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A FAKE COMPACT CONTRACTIBLE 4-MANIFOLD

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Here we construct a fake smooth structure on a compact contractible 4-manifold W^4, where W^4 is a well-known Mazur manifold obtained by attaching in two-handle to $S^1 \times B^3$ along its boundary as in Figure 1.* Here we use the conventions of [2].

The results of this paper imply:

Theorem 1. *There is a smooth contractible 4-manifold V with $\partial V = \partial W$, such that V is homeomorphic but not diffeomorphic to W relative to the boundary.*

Let α be the loop in ∂W given by $S^1 \times p \subset S^1 \times S^2 = \partial(S^1 \times B^3)$ as in Figure 2. Zeeman raised the question whether α is slice in W [12], i.e., if α bounds an imbedded smooth D^2 in W. Even though it turned out that α is slice in another smooth contractible manifold with the same boundary [2], the original question has remained open. Let $f: \partial W \to \partial W$ be the diffeomorphism, obtained by first surgering $S^1 \times B^3$ to $B^2 \times S^2$ in the interior of W, then surgering the other imbedded $B^2 \times S^2$ back to $S^1 \times B^3$ (i.e., replacing the dots in Figure 2.)

Clearly this diffeomorphism extends to a self-homotopy equivalence of W. In fact, by [9], f extends to a homeomorphism $F: W \to W$. In [2, p. 279] the question of whether f extends to a diffeomorphism of W was posed. If it did, α would be slice in W since $f(\alpha)$ is clearly slice in W. Here we answer these questions negatively:

Theorem 2. *α is not slice in W, in particular f does not extend to a self-diffeomorphism of W.*

Theorem 1 follows from Theorem 2 as follows: Let $F: W \to W$ be a homeomorphism extending f. Let V be the smooth structure on W obtained by pulling back the smooth structure of W by F. This gives a diffeomorphism $F: V \to W$ extending f on the boundary. If $G: W \to V$

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* Numbered figures appear at the end of the article (pp. 345–355).

were any diffeomorphism extending the identity on the boundary, then $F \circ G$ would be a diffeomorphism extending f, contradicting the existence of G. In particular, Theorem 1 implies that there is a nontrivial h-cobordism from V to W rel ∂. This h-cobordism will be explicitly discussed in §6. So V is diffeomorphic to W but no such diffeomorphism can extend the identity map on the boundary.

We now summarize the proof. We use the conventions $\simeq$ for homotopy equivalence and $\approx$ for diffeomorphism. The orientation of $\mathbf{C}P^2$ comes from the complex orientation, i.e., it has the intersection form $(+1)$. $\overline{\mathbf{C}P}^2$ denotes $\mathbf{C}P^2$ with the opposite orientation. We also use the convention that if M is an oriented manifold, then $-M$ is M with the opposite orientation.

We first construct a compact 1-connected smooth 4-manifold M_1 with a homology sphere boundary $\partial M_1 = \Sigma$. M_1 is even with signature 16 and has the second betti number $b_2(M_1) = 22$. We also construct two more compact smooth 4-manifolds W_1 and Q such that:

(1) $\partial W_1 = \partial Q = \Sigma$.

(2) W_1 is contractible.

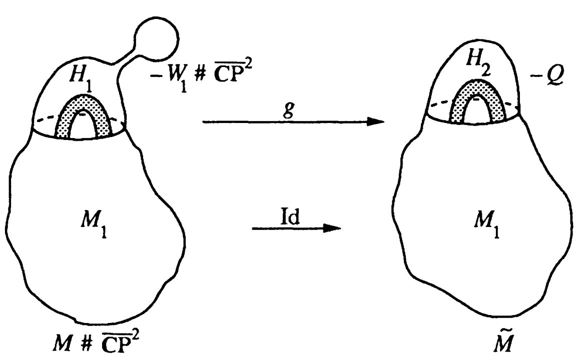
(3) $Q \simeq W_1 \# \mathbf{C}P^2$.

(4) If α is slice in W, then Σ bounds a contractible manifold W_2 with $Q \approx W_2 \# \mathbf{C}P^2$.

We define $M = M_1 \cup_\partial (-W_1)$ and $\widetilde{M} = M_1 \cup_\partial (-Q)$. Then we show:

(5) $\widetilde{M} \approx (3\mathbf{C}P^2) \# (20\overline{\mathbf{C}P}^2)$.

Clearly $\widetilde{M} \simeq M \# \overline{\mathbf{C}P}^2$, and M is homotopy equivalent to the Kummer surface. Let $g \colon \partial(-W_1) \to \partial(-Q)$ be the restriction of the identity map $M_1 \to M_1$ to its boundary (see the diagram). Hence if f extends to a diffeomorphism $-W_1 \# \overline{\mathbf{C}P}^2 \to -Q$, then $\widetilde{M}$ would be diffeomorphic to



$M\#\overline{\mathbf{C}P}^2$. It turns out that g extends across a two-handle; that is, there are two-handles $H_k = \mathrm{im}(h_k)$, $k = 1, 2$, where

$$h_1\colon (D^2 \times D^2, S^1 \times D^2) \to (-W_1 \# \overline{\mathbf{C}P}^2, -\Sigma),$$
$$h_2\colon (D^2 \times D^2, S^1 \times D^2) \to (-Q, -\Sigma)$$

such that g extends to a diffeomorphism $H_1 \to H_2$.

Then we show the surprising fact that

$$-W = (-W_1 \# \overline{\mathbf{C}P}^2) - \mathrm{int}(H_1) = (-Q) - \mathrm{int}(H_2)$$

and the map $\partial(-W) \to \partial(-W)$ induced by g is f. Hence, by turning $-W$ upside down, we conclude that if $f\colon \partial W \to \partial W$ extends to a diffeomorphism $W \to W$, then $\widetilde{M} \approx M\#\overline{\mathbf{C}P}^2$. More generally, if α is slice in W, then

$$\begin{aligned}\widetilde{M} &\approx M_1 \cup_\partial (-Q) \approx M_1 \cup_\partial (-W_2 \# \overline{\mathbf{C}P}^2)\\ &\approx [M_1 \cup_\partial (-W_2)] \# \overline{\mathbf{C}P}^2 = M' \# \overline{\mathbf{C}P}^2,\end{aligned}$$

where $M' = M_1 \cup_\partial (-W_2)$ is a homotopy Kummer surface. On the other hand this contradicts Gauge theoretic results discussed in §5 (the last paragraph of §5). More specifically, since $\widetilde{M}$ decomposes, its Donaldson polynomial invariants must vanish [4]. On the other hand in [7] Fintushel and Stern showed that a particular Donaldson invariant of a homotopy Kummer surface is nonzero if it contains a copy of the Brieskorn homology sphere $\Sigma(2, 3, 7)$. We show that M_1 contains $\Sigma(2, 3, 7)$. Hence M' has a nonzero Donaldson invariant. Since these invariants persist under connected summing with $\overline{\mathbf{C}P}^2$, $M'\#\overline{\mathbf{C}P}^2$ has a nontrivial Donaldson invariant. So $M'\#\overline{\mathbf{C}P}^2 \not\approx \widetilde{M}$ and, more generally,

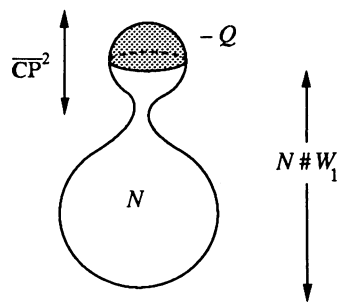
$$(M' \# \overline{\mathbf{C}P}^2) \# (k\overline{\mathbf{C}P}^2) \not\approx \widetilde{M} \# (k\overline{\mathbf{C}P}^2).$$

Because this argument is independent of the contractible manifold W_2 which Σ bounds. We conclude that Q cannot be decomposed as $V\#\mathbf{C}P^2$, where V is any contractible manifold. In particular Q and $W_1\#\mathbf{C}P^2$ are homeomorphic but not diffeomorphic to each other. We use this in the construction of [1]. An interesting corollary is the following.

Corollary. *M_1 cannot be diffeomorphic to $N\#W_1$ for any compact complex manifold N.*

This is because $Q_1 \cup (-W_1) = \mathbf{C}P^2$ (§6). Hence if $M_1 \approx N\#W_1$, then

$$N\#\overline{\mathbf{C}P}^2 \approx N\#(W_1 \cup -Q_1) \approx M_1 \cup (-Q_1) = \widetilde{M} \approx (3\mathbf{C}P^2)\#(20\overline{\mathbf{C}P}^2).$$



Again contradicting a result of Donaldson in §5, i.e., complex surfaces cannot admit such splitting (since N is complex, so is $N\#\overline{\mathbf{C}P}^2$). Notice also that, if we could extend f to W, then M could not be a complex manifold, since $M\#\overline{\mathbf{C}P}^2$ would have such a splitting. (Here the author would like to thank John Morgan for explaining Donaldson's results on relating his polynomial invariants to the Floer homology, and also thank R. Fintushel and R. Stern for making available their calculation of Donaldson's polynomial on the particular type of four-manifolds [7].)

1. Construction of Q and M_1

To simplify the steps in our construction, we first construct a diffeomorphism between the boundary of the four-manifold in Figure 3 and the boundary of the manifold in Figure 6. We accomplish this by first blowing up Figure 3 to Figure 4, then blowing it up more in a similar way to Figure 5, and then blowing down the three $+1$-framed circles (compare [10]).

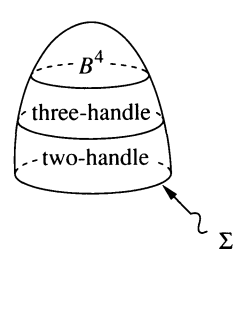
We can now start with our main construction. We define Q^4 to be the manifold in Figure 7. In Figures 7–33 we modify the interior of Q^4 to obtain M_1. Along the way we carry along a loop $\alpha \subset \partial Q$ as indicated in Figure 8 in order to see where it will end up in ∂M_1 (Figure 33) under the diffeomorphism $\partial Q \approx \partial M_1$. This diffeomorphism might twist the neighborhood of the loop α; to keep track of this we start out with a reference framing (zero) and denote it by $\alpha(0)$. Figures 8–18 are obtained either by blowing up, or an isotopy of the previous figure. We apply the modification of Figure $3 \to$ Figure 6 to Figure 18 to arrive at Figure 19. Notice that this diffeomorphism twisted the framing of the loop α to -1. Figure 20 is obtained by introducing of pair of two-handles to Figure 19 in order to change the left-handed twist to a right-hand twist. Blowing up $+1$ and blowing down -1 gives Figures 21 and 22. Sliding a -2-framed handle over the 0-framed handle gives Figure 23. We now modify the handle corresponding to the left-handed trefoil knot with $+1$-framing in Figure 23. To save space, from Figures 24–32 we only draw the modification of the part of the picture in Figure 23. Modification from Figures 24–32 is exactly the same modification as Figures 9–19, except here during the modification we carry along the other two two-handles tangled to the $+1$-framed left-handed trefoil knot. Figure 32 is only part of the picture. In Figure 33 we draw the whole picture; that is we go back to Figure 23 and install Figure 32 into it by recalling; the positions of the 0- and -2-framed handles (notice that the -2-framed handle was the -1-framed handle in Figure 23). We call the manifold in Figure 33 M_1^4.

2. $\widetilde{M} \approx (3\mathbf{C}P^2)\#(20\overline{\mathbf{C}P}^2)$

Recall $\widetilde{M} = M_1 \cup_\partial (-Q)$. To see the handlebody of $\widetilde{M}$ we must turn Q upside down and attach it to M_1. Upside down $-Q$ is obtained by attaching a two-handle to $\Sigma = \partial Q$ along α with 0-framing (Figure 8), and then capping it off with B^4. Hence $\widetilde{M}$ is obtained by attaching M_1 (Figure 33) a two-handle along the loop α with -1-framing. Then by sliding three two-handles (three loops that are linked to α) over the handle α we get Figure 34. By consequently sliding -2-framing handles over the -1-framed handles we get Figure 35 and by more handle slides we get Figure 36. Figure 36 is obviously $(3\mathbf{C}P^2)\#(20\overline{\mathbf{C}P}^2)$.

3. Construction W_1

We attach a two-handle to $\Sigma = \partial Q$ as in Figure 37, then we see that this makes the boundary $S^1 \times S^2$ (e.g., blow down $+1$-framed unknotted circle). Then attach a three-handle and a four-handle to get a compact manifold $-W_1$ with $\partial(-W_1) = \Sigma$:



Clearly W_1 is a contractible manifold. We turn W_1 upside down. We do this by turning the three-handle into a one-handle (i.e., $B^2 \times S^2$ of Figure 38 to $S^1 \times B^3$) and attaching a two-handle along $h(\gamma)$, where h is the diffeomorphism between the boundaries of Figure 37 and Figure 38 and γ is the dual circle. This gives us Figure 40. Similarly we turn $-Q$ upside down by attaching a two-handle to B^4 along the image $h'(\delta)$ as in Figure 42, which is the same as Figure 43. Again, here h' is the diffeomorphism between the boundaries of Figure 41 and B^4, and δ is the dual circle. We can now see the diffeomorphism $\partial Q \approx \partial W_1$ between Figures 43 and 44, namely surger Figure 44 (i.e., replace the dot on the unknotted circle with 0 and blow down -1).

4. Diffeomorphism f

We start with Q (Figure 45), which is clearly diffeomorphic to Figure 46 (by cancelling the one-handle with the $+1$-framed two-handle). Now the 'flip' diffeomorphism f described in the introduction induces a diffeomorphism between the boundaries of Figures 46 and 47. By a handle slide Figure 47 is diffeomorphic to Figure 48, which is $W_1 \# \mathbf{C}P^2$. Hence, if f extends to a self-diffeomorphism of W, it would induce a diffeomorphism between Figures 46 and 47, hence we would have $Q \approx W_1 \# \mathbf{C}P^2$.

It is also clear that if α was slice in W, the $+1$-framed handle in Figure 46 would be represented by an imbedded two-sphere. Hence Q would be $W_2 \# \mathbf{C}P^2$ for some contractible W_2 with $\partial W_2 = \partial Q$.

5. Donaldson polynomials

In this section we summarize Donaldson's theory of polynomial invariants of four-manifolds, and indicate a calculation of them due to Fintushel and Stern. We follow [4], [5], [11].

Let M^4 be a one-connected closed smooth four-manifold, and let $P \xrightarrow{\pi} M^4$ be a principal $\mathrm{SU}(2)$-bundle with $c_2(P) = k$. Let

$$\mathscr{A}^*(P) = \text{Space of all irreducible connections on } P;$$
$$\mathscr{G}(P) = \text{The gauge group } \mathrm{Aut}(P);$$
$$\mathscr{B}(P) = \mathscr{A}^*(P)/\mathscr{G}(P).$$

After completing with an appropriate Sobolev norm, $\mathscr{B}(P)$ becomes a Hilbert manifold. Fix a Riemannian metric g on M^4 and let $\mathscr{M}_k(M, g)$

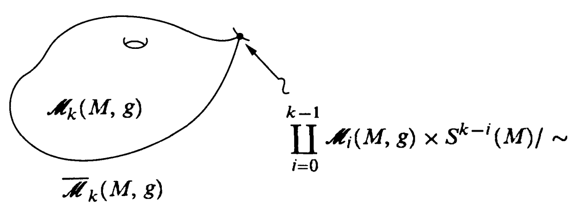
be the moduli space of anti-self-dual connections in $\mathscr{B}(P)$, i.e.,

$$\mathscr{M}_k(M, g) = \{[A] \in \mathscr{B}(P) \mid *F_A = -F_A\},$$

where F_A is the curvature of the connection A. According to [8], for generic metrics g, $\mathscr{M}_k(M, g)$ is a manifold of dimension $d(k) = 8k - 3(b_2^+ + 1)$, and usually it is not compact. Here b_2^+ denotes the number of positive terms in a diagonalization of the intersection form of $H_2(M)$. $\mathscr{M}_k(M, g)$ has a compactification

$$\overline{\mathscr{M}}_k(M, g) \subset \coprod_{i=0}^{k} \mathscr{M}_i(M, g) \times S^{k-i}(M)/\sim,$$

where $S^j(M)$ denotes the jth symmetric product and $\sim$ denotes identifications, which make $\overline{\mathscr{M}}_k(M, g)$ a stratified space with strata contained in $\mathscr{M}_i(M, g) \times S^{k-i}(M)$, $i = 0, 1, \cdots, k$:



When $k \geq \frac{1}{4}(3b_2^+ + 5)$ all strata except the top stratum $\mathscr{M}_k(M, g)$ has codimension ≥ 2 in $\overline{\mathscr{M}}_k(M, g)$. Also an orientation of $H_+^2(M, \mathbf{R})$ gives an orientation on $\mathscr{M}_k(M, g)$, hence it induces a fundamental class on $\overline{\mathscr{M}}_k(M, g)$. Furthermore every homology class $\alpha \in H_2(M)$ induces a codimension-two submanifold $V_\alpha \hookrightarrow \mathscr{M}_k(M, g)$. Now assume $k \geq \frac{1}{4}(3b_2^+ + 5)$ and b_2^+ is odd; then $d(k)$ is even. So if we choose any $\alpha_i \in H_2(M)$, $i = 1, 2 \cdots, d/2$, the corresponding submanifolds $V_i \subset \mathscr{M}_i(M, g)$, $i = 1, 2, \cdots, d/2$, generically intersect along points and give rise to the intersection number $V_1 \cap \cdots \cap V_{d/2}$. Denote this intersection number by $\gamma_k(M)(\alpha_1, \cdots, \alpha_{d/2})$; this turns out to be independent of the metric g. Hence it induces a map $\gamma_k(M)\colon S^{d/2}(H_2(M)) \to \mathbf{Z}$. Furthermore $\gamma_k(M)$ has the property that if $\gamma_k(M) \neq 0$, then $\gamma_k(M \# \overline{\mathbf{C}P^2}) \neq 0$.

If M has a homology sphere boundary $\Sigma = \partial M$, then as outlined in [3], the above process generalizes to give homomorphisms

$$\gamma_k(M)\colon S^r(H_2(M)) \to HF_*(\Sigma), \qquad r = 1, 2, \cdots, d/2,$$

where HF_* denotes the Flower homology group of Σ. Donaldson has shown that if $H_*(M)$ has the intersection form $E_8 \oplus i\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where $i = 0, 1$, or 2, then under some restrictions of Σ $HF_*(\Sigma) \neq 0$. Furthermore if $\partial M_1 = \Sigma$, $\partial M_2 = -\Sigma$, and $M = M_1 \cup_\partial M_2$, i.e., M is obtain by glueing two four-manifolds along a common homology sphere boundary Σ, then we can pair these invariants and get a homomorphism, as indicated in [3]:

$$\gamma_k(M) \colon S^r(H_2(M_1)) \otimes S^{d/2-r}(H_2(M_1)) \to \mathbf{Z}.$$

So for example if

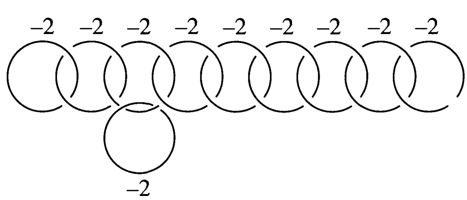
$$\gamma_k(M_1)(\alpha_1, \cdots, \alpha_r) = \sum n_\rho[\rho] \in HF_*(\Sigma)$$
$$\gamma_k(M_2)(\alpha_{r+1}, \cdots, \alpha_{d/2}) = \sum m_\rho[\rho] \in HF_*(-\Sigma),$$

then

$$\gamma_k(M) = \sum n_\rho m_\rho$$

(here $[\rho]$ denotes the conjugacy class of generic representations $\pi_1(\Sigma) \to SU(2)$). So having explicit knowledge of the Floer homology groups $HF_*(\Sigma)$ is helpful to compute $\gamma_k(M)$. In [6] Floer homology groups of Seifert homology spheres $\Sigma(p, q, r)$ have been explicitly computed. As a corollary, Fintushel and Stern [7] showed that if $\Sigma(2, 3, 7)$ imbeds into a homotopy Kummer surface M, then $\gamma_4(M) \neq 0$. In this case we take $k = 4$, $d = 20$, and $r = 4$. The idea is that such an imbedding decomposes $M = M_1 \cup_\partial M_2$, with $H_*(M_i)$ having the intersection form $E_8 \oplus i\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $i = 1, 2$. Now, the Donaldson type of arguments applied to each side M_i, $i = 1, 2$, coupled with the knowledge of $HF_*(\Sigma(2, 3, 7))$ gives the result [7].

The manifold M_1 of Figure 33 contains $\Sigma(2, 3, 7)$, since $\Sigma(2, 3, 7)$ is the boundary of the plumbing:

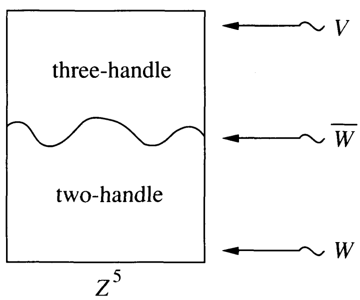


Hence if W_2 is any contractible manifold with $\partial W_2 = \Sigma$, the homotopy Kummer surface $M' = M_1 \cup_\partial (-W_2)$ contains $\Sigma(2, 3, 7)$. Consequently,

$\gamma_4(M') \neq 0$, so $\gamma_4(M' \# \overline{\mathbf{C}P}^2) \neq 0$. Since $\gamma_4(3\mathbf{C}P^2 \# 20\overline{\mathbf{C}P}^2) = 0$ [5], $3\mathbf{C}P^2 \# 20\overline{\mathbf{C}P}^2$ cannot be diffeomorphic to $M' \# \overline{\mathbf{C}P}^2$.

6. The nontrivial h-cobordism and the conclusion

Here we will describe the nontrivial h-cobordism Z^5 between W and V rel ∂ and prove some miscellaneous facts. The h-cobordism Z can be described by attaching an algebraically cancelling pair of two- and three-handles to the interior of W:



We first attach a two-handle to W along α (Figure 2), obtaining $\overline{W}^4$ (Figure 49). So $\overline{W}^4$ is obtained by surgering the loop α in W. We then attach a three-handle along the imbedded two-sphere ξ in $\overline{W}$ obtaining V. Hence V is obtained by surgering the two-sphere ξ in $\overline{W}$ as indicated in Figure 49. So, in particular, $V \approx W$, but this diffeomorphism cannot extend the identity on the boundary. In particular, $V \cup_\partial (-V) \approx S^4$.

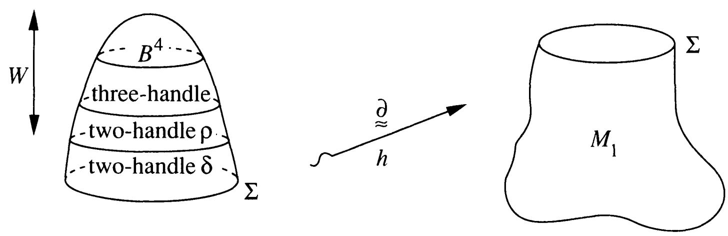
We also have

Proposition. (a) $V \cup_\partial (-W) \approx S^4$.

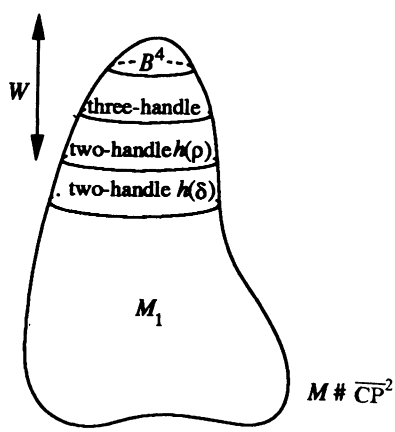
(b) $Q \cup_\partial (-W_1) \approx \mathbf{C}P^2$.

Proof. (a) holds since $V \cup_\partial (-W) = W \cup_f (-W)$ and the handlebody of $W \cup_f (-W)$ is obtained by attaching a two-handle to W along $\alpha \subset \partial W$, and a three-handle. So it is Figure 50 with a three-handle, which is the same as Figure 51 with a three-handle, which is S^4. To draw the handlebody of $W_1 \cup_\partial (-Q)$ we simply add a two-handle to Figure 44 along $\varphi(\beta)$, where φ is the diffeomorphism between the boundaries of Figures 43 and 44, and β is the dual circle in Figure 43. We get Figure 52, which is diffeomorphic to Figure 53, which is $\overline{CP}^2$. Hence we have shown (b). q.e.d.

$M \# \overline{\mathbf{C}P}^2$ is obtained from $\widetilde{M}$ by removing W and reglueing it with the diffeomorphism f (i.e., Gluck construction to W). This operation changes the smooth structure of $M \# \overline{\mathbf{C}P}^2$. To demonstrate this better we will draw the handlebody of $M \# \overline{\mathbf{C}P}^2$ and indicate this with pictures. Recall $M = M_1 \cup (-W_1)$. M is obtained by attaching upside down W_1 to M. Upside down W_1 is obtained by attaching a two-handle to Σ as in Figure 37, along with a three-handle and a four-handle. Therefore if we ignore three- and four-handles, M is obtained by attaching a two-handle to M_1 along $h(\delta)$, where h is the diffeomorphism between the boundaries of Q (Figure 7) and M_1 (Figure 33), where δ is the $+1$-framed loop in Figure 54. Hence the handlebody of $M \# \overline{\mathbf{C}P}^2$ is obtained by attaching two-handles to M_1 along $h(\delta)$ and $h(\rho)$, where ρ is the -1-framed unknot in Σ as indicated in Figure 55. By sliding the handle $h(\delta)$ over $h(\rho)$ we can assume that the positions of δ and ρ (in $\Sigma = \partial Q$) are as in Figure 56.



We claim that the two-handle $h(\rho)$ along the three-handle and the four-handle give W:



The effect of removing W from $M \# \overline{\mathbf{C}P}^2$ and putting it back with the diffeomorphism f is attaching M_1 two-handles along $h(\delta)$ and $h(\alpha)$ (instead of $h(\rho)$) (Figure 57). Figures 57 and 58 show how f throws the handle ρ to α. Here we view:

$$f: \partial[M_1 \cup (\text{2-handle } h(\delta))] \circlearrowleft$$

In Figure 59 we draw the pictures of $h(\alpha)$, $h(\delta)$, and $h(\rho)$ in ∂M_1 (Figure 33). For simplicity, in the figure we continued to denote them by α, δ, and ρ. We also indicated their change of framings under h. So Figure 33 along with two-handles attached to δ and ρ gives $M \# \overline{\mathbf{C}P}^2$, as indicated in Figure 59. Removing W from $M \# \overline{\mathbf{C}P}^2$ and regluing back with f gives Figure 33 along with two-handles attached to δ and α as in Figure 59. So twisting $M \# \overline{\mathbf{C}P}^2$ along W has the effect of throwing the handle ρ to α. The -1-framed handle α then decomposes the manifold! That is, by sliding the other handles over this handle (as in Figures 33–36) we get $3\mathbf{C}P^2 \# 20\overline{\mathbf{C}P}^2 \# S^2 \times B^2$, along with a three-handle, which is $3\mathbf{C}P^2 \# 20\overline{\mathbf{C}P}^2$.

Finally we point out that the map $f: \partial W \to \partial W$ is a branched covering involution. That is, there is a knot $K \subset S^3$ such that ∂W is the two-fold branched cover of S^3 branched along K, and f is the branched covering transformation. In fact, there is a properly imbedded $D^2 \subset \mathbf{C}P_0^2 = \mathbf{C}P^2 - \text{int}(B^4)$ such that the manifold $\overline{W}$ (recall $\partial \overline{W} = \partial W$) is the two-fold branched cover of $\mathbf{C}P_0^2$ branched along D^2. Figure 60 is the picture of $D^2 \hookrightarrow \mathbf{C}P_0^2$ (in particular, the picture of $K \subset S^3$). Figure 61 is the two-fold branched cover of $\mathbf{C}P_0^2$, which is $\overline{W}$.

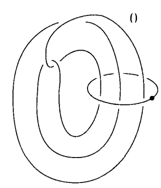


FIGURE 1

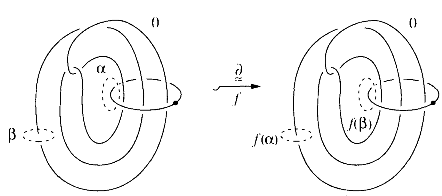


FIGURE 2

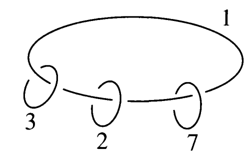


FIGURE 3

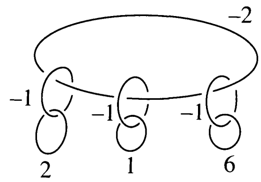
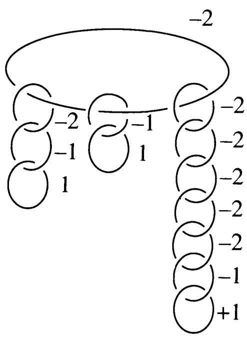


FIGURE 4



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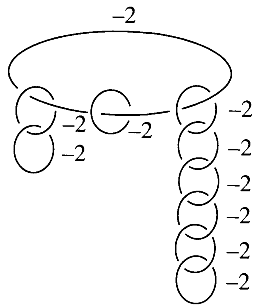

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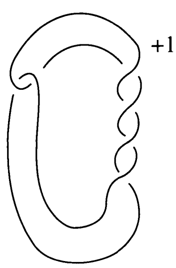

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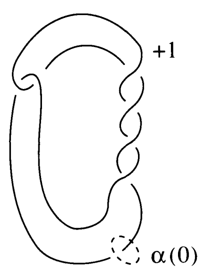

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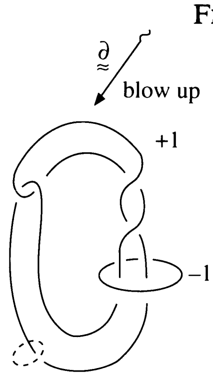

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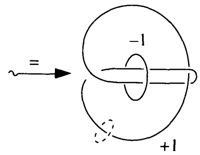

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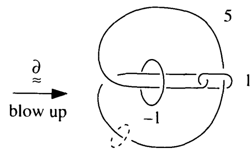

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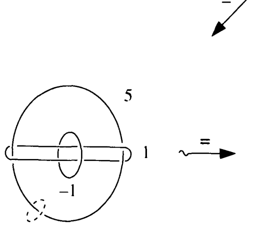

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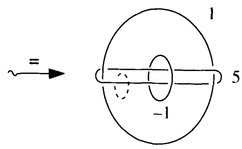

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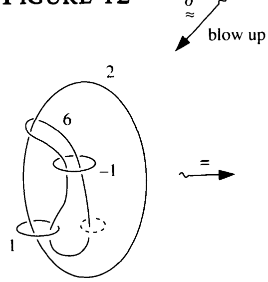

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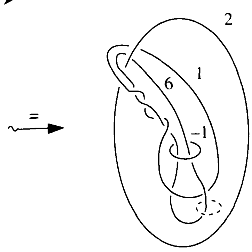

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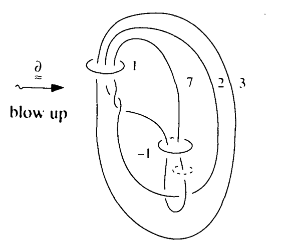

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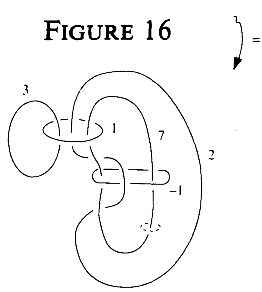

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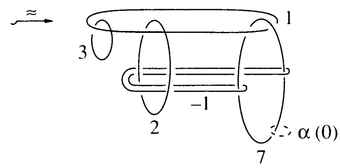

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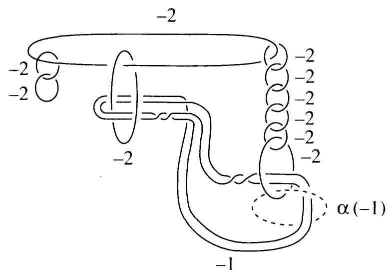

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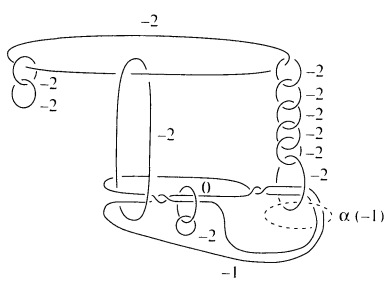

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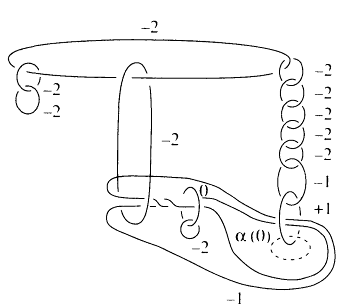

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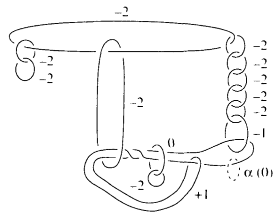

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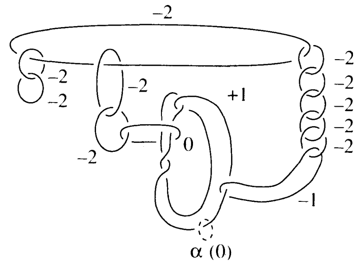

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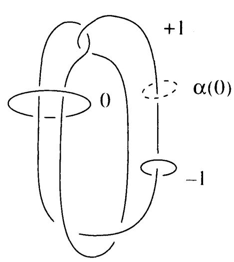

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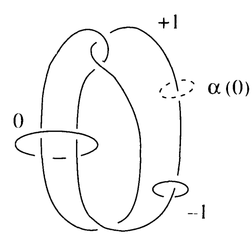

FIGURE 24

FIGURE 25

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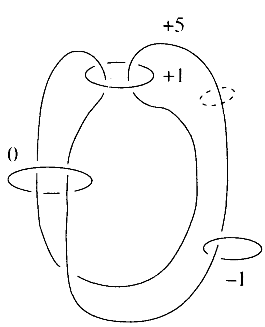


$\xrightarrow{=}$

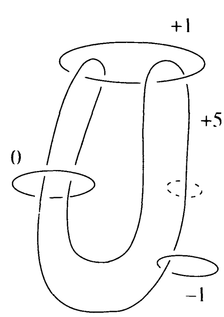


FIGURE 26

FIGURE 27

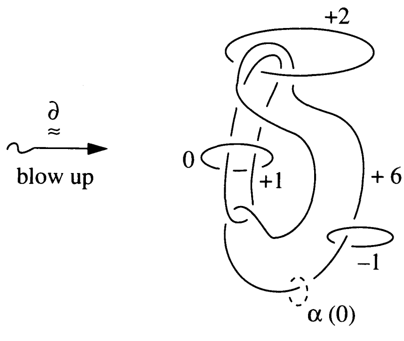
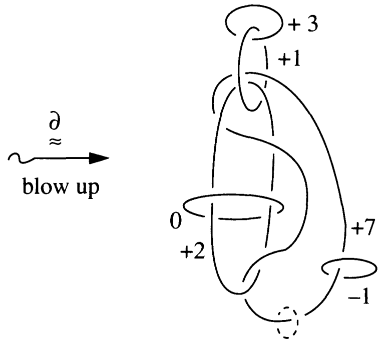


FIGURE 28



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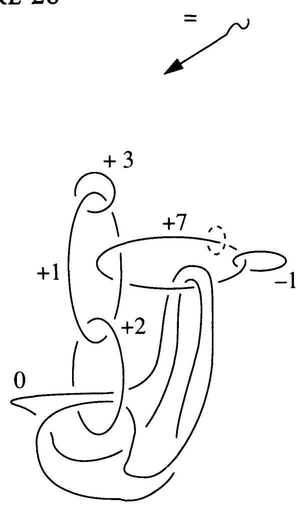

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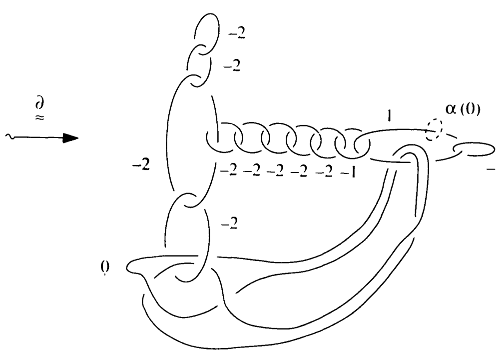

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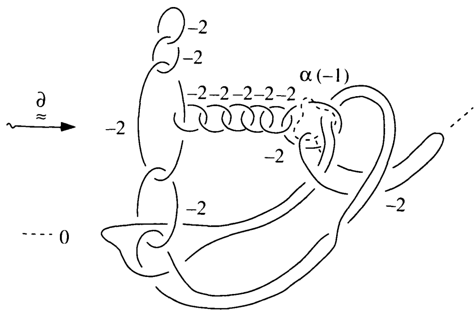

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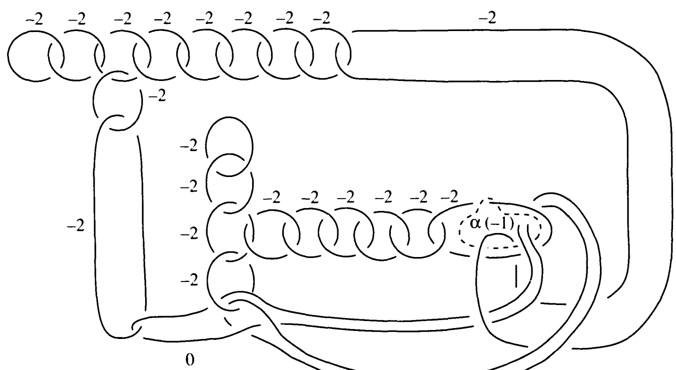

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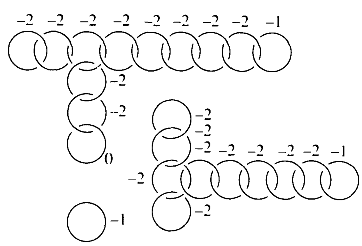

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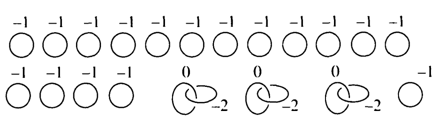

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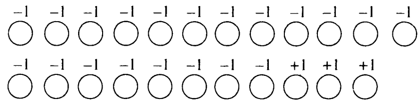

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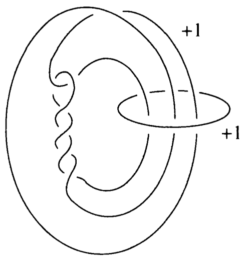
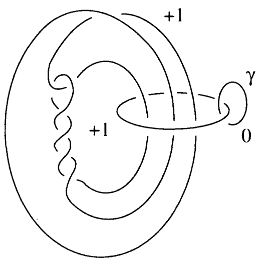


FIGURE 37 FIGURE 38



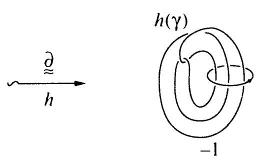
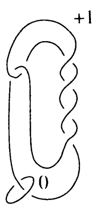


FIGURE 39 FIGURE 40



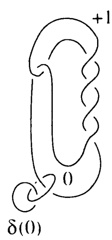
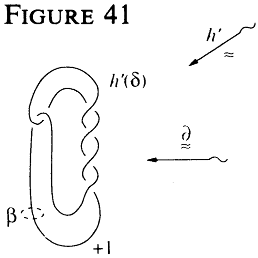


FIGURE 41 FIGURE 42



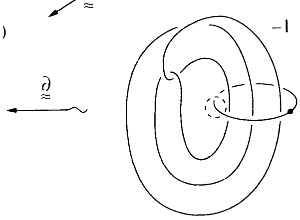
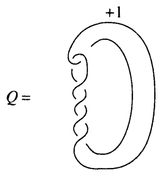


FIGURE 43 FIGURE 44



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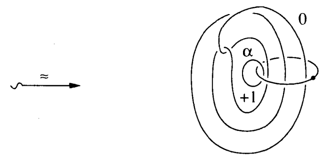

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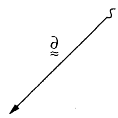

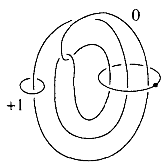

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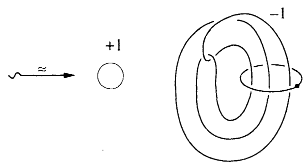

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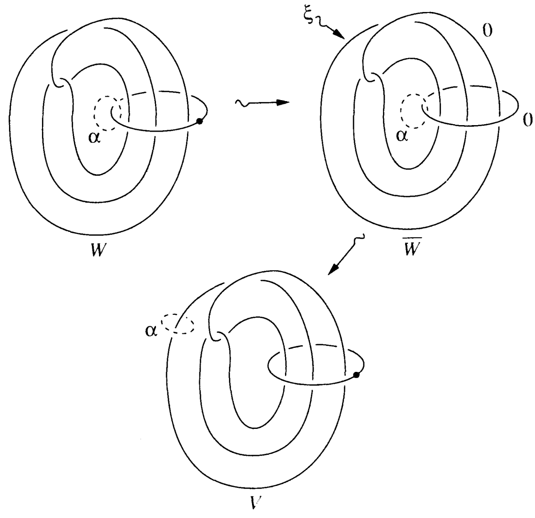

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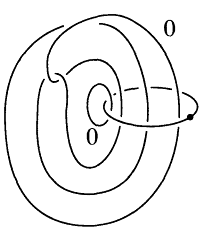

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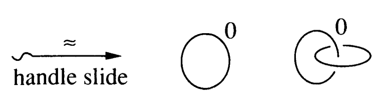

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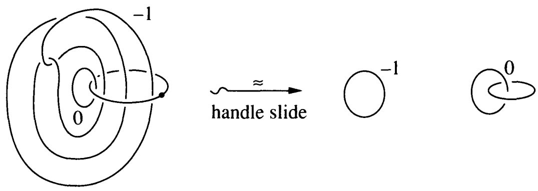
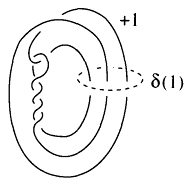


FIGURE 52 FIGURE 53



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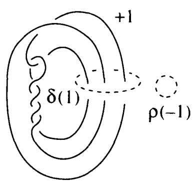

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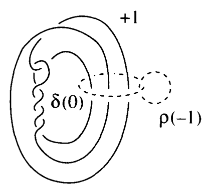

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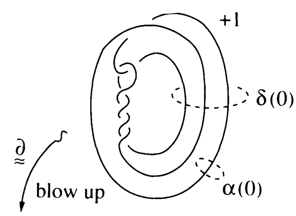

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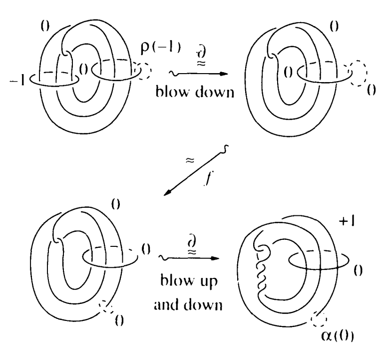

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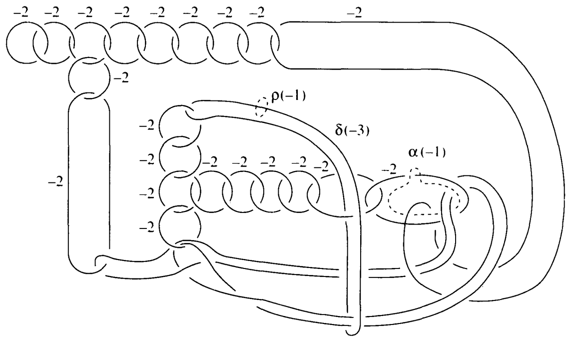

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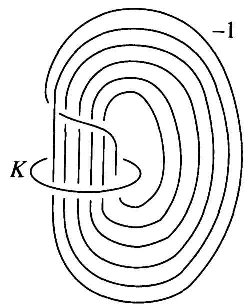

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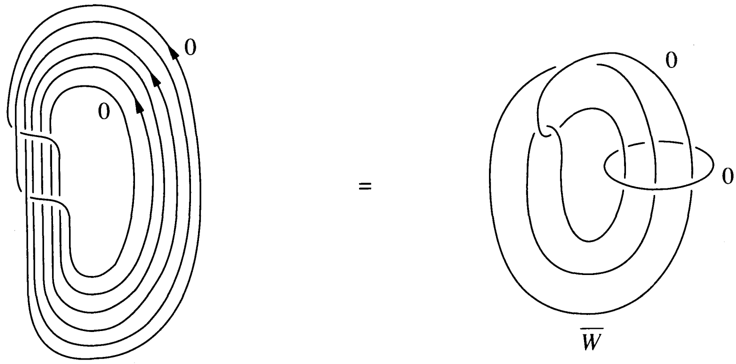

FIGURE 61

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