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EXISTENCE OF SMOOTH EMBEDDED SURFACES OF PRESCRIBED GENUS THAT MINIMIZE PARAMETRIC EVEN ELLIPTIC FUNCTIONALS ON 3-MANIFOLDS

BRIAN WHITE

Introduction

Let F be a smooth positive function on the boundary of the unit ball in a Euclidean 3-space R^3. Then F defines a functional on immersed surfaces M in R^3 by the formula

$$F(M) = \int_M F(\nu)\,dA,$$

where $\nu(x)$ is the unit normal to M at x and the integration is with respect to surface area. Such a functional is called a "constant coefficient parametric functional". In this paper we study the existence of smooth embedded surfaces (with given boundaries) that minimize F. In particular, we show:

Theorem 3.4 (condensed version). *If F is even and elliptic, and S is a smooth simple closed curve on the boundary of a convex set in R^3, then for each $g \geq 0$ there exists a smooth embedded surface that minimizes $F(M)$ among all embedded surfaces M with boundary $\partial M = S$ and* genus$(M) \leq g$.

Here "F is even" means that $F(\nu) \equiv F(-\nu)$, i.e., that $F(M)$ does not depend on the orientation of M. Ellipticity of F means that the set

$$\{x\colon |x|F(x/|x|) \leq 1\}$$

is uniformly convex; this is equivalent to ellipticity of the Euler-Lagrange equations for the corresponding functional on graphs.

More generally, F can depend on position as well as unit normal direction:

$$F\colon R^3 \times \partial B^3 \to R^+,$$

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where B^3 is a 3-ball, and R^+ is the set of positive real numbers. The functional on surfaces is then given by

$$F(M) = \int_{x \in M} F(x, \nu(x))\, dA.$$

Such an F is called a (nonconstant coefficient) parametric functional. We say that F is even and elliptic if and only if $F(x, \cdot)$ is even and elliptic for each x. The theorem above remains true for such an F defined on a compact 3-manifold-with-boundary N provided that S is a simple closed curve in ∂N and that ∂N is strictly F-convex (see §1.5).

There has been much work on related questions, especially for the area functional (the case $F(\nu) \equiv 1$). First, Douglas [6], [7] proved the existence of a branched immersion minimizing area under the hypotheses of Theorem 3.4, except he did not require S to be on the boundary of a convex set; Rado [29] also showed this in the case $g = 0$. Morrey [26] proved existence of smooth area minimizing maps of disks into 3-manifolds. Hildebrandt [21] and Heinz and Hildebrandt [19] showed that such maps (into Euclidean space and Riemannian manifolds, respectively) were branched immersions up to and including the boundary. (The regularity results of Douglas, Rado, and Morrey were for interior points.) Osserman [28] showed that such surfaces in R^3 have no true branch points except possibly at the boundary, and Gulliver [15], [16] extended Osserman's result to both true and false branch points and to arbitrary ambient 3-manifolds.

Now suppose that S is on the boundary of a convex set as in Theorem 3.4. It is fairly easy to see that the area minimizing disk it bounds has no boundary branch points either. Thus such an S bounds an area-minimizing immersed disk. For some time, however, it was not known whether such an S must bound an embedded minimal disk. In 1978 Tomi and Tromba [32] used degree theory and global analysis to show that S must bound an embedded disk that is minimal (i.e., is a critical point for the area functional). But the disk given by their proof is not necessarily area minimizing or even stable. Shortly afterward, Almgren & Simon [3] proved that S bounds a smooth embedded disk that minimizes area among all embedded disks. They used techniques of geometric measure theory to get a varifold solution and then to prove that the varifold was a smooth surface. Finally, Meeks and Yau [25] showed that for such an S the genus 0 Douglas-Rado solution is in fact embedded.

For general elliptic functionals F, Cesari [4], Danskin [5], and Sigalov [31] showed that there is a $W^{1,2}$ map u of the two-disk D into R^3 with

$u(\partial D) = S$ that minimizes

$$\begin{aligned} F(u(D)) &= \int_D F\left(\frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y} \Big/ \left|\frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y}\right|\right) dA \\ &= \int_D F\left(\frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y}\right) dx\, dy, \end{aligned}$$

where in the second line we have extended $F: \partial B^3 \to R^+$ to be homogeneous of degree 1 on R^3. Morrey [27] gave a simpler proof of the result. However, they were unable to prove smoothness of the weak solution.

By the compactness theorem for integral currents [10], [37] or for BV functions [14] there is an integral current that minimizes $F(M)$ among surfaces with $\partial M = S$. In 1977 Almgren, Schoen, and Simon [2] showed that the integral current is a smooth embedded surface away from its boundary, and Hardt [17] showed that if S is on the boundary of a convex set then it is also smooth at the boundary. However, those theorems say nothing about minimizing F among surfaces of specified topological types.

Recently Lin [23] and the author [36] independently used the Tomi-Tromba argument to show existence of smooth embedded disks that are stable for even elliptic functionals. However, the disks produced need not minimize the functional. Examples show that in manifolds the Tomi-Tromba approach fails even for the area functional [36, §3].

The main contribution of this paper is to prove Theorem 3.4 for disks (the $g = 0$ case); the higher genus result follows easily with the help of an elegant idea of Hass and Scott [18]. The proof here is not a generalization of the proofs for the area case. It is unusual in that it uses the Almgren-Schoen-Simon result on surfaces of unspecified genus. The proof starts with minimizing sequence of embedded disks. For each disk D_i in the sequence, we consider the family $\mathscr{C}_i$ of $C^{1,1}$ surfaces with boundary S that: (i) lie on a given side of D_i (let us say "above" D_i), (ii) are F-stable among surfaces with D_i as an obstacle, and (iii) have F-integral less than or equal to $F(D_i)$. (This is a slightly simplified account of the proof; for technical reasons a condition somewhat stronger than stability is used.) This family is not empty because (by the Almgren-Schoen-Simon theorem) it contains the F-minimizing integral current that lies above D_i. Of course this integral current may have high genus, but it turns out that the surface in the family $\mathscr{C}_i$ that is closest to D_i must be a disk D_i'. Note that D_i' will be F-stationary except at points x where it touches the obstacle D_i, and at those points the F analogue of the mean curvature points "downward" (toward D_i).

Now the process is repeated with D_i' instead of D_i and with "above" and "below" reversed. The result is a $C^{1,1}$ disk D_i'' that lies "below" D_i' and that is F-stationary except where it touches D_i'. But by the maximum principle it cannot touch D_i'. Also, by the construction,

$$F(D_i'') \leq F(D_i') \leq F(D_i).$$

Thus we have a minimizing sequence of smooth F-stationary embedded disks. By a priori curvature estimates there is a convergent subsequence; the limit is of course the F-minimizing disk D.

The a priori estimates come from two Bernstein-type theorems:

4.1. Theorem. *Let M be a complete connected orientable surface that is properly immersed in R^3 and stable with respect to a constant coefficient parametric elliptic functional F. Suppose that the density ratios are bounded above:*

$$\sup_{r>0} \frac{\operatorname{Area}(M \cap B_r(0))}{\pi r^2} = C < \infty.$$

Then M is a plane.

5.1. Theorem. *Let l be the line of intersection of two planes in R^3 and let $\mathscr{R}$ be one of the four regions into which the planes divide R^3. Let M be a complete connected orientable surface-with-boundary properly immersed in R^3 such that $\operatorname{int}(M) \subset \mathscr{R}$ and $\partial M = l$ where $\operatorname{int}(M)$ denotes the interior of M. Suppose that M is stable with respect to a constant coefficient parametric elliptic functional F and that the density ratios are uniformly bounded above:*

$$\sup_{r>0} \frac{\operatorname{Area}(M \cap B_r(0))}{\pi r^2} \leq C < \infty.$$

Then M is a half-plane.

For the case of the area functional, Theorem 4.1 was proved by Fischer-Colbrie and Schoen [11] and by doCarmo and Peng [8] without assuming bounded density ratios. Theorem 5.1 is apparently new even for the area functional.

The organization of the paper is as follows. §1 contains definitions and basic facts about variational problems with obstacles; the reader may wish to skip it. §2 gives the proof of the main result (the genus 0 case). In §3 we obtain the extension to higher genus and to arbitrary ambient manifolds. In §4 we establish a Bernstein type theorem and curvature estimates for stable surfaces, and in §5 we derive the analogous results at boundaries. In §6 we prove the regularity of F-minimizing currents with obstacles and extend Hardt's boundary regularity theorem to nonconstant coefficient functionals.

The referee has informed us that Theorem 4.1 and the results of §6 are contained in some unpublished work by F. H. Lin [22], [24]; Lin's methods for proving 4.1 are, however, different.

1. Preliminaries

1.1. In what follows, N will be a 3-manifold (with or without boundary), and F will be a smooth even elliptic parametric functional on N. If M is a C^1 submanifold of N, we will refer to $F(M)$ as the *F-area* of M.

1.2. Let M be a C^1 immersed surface in N and v be a Lipschitz vectorfield on M. We say that ϕ_t is a *one-parameter family with initial velocity* v if

(1) $t \mapsto \phi_t(\cdot)$ is a smooth map from R (or from a neighborhood of 0 in R) to the space of Lipschitz maps of M into N,

(2) $\phi_0(x) \equiv x$, and

(3) $(d/dt)_{t=0}\phi_t(x) = v(x)$.

For example, if N is Euclidean space, then $\phi_t(x) = x + tv(x)$ defines a one-parameter family with initial velocity v.

1.3. Let $M \subset N$ be a C^1 surface. We say that M has *F-mean curvature h* if h is a vectorfield on M such that

$$(d/dt)_{t=0}F(\phi_t(M)) = -\int_M h \cdot v\, dA,$$

whenever v is a Lipschitz vectorfield supported in a compact subset of $M \setminus \partial M$, and ϕ_t is a one-parameter family with initial velocity v. If M is C^2, then it has a continuous F-mean curvature vectorfield whose value at each point is determined by the second fundamental form of M at that point.

The surface M is said to be *F-stationary* if it has F-mean curvature 0. An F-stationary surface F is said to be *F-stable* if

$$(d/dt)^2_{t=0}F(\phi_t(M)) \geq 0$$

for every ϕ_t as above.

1.4. Let Ω be an open subset of N with C^1 boundary, and let M be a C^1 surface disjoint from Ω. Let ρ be a C^1 retraction from an open set containing $N \setminus \Omega$ onto $N \setminus \Omega$. We say that M is *F-stationary for the* *obstacle* Ω (or, by slight abuse of terminology, *for the obstacle* $\partial\Omega$) if

$$F(\rho(f_t(M))) \geq F(M) - o(t)$$

for every ϕ_t as in §1.3.

Proposition. *Let Ω be an open subset of N such that $\partial\Omega$ is C^1 and has bounded F-mean curvature, and let M be a C^1 surface that is F-stationary for the obstacle Ω. Then M has bounded F-mean curvature (with the same bound). If Ω is $C^{1,1}$, then so is M.*

Proof. By working locally we may assume that M is the graph of a C^1 function $u: \overline{B} \to R$ on the unit ball in R^2 and that Ω is the region below the graph of a C^1 function $f: \overline{B} \to R$ with $f \leq u$.

Define a functional $\Phi: C^{0,1}(\overline{B}) \to R$ by letting $\Phi(v)$ be the F-area of the graph of v. Then

$$\Phi(v) = \int_B L(x, v(x), Dv(x))\, dx,$$

where

$$L(x, y, p) = F\left((x, y), \frac{(-p, 1)}{\sqrt{1+|p|^2}}\right) \cdot \sqrt{1+|p|^2}.$$

Since L is smooth, Φ is a smooth functional, so

$$\Phi(f+v) = D\Phi(f)v + o(|v|_{0,1})$$

and, since $\partial\Omega$ has bounded F-mean curvature, we have

$$D\Phi(f)v = -\int_B v \cdot h,$$

where h is a bounded measurable function. Thus

$$\begin{aligned}
&\Phi(u) - \Phi(u+tv) \\
&\quad \leq [\Phi(u) - \Phi((u+tv-f)^+ + f)] \\
&\qquad + [\Phi(f) - \Phi((u+tv-f)^- + f)] \\
&\quad \leq o(t) + |\Phi((u+tv-f)^- + f) - \Phi(f)| \\
&\quad \leq o(t) + \left|\int_B (u+tv-f)^- \cdot h\right| + o(|(u+tv-f)^-|_{0,1}) \\
&\quad \leq o(t) + \|(u+tv-f)^-\|_1 \cdot \|h\|_\infty + o(|(u+tv-f)^-|_{0,1}) \\
&\quad \leq o(t) + \|tv\|_1 \cdot \|h\|_\infty + o(|tv|_{0,1}),
\end{aligned}$$

(where $a^+ = (a+|a|)/2$ and $a^- = (a-|a|)/2$) since $u - f \geq 0$. Hence,

$$-D\Phi(u)v \leq \|h\|_\infty \cdot \|v\|_1,$$

and by the Riesz-representation theorem, we have

$$D\Phi(u)v \equiv -\int_B g\cdot v$$

for some function g with $|g| \le |h|$.

Note that u is a weak solution of $H(u) = g$, where H is the Euler-Lagrange operator for L. Note also that if $w \ge f$, then

$$\langle H(u), w-u\rangle = (d/dt)_{t=0}\Phi(u + t(w-u)) \ge 0.$$

It follows that u is $C^{1,1}$ if f is $C^{1,1}$ [12].

Corollary. *Let $\Omega \subset N$ be an open set such that $\partial\Omega$ is C^1 and has bounded F-mean curvature, and let M be a C^1 surface in $N\backslash\Omega$. Then M is F-stationary for the obstacle Ω if and only if M has F-mean curvature h, where $h(x) = 0$ if $x \notin \partial\Omega$, and $h(x)$ points into Ω if $x \in \partial\Omega$.*

Remark 1. For this paper it is not really necessary to use the theorem that u is $C^{1,1}$ if f is $C^{1,1}$. For, as proved above (without assuming f to be $C^{1,1}$), u is a weak solution to $H(u) = g$ for some bounded measurable function g. But then by Theorem 13.1 of Gilbarg and Trudinger [13], u is $C^{1,\gamma}$, where γ depends only on F. (Gilbarg and Trudinger assume that u is C^2 and prove $C^{1,\gamma}$ estimates, but the proof can be slightly modified with an approximation argument to avoid the assumption.)

In the rest of the paper one can substitute $C^{1,\gamma}$ for $C^{1,1}$.

Remark 2. Note that the first conclusion of the proposition remains true for several disjoint obstacles Ω_i, even if their closures are not disjoint.

1.5. Let $\Omega \subset N$ be an open set such that $\partial\Omega$ is C^1 and has F-mean curvature $h(x) = u(x)n(x)$, where $n(x)$ is the unit normal to $\partial\Omega$ that points into Ω. We say that Ω is *F-convex* if $u(x) \ge 0$ for almost every $x \in \partial\Omega$, and *strictly F-convex* if $u(x) > 0$ for almost every $x \in \partial\Omega$.

Proposition. *Suppose that $\Omega_1 \subset \Omega_2$, where Ω_2 is F-convex, and $\partial\Omega_1$ has bounded F-mean curvature. Let M be a C^1 surface that is F-stationary for the obstacle $\Omega_1 \cup (N \setminus \overline{\Omega}_2)$. Then M is F-stationary for the obstacle Ω_1.*

Proof. By Proposition 1.4, M has a bounded F-mean curvature vectorfield h. By the maximum principle, at any point x where M touches $\partial\Omega_2$, $h(x)$ points into Ω_2. The result then follows immediately from Corollary 1.4.

2. Existence of smooth F-minimizing disks

Theorem. *Let B be a ball in R^3 (or more generally a bounded region diffeomorphic to a ball and having smooth boundary), and let F be an even* *parametric elliptic functional on $\overline{B}$ such that B is strictly F-convex. If S is a simple closed curve in ∂B, then there is a smooth disk D that minimizes $\int_D F(x, \nu(x))\, dA$ among all embedded disks in B with boundary S.*

If $j > 1$, and S is $C^{j,\alpha}$ near $x \in S$, then $\overline{D}$ is an embedded $C^{j,\alpha}$ manifold-with-boundary near x.

Proof. Suppose first that S is smooth. Let ∂B^+ and ∂B^- be the two connected components of $\partial B \setminus S$. For any embedded surface M in B with $\partial M = S$, we will say that x is *above* M if x is in the closed region bounded by M and ∂B^+, and that x is *below* M if x is in the closed region bounded by M and ∂B^-.

Let D be any smooth embedded disk in B with $\partial D = S$ and $F(D) = A$. We will first show that there exists an F-stable such disk $\widehat{D}$ with $F(\widehat{D}) \leq A$. Now if D is not F-stationary and F-stable, then we can modify it to get a new disk D' so that $F(D') < F(D)$. For technical reasons we modify D' to get a disk D'' so that D'' meets ∂B^+ tangentially along S; in other words, so that $D'' \cup \partial B^-$ is a smooth surface. Note that this modification can be done at arbitrarily small cost in F-area, in particular so that $F(D'') \leq F(D)$. Thus we may as well assume that $D'' = D$.

Let $\mathscr{M}$ be the set of embedded $C^{1,1}$ surfaces M such that:

(1) M lies above D,

(2) $\partial M = S$,

(3) $M \cup \partial B^-$ is F-stationary for the obstacle $D \cup \partial B^-$, and

(4) M is "one-sided F-minimizing" in that $F(M) \leq F(M')$ for every surface M' such that $\partial M' = S$ and M' lies in the closed region between M and D.

Let $M \in \mathscr{M}$. If $B(x, r) \subset B$, then by (4)

$$\begin{aligned} F(M \cap B(x, r)) &\leq F(\partial B(x, r) \cap \{\text{the region between } M \text{ and } D\}) \\ &\quad + F(B(x, r) \cap D) \\ &\leq cr^2, \end{aligned}$$

where c depends only on D and F. Note also that $M \setminus D$ is F-stable. (Otherwise a compact subset would be unstable, and, because the lowest eigenfunction does not change sign, we would be able to decrease the F-area of M by deforming it toward D, which would contradict (4).)

Therefore by Theorem 4.2 the surfaces M in $\mathscr{M}$ are uniformly $C^{1,\alpha}$.

The class $\mathscr{M}$ is nonempty since it contains the surface M of least F-area among surfaces above D and having boundary S. (Of course one must check that this surface does in fact belong to $\mathscr{M}$. Certainly $M \cup B^-$ is stationary given as obstacles both the region inside $D \cup B^-$ and the region outside ∂B; since B is F-convex, $M \cup B^-$ is in fact stationary given only the first obstacle (Proposition 1.5). See §6 for a proof that the surface is $C^{1,1/2}$; it then follows from Proposition 1.4 that it is $C^{1,1}$).

Hence among the surfaces in $\mathscr{M}$, there exists a surface Σ for which the volume of the region between Σ and D is as small as possible. We claim that Σ is a disk.

For suppose not. Then by topology (e.g. the Mayer-Vietoris sequence for the regions above and below Σ), there is a closed curve C in the interior of the region U above Σ such that C does not bound any surface (i.e., is not Z_2-homologically trivial) in U. On the other hand, C does bound a surface in the region above D. Let T be such a surface (flat chain mod 2) of at least possible F-area. Then T is a $C^{1,1}$ embedded surface away from C (by 1.4, 6.1, and 6.2); in particular the portion of T below Σ is $C^{1,1}$.

Now consider the class of surfaces in the closed region between Σ and D that have boundary S and do not cut across T. This class is nonempty since it contains D. Let M be the surface of least F-area in this class. Then M is $C^{1,1}$ (just as T was) and by the maximum principle M cannot touch Σ or T at any point not in D. Thus $M \in \mathscr{M}$. Since M lies below Σ, this contradicts the choice of Σ. Thus Σ must be a disk as claimed.

(To make the argument in the previous paragraph more precise, let $\widetilde{B}$ be the geodesic completion of $B \setminus T$, that is, the metric space completion of $B \setminus T$ with respect to the metric given by geodesic distance in $B \setminus T$. Then there is a natural map $\pi\colon \widetilde{B} \to B$ that is the identity on $B \setminus T$. Note that for each x in $T \setminus \partial T$, $\pi^{-1}(x)$ consists of two points. Consider the class of surfaces in $\widetilde{B}$ that have boundary S and lie between Σ and D. Note that D is in this class but that Σ is not. (Σ has extra boundary on $\pi^{-1}(T)$.) Let $\widetilde{M}$ be the support of the least F-area surface in this class, and let $M = \pi(\widetilde{M})$.)

Note by (4) that $F(\Sigma) \le F(D)$.

Now we redefine $\mathscr{M}$ to be the set of all $C^{1,1}$ surfaces M such that

(1) M lies below Σ,

(2) $\partial M = S$,

(3) M is F-stationary, and

(4) M is "one-sided F-minimizing" in that $F(M) \le F(M')$ for every surface M' such that $\partial M' = S$ and M' lies in the closed region between M and Σ.

The set $\mathscr{M}$ is not empty since it contains the surface M of least F-area between ∂B^- and Σ with boundary S. (See §6.3 for the proof that M is regular up to the boundary.) As with $\mathscr{M}$, the one-sided minimizing property implies that members of $\mathscr{M}$ are F-stable and satisfy a uniform upper bound on density ratios. Thus the curvature bounds of Theorem 5.2 imply that $\mathscr{M}$ is compact. Let $\widehat{D}$ be a member of $\mathscr{M}$ for which the volume between $\widehat{D}$ and Σ is as small as possible. Then (by the argument used above for Σ) $\widehat{D}$ is a disk. Also, by (4), the F-area of $\widehat{D}$ is less than or equal to the F-area of Σ, which, as already mentioned, is less than or equal to the F-area of D.

Thus we have shown that given any smooth disk D with boundary S, there is a smooth F-stationary (indeed F-stable) disk $\widehat{D}$ with $\partial\widehat{D} = S$ and $F(\widehat{D}) \le F(D)$.

Now let

$$\mathscr{D} = \{\text{smooth embedded disks } D \subset B \text{ with } \partial D = S\},$$
$$m = \inf\{F(D) : D \in \mathscr{D}\},$$

and let $D_i \in \mathscr{D}$ be a sequence with $F(D_i) \to m$. Let $\widehat{D}_i$ be the corresponding sequence of F-stable disks.

Unfortunately we cannot use the previously cited curvature bound (Theorem 5.2) to get a convergent subsequence because that theorem requires a uniform bound on density ratios, which we do not know how to prove. However, since the boundary S is smooth and the areas and genuses of the D_i are bounded, we do have curvature estimates and therefore a convergent subsequence (see [34]: the curvature estimate comes in part from the Gauss-Bonnet theorem). The limit is of course a smooth embedded disk with F-area m.

If $S \in \partial B$ is not smooth, let $S_i \in \partial B$ be a sequence of smooth embedded curves that approach S in the C^0 topology. Let D_i be a smooth embedded F-minimizing disk with $\partial D_i = S_i$. Let $x \in B$. In the next section (Corollary to Lemma 3.3) we will show that for sufficiently small $r > 0$, the intersection of each D_i with $B(x, r)$ consists of disjoint disks, each of which is F-minimizing. Note that each of these disks has F-area less than or equal to half the F-area of $\partial B(x, r)$. This upper density ratio bound (for connected components of $D_i \cap B(x, r)$) implies (by Theorem 5.2) that the principal curvatures of the D_i are uniformly bounded on compact subsets of B. Therefore there is a subsequence (which we will assume to be the original sequence) converging to a disk D that is smooth away from S.

If $\partial D' = S$, then

$$F(D_i) \leq F(D') + F(\text{the region in } \partial B \text{ between } S_i \text{ and } S).$$

Thus

$$F(D) \leq \liminf F(D_i) \leq F(D'),$$

hence D is F-minimizing.

If S is $C^{j,\alpha}$ near $x \in S$, then we can choose the S_i to coincide with S near x. It then follows from Theorem 5.2 that the curvatures of the D_i are uniformly bounded near x and thus that the limit surface D is $C^{1,\alpha}$ and (by PDE) $C^{j,\alpha}$ near x.

3. Surfaces of higher genus

In this section we extend the results of the previous section to surfaces of higher genus and to arbitrary ambient 3-manifolds. We begin with a proposition and two lemmas which are topological.

3.1. Proposition. *Let B, F, S and D be as in the Theorem of* §2. *If $D' \subset B$ is an immersed disk with $\partial D' = S$, then $F(D) \leq F(D')$. If S_i $(1 \leq i \leq k)$ are disjoint simple closed curves in ∂B, then there exist disjoint smooth embedded F-minimizing disks D_i with $\partial D_i = S_i$.*

Proof. For any $\epsilon > 0$ we can find a smooth transversally immersed disk D'' with $\partial D'' = S$ and $F(D'') < F(D') + \epsilon$. By cutting and pasting along the self-intersection set of D'' as in the proof of Dehn's lemma (see [25] or [20]), we can make an embedded disk $\tilde{D}$ with $F(\tilde{D}) \leq F(D'')$. But $F(D) \leq F(\tilde{D})$, so $F(D) < F(D') + \epsilon$ for every $\epsilon > 0$. This proves the first conclusion. The proof of the second is similar. q.e.d.

Recall that a *plane domain* is a compact 2-manifold-with-boundary that is homeomorphic to a submanifold of the 2-sphere. In particular, a connected plane domain is homeomorphic to the complement of a disjoint union of disks in S^2.

3.2. Lemma. *Let P be a connected plane domain contained in a disk D. Let P' be another plane domain with $\partial P' = \partial P$. Then $(D \setminus P) \cup P'$ is the union of a disk and several* (0 *or more*) *spheres.*

Proof. Attach another disk to D to get a sphere S. We must show that $(S \setminus P) \cup P'$ is a union of spheres. But $S \setminus P$ is a collection of disjoint disks D_i. Note that attaching a disk to a connected plane domain gives a plane domain or a sphere. Thus when we form $(S \setminus P) \cup P'$ by attaching the disks $\bigcup_i D_i = S \setminus P$ to P' we get a union of plane domains and spheres. But $(S \setminus P) \cup P'$ has no boundary, so it must consist entirely of spheres.

3.3. Replacement Lemma. *Let N be a compact 3-manifold with boundary, and let F be an even parametric elliptic functional on N such that N is F-convex. There is an $\epsilon > 0$ such that if $r \leq \epsilon$, and $D \subset N$ is a piecewise smooth immersed disk transverse to $\partial B(x, r)$ with $\partial D \subset \partial(B(x, r) \cap N)$ and $D \setminus B(x, r)$ embedded, then there is a smooth embedded disk $D' \subset B(x, r) \cap N$ such that $\partial D' = \partial D$ and $F(D') \leq F(D)$. If $D \not\subset B(x, r)$, then $F(D') < F(D)$.*

Proof. Choose $R > 0$ so that $B(x, \rho)$ is strictly F-convex whenever $\rho \leq 3R$. (An $\epsilon < R$ will be chosen later.)

Suppose D and $B(x, r)$ are as in the statement of Lemma 3.2. For simplicity let us suppose that $B(x, 3R) \cap \partial N = \varnothing$ (the other case is similar). If $D \not\subset B(x, r)$, let U be a connected component of $D \setminus B(x, r)$. Let T be the surface (flat chain mod 2) of least F-area in $N \setminus B(x, r)$ such that $\partial T = \partial U$. If T has a point y in $\partial B(x, 2R)$, then

$$F(T) \geq F(T \cap B(y, R)) \geq c_1 R^2,$$

since $\partial T \cap B(y, R) = \varnothing$; cf. [9, 5.1.6].

On the other hand, ∂U divides the sphere $\partial B(x, r)$ into two regions, each of which is a candidate for T and one of which has area $\leq 2\pi r^2$. Thus

$$F(T) \leq c_2(2\pi r^2) \leq 2\pi c_2 \epsilon^2,$$

hence under the assumption that $T \cap \partial B(x, 2R) \neq \varnothing$ we have

$$c_1 R^2 \leq 2\pi c_2 \epsilon^2 \quad \text{or} \quad \epsilon \geq R\sqrt{c_1/(2\pi c_2)}.$$

Conversely, if we choose $\epsilon < R\sqrt{c_1/(2\pi c_2)}$ (and we assume from now on that such an ϵ has been chosen), then $T \cap \partial B(x, 2R) = \varnothing$.

We claim that $T \subset \partial B(x, r)$. To see this, first note that $T \subset B(x, 2R)$; otherwise replace T by $T \cap B(x, 2R)$. Let ρ be the smallest radius such that $T \subset \overline{B}(x, \rho)$. If ρ were greater than r, then at the intersection of T and $\partial B(x, \rho)$ we would have a contradiction to the maximum principle by recalling that $\partial B(x, \rho)$ is strictly F-convex, and T is smooth away from its boundary [2].

Thus T is one of the two regions in $\partial B(x, r)$ with boundary ∂U. By Lemma 3.2, $(D \setminus U) \cup T$ is the union of a disk and zero or more spheres. Let D_1 be that disk. Then

$$F(D_1) \leq F(D).$$

Now deform D_1 by pushing T in a little to get a piecewise smooth immersed disk D_2 with $F(D_2) < F(D_1)$. Note that the set of connected components of $D_2 \setminus B(x, r)$ is a proper subset of the set of connected components of $D \setminus B(x, r)$. Thus by repeating the process we eventually get an immersed disk D_3 inside $B(x, r)$. Finally, by Proposition 3.1, we get an embedded disk $D' \subset B(x, r)$ with $F(D') \leq F(D_3) < F(D)$.

Corollary. *Let N and F be as above. There exist $\epsilon > 0$ and $c < \infty$ such that if D is an embedded F-minimizing disk in N with $\partial D \subset (N \setminus B) \cup \partial N$, where B is a ball of radius $r \leq \epsilon$ such that ∂B intersects D transversely, then $D \cap B$ is a disjoint union of embedded F-minimizing disks, each of which has area $\leq cr^2$.*

Proof. By transversality $D \cap \partial B$ is a disjoint union of simple closed curves $C_1, \cdots, C_n$. Each C_i bounds a unique disk D_i in D. If D_i is not a subset of any other D_j, let D_i' be an F-minimizing disk in B with boundary C_i. Otherwise let $D_i' = \varnothing$. Then $D' = (D \setminus \bigcup D_i) \cup \bigcup D_i'$ is a disk with boundary ∂D. By Proposition 3.1, D' is embedded and thus $F(D') \geq F(D)$. But by Lemma 3.3, $F(D')$ would be less than $F(D)$ if it were not the case that each D_i is contained in B and is F-minimizing.

Note that C_i divides ∂B into two regions, each of which is a comparison surface for D_i. Thus the F-area of D_i is less than or equal to half of the F-area of ∂B, so the area of D_i is less than or equal to cr^2.

3.4. Theorem. *Let N be a compact 3-manifold-with-boundary, F be an even parametric elliptic functional on N such that N is strictly F-convex, and S be a simple closed curve in ∂N. Let $\mathscr{M}_g$ be the set of all piecewise smooth embedded surfaces in N with genus g and boundary S, and let*

$$a(g) = \inf \left\{ F(M) \colon M \in \bigcup_{i \leq g} \mathscr{M}_g \right\}.$$

If $a(g) < a(g-1)$, then there is a smooth $M \in \mathscr{M}_g$ with $F(M) = a(g)$.

Proof. Suppose $a(g) < a(g-1)$ and let $\delta = (a(g-1) - a(g))/2$. We first prove the following lemma.

Modification Lemma. *There is an $\epsilon > 0$ with the following property. If $M \in \mathscr{M}_g$ with $F(M) \leq a(g) + \delta$, and $B \subset N$ is a ball of radius $r \leq \epsilon$ such that ∂B intersects M transversely, then there is a surface $M' \in \mathscr{M}_g$ such that*

(1) $F(M') \leq F(M)$,
(2) $M' \setminus B \subset M$, *and*
(3) $M' \cap B$ *is a union of embedded F-minimizing disks.*

Proof of Lemma. By transversality, the intersection of ∂B and M is a collection $C_1, C_2, \cdots, C_n$ of simple closed curves. Let D_k be an F-minimizing embedded disk in B with $\partial D_k = C_k$. Note that each D_k has F-area less than or equal to the F-area of each of the two components of $\partial B \setminus C_k$. In particular, $F(D_k) \leq cr^2 \leq c\epsilon^2$. Form a sequence $M_0, M_1, \cdots, M_n$ of surfaces as follows:

(1) Let $M_0 = M$.

(2) If C_k bounds a disk D in M_{k-1}, replace D by D_k to form M_k:

$$M_k = (M_{k-1} \setminus D) \cup D_k.$$

Then M_k and M_{k-1} have the same genus, and $F(M_k) \leq F(M_{k-1})$ by Lemma 3.3.

(3) If C_k bounds a nonsimply connected subset T of M_{k-1}, replace T by D_k to form M_k:

$$M_k = (M_{k-1} \setminus T) \cup D_k.$$

Then the genus of M_k is less than the genus of M_{k-1} and

$$F(M_k) \leq F(M_{k-1}) + F(D_k) \leq F(M_{k-1}) + c\epsilon^2.$$

(4) If C_k is not contained in M_{k-1}, let $M_k = M_{k-1}$.

(5) If $C_k \subset M_{k-1}$ but C_k is not the boundary of a region in M_{k-1}, then form M_k by cutting M_{k-1} along C_k and glueing in two copies of D_k. Thus the genus of M_k is less than the genus of M_{k-1} and

$$F(M_k) = F(M_{k-1}) + 2F(D_k) \leq F(M_{k-1}) + 2c\epsilon^2.$$

Let $M' = M_n$. Then clearly M' has boundary S as does each M_k. Conclusions (2) and (3) of the lemma are obvious. Now $M' \setminus B$ is embedded by conclusion (2). Also, $M' \cap B$ consists of disks, any two of which are either disjoint or else coincide (by Proposition 3.1). But two cannot coincide since $M' \setminus B$ is embedded. Thus M' is embedded.

It remains to show that M' has genus g and that $F(M') \leq F(M)$. Note that in each of the four cases (2)–(5) of the construction,

$$F(M_k) + 2c\epsilon^2 \operatorname{genus}(M_k) \leq F(M_{k-1}) + 2c\epsilon^2 \operatorname{genus}(M_{k-1}).$$

Thus

$$(*) \qquad \begin{aligned} F(M') &\leq F(M) + 2c\epsilon^2(\operatorname{genus}(M) - \operatorname{genus}(M')) \\ &\leq F(M) + 2c\epsilon^2 \operatorname{genus}(M) \\ &\leq a(g) + \delta + 2c\epsilon^2 \operatorname{genus}(M) \\ &= a(g-1) - \delta + 2c\epsilon^2 \operatorname{genus}(M). \end{aligned}$$

Now choose ϵ so that $2c\epsilon^2 \operatorname{genus}(M) < \delta$. Then $F(M') < a(g-1)$ so M' has genus g. Thus by $(*)$, $F(M') \le F(M)$. This completes the proof of the modification lemma. q.e.d.

Now let $B_1, \cdots, B_n$ be a collection of balls of radii $< \epsilon$ that cover N. Let $M_1, M_2, \ldots$ be a sequence of surfaces in $\mathscr{M}_g$ with $F(M_i) < a(g) + \delta$ and $F(M_i) \to a(g)$.

Increase the radius of B_1 slightly, if necessary, so that ∂B_1 intersects each M_i transversely. Now apply the modification lemma to B_1 and M_i to get a new surface M_i^1.

Then $F(M_i^1) \to a(g)$, and $M_i^1 \cap B_1$ consists of F-minimizing disks. Thus by the Corollary to Lemma 3.3 and Theorem 5.2, the curvatures of the M_i^1 are uniformly bounded on compact subsets of B_1. Hence a subsequence (which we will assume to be the original sequence) of the $M_i^1 \cap B_1$ converges smoothly on compact subsets of B_1 to a surface Σ_1.

Next increase the radius of B_2 slightly, if necessary, so that ∂B_2 is transverse to the M_i^1 and to Σ_1. Then

$$\partial B_2 \cap M_i^1 \cap B_1 \to \partial B_2 \cap \Sigma_1$$

smoothly on compact subset of B_1. Now apply the modification lemma to B_2 and M_i^1 to get a surface M_i^2. Then

$$F(M_i^2) \le F(M_i^1) \le F(M_i).$$

As before, the curvatures of $M_i^2 \cap B_2$ are uniformly bounded on compact subsets of B_2, and so we may assume that the $M_i^2 \cap B_2$ converge smoothly on compact subsets of B_2. Furthermore, by the Corollary to Lemma 3.3 and the boundary curvature estimate 5.2, the curvatures of the M_i^2 are uniformly bounded on compact subsets of $B_2 \cup (\partial B_2 \cap B_1)$. Thus a subsequence of the M_i^2 (which we will assume to be the original sequence) converges smoothly on compact subsets of $B_2 \cup (\partial B_2 \cap B_1)$. Note that we also still have smooth convergence of the M_i^2 on $B_1 \setminus B_2$.

Likewise, for $k = 3, 4, \cdots, n$ form sequences M_i^k by applying the modification lemma to M_i^{k-1} and B_k (and passing to subsequences as necessary). Let $M_i' = M_i^n$. Then $F(M_i') \to a(g)$, and the M_i' converge smoothly on compact subsets of

$$B_j \setminus \left(\bigcup_{k=j+1}^{n} B_k \right)$$

for each j. In particular, we have uniform convergence on all of N, and the limit surface M has F-area $a(g)$, is piecewise smooth, and is a limit of embedded surfaces of genus g. If M were not C^1 at some point x, then we could round M off slightly near x to reduce its F-area and then perturb slightly to get an embedded surface of genus g and F-area $< a(g)$, a contradiction. Thus M is C^1. It follows that M is a regular immersed F-minimal surface. But since M is a limit of embeddings, and M is embedded near ∂M, it follows from the strong maximum principle that M is embedded everywhere.

4. A Bernstein-type theorem and curvature estimates for F-stable surfaces

4.1. Theorem. *Let M be a complete connected orientable surface that is properly immersed in R^3 and stable with respect to a constant coefficient parametric elliptic functional F. Suppose that the density ratios are bounded above:*

$$\sup_{r>0} \frac{\operatorname{Area}(M \cap B_r(0))}{\pi r^2} = C < \infty.$$

Then M is a plane.

Proof. Suppose that M is not a plane. Without loss of generality, we may assume that $0 \in M$ and that the principal curvatures of M at 0 do not vanish. Let $n(x)$ be the unit normal to M at x. Let $J\colon C^{2,\alpha}(M) \to C^{0,\alpha}(M)$ be the second variation operator for F. Since J is a second-order self-adjoint elliptic operator (see [35] for basic properties of J), it has the form

$$Ju = -D_i^M a^{ij} D_j^M u + fu,$$

where $a^{ij}(x)$ is a 3×3 rank 2 positive semidefinite symmetric matrix with $\Sigma a^{ij} n_j = 0$ and

$$D_i^M u(x) = \nabla_{e_i - (e_i \cdot n)n} u(x).$$

Let Q be the associated quadratic form:

$$Q(u) = \int_M (a^{ij} D_i^M u D_j^M u + fu^2)\, dA.$$

We claim that the a^{ij} are bounded by a constant depending only on F. To see this, suppose that $0 \in M$. Let M_s be the surface obtained from

M by a dilation by s. Let u be a function with compact support on M, and u_s be the corresponding function on M_s. Define $\phi_{s,t}\colon M_s \to R^3$ by $\phi_{s,t}(x) = x + tsu_s(x)$. Finally, let a_s^{ij} and f_s be the coefficients of the second variation operator on F on M_s. Note that:

$$F(\phi_{s,t}(M_s)) = s^2 F(\phi_{0,t}(M)),$$

so

$$s^2 \left(\frac{d}{dt}\right)^2_{t=0} F(\phi_{0,t}(M)) = \left(\frac{d}{dt}^2_{t=0} F(\phi_{s,t}(M_s)\right)$$

or

$$\begin{aligned} s^2 &\int_M (a^{ij} D_i^M u D_j^M u + f u^2)\, dA \\ &= \int_{M_s} (a_s^{ij} D_i^{M_s}(su_s) D_j^{M_s}(su_s) + (f_s)(su_s)^2)\, dA_s \\ &= \int_M (a_s^{ij}(sx) D_i^M u(x) D_j^M u(x) + s^2 f_s(sx) u(x)^2)(s^2\, dA(x)). \end{aligned}$$

Thus $a^{ij}(x) \equiv a_s^{ij}(sx)$ and $f(x) \equiv s^2 f_s(sx)$. In particular, $a^{ij}(0) = a_s^{ij}(0)$. As $s \to \infty$, $M_s \to \mathrm{Tan}_0 M$, so $a^{ij}(0)$ depends only on $\mathrm{Tan}_0 M$. Likewise $a^{ij}(x)$ depends only on $\mathrm{Tan}_x M$. Since the set of planes is compact, the $a^{ij}(x)$ are bounded as claimed.

If $v \in R^3$, then $\{M + tv\}$ is a one-parameter family of stationary surfaces with initial velocity vectorfield v, so $Ju = 0$, where $u(x) = v \cdot n(x)$. Fix a unit vector v that is tangent to M at 0, so that u changes sign in $M \cap B_1(0)$.

Intuitively, u is an eigenfunction with eigenvalue 0, u changes sign, and the first eigenfunction should not change sign, so M should be unstable. If M were a compact surface in a flat torus, this would be a proof. But since M is not compact, we have to use a cutoff function.

Let

$$w(x) = \begin{cases} u(x) & \text{if } u(x) > 0, \\ 0 & \text{if not}, \end{cases}$$

and let $P = \{x \in M\colon u(x) > 0\}$. Let $\phi\colon M \to R$ be a smooth function with compact support. Then

$$
\begin{aligned}
&\int_M a^{ij} D_i^M(\phi w) D_j^M(\phi w)\, dA \\
&\quad = \int_P a^{ij} D_i^M(\phi u) D_j^M(\phi u)\, dA \\
&\quad = \int_P a^{ij}(\phi D_i^M u + u D_i^M \phi)(\phi D_u^M u + u D_j^M \phi)\, dA \\
&\quad = \int_P 2a^{ij} u D_i^M u \phi D_j^M \phi\, dA + \int_P a^{ij}\phi^2 D_i^M u D_j^M u\, dA + E \\
&\quad = \int_P a^{ij} u D_i^M u D_j^M(\phi^2)\, dA + \int_P a^{ij}\phi^2 D_i^M u D_j^M u\, dA + E \\
&\quad = \int_P (D_j^M(a^{ij} u D_i^M u \phi^2) - a^{ij} D_j^M u D_i^M u \phi^2 - u D_j^M(a^{ij} D_i^M u)\phi^2)\, dA \\
&\qquad + \int_P a^{ij}\phi^2 D_i^M u D_j^M u\, dA + E \\
&\quad = \int_P (D_j^M(a^{ij} u D_i^M u \phi^2) - u D_j^M(a^{ij} D_i^M u)\phi^2)\, dA + E \\
&\quad = \int_P D_j^M(a^{ij} u D_i^M u \phi^2)\, dA - \int_P u f u \phi^2\, dA + E \\
&\quad = -\int_P f\phi^2 u^2\, dA + E,
\end{aligned}
$$

(where $E = \int_P a^{ij} u^2 D_i^M \phi D_j^M \phi\, dA$) since ϕ has compact support and u vanishes on ∂P.

Thus $Q(\phi w) = E$. Now let $R > 1$ and let

$$
\phi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 1 - \ln|x|/\ln R & \text{if } 1 \le |x| \le R, \\ 0 & \text{if } |x| \ge R. \end{cases}
$$

Then

$$
\begin{aligned}
Q(\phi w) = E &= \int_P u^2 a^{ij} D_i^M \phi D_j^M \phi\, dA \\
&\le a \int_P |\nabla\phi|^2\, dA \\
&\le a \int_{1\le|x|\le R} (|x| \ln R)^{-2}\, dA
\end{aligned}
$$

$$
\begin{aligned}
&= a\int_{t=1}^{R}(t\ln R)^{-2}\,dA(t)\\
&= a\int_{t=1}^{R} d((t\ln R)^{-2}A(t)) + a\int_{t=1}^{R} 2t^{-3}(\ln R)^{-2}A(t)\,dt\\
&= a\frac{A(R)}{R^2(\ln R)^2} - a\frac{A(1)}{(\ln R)^2} + a\int_{t=1}^{R} 2t^{-3}(\ln R)^{-2}A(t)\,dt\\
&\le \frac{Ca}{(\ln R)^2} + Ca\int_{t=1}^{R} 2t^{-1}(\ln R)^{-2}\,dt\\
&= \frac{Ca}{(\ln R)^2} + \frac{2Ca}{\ln R},
\end{aligned}
$$

where $a = \sup|a^{ij}(x)|$, and $A(t)$ is the area of $M \cap B_t(0)$.

Now let w' minimize $Q(w')$ among all Lipschitz functions that coincide with ϕw outside of $B_1(0)$. Note that $w' \ne \phi w$ since ϕw vanishes on a proper open subset of $M \cap B_1(0)$. Thus

$$Q(\phi w) - Q(w') = \epsilon > 0,$$

hence

$$Q(w') = Q(\phi w) - \epsilon \le \frac{Ca}{(\ln R)^2} + \frac{2Ca}{\ln R} - \epsilon.$$

Note that $w'|M \cap B_1(0)$ and therefore ϵ is independent of the choice of $R > 1$. Thus by choosing R large we can make $Q(w') < 0$. Hence M is unstable.

4.2. Theorem. *Let F be a parametric elliptic functional on R^3, B be a ball in R^3, and $\Omega \subset R^3$ an open set with smooth boundary. Let $M \subset B \setminus \Omega$ be a C^1 immersed surface such that*

(1) $\partial M \cap B = \varnothing$,
(2) *M is F-stationary for the obstacle Ω,*
(3) *$M \setminus \overline{\Omega}$ is F-stable, and*
(4) *the density ratios of M are bounded above:*

$$\sup_{B(x,r)\subset B} \frac{\operatorname{Area}(M \cap B(x,r))}{\pi r^2} = b < \infty.$$

Then there exist $0 < \alpha < 1$ and $C < \infty$, depending only on F, B, Ω, and b (not on M), such that

$$\sup_{x,y\in M} d_B(x,y)^{\alpha}\frac{|n(x)-n(y)|}{|x-y|^{\alpha}} \le C,$$

where

$$d_B(x, y) = \min\{\operatorname{dist}(x, \partial B), \operatorname{dist}(y, \partial B)\}.$$

Proof. Fix an $\alpha \in (0, 1)$ (to be specified later), and suppose the result is false. Then there exists a sequence M_i of surfaces satisfying the hypotheses above and sequences $x_i, y_i \in M_i$ such that

$$d_B(x_i, y_i)\frac{|n_i(x_i) - n_i(y_i)|^{1/\alpha}}{|x_i - y_i|} \to \infty.$$

Choose $\epsilon_i > 0$ such that

$$(d_B(x_i, y_i) - \epsilon_i)\frac{|n_i(x_i) - n_i(y_i)|^{1/\alpha}}{|x_i - y_i|} \to \infty.$$

We may suppose that x_i and y_i have been chosen so that

$$f_i(x, y) = (d_B(x, y) - \epsilon_i)\frac{|n_i(x) - n_i(y)|^{1/\alpha}}{|x - y|}$$

attains its maximum (among $x, y \in M_i$) at $(x, y) = (x_i, y_i)$.

(Note that $f_i(x, y)$ is negative if x or y is near ∂B, and that $\lim_{x \to y} f_i(x, y) = 0$ because M_i is a $C^{1,1}$ surface (Proposition 1.4). Thus f_i does have a maximum.)

Translate x_i, y_i, M_i, B, and Ω by $-x_i$ and then dilate by

$$\mu_i = \frac{|n_i(x_i) - n_i(y_i)|^{1/\alpha}}{|x_i - y_i|}$$

to get $x_i' = 0$, y_i', M_i', B_i', and Ω_i'. Thus the inequality $f_i(x, y) \leq f_i(x_i, y_i)$ becomes

$$(d_B(x, y) - \epsilon_i)\frac{|n_i(x) - n_i(y)|^{1/\alpha}}{|x - y|} \leq (d_B(x_i, y_i) - \epsilon_i)\mu_i$$

or

(1)
$$\begin{aligned}
\frac{|n_i'(x') - n_i'(y')^{1/\alpha}}{|x' - y'|} &\le \frac{d_B(x_i, y_i) - \epsilon_i}{d_B(x, y) - \epsilon_i} \\
&\le \frac{d_B(x_i, y_i) - \epsilon_i}{d_B(x_i, y_i) - \epsilon_i - |x - x_i| - |y - x_i| - |x_i - y_i|} \\
&\qquad\qquad \text{(by the triangle inequality)} \\
&= \frac{\mu_i(d_B(x_i, y_i) - \epsilon_i)}{\mu_i(d_B(x_i, y_i) - \epsilon_i) - |x' - x_i'| - |y' - x_i'| - |x_i' - y_i'|} \\
&= \frac{f_i(x_i, y_i)}{f_i(x_i, y_i) - |x'| - |y'| - |y_i'|}.
\end{aligned}$$

Note that

$$\begin{aligned}
|y_i'| = |y_i' - x_i'| &= \mu_i|x_i - y_i| \\
&= (d_B(x_i, y_i) - \epsilon_i)|n_i(x_i) - n_i(y_i)|^{1/\alpha} \qquad (2) \\
&\le \operatorname{diam}(B)2^{1/\alpha},
\end{aligned}$$

so that (1) becomes

$$\frac{|n_i'(x') - n_i'(y')|^{1/\alpha}}{|x' - y'|} \le \frac{f_i(x_i, y_i)}{f_i(x_i, y_i) - |x'| - |y'| - 2^{1/\alpha}\operatorname{diam}(B)}.$$

Thus (since $f_i(x_i, y_i) \to \infty$), the M_i are uniformly $C^{1,\alpha}$ bounded on compact subsets of R^3. Hence there is a subsequence (which we may take to be the original sequence) of the M_i that converges uniformly in $C^{1,\beta}$ for every $\beta < \alpha$ on compact subsets of R^3 to a surface M.

Now the Ω_i' converge nicely on compact subsets of R^3 either to a half-space H or to the empty set by wandering off to ∞. If $x \in M \setminus H$, then the M_i converge smoothly to M near x. Thus $M \setminus H$ is a smooth F-stationary and F-stable surface. If $M \cap H \neq \varnothing$, then $M = \partial H$ by the Hopf maximum principle [13, 3.4 and 17.1]. If $M \cap H = \varnothing$, then M is a complete F-stable surface with bounded density ratios and hence is a plane by Theorem 4.1. Thus M is a plane. Without loss of generality we may assume it is the horizontal plane.

For every $R < \infty$, for i sufficiently large there is a $C^{1,\alpha}$ function u_i such that

$$\{(x, u_i(x)) : x \in B(0, R) \subset R^2\} \subset M_i$$

and $\|u_i\|_{1,\beta} \to 0$ for $\beta < \alpha$. Note also that $\partial\Omega_i'$ can (unless it moves off to infinity) be represented near 0 by the graph of a function ϕ_i such that the ϕ_i's converge smoothly to a linear function. Now u_i satisfies the Euler-Lagrange variational inequality corresponding to F_i' and the obstacle ϕ_i. Thus we have the estimate $\|u_i\|_{1,\gamma} \le C$, where γ and C depend only on F and Ω. (See Theorem 13.1 of [13] and Remark 1 after Proposition 1.4 of this paper.) Thus if we choose α to be less than γ, then u_i converges to 0 in $C^{1,\alpha}$ on compact subsets of R^2. But that is impossible because

$$\frac{|n_i'(y_i') - n_i'(0)|^{1/\alpha}}{|y_i' - 0|} = 1$$

by choice of μ_i, and $|y_i'|$ is bounded by (2).

5. Boundary estimates for F-stable surfaces

5.1. Theorem. *Let l be the line of intersection of two planes in R^3, and let $\mathscr{R}$ be one of the four regions into which the planes divide R^3. Let M be a complete connected orientable surface-with-boundary properly immersed in R^3 such that* $\mathrm{int}(M) \subset \mathscr{R}$ *and $\partial M = l$. Suppose that M is stable with respect to a constant coefficient parametric elliptic functional F and that the density ratios are uniformly bounded above*:

$$\sup_{r>0} \frac{\mathrm{Area}(M \cap B_r(x))}{\pi r^2} \le C < \infty.$$

Then M is a half-plane.

Proof. We may assume that ∂M is the z-axis, and that $\mathscr{R}$ is given in cylindrical coordinates by $-\theta_0 < \theta < \theta_0$ where $0 < \theta_0 < \pi/2$.

Let $n(x)$ be the unit normal to M at x, and let $v = (0, 0, 1)$ be the unit vector that points up along the z-axis. Then, as in the proof of Theorem 4.1, $u(x) = v \cdot n(x)$ is a Jacobi field. Note that $u(x) = 0$ for $x \in \partial M$. The proof of Theorem 4.1 shows that u cannot change sign in the interior of M. If $u \equiv 0$, then M is translation invariant in the z-direction, and it follows easily that M is a half-plane.

Thus suppose that $u(x) > 0$ for every x in the interior of M.

Without loss of generality we may assume that $(1, 0, 0) \in M$. Note that the projection Π: $M \subset R^3 \to R^2$ is locally a diffeomorphism. Let (a, b) be the largest open interval containing 1 such that there is a function ϕ: $(a, b) \to R$ with

$$(r, 0, \phi(r)) \in M, \qquad \phi(1) = 0.$$

For each $r \in (a, b)$, let $(\theta^-(r), \theta^+(r))$ be the largest open interval such that the path

$$\theta \in (\theta^-(r), \theta^+(r)) \mapsto (r\cos\theta, r\sin\theta)$$

in R^2 lifts to a path in M passing through $(r, 0, \phi(r))$.

Thus there is a smooth real-valued function w defined on

$$\Omega = \{(r, \theta): a < r < b, \ \theta^-(r) < \theta < \theta^+(r)\}$$

such that

$$(r\cos\theta, r\sin\theta, w(r, \theta)) \in M \quad \text{if } (r, \theta) \in \Omega.$$

Now suppose that $(r_i, \theta_i) \in \Omega$ and $(r_i, \theta_i) \to (r, \theta^+(r))$. If $Dw(r_i, \theta)$ stays bounded, then (since by Theorem 4.2 the curvature of M is uniformly bounded on compact subsets of $R^3 \setminus l$) there is a $\rho > 0$ such that (for sufficiently large i) a neighborhood in M of $(r_i\cos\theta_i, r_i\sin\theta_i, w(r_i, \theta_i))$ projects diffeomorphically onto a disk in R^2 of radius ρ centered at $(r_i\cos\theta_i, r_i\sin\theta_i)$. But this contradicts the choice of $\theta^+(r)$. Thus $|Dw(r_i, \theta_i)| \to \infty$. It also follows from the properness of M that $|w(r_i, \theta_i)| \to \infty$.

Note that by the curvature estimates 4.2, a subsequence of the surfaces $M_i = M - (0, 0, w(r_i, \theta_i))$ converges to a surface M'. Since $n' \cdot v \geq 0$ (where $n'(p)$ is the unit normal to M' at p) and

$$n'(r\cos\theta^+(r), r\sin\theta^+(r), 0) \cdot v = 0,$$

we have (by the maximum principle) that $n' \cdot v \equiv 0$ and thus M' is a half-plane with boundary l. It follows that the level set of w containing (r_i, θ_i) becomes closer and closer to the ray $\theta = \theta_i$ as $i \to \infty$. Hence θ^+ is constant. Likewise θ^- is also constant.

Thus

$$\Omega = \{(r, \theta): a < r < b, \ \theta^- < \theta < \theta^+\}.$$

The same reasoning shows that if $b < \infty$, then

$$\lim_{r\to b} |w(r, 0)| = \lim_{r\to b} |Dw(r, 0)| = \infty,$$

and that the level set of w containing $(r, 0)$ tends to the ray $\theta = 0$ as $r \to b$. But that would imply that $|w(r, 0)| \equiv \infty$, a contradiction. Thus $b = \infty$. Likewise $a = 0$.

Hence we have shown that $M \setminus l$ is the graph of a function defined on a wedge shaped region U of R^2. But the graph of any F-stationary function defined on a convex domain must be absolutely F-minimizing (by the argument of [9, 5.4.18] together with the convex hull property

[17, 4.2], or by [17, 4.3]). Thus M is F-minimizing and hence M is a half-plane by the work of R. Hardt (see (4) in the proof of Theorem 6.3).

5.2. Theorem. *Let N be a compact 3-manifold with boundary, F be a parametric elliptic functional such that N is strictly F-convex, and Γ be a (possibly empty) $C^{2,\alpha}$ embedded curve in $\partial N \cap B(x, R)$. Let M be a compact F-stable surface in N such that $\partial M \cap B(x, r) = \Gamma$ and such that the density ratios are bounded above:*

$$\sup_{B(x,r)\subset B} \frac{\operatorname{Area}(M \cap B(x, r))}{\pi r^2} = C < \infty.$$

Then the principal curvatures of $M \cap B(x, r/2)$ are bounded by a constant depending only on F, N, r, C, and Γ.

Proof. This follows from Theorem 5.1 in essentially the same way that Theorem 4.2 follows from Theorem 4.1.

6. Regularity of integral current solutions to obstacle problems

In this section we show that n-dimensional integral currents minimizing parametric elliptic functionals in R^{n+1} with smooth obstacles have small singular sets; in particular, the singular sets are empty if $n = 2$. We also extend Hardt's boundary regularity theorem to nonconstant coefficient functionals.

Definition. Let T and R be k-dimensional (integer multiplicity) rectifiable currents. We say that R is a *piece* of T if $\|T\| = \|R\| + \|T - R\|$, where $\|S\|$ is the radon measure determined by the current S.

Definition. Let $c > 0$, $\delta > 0$, and $0 < \alpha \le 1$, and let F be a parametric elliptic functional. We say that the rectifiable current T is *(F, ct^α, δ)-minimal* if for every piece R of T supported in a ball of radius $r \le \delta$, we have

$$F(R) \le (1 + cr^\alpha)F(S)$$

whenever $\partial S = \partial R$.

6.1. Proposition. *Let F be a smooth parametric elliptic functional on R^{n+1}. Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$, where each Ω_i is an open subset of R^{n+1} with $C^{1,\alpha}$ boundary. If T is a compactly supported rectifiable current that minimizes F among rectifiable currents supported in $R^{n+1} \setminus \Omega$, then T is (F, Ct^α, δ)-minimal for suitable C and δ.*

Proof. Choose $\delta > 0$ small enough (as in the proof of Lemma 3.3) that if B is a ball of radius $r \le \delta$ that intersects support (T), then the following hold:

(1) If S is an F-minimizing n-current with boundary in B, then S is supported in B, and

(2) for each i, there is a hyperplane P_i such that $B \cap \partial\Omega_i$ is the graph over P_i of a $C^{1,\alpha}$ function f_i with $|Df_i| \leq 1$.

Now let $B(x, r)$ be any ball of radius $\leq \delta$ that intersects Support (T). Let R be a piece of T in $B(x, r)$, and let S be the F-minimizing current with boundary ∂T. We must show that $F(R) \leq (1 + Cr^\alpha)F(S)$. By choice of δ, S is supported in $B(x, r)$. Thus there is an $(n+1)$-dimensional current V_0 in $B(x, r)$ such that $S = R + \partial V_0$. Now for $i = 1, \cdots, k$, let $V_i = V_{i-1} \cap \Omega_i^c$. We claim that

$$F(R + \partial V_i) \leq (1 + Cr^\alpha)F(R + \partial V_{i-1}),$$

from which it follows that

$$F(R + \partial V_k) \leq (1 + cr^\alpha)^k F(R + \partial V_0) = (1 + cr^\alpha)^k F(S).$$

Note that V_k and hence $R + \partial V_k$ are supported in Ω^c. Thus (assuming the claim) we have

$$F(R) \leq F(R + \partial V_k) \leq (1 + cr^\alpha)^k F(S) \leq (1 + Cr^\alpha)F(S)$$

as desired. Hence, it remains only to prove the claim.

It suffices to prove the claim for a single obstacle Ω, which we may assume to be of the form $\{(x, z) \in R^n \times R : z < f(x)\}$, where f is $C^{1,\alpha}$ and $|Df| \leq 1$. Let R be an n-current in $B(0, r) \setminus \Omega$, and let V be an $(n+1)$-current in $B(0, r)$. By the change of coordinates $(x, z) \mapsto (x, z - f(x))$, we may assume that $f \equiv 0$. Of course in the new coordinates F need not be smooth, but it still must be locally $C^{0,\alpha}$. Also, R and V will still be contained in some ball $B(p, 2r)$ of radius $2r$. Now by ellipticity,

$$F_p(R + \partial(V \cap \Omega^c)) \leq F_p(R + \partial V),$$

where F_p is the constant coefficient functional $F_p(v) = F(p, v)$. Hence

$$\begin{aligned}
&F(R + \partial(V \cap \Omega^c)) \\
&\quad \leq F(R + \partial V) + |F_p(R + \partial V) - F(R + \partial V)| \\
&\quad\quad + |F(R + \partial(V \cap \Omega^c)) - F_p(R + \partial(V \cap \Omega^c))| \\
&\quad \leq F(R + \partial V) + Kr^\alpha M(R + \partial V) + Kr^\alpha M(R + \partial(V \cap \Omega^c)) \\
&\quad \leq F(R + \partial V) + 2Kr^\alpha M(R + \partial V) \\
&\quad \leq F(R + \partial V) + K'r^\alpha F(R + \partial V) \\
&\quad = (1 + K'r^\alpha)F(R + \partial V).
\end{aligned}$$

Remark. More care in proving the claim shows that T is actually (F, Ct^{β}, δ)-minimal where $\beta = \min\{2\alpha, 1\}$.

For the next proposition we refer to the basic regularity theorem for parametric elliptic functionals (due to Almgren [1]) as proved in [30]. The proof there is for minimizing currents, but it is easily modified for (F, ct^{α}, δ)-minimal currents. Indeed, for the case $\alpha = 1$ (the case we need for the obstacle problem), one need only change the two inequalities preceding line (57) of [30]; the extra terms can easily be absorbed in line (57).

6.2. Theorem. *Let F be a smooth parametric elliptic functional on $B \subset R^{n+1}$. Let T be an n-dimensional (F, Ct^{α}, δ)-minimal rectifiable current in B with $\partial T \cap B = \varnothing$ and $0 < \alpha \le 1$. Then the support of T is the union of $C^{1,\alpha/2}$ submanifolds of B together with a singular set of Hausdorff dimension $< n-2$. No two of the submanifolds cross each other (though they may partially coincide). If T is the boundary of a set, then the submanifolds are disjoint.*

Proof. Note that there is a BV function f such that

$$T = \partial([B] \llcorner f) = \sum_{k=-\infty}^{\infty} \partial[f > k].$$

Each of the currents $\partial[f > k]$ is a piece of T and is therefore also (F, ct, δ)-minimal. Thus we may as well assume that T is one of these pieces, i.e., that $T = \partial[U]$ for some set $U \subset B$ of finite perimeter.

Without loss of generality we may take B to be the ball $B(0, 2)$. It suffices to show that the intersection of the singular set $\operatorname{sing}(T)$ of T with $B(0, 1)$ has Hausdorff dimension $< n-2$. Note that if $x_i \to x \in B(0, 1)$ and $r_i \to 0$, then a subsequence (which we will take to be the original sequence) of $\eta_{x_i, r_i \#} T$ converges to a $F(x, \cdot)$-minimizing current T_{∞}, where $\eta_{x,r}\colon R^{n+1} \to R^{n+1}$ is the map defined by $\eta_{x,r}(y) = r^{-1}(y-x)$. By [2] and [33, 5.2] we know that there is an $s < n-2$ such that the singular set of every such T_{∞} has Hausdorff s-dimensional measure 0. Also if W is an open set containing $\operatorname{sing}(T_{\infty})$, then for sufficiently large i

$$\operatorname{sing}((\eta_{x_i, r_i \#} T) \cap B(0, 1)) \subset W$$

(cf. 4.5 and 5.2 of [33] for a proof in the $\bmod p$ setting). The theorem is thus an immediate consequence of the following lemma.

Lemma. *Let K be a compact subset of R^n such that if $r_i \to 0$ and $\eta_{x_i, r_i} K \cap B(0, 1)$ converges in the Hausdorff topology to a set K', then $\mathscr{H}^s(K') = 0$ and $\mathscr{H}^s(K) = 0$.*

Remark. The conclusion of this lemma can be strengthened. It is easy to see that the set of all such $K' \cap B(0, 1)$ satisfy the hypotheses of the Work-Raccoon Theorem [33, 5.1], according to which there is then an $\epsilon > 0$ such that $\mathscr{H}^{s-\epsilon}(K') = 0$ for every such K'. Hence the lemma then implies that $\mathscr{H}^{s-\epsilon}(K) = 0$.

Proof of Lemma. Let

$$h^{s,\delta}(X) = \inf\left\{\sum_i r_i^s : X \subset \bigcup_i B(x_i, r_i),\ r_i < \delta\right\}.$$

Note that for each $0 < \delta \le \infty$, $\mathscr{H}^s(X) = 0$ if and only if $h^{s,\delta}(X) = 0$.

Note also that there is a $\delta > 0$ so that if $r \le \delta$, then

$$h^{s,1}((\eta_{x,r} K \cap B(0, 1))) < \tfrac{1}{3}.$$

For if not, there would be sequences x_i and $r_i \to 0$ with

$$h^{s,1}((\eta_{x_i,r_i} K \cap B(0, 1))) \ge \tfrac{1}{3}.$$

Since the collection of compact subsets of $B(0, 1)$ is compact in the Hausdorff topology, a subsequence (which we will take to be the original sequence) of the $(\eta_{x_i,r_i} K \cap B(0, 1))$ converges to a limit K'. Since $h^{s,1}(K') = 0$, we can find open balls $B(y_j, \rho_j)$ such that

$$K' \subset \bigcup B(y_j, \rho_j) \quad \text{and} \quad \sigma\rho_j^s < \tfrac{1}{3}.$$

For large enough i,

$$\eta_{x_i,r_i} K \cap B(0, 1) \subset \bigcup B(y_j, \rho_j),$$

so

$$h^{s,1}(\eta_{x_i,r_i} K \cap B(0, 1)) < \tfrac{1}{3},$$

a contradiction. Thus there does exist a δ as claimed.

Now suppose that $h^{s,\delta}(K) \ne 0$. Then we can find balls $B(x_i, r_i)$ of radii less than δ such that $K \subset \bigcup_i B(x_i, r_i)$ and $\sum_i r_i^s < 2h^{s,\delta}(K)$. By choice of δ,

$$h^{s,1}(\eta_{x_i,r_i} K \cap B(0, 1)) < \tfrac{1}{3},$$

i.e.,

$$h^{s,\delta}(K \cap B(x_i, r_i)) < \tfrac{1}{3} r_i^s.$$

Thus there are balls $B(x_{ij}, r_{ij})$ of radii less than δ such that

$$K \cap B(x_i, r_i) \subset \bigcup_j B(x_{ij}, r_{ij})$$

and

$$\sum_j r_{ij}^s < \tfrac{1}{3} r_i^s.$$

Thus

$$K \subset \bigcup (K \cap B(x_i, r_i)) \subset \bigcup_i \bigcup_j B(x_{ij}, r_{ij})$$

and

$$h^{s,\delta}(K) \le \sum_{ij} r_{ij}^s \le \sum_i \tfrac{1}{3} r_i^s \le \tfrac{2}{3} h^{2,\delta}(K),$$

a contradiction.

Remark. The definitions and propositions of this section apply equally well to flat chains mod 2 provided F is even. In this case the submanifolds in the statement of Theorem 6.2 are disjoint.

6.3. Theorem. *Let N be an $(n+1)$-manifold with boundary, and F be a smooth parametric elliptic functional on N such that ∂N is strictly F-convex. Let M be an n-dimensional F-minimizing surface (integral current or flat chain* mod 2) *such that* $\mathrm{spt}(\partial M)$ *is a smooth submanifold (with multiplicity* 1) *of ∂N. Then there is a neighborhood U of* $\mathrm{spt}(\partial M)$ *such that* $(\mathrm{spt}\, M) \cap U$ *is a smooth manifold with boundary.*

Proof. From the work of R. Hardt [17] we know the following facts:

(1) If the portion of M in a small neighborhood of $x \in \partial M$ is weakly close enough to an n-dimensional half-disk with multiplicity 1, then M is a smooth manifold with boundary in a neighborhood of x.

(2) Suppose that F_i is a sequence of functionals on an open set Ω converging smoothly to a parametric elliptic functional F_∞, M_i is a sequence of F_i-minimizing hypersurfaces converging weakly to M_∞, and that the ∂M_i and ∂M_∞ are smooth submanifolds (with multiplicity 1) such that $\partial M_i \to \partial M_\infty$ smoothly. Let U be an open set such that $\overline{U} \subset \Omega$ is a compact set not containing any singular points of M_∞. Then for sufficiently large i, the $M_i \cap U$ are smooth manifolds-with-boundary that converge smoothly to $M_\infty \cap U$.

(3) Let F_0 be a constant coefficient parametric elliptic functional on R^{n+1}. Let P_1 and P_2 be distinct nonparallel hyperplanes in R^{n+1}, and $\mathscr{R}$ be one of the connected components of $R^{n+1} \setminus (P_1 \cup P_2)$. Suppose that M_0 is an F_0-minimizing hypersurface in $\mathscr{R}$ such that ∂M_0 is $P_1 \cap P_2$ with multiplicity 1. Then in a neighborhood of ∂M_0, M_0 is a smooth manifold with boundary.

(4) If, in addition to the hypotheses of (3), we have

$$\sup_{r>0} \frac{\mathrm{Mass}(M_0 \cap B(0, r))}{r^n} < \infty,$$

then M_0 is a half-plane.

Fact (1) is Theorem 3.4 of Hardt's paper [17]; see the statement of the theorem for the precise meaning of "weakly close enough". Fact (2) is not stated in Hardt's paper, but follows in a standard way from (1) and the analogous interior estimates ([9, 5.3.15] or [30]; see also the proofs of [9, 5.3.18], [9, 5.3.19], or [33, 5.2]). Fact (3) is an immediate consequence of Theorems 3.6 and 4.6 of [17]. To see (4), let M_i be the surface obtained from M_0 by dilation by i^{-1} about the origin. The density hypothesis and the compactness theorem for integral currents [37] imply that a subsequence of the M_i converges weakly to a current M_∞. By (3) the surface M_∞ must be smooth near the origin. But if M were not totally geodesic, then the curvatures of the M_i near 0 would blow up, contradicting (2).

Since the result is local, we may assume that N is a subset of R^{n+1}. Let $x \in \partial M$ and choose a sequence S_i of $(n-1)$-dimensional submanifolds of ∂N such that $S_i \cap \partial M = \{x\}$ and so that if we dilate S_i to get a manifold S_i' of unit volume, then S_i' converge smoothly to a sphere S_∞ with multiplicity one. Let F_∞ be the corresponding constant coefficient functional on R^{n+1}. In other words, F_∞ is the functional obtained by freezing F at x: $F_\infty(z, v) \equiv F(x, v)$.

Let T_i be the F-minimizing surface in N with boundary S_i, and let T_i' be the corresponding dilated surface. Note that a flat n-disk is the unique F_∞-minimizing surface with boundary S_∞. Thus the T_i' must converge weakly to a flat disk. Hence by (2), for sufficiently large i, T_i' will be a smooth manifold with boundary that is very nearly a flat n-disk. Fix such an i and let $T = T_i$. Of course T is a nearly flat regular n-disk. A simple cut and paste argument shows that M cannot cross T.

Now let M_i be the surface obtained by dilating M by i about x. Note that the density of M at X is finite by the argument used in the proof of the Corollary to Lemma 3.3, so by the compactness theorem for integral currents, a subsequence of the M_i (for simplicity let us suppose the original sequence) converges weakly to an F_∞-minimizing surface M_∞. Note that by the F-convexity of ∂N and (for instance) the Hopf boundary point lemma [13, 3.4 and 17.1], T is not tangent to ∂N at x. It follows that M_∞ satisfies the hypotheses of (3) above. Thus M_∞ is regular at its boundary, and in particular near x.

By (2), M_i is regular near x for sufficiently large i. But M_i is just a dilation of M, so M is also regular near x.

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