M is the transitive one?

In the example above we can use the diffeomorphism between the quotient manifolds to carry the standard $SU(3)$-action on $N_{-4638661, 582656}$ over to the manifold $N_{-2594149, 5052965}$. This new action is not equivalently diffeomorphic to the standard $SU(3)$-action on $N_{-2594149, 5052965}$ since the corresponding isotropy subgroups are not conjugate.

Corollary. *There exists an embedding of* $H = S^1$ *into* $G = SU(3)$, *such that* G/H *admits at least two nonconjugate nontrivial* G-*actions.*

The structure of the paper is as follows. In §2 we recall the definition of the diffeomorphism (resp. homeomorphism) invariants $s_i \in \mathbf{Q}/\mathbf{Z}$ (resp. $\bar{s}_i \in \mathbf{Q}/\mathbf{Z}$), $i = 1, 2, 3$, from [13]. We use the opportunity to correct our previous definition of $\bar{s}_2(M)$, which is *not* a homeomorphism invariant if M is spin and the order of $H^4(M; \mathbf{Z})$ is even (see Remark 2.6). This mistake however does not affect the main results [13], since none of the manifolds $M_{k,l}$ considered there has both a spin structure and a fourth cohomology group of even order.

In §3 we show that the diffeomorphism (resp. homeomorphism) type of a closed smooth (resp. topological) 1-connected 7-manifold with a certain cohomology structure is determined by the invariants s_i (resp. $\bar{s}_i$), $i = 1, 2, 3$. This result may be of independent interest.

For comparison, in [13] we classified smooth manifolds of this type up to diffeomorphism (resp. homeomorphism) using the s_i (resp. $\bar{s}_i$) and additional invariants like the linking form and the first Pontrjagin class. These additional invariants turned out to be superflous as they can be expressed in terms of the $\bar{s}_i$ which led to a better version of Proposition 9.1 of [12] since [13] was published.

In §4 we compute the invariants s_i (resp. $\bar{s}_i$) for the Wallach spaces. In §5 we do the number theory leading to the theorem above and explain the number theoretic condition leading to homeomorphic, but not diffeomorphic Wallach spaces.

2. The invariants

In this section we recall the definition of the diffeomorphism (resp. homeomorphism) invariants $s_i(M) \in \mathbf{Q}/\mathbf{Z}$ (resp. $\bar{s}_i(M) \in \mathbf{Q}/\mathbf{Z}$) of [13]. These are invariants for manifolds M of the following type which in particular includes all Wallach spaces (see §4).

2.1. M is a 1-connected smooth (resp. topological) closed 7-manifold with $H^2(M; \mathbf{Z}) \cong \mathbf{Z}$, $H^3(M; \mathbf{Z}) = 0$, and $H^4(M; \mathbf{Z})$ a finite cyclic group generated by u^2, where u is a generator of $H^2(M; \mathbf{Z})$.

More generally there are invariants $s_i(M, u) \in \mathbf{Q}/\mathbf{Z}$ (resp. $\bar{s}_i(M, u) \in \mathbf{Q}/\mathbf{Z}$) for pairs (M, u), where M is a 7-manifold with $H^4(M; \mathbf{Q}) = 0$, and u is an element of $H^2(M; \mathbf{Z})$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u \bmod 2$ (nonspin case). It turns out that these invariants do not change if we replace u by $-u$. If M is a manifold of the type (2.1), and u is a generator of $H^2(M; \mathbf{Z})$, then $s_i(M) = s_i(M, u)$ (resp. $\bar{s}_i(M) = \bar{s}_i(M, u)$).

The invariants $s_i(M, u)$ (resp. $\bar{s}_i(M, u)$) are defined as certain characteristic numbers of a pair (W, z) whose boundary is (M, u), i.e., W is a smooth (resp. topological) 8-manifold with $\partial W = M$, and z is an element of $H^2(W; \mathbf{Z})$ restricting to u on the boundary such that $w_2(W) = 0$ in the spin case and $w_2(W) = z \bmod 2$ in the nonspin case. Alternatively, the invariants s_i can be defined analytically as in [13].

The following result is proved in §6.

Proposition 2.2. *Let M be a closed smooth (resp. topological) 7-dimensional manifold with an element $u \in H^2(M; \mathbf{Z})$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u \bmod 2$ (nonspin case). Then (M, u) is the boundary of a pair (W, z) in the sense explained above.*

Remark 2.3. In [13] we mentioned, but did not prove, that such a zero bordism (W, z) always exists. This was sufficient for the purposes of [13] since the manifolds $M_{k,l}$ considered there are sphere bundles, and the corresponding disk bundles turn out to provide the required zero bordism. The Wallach manifolds are sphere bundles, too. Unfortunately, they are spin manifolds, whereas the corresponding disk bundles are not, and hence the disk bundles do not qualify as zero bordisms in the above sense. Our main motivation behind proving that every pair (M, u) bounds is that we need this result for the proof of the classification Theorem 3.1 (even if we restrict the statement of the theorem to Wallach spaces!)

Let (W, z) be a pair as in the above proposition with $H^4(\partial W; \mathbf{Q}) = 0$. For such pairs (W, z) we define characteristic numbers $S_i(W, z) \in \mathbf{Q}$,

$i = 1, 2, 3$, as follows:

$$S_1(W, z) := \langle e^{d/2}\widehat{A}(W), [W, \partial W]\rangle,$$
$$S_2(W, z) := \langle \mathrm{ch}(\lambda(z) - 1)e^{d/2}\widehat{A}(W), [W, \partial W]\rangle,$$
$$S_3(W, z) := \langle \mathrm{ch}(\lambda^2(z) - 1)e^{d/2}\widehat{A}(W), [W, \partial W]\rangle.$$

Here $d = 0$ in the spin case, $d = z$ in the nonspin case, $\lambda(z)$ is the complex line bundle over W with first Chern class z, ch is the Chern character, $\widehat{A}(W)$ is the $\widehat{A}$-polynomial of W, and $[W, \partial W]$ is the fundamental class of W. The interpretation of the right-hand sides is the following. The cohomology classes $\mathrm{ch}(\lambda(z) - 1)e^{d/2}\widehat{A}(W)$ and $\mathrm{ch}(\lambda^2(z) - 1)e^{d/2}\widehat{A}(W)$ are linear combinations of p_1^2, z^2p_1, and z^4, where p_1 is the first Pontrjagin class of W. These classes can be regarded as elements of $H^8(W, \partial W; \mathbf{Q})$ and can thus be evaluated on the fundamental class $[W, \partial W]$ since p_1 and z^2 pull back from classes in $H^4(W, \partial W; \mathbf{Q})$. The class $e^{d/2}\widehat{A}(W)$ can be expressed as a rational linear combination of p_1^2, z^2p_1, z^4, and the L-polynomial $L(W) = \frac{1}{45}(7p_2 - p_1^2)$. We interpret $\langle L(W), [W, \partial W]\rangle$ as the signature of W.

For explicit calculations and future reference it is convenient to have the explicit formulas expressing $S_i(W, z)$ in terms of p_1^2, z^2p_1, z^4, and the signature of W. Here we abuse notation by writing p_1^2 instead of $\langle p_1^2, [W, \partial W]\rangle$, etc.

(2.4) Spin case:
$$S_1(W, z) = -\frac{1}{2^5 \cdot 7}\,\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2,$$
$$S_2(W, z) = -\frac{1}{2^4 \cdot 3}z^2p_1 + \frac{1}{2^3 \cdot 3}z^4,$$
$$S_3(W, z) = -\frac{1}{2^2 \cdot 3}z^2p_1 + \frac{2}{3}z^4,$$

Nonspin case:
$$S_1(W, z) = -\frac{1}{2^5 \cdot 7}\,\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}z^2p_1 + \frac{1}{2^7 \cdot 3}z^4,$$
$$S_2(W, z) = -\frac{1}{2^3 \cdot 3}z^2p_1 + \frac{5}{2^3 \cdot 3}z^4,$$
$$S_3(W, z) = -\frac{1}{2^3}z^2p_1 + \frac{13}{2^3}z^4.$$

To obtain an invariant of ∂W we need to know the values of S_i for closed manifolds.

Proposition 2.5. *Let* $S(W, z) = (S_1(W, z), S_2(W, z), S_3(W, z)) \in \mathbf{Q}^3$. *Then*

$$\{S(W, z) | W \text{ is a closed smooth manifold, } w_2(W) = 0\} = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z},$$
$$\{S(W, z) | W \text{ is a closed smooth manifold, } w_2(W) = z \bmod 2\} = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z},$$
$$\{S(W, z) | W \text{ is a closed topological manifold, } w_2(W) = 0\} = \tfrac{1}{28}\mathbf{Z} \oplus \tfrac{1}{2}\mathbf{Z} \oplus \mathbf{Z},$$
$$\{S(W, z) | W \text{ is a closed topological manifold, } w_2(W) = z \bmod 2\} = \tfrac{1}{28}\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}.$$

In the differentiable case the result was proved in [13, Lemma 3.2]. We defer the somewhat technical proof in the topological case to §6.

The proposition shows that for smooth manifolds $S_i(W, z) \bmod \mathbf{Z}$ depends only on the boundary of (W, z), and we define $s_i(M, u) \in \mathbf{Q}/\mathbf{Z}$ as $S_i(W, z) \bmod \mathbf{Z}$ if (M, u) is the boundary of (W, z). For the topological case it is convenient to define numbers $\overline{S}_i(W, z) \in \mathbf{Q}$ as follows: $\overline{S}_i(W, z) = 28 \cdot S_1(W, z)$, $\overline{S}_2(W, z) = 2 \cdot S_2(W, z)$ if $w_2(W) = 0$, $\overline{S}_2(W, z) = S_2(W, z)$ if $w_2(W) = z \bmod 2$, and $\overline{S}_3(W, z) = S_3(W, z)$. Then again the above proposition shows that we get invariants $\bar{s}_i(M, u) \in \mathbf{Q}/\mathbf{Z}$, where M is a topological manifold, by defining $\bar{s}_i(M, u) = \overline{S}_i(W, z) \bmod \mathbf{Z}$ if (M, u) is the boundary of (W, z).

Remark 2.6. As mentioned in the introduction, the invariant $\bar{s}_2(M)$ as defined in [13] is *not* a topological invariant if M is spin and the order of $H^4(M; \mathbf{Z})$ is even. The mistake occurs in the remark on p. 375, where we argue that if M, N are two manifolds of the type (2.1) which are homeomorphic, then there is a homotopy sphere Σ such that the connected sum $M \# \Sigma$ is diffeomorphic to N. The reason we give is $H^3(M; \mathbf{Z}/2) = \{0\}$, which is not true if $H^4(M; \mathbf{Z})$ is even. However, the statement is still true if M is nonspin. This follows from the classification result (3.1) below, and the fact that $\bar{s}_2(M)$ as defined in [13] agrees with the topological invariant $\bar{s}_2(M)$ defined above in the nonspin case.

In the case of the Wallach manifolds this difficulty does not arise since $H^4(M; \mathbf{Z})$ is odd for those manifolds. Still it seemed worthwhile to include Proposition 2.5 in this paper (whose proof constitutes the bulk of the technical §6) since we need it to prove the general classification Theorem 3.1 which fills the gap in [13] and may be of independent interest.

For the Wallach spaces we have not been able to find an explicit bordism (W, z) with the required properties. Here the following observation is useful. Let M be a smooth 7-manifold together with an element $u \in H^2(M;\mathbf{Z})$ as above. Assume that W is a smooth 8-manifold with $\partial W = M$, and there are classes $z, c \in H^2(W;\mathbf{Z})$ such that $z_{|M} = u$, $c_{|M} = 0$, and $w_2(W) = c \bmod 2$ in the spin case and $w_2(W) = z + c \bmod 2$ in the nonspin case. Define characteristic numbers

$$
\begin{aligned}
(2.7)\qquad S_1(W, z, c) &= \langle e^{(c+d)/2}\widehat{A}(W), [W, \partial W]\rangle,\\
S_2(W, z, c) &= \langle \operatorname{ch}(\lambda(z) - 1)e^{(c+d)/2}\widehat{A}(W), [W, \partial W]\rangle,\\
S_3(W, z, c) &= \langle \operatorname{ch}(\lambda^2(z) - 1)e^{(c+d)/2}\widehat{A}(W), [W, \partial W]\rangle,
\end{aligned}
$$

where $d = 0$ in the spin case, and $d = z$ in the nonspin case. The integrality of these characteristic numbers for closed manifolds [8, Theorem 26.1.1] implies that $S_i(W, z, c) \bmod \mathbf{Z}$ depends only on (M, u), in particular $s_i(M, u) = S_i(W, z, c) \bmod \mathbf{Z}$.

3. The classification result

In this section we give a diffeomorphism (resp. homeomorphism) classification of smooth (resp. topological) manifolds M of the type (2.1), i.e., M is a 1-connected closed 7-manifold with $H^2(M;\mathbf{Z}) \cong \mathbf{Z}$, $H^3(M;\mathbf{Z}) = 0$, and $H^4(M;\mathbf{Z})$ is a finite cyclic group generated by u^2, where u is a generator of $H^2(M;\mathbf{Z})$.

Theorem 3.1. *Let M and M' be two smooth (topological) manifolds of type* (2.1) *such that $|H^4(M;\mathbf{Z})| = |H^4(M';\mathbf{Z})|$ which are both spin or both nonspin. In the topological case assume furthermore that the Kirby-Siebenmann smoothing obstruction is trivial for both or nontrivial for both. Then M is diffeomorphic (homeomorphic) to M' if and only if $s_i(M) = s_i(M')$ (resp. $\bar{s}_i(M) = \bar{s}_i(M')$) for $i = 1, 2, 3$.*

The proof of this theorem is based on the following result which is proved using a slightly different language in [12].

Proposition 3.2 [12, Theorem 9.1]. *Let M and M' be two smooth (topological) manifolds as in Theorem* 3.1. *Then M is diffeomorphic (homeomorphic) to M' if and only if there is a smooth (topological) bordism W between M and N and a class $z \in H^2(W;\mathbf{Z})$ such that the relative characteristic numbers $\langle p_1^2(W), [W, \partial W]\rangle$, $\langle z^2 p_1(W), [W, \partial W]\rangle$, $\langle z^4, [W, \partial W]\rangle$, and the signature of W vanish.*

Proof of Theorem 3.1. Assume $s_i(M) = s_i(M')$ (resp. $\bar{s}_i(M) = \bar{s}_i(M')$) for $i = 1, 2, 3$. Our goal is to show that M and M' are diffeomorphic (resp. homeomorphic) by constructing a bordism (W, z) with vanishing characteristic numbers and vanishing signature. Proposition 2.2

shows that we can find a bordism (W, z) between M and M'. It follows from formulas (2.4) that $\langle p_1^2(W)[W, \partial W]\rangle$, $\langle z^2 p_1(W), [W, \partial W]\rangle$, $\langle z^4, [W, \partial W]\rangle$, and the signature of W vanish if and only if the invariants $S_i(W, z)$ vanish for $i = 1, 2, 3$ and $\operatorname{sign}(W) = 0$.

We have $S_i(W, z) = s_i(M) - s_i(M') = 0 \bmod \mathbf{Z}$ (resp. $\overline{S}_i(W, z) = \bar{s}_i(M) - \bar{s}_i(M') = 0 \bmod \mathbf{Z}$) for $i = 1, 2, 3$. Proposition 2.5 shows that there is a pair (W', z') where W' is a closed smooth (resp. topological) manifold such that $S_i(W', z') = -S_i(W, z)$, $i = 1, 2, 3$. Replacing W by the disjoint union of W and W' we obtain a new bordism between M and M' with $S_i = 0$ for $i = 1, 2, 3$. Finally adding a suitable number of copies of the quaternionic projective plane $\mathbf{HP}^2$ we obtain a bordism with vanishing signature. Note that adding $\mathbf{HP}^2$ does not change the characteristic numbers S_i since $S_i(\mathbf{HP}^2, 0) = 0$ for $i = 1, 2, 3$.

4. Computation of the invariants

Recall that $N_{k,l}$ is the homogeneous space $\mathrm{SU}(3)/i_{k,l}(S^1)$, where $i_{k,l}\colon S^1 \to T = S^1 \times S^1$ is defined by $z \to (z^k, z^l)$, and we identify T with a fixed maximal torus of $\mathrm{SU}(3)$. As explained in the introduction we assume that k and l are coprime, and we pick integers m, n such that $km + ln = 1$. The homotopy sequence of the fibration $\mathrm{SU}(3) \to \mathrm{SU}(3)/i_{k,l}(S^1)$ shows that $N_{k,l}$ is 1-connected since $\mathrm{SU}(3)$ is.

We compute the cohomology of $N_{k,l}$ using the fiber bundle

$$N_{k,l} = \mathrm{SU}(3)/i_{k,l}(S^1) \xrightarrow{p} \mathrm{SU}(3)/T. \tag{4.1}$$

This bundle is in fact a principal S^1-bundle if we let S^1 act on $\mathrm{SU}(3)/i_{k,l}(S^1)$ by restricting the obvious T-action to the image of $i_{-n,m}\colon S^1 \to T$. The cohomology of $\mathrm{SU}(3)/T$ looks as follows [5, Theorem 31.1]:

$$\begin{aligned} H^*(\mathrm{SU}(3)/T; \mathbf{Z}) &\cong \mathbf{Z}[w, x, y]/(\text{symmetric polynomials}) \\ &\cong \mathbf{Z}[x, y]/(x^2 + xy + y^2, x^2y + xy^2), \end{aligned}$$

where the generators w, x, y have degree 2. This isomorphism can be made more explicit by defining $x, y \in H^2(\mathrm{SU}(3)/T; \mathbf{Z})$ as follows.

We pick a generator $\iota \in H^1(S^1; \mathbf{Z})$ such that for a principal S^1-bundle $E \to B$ the image of ι under the transgression $\tau\colon H^1(S^1; \mathbf{Z}) \to H^2(B; \mathbf{Z})$ agrees with the first Chern class $c_1(L)$ of the corresponding complex line bundle L. Let $\mathrm{pr}_i\colon T = S^1 \times S^1 \to S^1$ be the projection of the ith factor, and define x (resp. y) to be the image of $\mathrm{pr}_1^* \iota$ (resp. $\mathrm{pr}_2^* \iota$) $\in H^2(T; \mathbf{Z})$ under the transgression $\tau\colon H^1(T; \mathbf{Z}) \to H^2(\mathrm{SU}(3)/T; \mathbf{Z})$ of the fiber bundle $T \to \mathrm{SU}(3) \to \mathrm{SU}(3)/T$.

Let L be the complex line bundle corresponding to the principal S^1-bundle (4.1).

Lemma 4.2. $c_1(L) = -lx + ky \in H^2(\mathrm{SU}(3)/T; \mathbf{Z})$.

Proof. Restricting to a fiber the free S^1-action on $\mathrm{SU}(3)/i_{k,l}(S^1)$ described above induces a homeomorphism $g\colon S^1 \to T/i_{k,l}(S^1)$. More explicitly, g is the composition of $i_{-n,m}\colon S^1 \to T$ and the projection map $\mathrm{pr}\colon T \to T/i_{k,l}(S^1)$. Its inverse $h\colon T/i_{k,l}(S^1) \to S^1$ is given by $[z_1, z_2] \to z_1^{-1} z_2^k$. It follows that $\mathrm{pr}^* h^* \iota = l\, \mathrm{pr}_l^* \iota + k\, \mathrm{pr}_2^* \iota$.

Consider the following commutative diagram whose rows are fibre bundles:

$$\begin{array}{ccccc}
T & \longrightarrow & \mathrm{SU}(3) & \longrightarrow & \mathrm{SU}(3)/T \\
{\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\
T/i_{k,l}(S^1) & \longrightarrow & \mathrm{SU}(3)/i_{k,l}(S^1) & \longrightarrow & \mathrm{SU}(3)/T
\end{array}$$

Using the naturality of the transgression we obtain $c_1(L) = \tau(h^* \iota) = \tau(\mathrm{pr}^* h^* \iota) = \tau(-l\, \mathrm{pr}_1^* \iota + k\, \mathrm{pr}_2^* \iota) = -lx + ky$. q.e.d.

The Gysin sequence

$$\begin{aligned}
&\to H^i(\mathrm{SU}(3).T; \mathbf{Z}) \xrightarrow{\cup c_1(L)} H^{i+2}(\mathrm{SU}(3)/T; \mathbf{Z}) \\
&\xrightarrow{p^*} H^{1+2}(N_{k,l}; \mathbf{Z}) \to H^{i+1}(\mathrm{SU}(3)/T; \mathbf{Z}) \to \cdots
\end{aligned}$$

of L implies

$$\begin{aligned}
&H^1(N_{k,l}; \mathbf{Z}) = 0, \\
&H^2(N_{k,l}; \mathbf{Z}) = \mathbf{Z}, \quad \text{generated by } u = p^* z, \\
&\qquad\qquad\qquad\qquad \text{where } z = mx + ny, \\
&H^3(N_{k,l}; \mathbf{Z}) = 0, \\
&H^4(N_{k,l}; \mathbf{Z}) = \mathbf{Z}/N(k, l) \quad \text{generated by } u^2, \\
&\qquad\qquad\qquad\qquad \text{where } N(k, l) = k^2 + kl + l^2.
\end{aligned}$$

Our next goal is to compute the invariants $s_i(N_{k,l})$. Denote by $W_{k,l}$ the disk bundle of L. Then $W_{k,l}$ is a zero bordism for $N_{k,l}$, and the class $z \in H^2(W_{k,l}; \mathbf{Z})$ restricts to $u \in H^2(N_{k,l}; \mathbf{Z})$ (we identify the cohomology of $W_{k,l}$ with the cohomology of $\mathrm{SU}(3)/T$ using the map induced by the inclusion of the zero section).

(4.3) Orientation conventions. We choose an orientation for $\mathrm{SU}(3)/T$ (resp. $W_{k,l}$) by the following requirement for the corresponding fundamental classes:

$$\langle x^2 y, [\mathrm{SU}(3)/T]\rangle = 1 \quad \text{and} \quad \langle s\,\mathrm{U}, [W_{k,l}, \partial W_{k,l}]\rangle = \langle s, [\mathrm{SU}(3)/T]\rangle$$

for any $s \in H^6(\mathrm{SU}(3)/T; \mathbf{Z})$. Here $\mathrm{U} \in H^2(W_{k,l}, \partial W_{k,l}; \mathbf{Z})$ is the Thom class of L. For $N_{k,l} = \partial W_{k,l}$ we pick the orientation induced by the orientation of $W_{k,l}$.

The tangent bundle of $\mathrm{SU}(3)/T$ is stably trivial. Hence the tangent bundle of $W_{k,l}$ is stably isomorphic to L, and the tangent bundle of $\mathrm{SU}(3)/S^1$ is stably trivial. Thus $w_2(W_{k,l}) = c_1(L) \bmod 2$ and $p_1(W_{k,l}) = c_1^2(L)$. In particular the manifold $W = W_{k,l}$ and the classes $z = mx+ny$ and $c = c_1(L)$ have the properties required in (2.7) and hence $s_i(M_{k,l}, u) = S_i(W, z, c) \bmod \mathbf{Z}$. Substituting c^2 for p_1 we get the following expressions for $S_i(W, z, c)$ (where c^4 is short for $\langle c^4, [W, \partial W]\rangle$, etc.):

$$S_1(W, z, c) = -\frac{1}{2^5\cdot 7}\operatorname{sign}(W) - \frac{1}{2^5\cdot 3\cdot 7}c^4,$$
$$S_2(W, z, c) = \frac{1}{2^3\cdot 3}z^4 + \frac{1}{2^2\cdot 3}cz^3 + \frac{1}{2^3\cdot 3}c^2z^2,$$
$$S_3(W, z, c) = \frac{2}{3}z^4 + \frac{2}{3}cz^3 + \frac{1}{2\cdot 3}c^2z^2.$$

The characteristic numbers involving c are already computed since $j^*\mathrm{U} = c$, where $j\colon W \to (W, \partial W)$ is the natural map:

$$c^4 = \langle c^3\mathrm{U}, [W, \partial W]\rangle = \langle c^3, [\mathrm{SU}(3)/T]\rangle = 3kl(k+l),$$
$$c^3z^2 = \langle cz^2, [\mathrm{SU}(3)/T]\rangle = km^2 - 2kmn - 2lmn + ln^2,$$
$$cz^3 = \langle z^3, [\mathrm{SU}(3)/T]\rangle = 3mn(m-n).$$

To compute z^4 we note that $z^2 = (ax+by)\cup c$, where

$$a = \frac{km^2 + lm^2 - 2kmn - ln^2}{-(k^2+kl+l^2)}, \quad b = \frac{km^2 - kn^2 + 2lmn - ln^2}{-(k^2+kl+l^2)}.$$

Then

$$z^4 = \langle z^2(ax+by), [\mathrm{SU}(3)/T]\rangle$$
$$= \frac{km^4 + 4kmn^3 - 6km^2n^2 - 6lm^2n^2 + 4lm^3n + ln^4}{-(k^2+kl+l^2)}.$$

Combining these calculations we obtain

Lemma 4.4. *Assume* $(k, l) = 1$, *and let* m, n *be integers such that* $km + ln = 1$ *and* m *or* n *is divisible by* 4. *Then*

$$s_1(N_{k,l}) = \frac{-kl(k+l)}{2^5 7}, \quad s_2(N_{k,l}) = \frac{-P+NS}{2^3 3N}, \quad s_3(N_{k,l}) + \frac{4P+NS}{6N},$$

where $N = k^2+kl+l^2$, $P = km^4+4kmn^3-6km^2n^2-6m^2n^2+4lm^3n+ln^4$, *and* $S = km^2 - 2kmn - 2lmn + ln^2$.

Remark. The condition that m or n is divisible by 4 guarantees that the summand $cz^3/2^2 3 = mn(m-n)/2^2$ in the formula for $S_2(W_{k,l}, z, c)$ is an integer.

5. Solving the equations

In this section we study the numerical conditions on $k, l, \tilde{k}, \tilde{l}$ given by the system of equations $s_i(N_{k,l}) = s_i(N_{\tilde{k},\tilde{l}})$ (resp. $\bar{s}_i(N_{k,l}) = \bar{s}_i(N_{k,l})$), $i = 1, 2, 3$.

Let T_i, $i = 1, 2, 3$, be the numerator in the expression for s_i in the lemma above, and denote by $\widetilde{N}, \widetilde{P}, \widetilde{R}, \widetilde{S}, \widetilde{T}_i$ the quantities obtained by replacing k, l, m, n by $\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}$ in the expressions for N, P, R, S, T_i.

Lemma 5.1. *Assume* $(k, l) = 1 = (\tilde{k}, \tilde{l})$ *and* $N = \widetilde{N}$. *Then*:

(i) $T_2 \equiv \widetilde{T}_2 \bmod 2^3 \cdot 3 \cdot N$ *is equivalent to* $T_1 \equiv \widetilde{T}_1 \bmod 2^3 \cdot 3 \cdot N$,

(ii) $\bar{s}_i(N_{k,l}) = \bar{s}_i(N_{\tilde{k},\tilde{l}})$ *for* $i = 1, 2, 3$ *iff* $T_1 \equiv \widetilde{T}_1 \bmod 2^3 \cdot 3 \cdot N$,

(iii) $s_i(N_{k,l}) = s_i(N_{\tilde{k},\tilde{l}})$ *for* $i = 1, 2, 3$ *iff* $T_1 \equiv \widetilde{T}_1 \bmod 2^5 \cdot 7^{\lambda(N)} \cdot 3 \cdot N$,

where

$$\lambda(N) = \begin{cases} 0 & \text{if } N \equiv 0 \bmod 7, \\ 1 & \text{otherwise}. \end{cases}$$

Proof. There are three main ingredients in the proof.

1. The observation that we can choose the numbers m, n such that $km + ln = 1$ at our convenience since $s_i(N_{k,l}) \in \mathbf{Q}/\mathbf{Z}$ is independent of this choice. In particular, if k is prime to some number, we can choose n to be divisible by it.

2. The observation that the values of N, $T_i \in \mathbf{Z}$ and $s_i(N_{k,l}) \in \mathbf{Q}/\mathbf{Z}$, $i = 1, 2, 3$, do not change if we permute k and l; remember that m and n should be permuted simultaneously in order to preserve the condition $km + ln = 1$. Hence, since k and l are coprime, we can always assume that k is not divisible by a given prime by making the above substitution if necessary.

3. The equality $T_1 = k(k^2 - N)$.

Ad (i). $N = k^2 + kl + l^2$ is coprime to k, since $(k, l) = 1$. Then we may choose n to be divisible by N. Hence $m \cdot k \equiv 1 \bmod N$, $T_1 = k(k^2 - N) \equiv k^3 \bmod N$, and $T_2 = -P + NS \equiv -P \equiv -km^4 \equiv -m^3 \bmod N$. It follows that $T_2 \equiv \widetilde{T}_2 \bmod N$ iff $T_1 \equiv \widetilde{T}_1 \bmod N$.

For the 2-primary analysis we note that N is always odd, and we assume without loss of generality that n is divisible by 2^3. Then $m \cdot k \equiv 1 \bmod 2^3$ and hence $m \equiv k \bmod 2^3$. It follows that $T_2 = -P + NS \equiv -km^4 + km^2N \equiv -m(m^2 - N) \equiv -k(k^2 - N) = -T_1 \bmod 2^3$.

For the 3-primary analysis we assume without loss of generality that n is divisible by 3^2. Then $m \cdot k = 1 \bmod 3^2$ and hence $m \equiv k \bmod 3$ and $m^3 = k^3 \bmod 3^2$. It follows that $T_2 = -P + NS = -km^4 + km^2N \equiv -m^3 + mN \equiv -k^3 + kN = -T_1$, where this congruence is modulo 3^2, if N is divisible by 3, and modulo 3 otherwise. Since N is never divisible by 3^2 (see Lemma 5.2(i) below), these statements imply part (i) of the lemma.

Ad (ii). It follows from part (i) that $T_1 \equiv \widetilde{T}_1 \bmod 2^3 \cdot 3 \cdot N$ is equivalent to $\bar{s}_2(N_{k,l}) = \bar{s}_2(N_{\tilde{k},\tilde{l}})$, and it is clear that $T_1 \equiv \widetilde{T}_1 \bmod 2^3 \cdot 3 \cdot N$ implies $\bar{s}_1(N_{k,l}) = \bar{s}_1(N_{\tilde{k},\tilde{l}})$. Thus it suffices to show that $T_2 \equiv \widetilde{T}_2 \bmod 2^3 \cdot 3 \cdot N$ implies $T_3 \equiv \widetilde{T}_3 \bmod 2 \cdot 3 \cdot N$. This follows from the congruences $T_3 \equiv T_3 + 3NS = -4P + 4NS = 4T_2 \bmod 3 \cdot NS$ and $S \equiv km^2 \equiv m \equiv 1 \bmod 2$ (here we assume without loss of generality n even).

Ad (iii). The equation $s_1(N_{k,l}) = s_1(N_{\tilde{k},\tilde{l}})$ is equivalent to $T_1 \equiv \widetilde{T}_1 \bmod 2^5 \cdot 7$ and hence (iii) follows from (ii) since N is odd.

Part (ii) (resp. part (iii)) of the above lemma then finishes the proof of the classification theorem of the introduction. It remains to construct examples of Wallach spaces which are homeomorphic but not diffeomorphic.

As mentioned in the introduction it is most convenient to think of the Wallach spaces as being parametrized by elements of $\alpha = k + l\omega \in \mathbf{Z}[\omega]$, where $\omega = (1 + \sqrt{-3})/2$ is a primitive sixth root of unity. We note that $(k, l) = 1$ if and only if α is a primitive element, and recall that $k^2 + kl + l^2 = \mathrm{N}(\alpha)$ and $kl(k+l) = \mathrm{Tr}(\alpha^3/3\sqrt{-3})$. Thus our classification result can be expressed nicely in terms of α. In particular finding Wallach manifolds which are homeomorphic but not diffeomorphic is equivalent to finding primitive elements $\alpha, \alpha' \in \mathbf{Z}[\omega]$ with $\mathrm{N}(\alpha) = N(\alpha')$ such that $\mathrm{Tr}(\alpha^3/3\sqrt{-3})$ is congruent to $\mathrm{Tr}(\alpha'^3/3\sqrt{-3})$ modulo $2^3 \cdot 3 \cdot \mathrm{N}(\alpha)$, but not modulo $2^5 \cdot 7^{\lambda(N)} \cdot 3 \cdot \mathrm{N}(\alpha)$. As a first step we want to determine for a given number N all primitives $\alpha \in \mathbf{Z}[\omega]$ such that $\mathrm{N}(\alpha) = N$.

Lemma 5.2. *Let α be a primitive element of $\mathbf{Z}[\omega]$, and let $\mathrm{N}(\alpha) = p_1^{e_1} \cdot \dots \cdot p_s^{e_s}$ be the prime decomposition of $\mathrm{N}(\alpha)$. Then the following hold*:

(i) $p_i = 1 \bmod 3$ *or* $p_i = 3$ *and in the latter case the exponent* e_i *is at most one.*

(ii) $\alpha = \varepsilon \cdot P_1^{e_1} \cdot \dots \cdot P_s^{e_s}$, *where* ε *is a unit in* $\mathbf{Z}[\omega]$ *and* P_i *is one of the two factors of the decomposition of* p_i *in* $\mathbf{Z}[\omega]$, *which are distinct for* $p_i \neq 3$. *In particular, the number of primitive elements* α' *such that* $\mathrm{N}(\alpha') = \mathrm{N}(\alpha)$ *is* $6 \cdot 2^n$, *where* n *is the number of prime divisors of* $\mathrm{N}(\alpha)$ *which are congruent to* $1 \bmod 3$ (*there are six units in* $\mathbf{Z}[\omega]$).

Proof. It is well known that $\mathbf{Z}[\omega]$ has unique factorization. Moreover the primes in $\mathbf{Z}[\omega]$ consist of (a) the square root of -3, (b) the prime numbers $p \in \mathbf{Z}$ with $p = 2 \bmod 3$, and (c) the factors P, $\overline{P}$ of a prime decomposition $p = P \cdot \overline{P}$ of a prime number $p \in \mathbf{Z}$ with $p = 1 \bmod 3$. Let $\alpha = P_1^{e_1} \cdot \dots \cdot P_s^{e_s}$ be the prime decomposition of α. It cannot contain any primes of type (b) since then α would not be primitive. The same argument shows that the exponent of $\sqrt{-3}$ is at most 1, and that P and $\overline{P}$ cannot occur both in the prime decomposition. This proves (ii). Part (i) follows from $\mathrm{N}(\alpha) = \mathrm{N}(P_1)^{e_1} \cdot \dots \cdot \mathrm{N}(P_s)^{e_s} = p_1^{e_1} \cdot \dots \cdot p_s^{e_s}$ and part (ii). q.e.d.

Let $\alpha, \alpha' \in \mathbf{Z}[\omega]$ be primitive elements with the same norm. Then part (ii) of the above lemma shows that there are $\gamma, \delta \in \mathbf{Z}[\omega]$, and a unit ε, such that $\alpha = \gamma\delta$ and $\alpha' = \varepsilon\bar{\gamma}\delta$. A short computation shows

$$\mathrm{Tr}\left(\frac{\alpha^3}{3\sqrt{-3}}\right) - \mathrm{Tr}\left(\frac{\alpha'^3}{3\sqrt{-3}}\right) = \mathrm{Tr}\left(\frac{\gamma^3}{3\sqrt{-3}}\right)\mathrm{Tr}(\delta^3)$$

if $\varepsilon^3 = 1$ (resp. $\mathrm{Tr}(\gamma^3)\,\mathrm{Tr}(\delta^3/3\sqrt{-3})$ if $\varepsilon^3 = -1$). Hence we conclude

Corollary 5.3. *There are Wallach manifolds which are homeomorphic, but not diffeomorphic whose fourth cohomology group has order* N *if and only if there are primitive elements* $\gamma, \delta \in \mathbf{Z}[\omega]$ *such that* $N = \mathrm{N}(\gamma)\,\mathrm{N}(\delta)$ *and* $2^3 \cdot 3 \cdot N$ *divides* $\mathrm{Tr}(\gamma^3/3\sqrt{-3})\,\mathrm{Tr}(\delta^3)$, *but* $2^5 \cdot 3 \cdot 7^\lambda \cdot N$ *does not, where* $\lambda = 0$ *if* N *is divisible by* 7 *and* $\lambda = 1$ *otherwise.*

A computer calculation by D. Zagier which was extended later by Andrew Odlyzko showed that primitive elements γ and δ, such that $2^3 \cdot 3 \cdot N$ divides $\mathrm{Tr}(\gamma^3/3\sqrt{-3})\,\mathrm{Tr}(\delta^3)$, are very rare. They found 30 examples—the first 14 of which are given by the following table, where $N = \mathrm{N}(\gamma)\,\mathrm{N}(\delta)$ and $T = \mathrm{Tr}(\gamma^3/3\sqrt{-3})\,\mathrm{Tr}(\delta^3)$.

γ	δ	N	$\frac{T}{2^3 \cdot 3 \cdot N}$	$T/2^3$ mod28
$127 + 186\omega$	$-226 + 151\omega$	2955367597	$29 \cdot 31$	21
$473 + 1240\omega$	$-135 + 131\omega$	41559275149	$2 \cdot 5 \cdot 11 \cdot 43$	14
$741 + 964\omega$	$-433 + 253\omega$	311251714249	$5 \cdot 7 \cdot 11 \cdot 13$	7
$196 + 237\omega$	$-2393 + 2053\omega$	709194540873	$3^3 \cdot 911$	7
$39 + 1646\omega$	$-2134 + 1503\omega$	10005549097453	$2 \cdot 5^2 \cdot 79$	14
$1279 + 1529\omega$	$-1727 + 1136\omega$	13703457081769	$2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 61$	14
$335 + 687\omega$	$-5330 + 4153\omega$	19153920223641	$2^3 \cdot 3^4 \cdot 5 \cdot 29$	0
$436 + 1205\omega$	$-3831 + 3749\omega$	31145131821643	$2 \cdot 5^2 \cdot 13 \cdot 193$	14
$2019 + 2921\omega$	$-4833 + 4312\omega$	390688534767037	$5^2 \cdot 17 \cdot 23 \cdot 59$	11
$2735 + 4017\omega$	$-3855 + 2789\omega$	411358875444559	$2^4 \cdot 5 \cdot 11 \cdot 13 \cdot 37$	0
$508 + 3008\omega$	$-10484 + 7877\omega$	966599827776793	$5^2 \cdot 13 \cdot 17 \cdot 53$	7
$3165 + 12719\omega$	$-2827 + 1567\omega$	1275987636279889	$5 \cdot 7 \cdot 11 \cdot 13 \cdot 23$	21
$997 + 2721\omega$	$-12906 + 11735\omega$	1697970197114737	$11 \cdot 41 \cdot 2011$	7
$553 + 10062\omega$	$-5735 + 3817\omega$	2738819764243641	$2^2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	0

Remarks. (i) Let M (resp. $\widetilde{M}$) be the Wallach manifolds corresponding to $\gamma\delta$ (resp. $\bar{\gamma}\delta$). If $2^3 \cdot 3 \cdot N$ divides T, then M, $\widetilde{M}$ are homeomorphic, and hence M is diffeomorphic to $\widetilde{M} \# s\Sigma(E_8)$, where $s\Sigma(E_8)$ is the connected sum of s copies of the Milnor sphere $\Sigma(E_8)$. Evaluating the s_1-invariant we obtain $s_1(M) = s_1(\widetilde{M} \# s\Sigma(E_8)) = s_1(\widetilde{M}) + s \cdot s_1(\Sigma(E_8))$. If M is the Wallach manifold corresponding to $\alpha = k + l\omega$, we find using the fact that $s_1(\Sigma(E_8)) = -\frac{1}{28}$ (see proof of Lemma 6.6) and $s_1(M) = -kl(k+l)/(2^5 \cdot 7) = -\operatorname{Tr}(\alpha^3/3\sqrt{-3})$

$$\frac{s}{28} = s_1(\widetilde{M}) - s_1(M) = \frac{1}{2^5 \cdot 7}\left(\operatorname{Tr}\left(\frac{(\gamma\delta)^3}{2\sqrt{-3}}\right) - \operatorname{Tr}\left(\frac{(\bar{\gamma}\delta)^3}{3\sqrt{-3}}\right)\right)$$
$$= \frac{1}{2^5 \cdot 7}\left(\operatorname{Tr}\left(\frac{\gamma^3}{3\sqrt{-3}}\right)\operatorname{Tr}(\delta^3)\right) = \frac{T}{2^5 \cdot 7} \quad \operatorname{mod} \mathbf{Z}.$$

So $s = T/2^3 \bmod 28$, and its value is given by the last column of the above table.

(ii) The subgroups of SU(3) corresponding to $\gamma\delta$ (resp. $\bar{\gamma}\delta$) are never conjugate, but in those cases where $T/2^3 = 0 \bmod 28$ the corresponding homogeneous spaces are diffeomorphic. The example mentioned in the introduction corresponds to line 7 of the above table.

6. Proofs of Propositions 2.2 and 2.5

Propositions 2.2 and 2.5 can both be rephrased as statements about certain bordism groups. We need some notation. If α is a vector bundle over some space X, we denote by $\Omega_n^{\mathrm{Spin}}(X;\alpha)$ the bordism group of triples (W, f, ω), where W is a smooth n-manifold, $f\colon W \to X$ is a map, and ω is a spin structure on $\nu(W) - f^*\alpha$; here $\nu(W)$ is the stable normal bundle of W. If α is trivial, we use the notation $\Omega_n^{\mathrm{Spin}}(X)$ instead of $\Omega_n^{\mathrm{Spin}}(X;\alpha)$, and if X is a point, we write Ω_n^{Spin}. Let $\Omega_n^{\mathrm{TopSpin}}(X;\alpha)$ be the bordism group defined analogously using topological instead of smooth manifolds.

We note that if X is complex projective space $\mathbf{CP}^\infty$, then a map $f\colon W \to \mathbf{CP}^\infty$ can be interpreted as a cohomology class $z \in H^2(W;\mathbf{Z})$, and there is a spin structure on $\nu(W) - f^*\alpha$ if and only if $w_2(W) = 0$ (if α is the trivial bundle), resp. $w_2(W) = z \bmod 2$ (if α is the Hopf bundle H). Suppressing ω in the notation we write $[W, z] \in \Omega_n^{\mathrm{Spin}}(\mathbf{CP}^\infty;\alpha)$ (resp. $[W, z] \in \Omega_n^{\mathrm{TopSpin}}(\mathbf{CP}^\infty;\alpha)$) for the element represented by such a pair (W, z). Note that the spin structure ω is uniquely determined if $H^1(W;\mathbf{Z}/2) = 0$. Using this language Propositions 2.2 and 2.5 can be rephrased as follows.

Lemma 6.1. *If α is the trivial bundle or a Hopf bundle, then*

$$\Omega_7^{\mathrm{Spin}}(\mathbf{CP}^\infty;\alpha) = 0 \quad \textit{and} \quad \Omega_7^{\mathrm{TopSpin}}(\mathbf{CP}^\infty;\alpha) = 0.$$

Lemma 6.2. *Let $S\colon \Omega_8^{\mathrm{TopSpin}}(\mathbf{CP}^\infty;\alpha) \to \mathbf{Q}^3$ be the homomorphism mapping $[W, z]$ to its characteristic numbers $(S_1(W, z), S_2(W, z), S_3(W, z)) \in \mathbf{Q}^3$. Then the image of S is $\frac{1}{28}\mathbf{Z} \oplus \frac{1}{2}\mathbf{Z} \oplus \mathbf{Z}$ if α is the trivial bundle and it is $\frac{1}{28}\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ if $\alpha = H$.*

For the proofs of 6.1 and 6.2 it is crucial that these bordism groups can be interpreted as stable homotopy groups as follows. Denote by BO (resp. $B\mathrm{Top}$) the classifying space for vectorbundles (resp. topological microbundles) and by $BO\langle 4\rangle$ (resp. $B\mathrm{Top}\langle 4\rangle$) its 3-connected cover, which classifies vectorbundles (resp. microbundles) with spin structure (note that $BO\langle 4\rangle = B\mathrm{Spin}$ and $B\mathrm{Top}\langle 4\rangle = B\mathrm{TopSpin}$). Let $MO[4]$ (resp. $M\mathrm{Top}[4]$) be the Thom spectrum corresponding to the universal bundle over $BO\langle 4\rangle$ (resp. $B\mathrm{Top}\langle 4\rangle$) and let $M\alpha$ be the Thom spectrum of α. The convention concerning Thom spectra we use is that the Thom class is always in dimension zero.

Then the Pontrjagin Thom construction (and topological transversality [6]) leads to isomorphisms $\Omega_n^{\mathrm{Spin}}(X;\alpha) \cong \pi_m MO[4] \wedge M\alpha$ and $\Omega_n^{\mathrm{TopSpin}}(X;\alpha) \cong \pi_n M\mathrm{Top}[4] \wedge M\alpha$. This implies the existence of the Atiyah Hirzebruch spectral sequences (short: AHSS):

$$(6.3)\qquad \begin{aligned} H_{\mathrm{S}}(M\alpha;\Omega_t^{\mathrm{Spin}}) &\Rightarrow \Omega_{s+t}^{\mathrm{Spin}}(X;\alpha),\\ H_{\mathrm{S}}(M\alpha;\Omega_t^{\mathrm{TopSpin}}) &\Rightarrow \Omega_{s+t}^{\mathrm{TopSpin}}(X;\alpha).\end{aligned}$$

The groups Ω_n^{Spin} are well known [1].

n	0	1	2	3	4	5	6	7	8
Ω_n^{Spin}	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}\oplus\mathbf{Z}$

The groups $\Omega_n^{\mathrm{TopSpin}}$ are certainly known to the experts in the above range but we have not been able to find a reference in this literature.

Lemma 6.4. *The groups $\Omega_n^{\mathrm{TopSpin}}$ are given by the following table, where T denotes a torsion group.*

n	0	1	2	3	4	5	6	7	8
$\Omega_n^{\mathrm{TopSpin}}$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}\oplus\mathbf{Z}\oplus T$

Moreover, the natural map $\Omega_n^{\mathrm{Spin}} \to \Omega_n^{\mathrm{TopSpin}}$ is an isomorphism in dimensions $0, 1, 2$ and multiplication by 2 in dimension 4.

Proof. The statement about $\pi_8 M\mathrm{Top}[4]$ follows from the fact that $MO[4] \to M\mathrm{Top}[4]$ is a rational equivalence. Let BPL be the classifying space for piecewise linear bundles, $BPL\langle 4\rangle$ its 3-connected cover and $MPL[4]$ the corresponding Thom spectrum. Then the natural map $BO \to BPL$ is a 7-equivalence [11], and hence induces an isomorphism $\pi_n MO[4] \to \pi_n MPL[4]$ for $n < 7$, and a surjection for $n = 7$. The fiber of the natural map $BPL \to B\mathrm{Top}$ has trivial homotopy groups except for the third homotopy group which is isomorphic to $\mathbf{Z}/2$ [11, Essay V, Theorem 5.5]. This implies the existence of a fibration $BPL\langle 4\rangle \to B\mathrm{Top}\langle 4\rangle \to K(\mathbf{Z}/2, 4)$, where $K(\mathbf{Z}/2, 4)$ is the Eilenberg-Mac Lane space with $\pi_4 K(\mathbf{Z}/2, 4) = \mathbf{Z}/2$. It follows from the associated Serre spectral sequence that up to and including dimension 7 the cofiber of $MPL[4] \to M\mathrm{Top}[4]$ is the Eilenberg-Mac Lane spectrum $\Sigma^4 K\mathbf{Z}/2$, i.e., its homotopy groups vanish except π_4 which is isomorphic to $\mathbf{Z}/2$. We look at the portion

$$0 \to \pi_4 MPL[4] \to \pi_4 M\,\mathrm{Top}[4] \to \pi_4 \Sigma^4 K\mathbf{Z}/2 \to 0$$

of the exact homotopy sequence of this cofibration. The group $\pi_4 MPL[4] = \pi_4 MO[4] = \Omega_4^{\mathrm{Spin}}$ is isomorphic to $\mathbf{Z}$ by mapping a spin 4-manifold to its signature divided by 16. According to Freedman there is a closed topological spin 4-manifold with signature 8 [6, Theorem 1.7]. Hence the above sequence is nonsplit and $\pi_4 M\,\mathrm{Top}[4] = \Omega_4^{\mathrm{TopSpin}}$ is isomorphic to $\mathbf{Z}$, the isomorphism being given by the signature divided by 8. q.e.d.

Proof of Lemma 6.1. We use the AHSS (6.3) to compute $\Omega_7^{\mathrm{Spin}}(\mathbf{CP}^\infty; \alpha)$ for α = trivial bundle and α = Hopf bundle. Since the cohomology of $\mathbf{CP}^\infty$ and hence the homology of $M\alpha$ are concentrated in even dimensions, the only nontrivial group in the E_2-term contributing to $\Omega_7^{\mathrm{Spin}}(\mathbf{CP}^\infty; \alpha)$ is $H_6(M\alpha; \Omega_1^{\mathrm{Spin}}) \cong H_6(M\alpha; \mathbf{Z}/2) \cong \mathbf{Z}/2$. This term, however, does not survive to the E_∞-term for the following reason.

If α is the trivial bundle, then $\mathrm{Sq}^2\colon H^6(M\alpha; \mathbf{Z}/2) \to H^8(M\alpha; \mathbf{Z}/2)$ is nontrivial, and therefore the differential

$$d_2\colon H_8(M\alpha; \Omega_0^{\mathrm{Spin}}) \cong \mathbf{Z} \to H_6(M\alpha; \Omega_1^{\mathrm{Spin}}) \cong \mathbf{Z}/2$$

is surjective since the first k-invariant of $MO[4]$ is given by Sq^2.

If $\alpha = H$, then $\mathrm{Sq}^2\colon H^4(M\alpha; \mathbf{Z}/2) \to H^6(M\alpha; \mathbf{Z}/2)$ is nontrivial, and therefore the differential $d_2\colon H_6(M\alpha; \Omega_1^{\mathrm{Spin}}) \cong \mathbf{Z}/2 \to H_4(M\alpha; \Omega_2^{\mathrm{Spin}}) \cong \mathbf{Z}/2$ is an isomorphism since the k-invariant of $MO[4]$ which relates $\Omega_1^{\mathrm{Spin}} = \pi_1 MO[4] = \mathbf{Z}/2$ and $\Omega_2^{\mathrm{Spin}} = \pi_2 MO[4] = \mathbf{Z}/2$ is also Sq^2.

The same argument shows $\Omega_7^{\mathrm{TopSpin}}(\mathbf{CP}^\infty; \alpha) = 0$. q.e.d.

For the proof of Lemma 6.2 we need some lemmas.

Lemma 6.5. *There is an element $W_4 \in H^4(B\,\mathrm{Top}\langle 4\rangle; \mathbf{Z})$ such that $2W_4 = p_1$ and $W_4 = w_4 \bmod 2$, where $p_1 \in H^4(B\,\mathrm{Top}\langle 4\rangle; \mathbf{Z})$ is the first Pontrjagin class, and $w_4 \in H^4(B\,\mathrm{Top}\langle 4\rangle; \mathbf{Z}/2)$ is the fourth Stiefel-Whitney class.*

Proof. The fibration $K(\mathbf{Z}/2, 3) \to BPL \to B\,\mathrm{Top}$ mentioned in the proof of 6.4 and the fact that $\pi_n BPL \cong \pi_n BO$ for $n \leq 7$ gives a short exact sequence

$$0 \to \pi_4 BO \to \pi_4 B\,\mathrm{Top} \to \mathbf{Z}/2 \to 0.$$

This sequence splits since if $f\colon S^4 \to BO$ is a generator then $\langle f^* w_4, [S^4]\rangle \neq 0$ showing that the compositions $S^4 \to BO \to B\,\mathrm{Top}$ cannot be divisible by 2. Hence $H^4(B\operatorname{Top}\langle 4\rangle; \mathbf{Z}) \cong H^4(BO\langle 4\rangle; \mathbf{Z}) \cong \mathbf{Z}$ and it is easily checked that $W_4 = p_1/2$ is a generator.

To prove that $W_4 \bmod 2$ and w_4 agree in $H^4(B\operatorname{Top}\langle 4\rangle; \mathbf{Z}/2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ we observe that there is a unique nontrivial element in this cohomology group which is annihilated by Sq^1. Obviously $W_4 \bmod 2$ is such an element since it is the reduction of an integral class. The calculation

$$\begin{aligned}(\mathrm{Sq}^1 w_4)\mathrm{U} &= \mathrm{Sq}^1(w_4\mathrm{U}) = \mathrm{Sq}^1\mathrm{Sq}^4\mathrm{U} = \mathrm{Sq}^5\mathrm{U} \\ &= \mathrm{Sq}^4\mathrm{Sq}^1\mathrm{U} + \mathrm{Sq}^2\mathrm{Sq}^3\mathrm{U} = 0\end{aligned}$$

in $H^*(M\operatorname{Top}[4]; \mathbf{Z}/2)$, where U denotes the Thom class, shows that also w_4 has this property.

Lemma 6.6. *The image of the composition of the natural map* $\Omega_8^{\mathrm{TopSpin}} \to \Omega_8^{\mathrm{TopSpin}}(\mathbf{CP}^\infty; \alpha)$ *and* $S_1\colon \Omega_8^{\mathrm{TopSpin}}(\mathbf{CP}^\infty; \alpha) \to \mathbf{Q}$ *is* $\frac{1}{28}\mathbf{Z}$.

Proof. If W is an 8-dimensional topological spin manifold, then denote by $W_4(W)$ the pull back of the class W_4 via the classifying map of the tangent bundle $W \to B\operatorname{Top}\langle 4\rangle$. The previous lemma shows that the mod2 reduction of $W_4(W)$ agrees with the Stiefel-Whitney class $w_4(W)$ which is equal to the Wu class $v_4(W)$ since the lower Stiefel-Whitney classes of W vanish. Hence $\langle W_4(W)x, [W]\rangle = \langle x^2, [W]\rangle \bmod 2$ for all classes $x \in H^4(W; \mathbf{Z})$. It follows that $\operatorname{sign}(W) = \langle W_4(W)^2, [W]\rangle \bmod 8$. This is a generalization of the well-known fact that the signature of an even form is divisible by 8. Thus

$$\begin{aligned}S_1(W, z) &= -\frac{1}{2^5\cdot 7}\operatorname{sign}(W) + \frac{1}{2^7\cdot 7}p_1^2 \\ &= \frac{1}{2^5\cdot 7}(-\operatorname{sign}(W) + W_4^2) \in \frac{1}{28}\mathbf{Z}.\end{aligned}$$

To show that $\frac{1}{28}\mathbf{Z}$ is contained in the image of the composition consider the 8-dimensional Milnor manifold $M^8(E_8)$ which is the smooth parallelizable manifold of signature 8 whose boundary generates the group of 7-dimensional homotopy spheres. Let N be the closed topological manifold $M^8(E_8) \cup D^8$. Then $\operatorname{sign}(N) = 8$ and

$$S_1(N, 0) = -\frac{1}{2^5\cdot 7}\operatorname{sign}(N) + \frac{1}{2^7\cdot 7}p_1^2 = -\frac{1}{28}.$$

Proof of Lemma 6.2. We proceed inductively using the skeletal filtration of $\Omega_8^{\mathrm{TopSpin}}(\mathbf{CP}^\infty; \alpha)$. Recall that the sth filtration group $F_s\Omega_{t+s}^{\mathrm{TopSpin}}(X; \alpha)$ consists of the elements represented by triples (M, f, ω), where f factors through the s-skeleton of X, and that the filtration quotients $F_s\omega_{t+s}/F_{s-1}\Omega_{t+s}$ are isomorphic to the group $E^\infty_{s,t}$ in the E^∞-term of the Atiyah-Hirzebruch spectral sequence. We write F_s for $F_s\Omega_8^{\mathrm{TopSpin}}(\mathbf{CP}^\infty;\alpha)$.

The previous lemma shows that the image of S_1 restricted to F_0 is $\frac{1}{28}\mathbf{Z}$. The images of S_2 and S_3 restricted to F_0 are trivial since S_2 and S_3 are linear combinations of the characteristic numbers z^2p_1 and z^4 which vanish for elements in F_0. Hence the image of S restricted to F_0 is $\frac{1}{28}\mathbf{Z}\oplus\{0\}\oplus\{0\}$.

The AHSS shows that the next interesting filtration group is F_4 with filtration quotient $F_4/F_3 = E^\infty_{4,4} = E^2_{4,4} = H_4(M\alpha;\Omega_4^{\mathrm{TopSpin}}) \cong \mathbf{Z}$. To compute S for a generator of this filtration quotient we consider the spin case and the nonspin case separately.

In the nonspin case, i.e., if α is the Hopf bundle, a generator is represented by the triple $(\mathbf{CP}^2\times M(E_8), z)$, where $M(E_8)$ is the Freedman manifold, i.e., a topological 4-manifold with signature 8, and z is a generator of $H^2(\mathbf{CP}^2;\mathbf{Z})$. A calculation shows $S(\mathbf{CP}^2\times M(E_8), z) = (0, -1, -3)$, and hence the image of S restricted to F_4 is contained in $\frac{1}{28}\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$.

In the spin case we argue more indirectly by comparing with the smooth case. Let $X_1, X_2\in\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^\infty)$ be the elements represented by $S^2\times S^2\times S^2\times S^2$ with $z=\sum_{i=1}^4 x_i$, where x_i generates $H^2(S^2;\mathbf{Z})$ (resp. by a degree 2 hypersurface in $\mathbf{CP}^5$ with $z=$ restriction of the generator of $H^2(\mathbf{CP}^5;\mathbf{Z})$), and let X be the linear combination $X = X_1 - 12\cdot X_2$. This linear combination is chosen such that the characteristic number z^4 vanishes showing that X has filtration ≤ 7. The vanishing of the groups $E^\infty_{7,1}$, $E^\infty_{6,2}$, and $E^\infty_{5,3}$ in the E^∞-term of the AHSS converging to $\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^\infty)$ implies that X has filtration ≤ 4. It follows from the proof of Lemma 3.2 in [13] that $\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^\infty)/F_3\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^\infty)\cong\mathbf{Z}^2$ is generated by X_1 and X_2, and hence X generates the filtration quotient $F_4\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^\infty)/F_3\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^\infty) = E^\infty_{4,4}\cong\mathbf{Z}$.

Let Y be an element of $\Omega_8^{\mathrm{TopSpin}}(\mathbf{CP}^\infty)$ which represents a generator of the corresponding filtration for that group. Then $X=\pm 2\cdot Y + Z$, where Z is an element of lower filtration since the map on the E_2-term

$$H_4(\mathbf{CP}^\infty;\Omega_4^{\mathrm{Spin}})\cong\mathbf{Z}\longrightarrow H_4(\mathbf{CP}^\infty;\Omega_4^{\mathrm{TopSpin}})\cong\mathbf{Z}$$

is multiplication by 2. The characteristic numbers S_2 and S_3 vanish for Z, and hence $S_i(Y) = \pm\frac{1}{2}S_i(X)\equiv\frac{1}{2}S_i(X_i)\equiv\frac{1}{2} \bmod \mathbf{Z}$ (resp. $0 \bmod \mathbf{Z}$)

for $i = 2$ (resp. $i = 3$) [13, p. 382]. Lemma 6.6 implies $S_1(Y) \in \frac{1}{28}\mathbf{Z}$ since in the spin case $S_1(W, z) = S_1(W, 0)$. Thus, the image of S restricted to F_4 is contained in $\frac{1}{28}\mathbf{Z} \oplus \frac{1}{2}\mathbf{Z} \oplus \mathbf{Z}$.

The elements of all higher filtration quotients can be represented by smooth manifolds since the natural map $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{TopSpin}}$ induces an isomorphism between the groups $E_{s,t}^{\infty}$ in the E^{∞}-term of the AHSS converging to $\Omega_*^{\mathrm{TopSpin}}(\mathbf{CP}^{\infty}; \alpha)$ and the corresponding groups in the AHSS converging to $\Omega_*^{\mathrm{Spin}}(\mathbf{CP}^{\infty}; \alpha)$ for $t \leq 3$. We know that the image of S restricted to $\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^{\infty}; \alpha)$ is $\mathbf{Z}^3$, and hence the image of S is contained in $\frac{1}{28}\mathbf{Z} \oplus \frac{1}{2}\mathbf{Z} \oplus \mathbf{Z}$ (resp. $\frac{1}{28}\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$) if α is the trivial bundle (resp. Hopf bundle).

On the other hand, $\frac{1}{28}\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ is contained in the image of S since $\frac{1}{28}\mathbf{Z} \oplus \{0\} \oplus \{0\}$ (resp. $\mathbf{Z}^3$) is the image of S restricted to F_0 (resp. $\Omega_8^{\mathrm{Spin}}(\mathbf{CP}^{\infty}; \alpha)$). Moreover, in the spin case $\frac{1}{28}\mathbf{Z} \oplus \frac{1}{2}\mathbf{Z} \oplus \mathbf{Z}$ is in the image of S since $S_2(Y) = \frac{1}{2} \bmod \mathbf{Z}$ with Y as above.

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UNIVERSITÄT MAINZ, WEST GERMANY
UNIVERSITY OF NOTRE DAME, INDIANA