

AN ESTIMATE FOR HEXAGONAL CIRCLE PACKINGS

ZHENG-XU HE

1. Introduction

Let P be a circle packing in the complex plane \mathbb{C} , i.e., a collection of circles in \mathbb{C} with disjoint interiors, and let c_0 be a circle of P . Suppose that for some positive integer $n \geq 2$, the n generations P_n of P about c_0 (defined successively by $P_0 = \{c_0\}$, $P_k = \{c \in P; c \in P_{k-1} \text{ or } c \text{ is tangent to some circle of } P_{k-1}\}$, $k \geq 1$) is combinatorially equivalent to the n generations H_n of a regular hexagonal circle packing about one of its circles. Then the ratio of radii of any two circles of P tangent to c_0 is bounded by $1 + s_n$, where s_2, s_3, \dots is some decreasing sequence of positive numbers. We will denote by s_n the smallest possible constant with this property. In [7], B. Rodin and D. Sullivan showed that any circle packing which is combinatorially equivalent to an infinite regular hexagonal circle packing is also regular hexagonal, and as a consequence, s_n converges to 0. They conjectured that $s_n \leq C/n$ for some constant C . In this paper, we will prove this conjecture. This estimate for s_n is best possible as (we will see later) $s_n \geq 4/n$.

One may use our result to estimate the rate of convergence of the circle packing solutions f_ε to the Riemann Mapping Theorem given in [7], where ε is the size of the preimage circles, and of the approximating solutions f_δ to the Beltrami equations constructed in [4]. This shows that these solutions are constructive. Moreover, for the circle packing solutions f_ε of [7], we may combine with [6, Theorems 5 and 8] to conclude that the rate of convergence on compact subsets is of order at most $\varepsilon^{\alpha/8}$ for any fixed $\alpha < 1$, and their derivatives converge in L^∞ on compact subsets.

The proof of $s_n \leq C/n$ will be given in §2 with the assistance of an area estimate on the union of the images of the interstices bounded by the circles of H_n under the Schottky group generated by inversions of the circles of H_n (Lemma 2.2). In §3 we will prove this estimate. The argument also leads to vanishing of the Lebesgue measure of the limit

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set of a class of infinitely generated Schottky groups. §4 is independent of the main subject, and discusses the globally uniform convergence of quasiconformal mappings. We will see that when the domain R (or Ω in [4]) is a Jordan domain, f_ε of [7] (or f_δ of [4]) converges globally uniformly.

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2. Proof of the estimate $s_n \leq C/n$

For any positive integer n , let H_n be n generations of some regular hexagonal circle packing about one of its circles, say c_0 . We may normalize H_n so that c_0 is the unit circle and 1 is a point of tangency of c_0 with some neighboring circle. In this way, the circles of H_n have radius 1 and are centered at points $2(k_1 + k_3) + 2e^{\pi i/3}(k_2 - k_3)$, where k_1, k_2 , and k_3 are integers with $|k_1| + |k_2| + |k_3| \leq n$.

Let H'_n be n generations of a circle packing P about some circle c'_0 of P such that H'_n is combinatorially equivalent to H_n . By this, we mean that there is a one-to-one correspondence of H'_n and H_n so that two circles of H'_n are tangent if and only if their corresponding circles in H_n are tangent. Let c'_1, c'_2, \dots, c'_6 be the six circles of H'_n corresponding to the circles $c_k = \{|z - 2e^{\pi(k-1)i/3}| = 1\}$, $k = 1, 2, \dots, 6$, of H_n which are tangent to $c_0 = \{|z| = 1\}$. Then

$$(2.1) \quad s_n = \sup_{(P, c_0)} \max_{1 \leq j, k \leq 6} \left(\frac{\text{radius}(c'_j)}{\text{radius}(c'_k)} - 1 \right),$$

where (P, c_0) is any pair satisfying the above property.

The estimation of s_n is briefly described as follows. First, there is a quasiconformal mapping ψ from plane to plane which maps the subpacking H_m ($m \sim n/2$ for n large) of H_n to the corresponding subpacking H'_m of H'_n . This mapping will be made conformal on the union I_m of interstices bounded by the circles of H_m . Normalize H'_m so that $c'_0 = c_0$

and $c'_0 \cap c'_1 = c_0 \cap c_1 = \{1\}$. We wish to show that ψ restricted to c_0 is $O(1/m)$ -close to the identity (and this implies that the points of tangency $c'_0 \cap c'_j, j = 1, 2, \dots, 6$, are almost equidistributed on the unit circle c'_0 , a fact which clearly yields an estimate for s_n). Using the Schottky group G_m generated by inversions on the circles of H_m , one may modify ψ in the interiors of circles of H_m so it becomes conformal on the images of I_m by the transformations of G_m . These images fill up most of the area of the unit disk D ($=$ interior of c_0). As a result, the above quasiconformal mapping restricted to D (which maps D onto $D =$ interior of c'_0) will be close to the identity. It follows that $\psi|_{c'_0}$ is close to the identity.

In the following, let us denote by δ_j and $C_j, j = 1, 2, \dots$, some positive universal constants. δ_j will be used for lower bounds and C_j for upper bounds, so we will always assume $0 < \delta_j \leq 1$ and $1 \leq C_j < \infty$.

Let δ_1 be the constant in the Ring Lemma of [7, §4] for hexagonal packing. This means that if six circles surround a circle of radius r , then each circle has radius at least $\delta_1 r$.

Lemma 2.1. *For any three mutually tangent circles c'_0, c'_1 , and c'_2 with disjoint interiors such that the ratio of the radii of any two circles is between δ_1 and $1/\delta_1$, there is an orientation-preserving Möbius transformation g which maps $c_0 = \{|z| = 1\}$, $c_1 = \{|z - 2| = 1\}$, and $c_2 = \{|z - 2e^{\pi i/3}| = 1\}$ onto c'_0, c'_1 , and c'_2 respectively. Moreover, g is C_1 -bi-Lipschitz on c_0 if c'_0 is normalized to have radius 1.*

Proof. Let g be the orientation-preserving Möbius transformation sending $c_j \cap c_k$ to $c'_j \cap c'_k$, where (j, k) is any pair of $\{(0, 1), (0, 2), (1, 2)\}$. Then g satisfies the requirements of the lemma. q.e.d.

Note that g maps the interstice bounded by c_0, c_1 , and c_2 to that bounded by c'_0, c'_1 , and c'_2 . Now for any three mutually tangent circles in H_{n-1} , the Ring Lemma of [7] implies that the corresponding circles in H'_{n-1} satisfy the conditions of Lemma 2.1, so there is a conformal mapping from each interstice bounded by circles of H_{n-1} to the interstice bounded by corresponding circles of H'_{n-1} . These conformal mappings may be glued together to form a conformal mapping from the union of interstices bounded by circles of H_{n-1} to the union of interstices bounded by circles of H'_{n-1} . Furthermore, this mapping maps each circle of H_{n-2} to the corresponding circle of H'_{n-2} and, by Lemma 2.1, it is C_1 -bi-Lipschitz if both circles were normalized. So we can extend the mapping radially on each disk bounded by circles of H_{n-2} and the result is a C_1 -quasiconformal mapping ϕ from the union of interstices and disks bounded by circles of H_{n-2} to the corresponding union bounded by circles of H'_{n-2} .

It is well known (see e.g. [2, p. 96]) that for any C_1 -quasiconformal mapping of the unit disk to some region in \mathbb{C} , its restriction to $\{|z| \leq 1/\sqrt{3}\}$ may be extended to some C_2 -quasiconformal homeomorphism of \mathbb{C} . As the union of interstices and disks bounded by circles of H_{n-2} contains the disk $\{|z| < (n-2)\sqrt{3}\}$, the restriction of φ to $\{|z| \leq n-2\}$ has a C_2 -quasiconformal extension $\psi: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with

$$(2.2) \quad \psi(\infty) = \infty.$$

Let

$$(2.3) \quad m = m(n) = \left\lfloor \frac{n-3}{2} \right\rfloor = \text{the integer part of } \frac{n-3}{2}.$$

Then all circles of H_m lie in $\{|z| \leq n-2\}$; it follows that ψ equals φ on the union of interstices and disks bounded by circles of H_m . Particularly, ψ is conformal on the union I_m of interstices bounded by the circles of H_m .

For any circle c in $\widehat{\mathbb{C}}$, we will denote by γ_c the inversion on c . Let c be a circle of H_m , let Δ be the disk bounded by c , and let c' be the circle of H'_m which corresponds to c . Then since ψ maps c onto c' , we may replace $\psi|\Delta$ by $\gamma_{c'} \circ \psi \circ \gamma_c|\Delta$. When this is done for each $c \in H_m$, we obtain a C_2 -quasiconformal mapping $\psi^1: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ which is conformal on

$$(2.4) \quad I_m^1 = I_m \cup \left(\bigcup_{c \in H_m} \gamma_c(I_m) \right),$$

and maps each circle of

$$(2.5) \quad H_m^1 = \bigcup_{c \in H_m} \gamma_c(H_m \setminus \{c\})$$

to a corresponding circle of

$$H_m'^1 = \bigcup_{c' \in H_m'} \gamma_{c'} \setminus \{c'\}.$$

Similarly, for each circle c of H_m^1 , let Δ be the disk it bounds, let c' be the corresponding circle of $H_m'^1$, and replace $\psi^1|\Delta$ by $\gamma_{c'} \circ \psi^1 \circ \gamma_c|\Delta$. We obtain a C_2 -quasiconformal mapping $\psi^2: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ which is conformal on $I_m^2 = I_m^1 \cup (\bigcup_{c \in H_m^1} \gamma_c(I_m^1))$, and maps each circle of $H_m^2 = \bigcup_{c \in H_m^1} \gamma_c(H_m^1 \setminus \{c\})$ to the corresponding circle of $H_m'^2 = \bigcup_{c' \in H_m'^1} \gamma_{c'}(H_m'^1 \setminus \{c'\})$. Continuing in

this way, we may find for each k a C_2 -quasiconformal mapping $\psi^k: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ which is conformal on

$$(2.6) \quad I_m^k = I_m^{k-1} \cup \left(\bigcup_{c \in H_m^{k-1}} \gamma_c(I_m^{k-1}) \right),$$

and maps each circle of

$$(2.7) \quad H_m^k = \bigcup_{c \in H_m^{k-1}} \gamma_c(H_m^{k-1} \setminus \{c\})$$

to the corresponding circle of

$$H_m'^k = \bigcup_{c' \in H_m'^{k-1}} \gamma_{c'}(H_m'^{k-1} \setminus \{c'\}).$$

It is easy to see that ψ^k converges to some C_2 -quasiconformal mapping $f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ which is conformal on the set

$$(2.8) \quad J_m = \bigcup_{k=1}^{\infty} I_m^k.$$

Note that the set J_m is equal to the union of images of I_m by the elements of the Schottky group G_m generated by the inversions $\gamma_c, c \in H_m$. This implies particularly that $\gamma(J_m) = J_m$ for $\gamma \in G_m$.

From (2.2), we see that $\psi^1(0) = \gamma_{c'_0} \circ \psi \circ \gamma_{c_0}(0) = \gamma_{c'_0}(\infty)$ is the center of c'_0 . We may normalize H'_n so that $c'_0 = c_0 =$ the unit circle, and c'_0 is tangent to c'_1 at 1. Then $\psi^1(0) = 0$ and $\psi(1) = 1$, from which it follows that $\psi^k(0) = 0 \ \forall k \geq 2$ and hence that

$$(2.9) \quad f(0) = 0.$$

On the other hand, $f(e^{i\theta}) = \psi^k(e^{i\theta}) = \psi(e^{i\theta})$ for any $\theta \in \mathbf{R}$, particularly,

$$(2.10) \quad f(1) = \psi(1) = 1.$$

The following lemma gives a critical estimate on the measure of the set $D \setminus J_m$. Its proof will be given in §3.

Lemma 2.2. *For each positive integer m we have*

$$(2.11) \quad |D \setminus J_m| \leq C_3/m^2,$$

where $|\cdot|$ denotes the Lebesgue measure in the plane.

Consider the restriction on the unit disk D of f , still denoted by f . Then f is a C_2 -quasiconformal self-homeomorphism of D . We wish

to show that f (and hence ψ) restricted to the unit circle $\partial D = c_0$ is $O(1/n)$ -close to the identity id . The following is an argument based on an idea of Dennis Sullivan. Consider the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ endowed with the spherical metric induced by stereographic projection. Define $F: \widehat{\mathbf{C}} \rightarrow \widehat{\mathbf{C}}$ by “doubling” f :

$$(2.12) \quad F(z) = \begin{cases} f(z) & \text{if } |z| \leq 1, \\ 1/\overline{f(1/\bar{z})} & \text{if } |z| \geq 1. \end{cases}$$

Then F is a C_2 -quasiconformal self-homeomorphism of $\widehat{\mathbf{C}}$. By (2.9) and (2.10), F fixes $0, 1,$ and ∞ . On the other hand, Lemma 2.2 implies that F is conformal except on a subset of spherical area $\leq O(1/m^2)$.

Take a point z_0 in $\widehat{\mathbf{C}}$ whose spherical distance from $0, 1,$ and ∞ is uniformly bounded from below, say $\geq 1/10$. Then F maps the four-punctured sphere $\widehat{\mathbf{C}} \setminus \{0, 1, \infty, z_0\}$ onto the four-punctured sphere $\widehat{\mathbf{C}} \setminus \{0, 1, \infty, F(z_0)\}$. These punctured spheres are doubly covered (via some elliptic functions π_1 and π_2) by some four-punctured tori T_1 and T_2 , respectively. Then F lifts to a C_2 -quasiconformal homeomorphism \bar{F} of T_1 and T_2 which is conformal except on the preimage by π_1 of the set of nonconformality of F . This last set also has spherical area $\leq O(1/m^2)$, and the covering mapping π_1 behaves like $z \rightarrow z^2$ near each of the four punctures. So, if T_1 is endowed with the flat metric of total volume uniformly bounded from above, the area of the subset of T_1 where \bar{F} fails to be conformal is bounded by $(O(1/m^2))^{1/2} = O(1/m)$.

Let ζ_1 and ζ_2 be the conformal moduli of T_1 and T_2 respectively. Then we may identify T_j with $\mathbf{C}/(z \sim z + 1 \sim z + \zeta_j), j = 1, 2$. Since z_0 is bounded away from $0, 1,$ and ∞ , ζ_1 should fall into a compact subset of the upper half-plane. We claim that

$$(2.13) \quad |\zeta_2 - \zeta_1| \leq O(1/m).$$

In fact, let $\tilde{F}: \mathbf{C} \rightarrow \mathbf{C}$ be the lift of $\bar{F}: T_1 \rightarrow T_2$. Then $\tilde{F}(z+1) = \tilde{F}(z)+1$ and $\tilde{F}(z + \zeta_1) = \tilde{F}(z) + \zeta_2$. Let $K: \mathbf{C} \rightarrow [1, C_2]$ be the pointwise linear dilatation of \tilde{F} . Then we have

$$\begin{aligned} 1 &= \tilde{F}(iy + 1) - \tilde{F}(iy) = \int_0^1 \frac{\partial \tilde{F}}{\partial x}(x + iy) dx \\ &\leq \int_0^1 \left| \frac{\partial \tilde{F}}{\partial x}(x + iy) \right| dx \leq \int_0^1 K(x + iy)^{1/2} J(x + iy)^{1/2} dx, \end{aligned}$$

where $J(x + iy)$ denotes the Jacobian of \tilde{F} as a function from \mathbf{R}^2 to \mathbf{R}^2 . Integrate the above inequality over $y \in [0, y_1]$, where $y_1 = \text{Im}(\zeta_1)$. We get

$$y_1 \leq \int_0^{y_1} \int_0^1 K^{1/2} J^{1/2} dx dy = \iint_{T_1} K^{1/2} J^{1/2} dA.$$

Using the Schwarz inequality, we obtain

$$y_1^2 \leq \iint_{T_1} K dA \iint_{T_1} J dA = \iint_{T_1} K dA \cdot \text{Area}(T_2) = \iint_{T_1} K dA \cdot (\text{Im}(\zeta_1)).$$

Recall that $K = 1$ on T_1 except on a subset of area $\leq O(1/m)$. Therefore

$$\iint_{T_1} K dA = \text{Area } T_1 + \iint (K - 1) dA \leq \text{Im}(\zeta_1) + (C_2 - 1)O(1/m),$$

which implies

$$\text{Im}(\zeta_1)^2 = y_1^2 \leq [\text{Im}(\zeta_1) + O(1/m)] \cdot \text{Im}(\zeta_2).$$

As $\text{Im}(\zeta_1) \in (0, \infty)$ lies on a compact subset, we obtain

$$(2.14) \quad \text{Im}(\zeta_1) \leq \text{Im} \zeta_2 + O(1/m).$$

Similarly, let α_1 and α_2 be integers. Then

$$\begin{aligned} |\alpha_1 + \alpha_2 \zeta_1| &\leq \int_0^1 \left| \frac{\partial \tilde{F}(x + t(\alpha_1 + \alpha_2 \zeta_1))}{\partial t} \right| dt \\ &\leq |\alpha_1 + \alpha_2 \zeta_1| \int_0^1 K(x + t(\alpha_1 + \alpha_2 \zeta_1))^{1/2} \\ &\quad \cdot J(x + t(\alpha_1 + \alpha_2 \zeta_1))^{1/2} dt. \end{aligned}$$

Integrating this inequality over $x \in [0, 1]$ yields

$$\begin{aligned} |\alpha_1 + \alpha_2 \zeta_2| &\leq |\alpha_1 + \alpha_2 \zeta_1| \int_0^1 \int_0^1 K(x + t(\alpha_1 + \alpha_2 \zeta_1))^{1/2} \\ &\quad \cdot J(x + t(\alpha_1 + \alpha_2 \zeta_1))^{1/2} dt dx. \end{aligned}$$

Then by the Schwarz inequality, we find

$$\begin{aligned} |\alpha_1 + \alpha_2 \zeta_2|^2 &\leq |\alpha_1 + \alpha_2 \zeta_1|^2 \int_0^1 \int_0^1 K(x + t(\alpha_1 + \alpha_2 \zeta_1)) dt dx \\ &\quad \cdot \int_0^1 \int_0^1 J(x + t(\alpha_1 + \alpha_2 \zeta_1)) dt dx \\ &= |\alpha_1 + \alpha_2 \zeta_1|^2 \frac{\text{Im}(\zeta_2)}{\text{Im}(\zeta_1)} \int_0^1 \int_0^1 K(x + t(\alpha_1 + \alpha_2 \zeta_1)) dt dx. \end{aligned}$$

Therefore

$$|\alpha_1 + \alpha_2 \zeta_2|^2 \leq |\alpha_1 + \alpha_2 \zeta_1|^2 \frac{\operatorname{Im}(\zeta_2)}{\operatorname{Im}(\zeta_1)} (1 + O(1/m)),$$

which implies

$$(2.15) \quad \frac{|\alpha_1 + \alpha_2 \zeta_2|^2}{\operatorname{Im}(\zeta_2)} \leq \frac{|\alpha_1 + \alpha_2 \zeta_1|^2}{\operatorname{Im}(\zeta_1)} (1 + O(1/m)).$$

Note that (2.15) holds for any rationals α_1 and α_2 and hence for any $\alpha_1, \alpha_2 \in \mathbf{R}$. Take $\alpha_1 = -\operatorname{Re}(\zeta_1)$, $\alpha_2 = 1$, and obtain

$$\operatorname{Im}(\zeta_2) \leq \operatorname{Im}(\zeta_1)(1 + O(1/m)),$$

which together with (2.14) yields

$$(2.16) \quad |\operatorname{Im}(\zeta_2) - \operatorname{Im}(\zeta_1)| \leq O(1/m).$$

Thus using (2.16), from (2.15) we deduce that

$$|\alpha_1 + \alpha_2 \zeta_2|^2 \leq |\alpha_1 + \alpha_2 \zeta_1|^2 (1 + O(1/m)),$$

which together with (2.16) implies $|\zeta_2 - \zeta_1| \leq O(1/m)$.

Since z_0 and $F(z_0)$ depend smoothly (in fact analytically) on ζ_1 and ζ_2 respectively, from the closeness of ζ_2 and ζ_1 we obtain $|F(z_0) - z_0| \leq O(1/m)$. Taking $z_0 = -1$, we get

$$(2.17) \quad |F(-1) - (-1)| \leq O(1/m).$$

Now let z_0 be an arbitrary point on $\widehat{\mathbf{C}}$. If z_0 is bounded away from $0, 1, \infty$, we have shown that $|F(z_0) - z_0| \leq O(1/m)$. But if z_0 is close to one of the points $0, 1$, or ∞ , say 0 , then we may apply the above argument to the four-punctured spheres $\widehat{\mathbf{C}} \setminus \{-1, 1, \infty, z_0\}$ and $\widehat{\mathbf{C}} \setminus \{F(-1), 1, \infty, F(z_0)\}$ to conclude that the cross ratio of $(F(-1), 1, \infty, F(z_0))$ is $O(1/m)$ -close to the cross ratio of $(-1, 1, \infty, z_0)$, a fact which implies $|F(z_0) - z_0| \leq O(1/m)$ by (2.17). In this way, we conclude that \tilde{F} is $O(1/m)$ -close to the identity, and f is $O(1/m)$ -close to the identity. Therefore, ψ is $O(1/m)$ -close to the identity on c_0 and $s_n \leq O(1/m)$. Since $m \sim n/2$, the estimate follows.

Remark 1. There is an obvious lower bound for s_n . First note that $2n + 2$ lies in the unbounded component in the complement of the union of circles of H_n . The Möbius transformation

$$g: z \rightarrow \frac{2(n+1)z - 1}{2(n+1) - z}$$

sends $2n + 2$ to ∞ , and hence $g(H_n)$ is a circle packing in \mathbb{C} combinatorially equivalent to H_n . By (2.1),

$$s_n \geq \frac{\text{radius}(g(c_1))}{\text{radius}(g(c_4))} - 1,$$

where we recall that $c_1 = \{|z - 2| = 1\}$ and $c_4 = \{|z + 2| = 1\}$. A direct computation yields

$$s_n \geq \frac{16n + 16}{4n^2 - 1} > 4/n.$$

This shows that the $O(1/n)$ -estimate for s_n is best possible.

By our estimate on s_n , we obtain (see [6]):

Corollary 2.3. *The circle packing solutions f_ϵ to the Riemann mapping given in [7] have derivatives which converge uniformly on compact subsets to the derivatives of the Riemann mapping.*

Remark 2. From the structure of the subset of D where $f: D \rightarrow D$ fails to be conformal, one can prove that there is a Möbius transformation h of the unit disk such that $h \circ f$ is $O(1/n^2)$ -close to the identity, which is stronger than the $O(1/n)$ -closeness. This fact leads us to conjecture the circle packing solution f_ϵ of [7] has “second derivatives” defined in an appropriate sense, and they converge to the second derivatives of the Riemann mapping. This would lead to the discovery of the (existence and the) value of $\lim_{n \rightarrow \infty} ns_n$.

3. Estimation of $|D \setminus J_m|$

In this section we prove

Lemma 3.1. *For any (not necessarily orientation preserving) Möbius transformation h of the unit disk D , we have*

$$(3.1) \quad |D \setminus h(J_m)| \leq C_3/m^2.$$

Taking $h = \text{id}$ in (3.1), we obtain Lemma 2.2.

We begin with

Lemma 3.2. *There is some $\delta_2 > 0$ such that for any Möbius transformation h of D we have*

$$(3.2) \quad |h(J_1)| \geq \delta_2 \pi.$$

Proof. First, observe that the disk Δ_{jj+1} bounded by the circle c_{jj+1} passing through $c_0 \cap c_j$, $c_0 \cap c_{j+1}$, and $c_j \cap c_{j+1}$ is in J_1 (see Figure 1, next page). This follows from the fact that on the disk Δ_{jj+1} , the Fuschian group generated by the inversions γ_{c_0} , γ_{c_j} , $\gamma_{c_{j+1}}$ has the interstice bounded

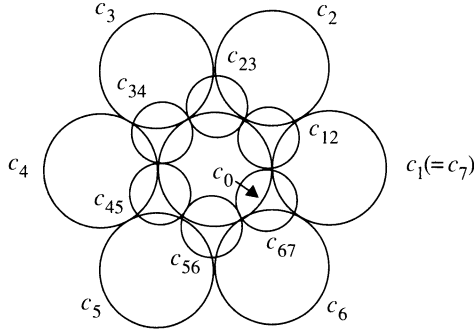


FIGURE 1

by the circles c_0 , c_j , and c_{j+1} as a fundamental domain. Thus, after applying a Möbius transformation h of D , at least one image of the arcs $c_0 \cap \Delta_{jj+1}$ on ∂D has length at least $\pi/3$. Then the image by h of at least one of the regions $D \cap \Delta_{jj+1} \subseteq D \cap J_1$ has area bounded from below since c_{jj+1} is orthogonal to c_0 . This implies (3.2). q.e.d.

Consider the Schottky group G_1 generated by the inversions γ_c , $c \in H_1$. Let U_1 be the complement in \widehat{C} of the union of I_1 and the disks bounded by the circles of H_1 . Then $I_1 \cup U_1$ is a finite-sided fundamental domain for G_1 and, by [2], the limit set of G_1 has measure zero. It follows particularly that for any Möbius transformation h of D , we have

$$\left| \bigcup_{g \in G_1} (D \cap h \circ g(I_1)) \right| + \sum_{h \in G_1} |D \cap h \circ g(U_1)| = \pi.$$

But $\bigcup_{g \in G_1} g(I_1) = J_1$, hence

$$|D \cap h(J_1)| + \sum_{g \in G_1} |D \cap h \circ g(U_1)| = \pi.$$

Using Lemma 3.2, this implies that

$$(3.3) \quad \sum_{g \in G_1} |D \cap h \circ g(U_1)| \leq (1 - \delta_2)\pi.$$

Lemma 3.1 will be proved by induction on m , with C_3 to be determined at the end of the proof. Obviously, (3.1) holds for $m = 1, 2$ if we choose $C_3 \geq 4\pi$. Now assume that (3.1) holds for $m \leq l - 1$. As J_l is invariant by the maps of $G_l \supseteq G_1$, $D \setminus h(J_l) = h(D \setminus J_l)$ is equal (up to a measure zero subset) to the disjoint union of $D \cap h(g(U_1) \setminus J_l) = h(D \cap g(U_1 \setminus J_l))$, $g \in G_1$. But for $g \in G_1$ we have either $g(U_1) \subseteq D$ or

$g(U_1) \cap D = \emptyset$, so

$$(3.4) \quad |D \setminus h(J_l)| = \sum_{\substack{g \in G_1 \\ g(U_1) \subseteq D}} |h \circ g(U_1 \setminus J_l)|.$$

On the other hand, (3.3) means that

$$(3.5) \quad \sum_{\substack{g \in G_1 \\ g(U_1) \subseteq D}} |h \circ g(U_1)| \leq (1 - \delta_2)\pi.$$

So (3.1) follows if we prove

$$(3.6) \quad \frac{|h \circ g(U_1 \setminus J_l)|}{|h \circ g(U_1)|} \leq \frac{C_3}{\pi(1 - \delta_2)l^2}$$

for suitably chosen C_3 . This is the goal of the next lemma.

Lemma 3.3. *Let $l \geq 3$. Suppose there is some $C_3 \geq 1$ so that (3.1) holds for any $m \leq l - 2$. Then for any Möbius transformation γ which maps U_1 to a bounded subset of \mathbf{C} , we have*

$$(3.7) \quad \frac{|\gamma(U_1 \setminus J_l)|}{|\gamma(U_1)|} \leq \frac{9}{(1 - \sqrt{1 - \delta_2/2})^2 l^2} + \frac{C_3}{\pi(1 - \delta_2/2)l^2}.$$

Proof. U_1 is the union of $I_l \setminus I_1$, the disks Δ bounded by circles of $H_l \setminus H_1$ and the unbounded connected component U_l in the complement of the circles of H_l . Define the “density” function $\eta_\gamma: \gamma(U_1) \rightarrow [0, 1]$ by

$$\eta_\gamma(z) = \begin{cases} 0 & \text{if } z \in \gamma(I_l \setminus I_1), \\ |\gamma(\Delta \setminus J_l)|/|\gamma(\Delta)| & \text{if } z \in \gamma(\Delta), \partial\Delta \in H_l \setminus H_1, \\ 1 & \text{if } z \in \gamma(U_l). \end{cases}$$

Then

$$(3.8) \quad \frac{|\gamma(U_1 \setminus J_l)|}{|\gamma(U_1)|} = \frac{1}{|\gamma(U_1)|} \iint_{\gamma(U_1)} \eta_\gamma(z) dx dy.$$

Let Δ be a disk bounded by some circle $c = \partial\Delta$ of $H_k - H_{k-1}$, $2 \leq k \leq l - 1$, and let z_Δ be its center. Since the $l - k$ generations of the circle packing H_l about $\partial\Delta$ is the translation $z_\Delta + H_{l-k}$, $z_\Delta + J_{l-k} \subseteq J_l$. Then by (3.1) for $m = l - k$ ($\leq l - 2$), we have

$$|\gamma(\Delta \setminus J_l)| \leq \frac{C_3}{\pi(l - k)^2} |\gamma(\Delta)|.$$

So if we define $\eta: U_1 \rightarrow [0, 1]$ by

$$\eta(z) = \begin{cases} 0 & \text{if } z \in I_l - I_1, \\ \min(C_3/\pi(l-k)^2, 1) & \text{if } z \in \Delta, \partial\Delta \in H_k - H_{k-1}, \\ & 2 \leq k \leq l-1, \\ 1 & \text{if } z \in U_1 \cup (\cup \Delta), \partial\Delta \in H_l - H_{l-1}, \end{cases}$$

then $\eta_\gamma(z) \leq \eta(\gamma^{-1}(z)) \quad \forall z \in \gamma(U_1)$.

If $z \in \Delta, \partial\Delta \in H_k \setminus H_{k-1}, 2 \leq k \leq l-1$, then $|z| \geq k\sqrt{3} - 1 > k$, and thus

$$\frac{C_3}{\pi(l-k)^2} \leq \frac{C_3}{\pi[(l-|z|)^+]^2},$$

where $j^+ = \max(j, 0)$. Let $\rho: U \rightarrow [0, 1]$ be the following function:

$$(3.9) \quad \rho(z) = \rho(|z|) = \min\left(\frac{C_3}{\pi[(l-|z|)^+]^2}, 1\right).$$

Then $\eta \leq \rho$ on U_1 , and hence

$$(3.10) \quad \eta_\gamma \leq \eta \circ \gamma^{-1} \leq \rho \circ \gamma^{-1}(z).$$

Let $V = \{|z| > 3\} \cup \{\infty\}$. Then $V \subseteq U_1$, and for any $z \in U_1 \setminus V, z' \in V$, we have

$$\rho(z) \leq \min\left(\frac{C_3}{\pi(l-3)^2}, 1\right) \leq \rho(z').$$

Therefore

$$(3.11) \quad \frac{1}{|\gamma(U_1)|} \iint_{\gamma(U_1)} \rho \circ \gamma^{-1}(z) dx dy \leq \frac{1}{|\gamma(V)|} \iint_{\gamma(V)} \rho \circ \gamma^{-1}(t) dx dy.$$

By (3.8), (3.10), and (3.11), we obtain

$$\frac{|\gamma(U_1 \setminus J_l)|}{|\gamma(U_1)|} \leq \frac{1}{|\gamma(V)|} \iint_{\gamma(V)} \rho \circ \gamma^{-1}(z) dx dy.$$

We will prove

$$(3.12) \quad \frac{1}{|\gamma(V)|} \iint_{\gamma(V)} \rho \circ \gamma^{-1}(z) dx dy \leq \frac{9}{(1 - \sqrt{1 - \delta_2/2})^2 l^2} + \frac{C_3}{\pi(1 - \delta_2/2)l^2}$$

and hence (3.7) holds. For this, consider first the special case $\gamma = \gamma_1: V \rightarrow D$, $\gamma_1(z) = 3/z$. Then,

$$\begin{aligned} & \frac{1}{|\gamma_1(V)|} \iint_{\gamma_1(V)} \rho \circ \gamma_1^{-1} dx dy \\ &= \frac{1}{\pi} \iint_D \min \left(\frac{C_3}{\pi[(l - 3/|z|)^+]^2}, 1 \right) dx dy \\ &\leq \frac{1}{\pi} \iint_{\{|z| < 3/(1 - \sqrt{1 - \delta_2/2})l\}} 1 \cdot dx dy \\ &\quad + \frac{1}{\pi} \iint_{\{1/(1 - \sqrt{1 - \delta_2/2})l \leq |z| < 1\}} \frac{C_3 dx dy}{\pi[(l - 3/|z|)^+]^2} \\ &\leq \frac{9}{(1 - \sqrt{1 - \delta_2/2})^2 l^2} + \frac{1}{\pi^2} \iint_D \frac{C_3}{(1 - \delta_2/2)l^2} dx dy. \end{aligned}$$

Thus (3.12) holds for $\gamma = \gamma_1$. For an arbitrary γ , as $\gamma(U_1) \supseteq \gamma(V)$ is bounded, we may assume (after composing γ with an affine mapping) that $\gamma(V) = D$. Let $h = \gamma_1 \circ \gamma^{-1}$, and let $\rho_1 = \rho \circ \gamma_1^{-1}$. Then h is a Möbius transformation of D , and $\rho \circ \gamma^{-1} = (\rho \circ \gamma_1^{-1}) \circ h = \rho_1 \circ h$. The function $\rho_1: D \rightarrow [0, 1]$ depends only on $|z|$ and is nonincreasing in $|z|$. Thus (3.12) (and hence (3.7)) follows by the following lemma.

Lemma 3.4. *Let $\rho_1(z) = \rho_1(|z|): D \rightarrow [0, 1]$ be a function which depends only on $|z|$ and is nonincreasing in $|z|$. For any Möbius transformation h of D , we have*

$$(3.13) \quad \iint_D \rho_1 \circ h(z) dx dy \leq \iint_D \rho_1(z) dx dy.$$

Proof. Since $h^{-1}(\{|z| \leq r\})$ has (Euclidean) radius $\leq r$ for any $r \in (0, 1]$, we deduce that (3.13) holds for the characteristic functions of disks centered at 0, and hence for their linear combinations with positive coefficients. But any function ρ_1 described in the lemma can be approximated in L^1 -norm by these linear combinations, so (3.13) holds for ρ_1 .

Proof of Lemma 3.1. Let

$$(3.14) \quad C_3 = \max \left\{ 4\pi, \frac{18\pi(1 - \delta_2)(1 - \delta_2/2)}{(1 - \sqrt{1 - \delta_2/2})^2 \delta_2} \right\}.$$

Then (3.1) is trivial for $m \leq 2$. Let $l \geq 3$, and assume that (3.1) holds for $m < l$. We prove it holds for $m = l$. By Lemma 3.3 and (3.14), we obtain (3.6), which implies (3.1) in virtue of (3.4) and (3.5).

Remark. Let G_∞ be the Schottky group generated by inversions on the circles of the infinite regular hexagonal circle packing $H_\infty = \bigcup_{n=1}^\infty H_n$.

Then it is clear that G_∞ is a Kleinian group with a fundamental domain I_∞ formed by all interstices bounded by the circles of H_∞ . Clearly $J_n \subseteq \bigcup_{h \in G_\infty} g(I_\infty)$ and by Lemma 3.1 we deduce that $|\mathbb{C} \setminus \bigcup_{g \in G_\infty} g(I_\infty)| = 0$, so the limit set of G_∞ has measure zero. More generally, let P be any circle packing on the sphere $\widehat{\mathbb{C}}$ which satisfies the following conditions:

- (i) there is no circle of P lying in the interstice bounded by any three mutually tangent circles of P and,
- (ii) the circles of P which are tangent to any given circle of P form a closed chain and their number is bounded by some uniform constant.

Let G be the Schottky group generated by the inversions on the circles of P . Then the limit set of G has measure zero. To prove this fact, let I be the complement (in $\widehat{\mathbb{C}}$) of the union of the disjoint disks bounded by circles of P . Clearly I is a nonvoid fundamental domain for G . By an argument similar to the proof of Lemma 3.2, we may deduce that for any Möbius transformation h , the measure of the image by h of $\bigcup_{g \in G} g(I)$ in any disk bounded by a circle of P has a positive ratio uniformly bounded away from zero. But any limit point of G lies in infinitely many images of these disks by the elements of G , and therefore any limit point is not a Lebesgue point for the limit set. So the limit set has measure zero. It is worth pointing out that condition (i) is essential, as is shown by the fact that the limit set of the Apollonian packing has full measure on S^2 .

4. Globally uniform convergence of quasiconformal mappings

Consider a sequence of C -quasiconformal mappings $f_n: \Omega_n \rightarrow \Omega'_n$, where $C \geq 1$, and Ω_n and Ω'_n are some sequences of (open) Jordan domains in $\widehat{\mathbb{C}}$ converging in the sense of Carathéodory to some Jordan domains Ω and Ω' respectively. Suppose that f_n converges uniformly on compact subsets of Ω to some quasiconformal mapping $f: \Omega \rightarrow \Omega'$, and the complex dilations λ_n of f_n converge almost everywhere pointwise to the complex dilation λ of f . One would ask under what conditions does f_n converge globally uniformly to f ? Here, the globally uniform convergence of f_n means that for any $\varepsilon > 0$, there is some $n(\varepsilon)$ and $\delta(\varepsilon) > 0$ such that for any $n \geq n(\varepsilon)$, $z \in \Omega_n$, and $w \in \Omega$ with $|z - w| \leq \delta(\varepsilon)$, we have

$$(4.1) \quad |f_n(z) - f(w)| \leq \varepsilon.$$

If $f_n: \Omega_n \rightarrow \Omega'_n$ and $g_n: \Omega'_n \rightarrow \Omega''_n$ converge globally uniformly to $f: \Omega \rightarrow \Omega'$ and $g: \Omega' \rightarrow \Omega''$ respectively, then $g_n \circ f_n$ converges globally

uniformly to $g \circ f$. If $f_n: \Omega_n \rightarrow \Omega'_n$ converges to $f: \Omega \rightarrow \Omega'$ which extends to a homeomorphism between $\bar{\Omega}$ and $\bar{\Omega}'$, then $f_n^{-1}: \Omega'_n \rightarrow \Omega_n$ converges to $f^{-1}: \Omega' \rightarrow \Omega$.

We will give a sufficient condition for the globally uniform convergence of f_n . As a consequence, we will show that in the case of Jordan domains the mappings f_ε constructed in [7] converge globally uniformly to the Riemann mapping, and the approximating solutions f_δ of [4] also converge globally uniformly to the solution of the Beltrami equation.

For any two Jordan domains Ω_0 and Ω_1 in $\hat{\mathbb{C}}$, we define the distance $\rho(\Omega_0, \Omega_1)$ by

(4.2)

$$\rho(\Omega_0, \Omega_1) = \inf \left\{ \sup_{z \in \partial\Omega_0} d(\psi(z), z); \psi: \partial\Omega_0 \rightarrow \partial\Omega_1 \right. \\ \left. \text{is an orientation preserving homeomorphism} \right\},$$

where $d(\cdot, \cdot)$ denotes the spherical distance between two points of $\hat{\mathbb{C}} = S^2$. It is easy to check that ρ defines a metric on the set of all Jordan domains of \mathbb{C} . Remark that $\rho(\Omega_n, \Omega) \rightarrow 0$ (so-called ‘‘Frechet convergence’’) is stronger than Carathéodory convergence. The following lemma follows immediately by [8, Theorem V].

Lemma 4.1. *Let Ω_n and Ω be Jordan domains such that $\rho(\Omega_n, \Omega) \rightarrow 0$, and let $f_n: D \rightarrow \Omega_n$ and $f: D \rightarrow \Omega$ be some conformal mappings such that f_n converges locally uniformly to f . Then f_n converges globally uniformly to f .*

Corollary 4.2. *Let $\Omega_n, \Omega'_n, \Omega$, and Ω' be some Jordan domains such that $\rho(\Omega_n, \Omega) \rightarrow 0$ and $\rho(\Omega'_n, \Omega') \rightarrow 0$. Then any sequence of conformal mappings $f_n: \Omega_n \rightarrow \Omega'_n$ which converges locally uniformly to some conformal mapping $f: \Omega \rightarrow \Omega'$ converges globally uniformly to f .*

Proof. Let $g_n: D \rightarrow \Omega_n$ and $g: D \rightarrow \Omega$ be some conformal mappings such that $g(0) \neq \infty$, $\partial_z g(0) > 0$, $g_n(0) \rightarrow g(0)$, and $\partial_z g_n(0) > 0$. Then g_n converges locally to g . Since $\rho(\Omega_n, \Omega) \rightarrow 0$, it follows that Lemma 4.1 that g_n converges globally uniformly to g . As g is a conformal homeomorphism between Jordan domains, g extends to a homeomorphism of \bar{D} and $\bar{\Omega}$, so g_n^{-1} converges uniformly to g^{-1} .

On the other hand, the mappings $f_n \circ g_n: D \rightarrow \Omega'_n$ converge locally to $f \circ g$, and $\rho(\Omega'_n, \Omega') \rightarrow 0$. Again by Lemma 4.1 we deduce that $f_n \circ g_n$ converges globally uniformly to $f \circ g$. So $f_n = (f_n \circ g_n) \circ g_n^{-1}$ converges globally uniformly to f . q.e.d.

The main result of this section is the following:

Theorem 4.3. *Let $\Omega_n, \Omega'_n, \Omega,$ and Ω' be Jordan domains such that $\rho(\Omega_n, \Omega) \rightarrow 0$ and $\rho(\Omega'_n, \Omega') \rightarrow 0$. Let $f_n: \Omega_n \rightarrow \Omega'_n$ be a sequence of C -quasiconformal homeomorphisms which converges locally to a quasiconformal homeomorphism $f: \Omega \rightarrow \Omega'$. If the complex dilations λ_n of f_n converge almost everywhere pointwise to the complex dilation λ of f , then f_n converges globally uniformly to f .*

Proof. We may assume without loss of generality that all domains $\Omega_n, \Omega'_n, \Omega,$ and Ω' are contained in some disk, say the unit disk D . Define $\mu_n, \mu: \mathbb{C} \rightarrow \{|\zeta| < 1\}$ by

$$\mu_n(z) = \begin{cases} \lambda_n(z) & \text{if } z \in \Omega_n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu(z) = \begin{cases} \lambda(z) & \text{if } z \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Then since $\lambda_n(z) \rightarrow \lambda(z)$ almost everywhere $z \in \Omega$ and $\|\lambda_n\| < 1$, we deduce that for any $p > 2$,

$$(4.3) \quad \|\mu_n - \mu\|_p = \left(\int \int_{\mathbb{C}} |\mu_n(z) - \mu(z)|^p dx dy \right)^{1/p} \rightarrow 0.$$

Define for each $p \geq 2$ the operators $P: L^p(\mathbb{C}; \mathbb{C}) \rightarrow C^{1-2/p}(\mathbb{C}; \mathbb{C})$ and $T: L^p(\mathbb{C}; \mathbb{C}) \rightarrow L^p(\mathbb{C}; \mathbb{C})$ by

$$(Ph)(\zeta) = -\frac{1}{\pi} \iint_{\mathbb{C}} h(z) \left(\frac{1}{z-\zeta} - \frac{1}{z} \right) dx dy$$

and

$$(TH)(\zeta) = \partial_{\bar{\zeta}}(Ph)(\zeta) = \lim_{\varepsilon \rightarrow 0} -\frac{1}{\pi} \iint_{|z-\zeta| > \varepsilon} \frac{h(z)}{(z-\zeta)^2} dx dy.$$

It is well known that $\|T\|_2 = 1$ and $\|T\|_p \rightarrow 1$ for $p \rightarrow 2$ (see [1], [3]).

Fix some $p > 2$ small enough so that $\|T\|_p \|\mu^h\|_{\infty}$ are uniformly bounded by a constant smaller than 1. For any complex dilation $\mu \in L^{\infty}(\mathbb{C}; \mathbb{C})$, denote by $h^{\mu} \in L^p(\mathbb{C}; \mathbb{C})$ the solution to

$$(4.4) \quad h^{\mu} = T(\mu h^{\mu}) + T\mu.$$

By [1, p. 90–92], h exists and is unique, and the mapping $f^{\mu}(z) = z + P(\mu(h^{\mu} + 1))(z)$ is a quasiconformal homeomorphism of \mathbb{C} with complex dilation μ . We have

$$\partial_{\bar{z}} f^{\mu} = \mu(h^{\mu} + 1) \quad \text{and} \quad \partial_z f^{\mu} = h^{\mu} + 1.$$

Now let $h_n = h^{\mu_n}$, $h = h^\mu$, $g_n = f^{\mu_n}$, and $g = f^\mu$. Then

$$\partial_z(g_n(z) - g(t)) = h_n - h$$

and

$$\partial_{\bar{z}}(g_n(z) - g(z)) = \mu_n h_n - \mu h.$$

But as $\mu_n \rightarrow \mu$ in $L^p(\mathbb{C}, \mathbb{C})$ we have $h_n \rightarrow h$ in $L^p(\mathbb{C}, \mathbb{C})$. Then $\partial_z(g_n(z) - g(z)) \rightarrow 0$ in $L^p(\mathbb{C}, \mathbb{C})$. On the other hand, $g_n(0) - g(0) = 0$. Hence g_n converges to g in $C^{1-2/p}(\bar{D})$ and $g \in C^{1-2/p}(\bar{D})$. This implies that $g_n|_{\bar{D}}$ converges globally uniformly to $g|_{\bar{D}}$. By the assumption $\rho(\Omega_n, \Omega) \rightarrow 0$, it follows that $\rho(g_n(\Omega_n), g(\Omega)) \rightarrow 0$. Let $h_n = f_n \circ g_n^{-1}: g_n(\Omega_n) \rightarrow \Omega'_n$ and let $h = f \circ g^{-1}: g(\Omega) \rightarrow \Omega'$. Clearly h_n is a sequence of conformal mappings which converges locally to h . Using Corollary 4.2, we deduce that h_n converges globally uniformly to h . So $f_n = h \circ g_n$ converges globally uniformly to f .

Corollary 4.4. (i) Suppose R is a Jordan domain. The circle packing solutions f_ϵ of [7] converge globally uniformly to the Riemann mapping.

(ii) The approximating solutions $f_\delta: \Omega \rightarrow \mathbb{C}$ of [4] converge globally uniformly to the solution of the Beltrami equation.

Proof. (i) It is shown in [7] that $f_\epsilon: R_\epsilon = |T_\epsilon| \rightarrow D_\epsilon = |T'_\epsilon|$ converges uniformly on compact subsets to the Riemann mapping $f: R \rightarrow D$, and the complex dilations of f_ϵ converge almost everywhere pointwise to 0. On the other hand, since R is a Jordan domain, R can be approximated (in the ρ -metric) by polyhedral domains. From the construction of R_ϵ it is then easy to show that $\rho(R_\epsilon, R) \rightarrow 0$. Similarly, by the Length-Area Lemma of [7], we have $\rho(D_\epsilon, D) \rightarrow 0$. We may apply Theorem 4.3 to conclude that f_ϵ converges globally uniformly to f .

(ii) As Ω is a Jordan domain in \mathbb{C} , from the construction of $\bar{\Omega}_\delta$ it is elementary to show that $\rho(\Omega_\delta, \Omega) \rightarrow 0$. As in the proof of (i), we have $\rho(D_\delta, D) \rightarrow 0$, where $D_\delta = f'_\delta(\Omega_\delta)$ and $f_\delta = f'_\delta|_{\Omega}$. As $f'_\delta: \Omega_\delta \rightarrow D_\delta$ converges locally uniformly to the solution f of the Beltrami equation and their complex dilations converge to λ (see [4]), we conclude from Theorem 4.3 that f'_δ (and hence f_δ) converge globally uniformly to f . q.e.d.

The conditions of Theorem 4.3 are quite general. In fact, it can be shown directly that: If $f_n: \Omega_n \rightarrow \Omega'_n$ converges globally uniformly to $f: \Omega \rightarrow \Omega'$, then $\rho(\Omega_n, \Omega) \rightarrow 0$ implies $\rho(\Omega'_n, \Omega) \rightarrow 0$. In particular, if $\Omega_n = \Omega$ for all n , then the globally uniform convergence of f_n implies $\rho(\Omega'_n, \Omega) \rightarrow 0$.

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