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SOME SPACES OF HOLOMORPHIC MAPS TO COMPLEX GRASSMANN MANIFOLDS

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Abstract

In this paper we study the topology of the space of based holomorphic maps of degree $-k$ from the Riemann sphere to complex Grassmann manifolds, which we denote by $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$. We compute $H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}))$ for all k, n and m as well as the natural inclusion $i(k,n,m)_*: H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}))\longrightarrow H_*(\Omega_k^2(\mathbb{G}_{n,n+m}))$ induced by forgetting the complex structures. These results also give the geometry of the moduli spaces of observable and controllable solutions to the linear control equations.

1. Introduction

Let $S^2 = \mathbb{CP}(1)$ denote the Riemann sphere and $\mathbb{G}_{n,n+m}$ the Grassmannian of all complex n-dimensional planes through the origin in $\mathbb{C}^{n+m}$. Both spaces are naturally complex manifolds and have natural base points (∞ and $\mathbb{C}^n\times\vec{0}\subset\mathbb{C}^{n+m}$, respectively). Let $\mathrm{Rat}(\mathbb{G}_{n,n+m})$ denote the space of all based holomorphic maps from (S^2,∞) to $(\mathbb{G}_{n,n+m},\mathbb{C}^n\times\vec{0})$ with the compact open topology. It is well known that every such holomorphic map is rational; that is, it is given by a series of zeros, poles, and residues, hence the terminology "Rat". In addition, associated to each element $f\in\mathrm{Rat}(\mathbb{G}_{n,n+m})$ is an integer $c(f)=k$, the total Chern number, given by the topological degree of $f: S^2\longrightarrow\mathbb{G}_{n,n+m}$. Thus $\mathrm{Rat}(\mathbb{G}_{n,n+m})$ breaks into components and it is known [5] that each component $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is a connected complex manifold of complex dimension $(n+m)k$.

By forgetting the complex structure one obtains based continuous maps from S^2 to $\mathbb{G}_{n,n+m}$; that is, elements in the two-fold loop space $\Omega^2(\mathbb{G}_{n,n+m})$ whose components are also indexed by the degree $c(f)$. We

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denote these natural inclusions

$$(1.1) \qquad i(k, n, m)\colon \mathrm{Rat}_k(\mathbb{G}_{n,n+m}) \longrightarrow \Omega^2_k(\mathbb{G}_{n,n+m})$$

where, for each fixed n and m, $\Omega^2_k(\mathbb{G}_{n,n+m})$ is a well-understood infinite-dimensional but locally finite CW complex (see §2), whose topology is independent of k, whereas $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is a finite-dimensional complex manifold whose global topology obviously depends on k.

There is a natural anti-holomorphic equivalence between $\mathbb{G}_{n,n+m}$ and $\mathbb{G}_{m,n+m}$ given by $P \mapsto P^{\perp}$ and thus, for notational convenience, we assume $1 \le n \le m$. When $n = 1$ Segal [22] obtained partial information about $\mathrm{Rat}_k(\mathbb{CP}(m))$ by showing $i(k, 1, m)$ is a homotopy equivalence through a range that increases as k increases. In joint work with F. Cohen and R. Cohen [6], [7] we completed the topological analysis by determining the stable homotopy type of $\mathrm{Rat}_k(\mathbb{CP}(m))$ in terms of pieces of the May-Milgram filtration of iterated loop spaces.

Specifically, we have a stable splitting (see 2.2)

$$\Omega^2\mathbb{CP}(m) \simeq_s \bigvee_j \Sigma^{(2m-2)j} D_j,$$

where the $D_j \simeq \mathbb{F}(\mathbb{C}, j)_+ \wedge_{\Sigma_j} (S^1)^{(j)}$, $\mathbb{F}(\mathbb{C}, j)$ is the space of j-tuples of distinct points in $\mathbb{C}$, $\mathbb{F}(\mathbb{C}, j) = \{(z_1, \cdots, z_j) \in \mathbb{C}^j \mid z_i \neq z_l \text{ for } i \neq l\}$, and Σ_j is the symmetric group on j letters acting by permutation on both the coordinates of the points in $\mathbb{F}(\mathbb{C}, j)$ and $(S^1)^{(j)}$. Here X_+ denotes the one-point union of X with a disjoint basepoint $+$, $\wedge$ denotes the smash product, and $(S^1)^{(j)}$ is the Σ_j equivariant j-fold smash product of S^1.

For example $D_1 \simeq S^1$, and D_2 is the Thom space of the bundle $(\xi + \epsilon)$ over S^1 where ξ is the nontrivial (real) line bundle, and ϵ is the trivial one.

The homology structure of D_j is known [17], [14]. In particular, $H_*(D_j; \mathbb{Z})$ is *torsion* for $j > 1$, and, if $p > j$ is a prime, then $H_*(D_j; \mathbb{F}_p) = 0$ as long as $j \neq 1$. With this notation, one of the main results of [6] is

Theorem [6]. $\mathrm{Rat}_k(\mathbb{CP}(m)) \simeq_s \bigvee_{j=1}^k \Sigma^{(2m-2)j} D_j$.

This is a special case of the situation we are interested in since $\mathbb{CP}(m) = \mathbb{G}_{1,1+m}$. We now turn our attention to the general situation. Kirwan [13] extended Segal's result by showing $i(k, n, m)$ is always a homotopy equivalence through a range that depends on k, n, and m. In this paper we prove

Theorem A. *For all k, n, m, and any coefficient ring, A,*

$$H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m});A) \cong \bigoplus_K H_*(\Sigma^{2t(K)}(\mathrm{Rat}_{k_1}(\mathbb{CP}(m))\times\ldots\times\mathrm{Rat}_{k_n}(\mathbb{CP}(m)))_+;A),$$

where $K=(k_1,\ldots,k_n)$ runs over all partitions of k, and $t(K)=(n-1)k-\sum_{j=2}^n(j-1)k_j$. Furthermore, the map

$$i(k,n,m)_*\colon H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m};A)\longrightarrow H_*(\Omega^2_k(\mathbb{G}_{n,n+m});A)$$

is an injection in homology for all coefficients.

The image of $i(k,n,m)$ is the subgroup which, in an explicit (multiplicative) filtration of $H_*(\Omega^2_*(\mathbb{G}_{n,n+m});A)$ consists of all elements having filtration degree $\leq k$. We will explain the geometry behind this filtration more precisely in the body of the text (see §§4–6). It is closely related to the filtration of $\Omega U(n)$ studied in [20]. In the process of proving Theorem A we construct a stratification of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ by open manifolds (each with a trivial normal bundle, and each contained in the closure of higher dimensional strata) whose homotopy types we understand. These strata are then organized into a filtration, and the resulting spectral sequence is shown to collapse.

The spaces $\mathrm{Rat}_k(\mathbb{CP}(m))$ become more and more highly connected as m increases until, in the limit, they become contractible. Moreover, the natural inclusions $\mathbb{CP}(m)\subset\mathbb{CP}(m+1)$ induce inclusions of Rat_k spaces, so the process of passing to a limit makes sense geometrically. Consequently, in the limit over m,

$$\Sigma^{2t(K)}\left(\mathrm{Rat}_{k_1}(\mathbb{CP}(m))\times\cdots\times\mathrm{Rat}_{k_n}(\mathbb{CP}(m))\right)_+ \simeq \Sigma^{2t(K)}(S^0)=S^{2t(K)},$$

and the space $\lim_{m\mapsto\infty}\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ has the same homology groups as a wedge of even-dimensional spheres.

There are mappings $\phi_{n,m}\colon\Omega^2(\mathbb{G}_{n,n+m})\longrightarrow\Omega(U(n))$, which, in the limit over m, become a homotopy equivalence. The composite maps

$$\mathrm{Rat}_k(\mathbb{G}_{n,n+m})\longrightarrow\Omega^2(\mathbb{G}_{n,n+m})\longrightarrow\Omega(U(n))$$

now fit together, taking limits over both m and k to define the loop group, and the image of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$, under this composite, i.e. $\lim_{m\mapsto\infty}\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is, in homology, precisely Mitchell's k^{th} filtering space for $\Omega U(n)$ [20] (see Remark 6.11).

In general, Theorem A shows that the homology of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is "tame" or taut inside of $\Omega^2(\mathbb{G}_{n,n+m})$ in that $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ very efficiently builds $\Omega_k^2(\mathbb{G}_{n,n+m})$ as k increases with classes appearing "exactly when they should". This statement is made precise in the filtration arguments of §6.

These spaces of holomorphic maps are of interest in linear control theory because $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is exactly the moduli space of (McMillan) degree[1] k observable and controllable solutions of the linear control equations with n inputs and m outputs,

$$\begin{aligned} \dot{X} &= AX + BU, \\ Y &= CX. \end{aligned} \tag{1.2}$$

In fact it was this connection which was the original impetus for the current work.

Remark. The full control equations have the form

$$\begin{aligned} \dot{X} &= AX + BU, \\ Y &= CX + DU. \end{aligned}$$

The extra term D just serves to make the base point wander over the interior of the Schubert cell which contains the original base point $\mathbb{C}^n \times \vec{0}$ in $\mathbb{G}_{n,n+m}$. (See Remark 3.4 for a description of these cells.) Hence, the full moduli space is a fibration over a contractible space with fiber the based rational maps $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$, and consequently is homeomorphic to the product $\mathbb{C}^{nm} \times \mathrm{Rat}_k(\mathbb{G}_{n,n+m})$.

Preliminary results on the topology of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ include [3], [4], [9], [11], and, of course, [22] and [13]. The more computational results cited seem to concentrate on the torsion-free part of $H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}); \mathbb{Z})$ for very small values of k with respect to m and n; however, Theorem A shows that in general $H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}); \mathbb{Z})$ is almost exclusively torsion. Moreover, one cannot ignore torsion classes. If we are interpreting the control theory literature correctly, then the measure of complexity of the moduli spaces above which interests control theorists is (roughly) the number of open cells necessary to construct the Rats. However, a torsion-free class needs only one cell, but each torsion class requires at least two cells.

Of course, it is the entire geometry of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ that primarily interests the control theorists, and not just the homology as computed in Theorem A. In the body of this paper (§§3–5) we show that the geometry of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is determined by a series of open submanifolds, each

[1] The McMillan degree of a rational map $f\colon S^2 \longrightarrow \mathbb{G}_{n,n+m}$ is exactly the absolute value of the Chern number $|\langle c_1(\zeta), [S^2]\rangle|$ where $\zeta = f^*(\gamma_n)$, γ_n is the canonical n-plane bundle over $\mathbb{G}_{n,n+m}$, and $c_1(\zeta)$ is the first Chern class.

given explicitly in terms of fixing a pattern of canonical forms for a matrix in the control theorist's resolution of the Laplace transform of the control equations (1.2). We show that each of these submanifolds is naturally an iterate fibration, with each fiber a copy of $\mathrm{Rat}_{k_i}(\mathbb{CP}(m))$, the k_i being determined by the degrees of the diagonal elements in the canonical form. To really understand the geometry of the moduli space, one must understand how these submanifolds fit together, and this we do by describing the normal bundle of each manifold in an appropriate filtration. Each of these submanifolds is open and does not contribute directly to the homology of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$; rather, the Thom spaces of the normal bundles to these submanifolds carry the homology classes given in Theorem A. Of course, if one compactifies $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$, then Poincaré duality implies that the open manifolds themselves do contribute to the homology (the analog for the case when $n = 1$ is made explicit in [6, §§5 and 6]). Finally, we believe that the geometry of the normal bundles described here (see §4) should be useful in a further understanding of certain geometric questions in control theory, such as stability of a system.

This paper is organized as follows: in §2 we review preliminary material, including the topology of symmetric products, $\mathrm{Rat}_k(\mathbb{CP}(m))$ and $\Omega^2_k(\mathbb{G}_{n,n+m})$. §3 then explains the normal forms which we use to begin to decompose $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$. In §§4 and 5 we filter $\mathrm{Rat}(\mathbb{G}_{n,n+m})$ into strata which we are able to analyze using the fundamental results of [6]. Finally, in §6 we reassemble the strata of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ to complete the proof of Theorem A.

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2. Topological preliminaries

In this section we recall known results on iterated loop spaces and symmetric products, as well as the homology of $\Omega^2(\mathbb{G}_{n,n+m})$ and $\mathrm{Rat}_k(\mathbb{CP}^n)$ which we will need in later sections to prove Theorem A. For further details we encourage the reader to consult §§2–6 of [6].

The space of all unordered k-tuples of points in a CW complex X is called the k-fold symmetric product and is denoted $SP^k(X)$. $SP^k(X)$ inherits the quotient (compactly generated) topology from the natural projection $X^k \xrightarrow{\pi} SP^k(X)$, and there is a commutative pairing $SP^k(X) \times SP^l(X) \to SP^{k+l}(X)$. Once a base point $* \in X$ is chosen, the direct limit

$SP^{\infty}(X, *)$ is defined and easily seen to be a free commutative, associative H-space with unit $*$. (Here the inclusion $SP^{i}(X) \subset SP^{i+1}(X)$ is given by $\langle x_1 \cdots x_i \rangle \mapsto \langle x_1 \cdots x_i * \rangle$.) The homology structure of $SP^{k}(X)$ is well known [10], [24], [15] and there is a strong duality between the topology of $SP^{\infty}(X)$ and the topology of iterated loop spaces [16], [17]. In fact, $\mathrm{Rat}_k(\mathbb{CP}(m))$ embeds as an open submanifold in $(SP^{k}(S^2))^{m+1}$ and it was this duality that was exploited in [6] to compute the stable homotopy type of $\mathrm{Rat}_k(\mathbb{CP}(m))$. To state the final result we need to recall the May-Milgram model $J_2(X)$ for $\Omega^2\Sigma^2 X$ [14], [17].

Definition 2.1.

$$J_2(S^{2n-1}, *) = \coprod_{j=1}^{\infty} \mathbb{F}(\mathbb{C}, j) \times_{\Sigma_j} (S^{2n-1})^j / \{equivalence\},$$

where

(a) $\mathbb{F}(\mathbb{C}, j)$ is the set of all j-tuples of distinct points in $\mathbb{C}$ with the natural free Σ_j action. The quotient $\mathbb{F}(\mathbb{C}, j)/\Sigma_j = DP(\mathbb{C})$ is called the deleted symmetric product.

(b) The equivalence relation is given by

$$(z_1, \cdots, z_j, \phi_1, \ldots \phi_j) \sim (z_1, \cdots, \hat{z}_l, \cdots, z_j, \phi_1, \cdots, \hat{\phi}_l, \cdots, \phi_j)$$

exactly when $\phi_l = *$, the base point in S^{2n-1}.

$J_2(S^{2n-1}, *)$ is homotopy equivalent to $\Omega^2\Sigma^2 S^{2n-1} = \Omega^2 S^{2n+1}$ [14], [17], and the Snaith splitting theorem shows that stably $\Omega^2 S^{2n+1}$ splits into a wedge product

Theorem 2.2 [23].

$$J_2(S^{2n-1}) \simeq_s \bigvee_{j=0}^{\infty} \mathbb{F}(\mathbb{C}, j)_+ \wedge_{\Sigma_j} (S^{2n-1})^{(j)} \simeq_s \bigvee_{j=0}^{\infty} D_j(S^{2n-1}).$$

We write D_j for $D_j(S^1)$ and recall

Theorem 2.3 [8].

$$D_j(S^{2n-1}) \simeq \Sigma^{(2n-2)j} D_j(S^1) = \Sigma^{(2n-2)j} D_j.$$

Remark. D_j is the Thom space of the vector bundle $\gamma_j\colon \mathbb{F}(\mathbb{C}, j) \times_{\Sigma_j} \mathbb{R}^j \to DP^j(\mathbb{C})$, and $DP^j(\mathbb{C}) \simeq K(\beta_j, 1)$ the Eilenberg-MacLane space for Artin's braid group on k-strings.

This is tied into the structure of the Rat_k spaces, because using the duality hinted at above one can show

Theorem 2.4 [6].

$$\mathrm{Rat}_k(\mathbb{CP}(m)) \simeq_s \bigvee_{j=1}^{k} \Sigma^{(2m-2)j} D_j.$$

Thus, we have stable decompositions of $\Omega^2 S^{2n+1}$ in terms of suspension of the D_j's and can read off $H_*(\Omega^2 S^{2n+1}; A)$ from the known homology of D_j [17], and this gives the homology structure of the first Rat_k spaces as well. To explicitly describe the homology of the D_j's we examine the Snaith splitting 2.2 for $J_2(S^1)$ more closely. First recall the definition of a multi-graded algebra over the field $\mathbb{F}$.

Definition 2.5. Let $\mathcal{I}$ be commutative monoid with unit, then the $\mathbb{F}$-algebra A is an $\mathcal{I}$-graded algebra if $A = \coprod_{I \in \mathcal{I}} A_I$ as an additive group, and $A_I \cdot A_J \subset A_{I+J}$.

Also, if $\phi \colon \mathcal{I} \to \mathbb{Z}$ is a monoid homomorphism, then A is said to be ϕ-commutative if $a_J \cdot a_I = (-1)^{\phi(I)\phi(J)} a_I \cdot a_J$. An $\mathcal{I}$-graded $\mathbb{F}$-algebra A is said to be free ϕ-commutative if it is the tensor product of a polynomial algebra on generators b_I with $\phi(b_I)$ even, and an exterior algebra on generators e_J with $\phi(e_J)$ odd, provided the characteristic of $\mathbb{F}$ is either 0 or odd. If the characteristic is 2, then A is simply the polynomial algebra on the stated generators. The number $\phi(e)$ is usually called the dimension of e.

The Snaith splitting implies we may write

$$H_*(J_2(S^1); \mathbb{F}) \cong \coprod_1^{\infty} H_*(J_2^j(S^1), J_2^{j-1}(S^1); \mathbb{F}), \tag{2.6}$$

and that

$$H_*(D_j; \mathbb{F}) \cong H_*(J_2^j(S^1), J_2^{j-1}(S^1); \mathbb{F}). \tag{2.7}$$

(2.6) makes $H_*(J_2(S^1); \mathbb{F})$ into an algebra over the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ where the first integer indexes the homological dimension and the second integer indexes the splitting filtration.

The calculation for the homology of $J_2(S^1)$ (as a bigraded algebra over $\mathbb{Z}^+ \times \mathbb{Z}^+$) is given in [17] where it is shown to be the free commutative algebra on the following generators:

$$H_*(J_2(S^1); \mathbb{Z}/2) = \mathbb{Z}/2[e_{(1,1)}, q_{(3,2)}, \cdots, q_{(2^{i+1}-1, 2^i)}, \dots] \tag{2.8}$$

for $p = 2$, while at odd primes

$$\begin{aligned} H_*(J_2(S^1); \mathbb{Z}/p) = E[&e_{(1,1)}, \cdots, e_{(2p^i-1, p^i)}, \dots] \\ &\otimes \mathbb{Z}/p[q_{(2p-2,p)}, \cdots, q_{(2p^i-2, p^i)}, \dots]. \end{aligned} \tag{2.9}$$

Here E means exterior algebra and $q_{(2p^i-2,p^i)}$ is the homology Bochstein of $e_{(2p^i-1,p^i)}$.

Hence, the mod p homology groups of the spaces D_j are given explicitly by the subgroups of (2.8) and (2.9) consisting of elements with second grading degree exactly j.

Finally, as Kirwan [13] has shown

$$i(k, n, m)\colon \mathrm{Rat}_k(\mathbb{G}_{n,n+m}) \to \Omega^2_k(\mathbb{G}_{n,n+m})$$

is an equivalence through a range which grows with k, it is useful to recall the structure of $H_*(\Omega^2_k(\mathbb{G}_{n,n+m}); A)$. As all the components of $\Omega^2\mathbb{G}_{n,n+m}$ are homotopy equivalent it suffices to consider the zero component.

Recall there is a fibration $V_{n,n+m} \to \mathbb{G}_{n,n+m} \to BU(n)$ which generalizes the Hopf fibration $S^{2i+1} \to \mathbb{CP}(i) \to BS^1$. Here $V_{n,n+m} = \dfrac{U(n+m)}{U(m)}$ is the Stiefel manifold of n-frames in $\mathbb{C}^{n+m}$. Looping down twice one obtains

$$\Omega^2(V_{n,n+m}) \longrightarrow \Omega^2_0(\mathbb{G}_{n,n+m}) \longrightarrow \Omega^2(BU(n)) \simeq \Omega_0(U(n)). \tag{2.10}$$

Lemma 2.11. *For all coefficients A and $n \le m$,*

$$\begin{aligned} &H_*(\Omega^2(V_{n,n+m}); A) \\ &\quad \cong H_*\left(\Omega^2(S^{2m+1}) \times \cdots \times \Omega^2(S^{2(m+n)-1}); A\right), \end{aligned} \tag{2.12}$$

$$H_*(\Omega_0(U(n)); A) \cong H_*(\Omega(S^3) \times \cdots \times \Omega(S^{2n-1}); A). \tag{2.13}$$

Proof. (2.13) is direct from the well-known theorem of Bott-Samelson that the loop space of a Lie group has torsion-free homology. However, (2.12) is not quite so clear. The point is that there is a well-known embedding $\mu\colon \Sigma[\mathbb{CP}(n+m-1)/\mathbb{CP}(m-1)] \subset V_{n,n+m}$ [18], [25], and $H^*(V_{n,n+m}; \mathbb{Z})$ is an exterior algebra, generated by classes $e_{2m+1}, \cdots, e_{2m+2n-1}$ with $\mu^*(e_{2m+2j-1})$ equal to the corresponding generator in

$$H^{2m+3j-1}(\Sigma[\mathbb{CP}(n+m-1)/\mathbb{CP}(m-1)]; \mathbb{Z}).$$

But $\Sigma[\mathbb{CP}(n+m-1)/\mathbb{CP}(m-1)]$ is actually a 2-fold suspension since $n \le m$, so there is a space Y with $\Sigma^2 Y = \Sigma[\mathbb{CP}(n+m-1)/\mathbb{CP}(m-1)]$. From this it follows that the homology duals of the cohomology classes above are in the image of the double suspension map

$$\sigma^2\colon H_*(\Omega^2 V_{n,n+m}; \mathbb{Z}) \to H_{*+2}(V_{n,n+m}; \mathbb{Z}).$$

It follows from this that the Serre spectral sequence of the fibering

$$\Omega^2(V_{n,n+m-1}) \to \Omega^2(V_{n,n+m}) \to \Omega^2(S^{2n+2m-1})$$

collapses. Hence, $E^2 = E^\infty$, and (2.12) follows by a simple induction. q.e.d.

Finally, we have

Theorem 2.14. *For all coefficients* A *and* $n \le m$,

$$H_*(\Omega_0^2(\mathbb{G}_{n,n+m}); A) \cong H_*(\Omega^2 V_{n,n+m}) \otimes H_*(\Omega_0 U(n); A).$$

Proof. The homology Serre spectral sequence for (2.5) is multiplicative where the base is a polynomial algebra on generators $x_2, \ldots, x_{2n-2}$ of dimension $2, \ldots, 2n-2$. Since $n \le m$, the lowest dimensional class on the fiber is in dimension $2m-1$. Hence the generators on the base are infinite cycles and the spectral sequence must collapse.

Corollary 2.15. *For all coefficients* A *and* $n \le m$, *the inclusion*

$$\Omega_0^2(\mathbb{G}_{n-1,n-1+m}) \hookrightarrow \Omega_0^2(\mathbb{G}_{n,n+m})$$

induces an injection in homology.

Proof. There is an inclusion of the fibration sequence (2.10) for $n-1$ into the fibration sequence for n. The corollary then follows from (2.14) and the observation that the base and fiber both inject in homology.

3. Normal forms

In this section we establish a normal form for elements of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$. While such normal forms abound in control theory (e.g. [12], [21]), our choice is governed by considerations of global geometry, and hence is slightly nonstandard.

Recall from [12] that the holomorphic maps $f: S^2 \to \mathbb{G}_{n,n+m}$ of Chern degree $-k$ arise from the matrices of rational functions (the so-called transfer matrices)

$$T(z) = C(zI - A)^{-1}B \in \mathrm{Mat}_{n\times m}(\mathbb{C}(z)), \tag{3.1}$$

where $T(z)$ is the Laplace transform of the first-order linear differential system (1.2), and

(a) A is $k \times k$.

(b) The system is observable and controllable; that is, (A, B, C) is a minimal realization [12, p. 363].

(c) $\lim_{z\to\infty} T(z) = 0$; that is, T is strictly proper [12, p. 382].

It follows ([12, 6.2.3, pp. 367–371, and 6.5, pp. 439–469]) that we may write

$$T(z) = D^{-1}(z)N(z), \tag{3.2}$$

where

(a) $N(z) \in \mathrm{Mat}_{n,m}(\mathbb{C}[z])$ and $D(z) \in \mathrm{Mat}_{n,n}(\mathbb{C}[z])$ are both polynomial matrices. (As is standard we denote by $\mathbb{C}[z]$ the ring of polynomials in the field of rational functions $\mathbb{C}(z)$.)

(b) $N(z)$ and $D(z)$ are relatively prime; that is, there are matrices $A(z) \in \mathrm{Mat}_{n,n}(\mathbb{C}[z])$, $B(z) \in \mathrm{Mat}_{m,n}(\mathbb{C}[z])$ so that $N(z)A(z) + D(z)B(z) = I \in \mathrm{Mat}_{n,n}(\mathbb{C}(z))$. It follows that, for every z, the rectangular matrix of $n+m$-vectors $[D(z), N(z)]$ has rank n, and hence defines a unique n-plane in $\mathbb{C}^{n+m}$, namely the span of the row vectors. This is a point in the Grassmannian, and consequently we have defined a map $f_T: S^2 \longrightarrow \mathbb{G}_{n,n+m}$ given by

$$f_T(u) = [D, N](u). \tag{3.3}$$

(c) The condition that the total Chern class $c(f) = -k$; equivalently, that the McMillan degree is k, is given by degree $(\det D(z)) = k$.

As $z \longrightarrow \infty$, $T(z) = D^{-1}(z)N(z) \longrightarrow 0$ and therefore $T(z)$ has no poles in a neighborhood of ∞. Thus, $T(z) \cong [I, D^{-1}N] = [I, T(z)]$, and $\lim_{z\to\infty} f_T(z) = [I, 0] \in \mathbb{G}_{n,n+m}$, our chosen base point.

Remark 3.4. There is a standard decomposition of $\mathbb{G}_{n,n+m}$ into Schubert cells defined for example in [19]. If we fix a basis for $\mathbb{C}^{n+m}$ and look at an n-plane $W \subset \mathbb{C}^{n+m}$, there is a unique nonzero vector $w_1 \in W$ with its first $j_1 - 1$ coordinates vanishing, where j_1 is maximal and its j_1^{st} coordinate $w_{j_1} = 1$. Next, among the remaining vectors of W consider those w which satisfy $w_{j_1} = 0$, but which are otherwise nonzero. Among these there is a unique one, w_2, with its first $j_2 - 1$ coordinates vanishing where j_2 is maximal and $w_{j_2} = 1$. It is clearly independent of w_1. The next set to look at is those vectors in the previous set with their j_2^{st} coordinate equal to zero, and we continue the process obtaining a unique basis for W. This produces an $n \times n+m$ matrix in reduced row echelon form and pattern $m+n \geq j_1 > j_2 > \cdots > j_n \geq 1$ when we look at the coordinates of these basis vectors. Fixing the pattern and varying the terms other than the leading coefficients in each row (always 1), except, leaving the 0's in the column below each leading 1 alone, we obtain an open cell in the Grassmannian. The closures of these cells are the Schubert cycles.

We are permitted to change the representation of $f_T \in \mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ given in (3.3) by multiplying f_T on the left by unimodular polynomial matrices. A unimodular matrix U is a polynomial valued matrix such that $\det(U) = \text{constant} \neq 0$. For example, let r_1, r_2, s_1, s_2 be polynomials, $g = g.c.d(r_1, s_1)$, and $d = \det\begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix}$. There exist polynomials a, b, such that $ar_1 + bs_1 = g$. Then $U = \begin{pmatrix} a & b \\ -s_1/g & r_1/g \end{pmatrix}$ is unimodular and $\begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix}$, when multiplied on the left by U, becomes $\begin{pmatrix} g & ar_2 + bs_2 \\ 0 & d/g \end{pmatrix}$. A further unimodular matrix of the form $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ can now be used to reduce $\begin{pmatrix} r_1 & r_2 \\ s_1 & s_2 \end{pmatrix}$ to the form $\begin{pmatrix} g & t \\ 0 & d/g \end{pmatrix}$ where degree (t) $<$ degree (d/g). In general, by multiplying on the left by a unitary matrix U we can bring f_T to the following normal form:

$$
\begin{aligned}
U \circ f_T &= U[D, N] \\
&= [P, Q] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} & q_{11} & \cdots & q_{1m} \\ 0 & p_{22} & \cdots & p_{2n} & q_{21} & \cdots & q_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{nn} & q_{n1} & \cdots & q_{nm} \end{bmatrix},
\end{aligned} \tag{3.5}
$$

where

(a) P is upper triangular with monic diagonal terms;
(b) degree (p_{ij}) $<$ degree $(p_{ii}) = k_i$ for $i < j$;
(c) $\sum_{i=1}^{n} k_i = c(f) = k$;
(d) degree (q_{nj}) $<$ degree (p_{nn}).

Conditions (a), (b), and (c) follow from applying U to D. Condition (d) follows from the fact that $(UD)^{-1}UN = D^{-1}N$, so, whatever the unitary matrix U, the resulting rational matrix $(UD)^{-1}UN$ must go to 0 as $z \to \infty$. In general, this means that each term t_{ij} may be written as $r_{ij}(z)/s_{ij}(z)$ for appropriate coprime polynomials $r_{ij}(z)$ and $s_{ij}(z)$, which must satisfy degree (r_{ij}) $<$ degree (s_{ij}). Hence, the relations between the remaining q_{ij} and the p_{tw}, q_{lj} for $l > i$, while somewhat involved, are inductively determined.

It is instructive to work out these conditions to understand the meaning of Clark's proof [5] that the complex dimension of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m}) = (n+m)k$. It follows directly from (3.5a), (3.5b) and (3.5c), that, for fixed degree k, the i^{th} column of the upper triangular matrix P has ik_i degrees of freedom in the coefficients of the p_{ij} polynomials. Furthermore, each of the m columns of the Q matrix then contributes precisely k

more degrees of freedom as the p_{ij} and $q_{i+h,j}$ terms uniquely fix the higher order entries in q_{ij} modulo the lowest k_i coefficients.

Explicitly, assume $[P, Q]$ given in 3.5 is in normal form. The inverse of P is given by

$$(3.6)\qquad P^{-1} = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ 0 & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{nn} \end{bmatrix},$$

where

(a) $e_{ii} = p_{ii}^{-1}$.

(b) $e_{i,i+1} = p_{ii}^{-1} p_{i+1,i+1}^{-1} p_{i,i+1} = e_{ii} e_{i+1,i+1} p_{i,i+1}$.

(c) In general, for $i < j$,

$$\begin{aligned} e_{ij} &= p_{ii}^{-1} p_{jj}^{-1} p_{ij} \pm \ldots \pm p_{ii}^{-1} p_{i+1,i+1}^{-1} \cdots p_{jj}^{-1} p_{i,i+1} p_{i+1,i+2} \cdots p_{j-1,j} \\ &= e_{ii} e_{jj} p_{ij} \pm \ldots \pm e_{ii} e_{i+1,i+1} \cdots e_{jj} p_{i,i+1} p_{i+1,i+2} \cdots p_{j-1,j}, \end{aligned}$$

where the intermediate terms all involve one more e_{ll} term than mixed term $p_{l,h}$, and only the last term contains the maximal number of occurrences $(j - i + 1)$ of e_{ll} type terms.

Now, applying the condition that $P^{-1}Q$ goes to zero as $z \to \infty$ we obtain, for all $1 \le j \le m$,

$$(3.8)\qquad e_{nn} q_{nj} = p_{nn}^{-1} q_{nj} \to 0$$

as $z \to \infty$ and thus q_{nj} has degree less than k_n. This is precisely condition (3.5.d).

Next we have

$$(3.9)\qquad e_{n-1,n-1} q_{n-1,j} - e_{nn} e_{n-1,n-1} p_{n-1,n} q_{nj} \to 0$$

as $z \to \infty$. Observe that every term in this equation except $q_{n-1,j}$ is determined in our previous calculations (3.5), (3.6) or (3.8). We do not claim that degree$(q_{n-1,j}) < k_{n-1}$; in fact, in general it is not. However, and this is the key point, the limiting behavior as z tends to infinity and the inductive knowledge of all the remaining terms, (3.5), (3.6), (3.8) and (3.9), imply that $q_{n-1,j}$ is completely determined modulo a polynomial of degree less than k_{n-1}.

In the general case for $1 \le i \le n$ we have

$$(3.10)\qquad \begin{aligned} & e_{i,i} q_{i,j} - e_{nn} e_{n-1,n-1} p_{n-1,n} q_{nj} \\ & \qquad + \ldots \pm e_{ii} e_{i+1,i+1} \cdots e_{nn} p_{i,i+1} p_{i+1,i+2} \cdots p_{n-1,n} q_{nj} \end{aligned}$$

tends to zero as $z \longrightarrow \infty$. Again observe that every polynomial in (3.10) except the $q_{i,j}$ polynomial appearing in the lowest term is determined by the previous calculations (3.5), (3.6), (3.8), (3.9) and iterates of (3.10) for larger values of i). Once more, although degree $(q_{i,j})$ is not, in general, less than k_i, the limiting behavior and the inductive knowledge of all the remaining terms imply that q_{ij} is completely determined modulo a polynomial of degree less than k_i.

The freedom to choose the lowest k_i coefficients in each q_{ij} as derived in equations (3.7), (3.8), (3.9) and (3.10) thus fill out the local manifold coordinates as desired. We shall refer to this choice of coordinates as the "normal" coordinates for $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$.

4. The Decomposition of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$

We now use the normal form constructed in the last section in order to decompose $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ into smooth strata. Let $K = (k_1, \cdots, k_n)$ be a partition of k. That is, each $k_j \geq 0$ and $\sum_{j=1}^{n} k_j = k$. We lexicographically order the partitions of k by setting $L = (l_1, \cdots, l_n) < K = (k_1, \cdots, k_n)$ if $l_n < k_n$ or, if $l_i = k_i$ for $i > j$ then $l_j < k_j$.

Definition 4.1. For each partition of k let

$$X(k_1, \cdots, k_n) = \left\{ f_T \in \mathrm{Rat}_k(\mathbb{G}_{n,n+m}) \;\middle|\; \begin{array}{l} \text{The normal form of } T = [P, Q] \\ \text{satisfies degree}(p_{ii}) = k_i \end{array} \right\}.$$

Proposition 4.2. *$X(K) = X(k_1, \cdots, k_n)$ is a complex submanifold of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ of complex dimension $(m+1)k + \sum_{j=2}^{n}(j-1)k_j$.*

Proof. The local coordinates are determined by the free coefficients of the polynomials in the normal form described in §3. The degrees of freedom for these coefficients are also given at the end of §3 where it is observed that each of the m columns of Q contributes k complex dimensions and each column of the P matrix contributes another ik_i. That $X(K)$ is a regularly embedded submanifold now follows from the proof of Proposition 4.4 below. q.e.d.

The only stratum of complex dimension $k(n+m)$ is $X(0, \cdots, 0, k)$, and, as the last coefficient decreases, the dimension gets smaller. Moreover, each of $X(0, \cdots, 1, 0, \cdots, 0, k-1)$ with the 1 in the i^{th} position is in the closure of $X(0, \cdots, 0, k)$. Similarly, each of $X(0, \cdots, 2, 0, \cdots, 0, k-2)$ with the 2 in the i^{th} position, as well as

$$X(0, \cdots, 1, 0, \cdots, 1, 0, \cdots, k-2)$$

with 1's in the i and j position is in the closure of $X(0, \cdots, 1, 0, \cdots, 0, k-1)$ with the 1 in the i^{th} position. There are similar closure properties as the partitions become more complex.

In fact, associated to k, there is a filtration of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ by open manifolds,

$$\mathbb{F}_r\left[\mathrm{Rat}_k(\mathbb{G}_{n,n+m})\right] = \bigcup_{k_n \geq r} X(k_1, \cdots, k_n).$$

To reassemble $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ from this stratification we will need to know the structure of the natural inclusions

$$(4.3) \qquad i(k_1, \cdots, k_n)\colon X(k_1, \cdots, k_n) \hookrightarrow \mathbb{F}_{k_n}\left[\mathrm{Rat}_k(\mathbb{G}_{n,n+m})\right]$$

or, more precisely, the normal bundle of $X(k_1, \cdots, k_n)$ in $\mathbb{F}_{k_n}\left[\mathrm{Rat}_k(\mathbb{G}_{n,n+m})\right]$.

Proposition 4.4. *The normal bundle $\nu(i(k_1, \cdots, k_n))$ is trivial.*

Proof. We begin by considering normal coordinates. Let $T \in X(k_1, \cdots, k_n)$ be given by

$$(4.5) \qquad T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} & q_{11} & \cdots & q_{1m} \\ 0 & p_{22} & \cdots & p_{2n} & q_{21} & \cdots & q_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{nn} & q_{n1} & \cdots & q_{nm} \end{bmatrix}.$$

To compute the normal bundle we must infinitesimally perturb

$$T \mapsto T + \epsilon S \in \mathrm{Rat}_k(\mathbb{G}_{n,n+m})$$

and analyze the first order effect on the tangent level. However, if one is not sufficiently careful when choosing S, one may leave the degree k component (which is given by the region where the $n \times n$ minors all have degree $< k$ except for the minor P itself). The proposition actually follows from the fact that we can vary the coordinates of P by small polynomials in the lower triangular region, taking care to keep the degrees sufficiently under control that $T + \epsilon S$ remains of degree exactly k, and, having done that, checking that all the coordinates in the new Q are uniquely determined, modulo the tangent direction to $X(K)$, by this procedure.

More precisely, let $S = (L, R)$ where $L = (l_{ij})$ is an n by n strictly lower triangular polynomial matrix ($l_{ij} \neq 0$ only if $i > j$) and R is an m by n polynomial matrix. Generically, if degree $(l_{ij}) >$ degree $(p_{jj}) + 1$ for some $j < i$ then the resulting determinant of $T + \epsilon S$ has degree $> k$. Consequently, if we are to stay in the tangent space to $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ we must generically have degree $(l_{ij}) \leq$ degree $(p_{jj}) + 1$ for all $j < i$. In fact, more is true and we shall see that every perturbation by S agrees, modulo tangents to $X(K)$, to a perturbation given by (L, R) where degree (l_{ij}) $<$ degree (p_{jj}).

Recall, as $z \longrightarrow \infty$, we must have $T_\epsilon(z) = [P + \epsilon L]^{-1}(z)[Q + \epsilon R](z) \longrightarrow 0$ in order to stay in $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ as we vary ϵ. Computing $\frac{dT_\epsilon}{d\epsilon}$ at $\epsilon = 0$ we obtain the relation

$$P^{-1}(LP^{-1}Q - R) \longrightarrow 0 \tag{4.6}$$

as $z \longrightarrow \infty$. Of course, if $L = 0$ and $P^{-1}R \longrightarrow 0$ as $z \longrightarrow \infty$ we remain in the tangent direction to $X(K)$ and $[0, R]$ satisfies (4.6).

First consider the case when L satisfies

$$\mathrm{degree}(l_{ij}) < \mathrm{degree}(p_{jj}) \tag{4.7}$$

for all $i > j$. In this case an inductive calculation similar to the one at the end of §3 shows there exists an R depending on P, Q, and L satisfying (4.6). Notice that if R_1 and R_2 both satisfy (4.6) for the same P, Q and L, then $P^{-1}(R_1 - R_2) \longrightarrow 0$ as $z \longrightarrow \infty$ and thus R is unique up to changes in the directions tangent to $X(K)$. Therefore, for each such L we have an element of

$$T(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}))/T(X(K)) \cong \nu(i(k_1, \cdots, k_n)). \tag{4.8}$$

Next, we consider L's that do not satisfy (4.7) but still satisfy the critical condition that the degree of the perturbed determinant is k. The simplest case not satisfying (4.7) occurs when $\mathrm{degree}(l_{ij}) \leq \mathrm{degree}(p_{jj})$ for all $i > j$ and $\mathrm{degree}(l_{ab}) = \mathrm{degree}(p_{bb})$ for some a and b. But then we can use a unimodular matrix U to write $T + \epsilon(L, R)$ as

$$T + \epsilon(L_1 + V_1, R_1), \tag{4.9}$$

where

(a) L_1 is strictly lower triangular and satisfies 4.7.
(b) V_1 is upper triangular with $\mathrm{degree}(v_{ij}) < \mathrm{degree}(p_{ij})$ for all i, j.
(c) R_1 is the m by n matrix obtained from R by applying U.

Since transformation by unimodular matrices preserves our determinantal conditions we have that $T + \epsilon(L_1 + V_1, R_1)$ is a perturbation in $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ if and only if $T + \epsilon(L, R)$ is also. But on the tangent space level the tangent vector v_1 associated to the perturbation (4.9) is the sum of the tangent vectors associated to the following two perturbations:

(1) $T + \epsilon(L_1, R_1)$ with tangent vector v_2.

(2) $T + \epsilon(V_1, 0)$ with tangent vector v_3 which is manifestly tangent to $X(K)$.

Thus, $v_1 = v_2 + v_3$ and v_1 and v_2 have the same image in the normal bundle (4.8). In the same manner, by induction, we may reduce any perturbation of T by (L, R), which stays in $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ with tangent vector v_1, to a perturbation of the form $(L_1 + V_1, R_1)$ where L_1 satisfies (4.7) with tangent vector v_2. Again, by induction, the deviation terms will construct a tangent vector v_3, which is actually tangent to $X(K)$, so that $v_1 = v_2 + v_3$. Thus, in general, v_1 and v_2 have the same image in the normal bundle (4.8).

Finally, if L is not identically zero and satisfies (4.7) then the Euclidean algorithm shows that $T + \epsilon(L, R)$ lies in higher strata for all $\epsilon \neq 0$. Furthermore, the assignment $L \mapsto \frac{dT_\epsilon}{d\epsilon}$ in (4.8) is linear. These facts, along with a simple dimension count and the reduction argument described above, imply that the strictly lower triangular matrices L, which satisfy (4.7), precisely fill out the normal directions. Hence, the coefficients of all L satisfying (4.7) give a trivialization of the normal bundle as required. q.e.d.

As noted above in the proof of (4.4), Euclidean algorithm implies that when $T+\epsilon S$ is written in normal form $[\hat{D}, \hat{N}]$, we must have degree $(\hat{d}_{ii})$ $>$ degree (p_{ii}) for the largest value of i where the degrees of the diagonal entries disagree. This implies that the normal directions for $X(K)$ in $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ lie in the union of the higher dimensional strata $X(L)$ where $K < L$ in the lexicographic ordering given above. We shall make use of this observation in §6.

5. The Decomposition of $X(k_1, \cdots, k_n)$

We now analyse the geometry of the individual strata $X(k_1, \cdots, k_n)$.

Proposition 5.1. *There is a sequence of fibrations*:

$$\begin{array}{ccc}
\mathrm{Rat}_{k_1}(\mathbb{CP}(m)) & \longrightarrow & X(k_1, \dots, k_n) \\
& & \downarrow \pi_1 \\
\mathrm{Rat}_{k_2}(\mathbb{CP}(m)) & \longrightarrow & X(k_2, \dots, k_n) \\
& & \downarrow \pi_2 \\
& & \vdots \\
& & \downarrow \pi_{n-2} \\
\mathrm{Rat}_{k_{n-1}}(\mathbb{CP}(m)) & \longrightarrow & X(k_{n-1}, k_n) \\
& & \downarrow \pi_{n-1} \\
\mathrm{Rat}_{k_n}(\mathbb{CP}(m)) & \xrightarrow{=} & X(k_n).
\end{array} \tag{5.2}$$

Proof. By induction on n. (5.1) is trivially true when $n = 1$ as $X(k_1)$ is precisely $\mathrm{Rat}_{k_1}(\mathbb{G}_{1,1+m}) = \mathrm{Rat}_{k_1}(\mathbb{CP}(m))$. Now assume that Proposition

5.1 holds for

$$
(5.3)\qquad
\begin{array}{ccc}
\mathrm{Rat}_{k_2}(\mathbb{CP}(m)) & \longrightarrow & X(k_2, \dots, k_n) \\
 & & \downarrow \pi_2 \\
 & & \vdots \\
 & & \downarrow \pi_{n-2} \\
\mathrm{Rat}_{k_{n-1}}(\mathbb{CP}(m)) & \longrightarrow & X(k_{n-1}, k_n) \\
 & & \downarrow \pi_{n-1} \\
\mathrm{Rat}_{k_n}(\mathbb{CP}(m)) & \xrightarrow{=} & X(k_n)
\end{array}
$$

and consider $\pi_1 : X(k_1, \dots, k_n) \to X(k_2, \dots, k_n)$. π_1 is defined by writing each element of $X(k_1, \dots, k_n)$ in normal form and forgetting the first row. It is therefore a smooth submersion onto $X(k_2, \dots, k_n)$. Of course, if π_1 were a proper map, then it would immediately follow that π_1 is also a locally trivial fibration. While the fibers are not compact, it is possible to modify $X(k_1, \dots, k_n)$, without changing its homotopy type, so as to verify that π_1 is a fibration. We proceed as follows:

First embed $X(k_1, \dots, k_n)$ as an open submanifold in a product of symmetric products of the two-sphere

$$
(5.4)\qquad X(k_1, \dots, k_n) \hookrightarrow \prod SP^{l_{ab}}(\mathbb{C}) \hookrightarrow \prod SP^{l_{ab}}(S^2).
$$

Here the last space is a closed, compact, smooth manifold [1] (in fact, in this case, a product of complex projective spaces) and the first embedding catalogs the normal manifold coordinates for the elements of $X(k_1, \dots, k_n)$ as described in §3. To see this is an open embedding notice that the composite inclusion exhibits $X(k_1, \dots, k_n)$ as the complement of a closed subspace, namely the union of the "singular set" given by normal coordinates for those holomorphic maps from S^2 into $\mathrm{Mat}_{n,n+m}(\mathbb{C})$ that drop rank at some point and the "set at infinity" where at least one of the entries in the symmetric product is $\infty \in S^2$. The "singular set" is closed because it is the union of a finite number of varieties given by the zeros of various determinants while the "set at infinity" is manifestly closed.

By thickening the closed complement of $X(k_1, \dots, k_n)$ in $\prod SP^{l_{ab}}(S^2)$ we see that $X(k_1, \dots, k_n)$ deformation retracts onto a subspace $(M(k_1, \dots, k_n), \partial M(k_1, \dots, k_n))$ that is a compact manifold pair. Notice that π_1 restricts to a map

$$
(5.5)\qquad
\begin{array}{c}
(M(k_1, \dots, k_n), \partial M(k_1, \dots, k_n)) \\
\downarrow \pi_1 \\
X(k_2, \dots, k_n).
\end{array}
$$

π_1 clearly restricts to a submersion on the interior of $M(k_1, \ldots, k_n)$. Furthermore, as $\partial M(k_1, \ldots, k_n)$ has real codimension 1 in $M(k_1, \ldots, k_n)$ whereas π_1 must drop real rank by at least 2 where it fails to be a submersion, it follows that π_1 restricted to $\partial M(k_1, \ldots, k_n)$ is also a smooth submersion. This implies that (5.5) is a fibration and hence, up to homotopy, our original π_1 is a locally trivial fibration.

It remains to identify the homotopy fiber which can be done by analyzing π_1^{-1} above any point we choose. We work with normal coordinates. First consider the case where k_2 through k_n are all strictly greater than zero and take the point

$$x = [P, Q] = [P_1, I_{n-1}, 0_{n-1, m-n+1}] \tag{5.6}$$

in $X(k_2, \ldots, k_n)$ where

(a) $P = P_1$ is the diagonal $n-1 \times n-1$ matrix

$$\begin{bmatrix} (z-1)^{k_2} & 0 & \cdots & 0 \\ 0 & (z-1)^{k_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (z-1)^{k_n} \end{bmatrix},$$

(b) Q is the block sum of the $n-1 \times n-1$ identity matrix $[I_{n-1}]$ and the $n-1 \times m-n+1$ zero matrix $[0_{n-1, m-n+1}]$.

Let $(p_{11}, p_{12}, \ldots, p_{1n}, q_{11}, \ldots, q_{1m})$ be the first row in $y \in \pi_1^{-1}(x)$. For all $z \in \mathbb{C}$,

$$(p_{11}, s_{12}, \ldots, s_{1n}, 0, \ldots, 0, q_{1n}, \ldots, q_{1m}) \tag{5.7}$$

is a nontrivial line through the origin that does not lie in the plane determined by $x(z)$. Here

$$s_{1j} = p_{1j} - (z-1)^{k_j} q_{1, j-1} \tag{5.8}$$

for $2 \le j \le n$. Furthermore, using the Euclidean algorithm, there are unique r_j's such that degree $(r_j) < k_1$ and $s_{1j} = t_j p_{11} + r_j$. But for these r_j's,

$$(p_{11}, r_2, \ldots, r_n, 0, \ldots, 0, q_{1n}, \ldots, q_{1m}) \tag{5.9}$$

represents a nontrivial line through the origin that does not lie in the plane determined by $x(z)$ for all $z \in \mathbb{C}$ if and only if the same is true for the line given by (5.7). Notice that the space of elements of the form (5.9) satisfying the rank condition is precisely a copy of $\mathrm{Rat}_{k_1}(\mathbb{CP}(m))$ in normal form. Therefore, the embedding

$$\mathrm{Rat}_{k_1}(\mathbb{CP}(m)) \hookrightarrow \pi_1^{-1}(x) \tag{5.10}$$

given by

$$(5.11)\qquad \begin{aligned}&(p_{11}, r_{12}, \dots, r_{1n}, q_{1n}, \dots, q_{1m})\\ &\quad\mapsto (p_{11}, r_{12}, \dots, r_{1n}, 0, \dots, 0, q_{1n}, \dots, q_{1m})\end{aligned}$$

is the desired homotopy equivalence.

Finally, if some $k_j = 0$, then replace the $(j-1)^{st}$ row of x in (5.6) by $(0, \dots, 0, 1, 0, \dots, 0)$, the vector with a 1 in the $(j-1)^{st}$ coordinate and 0 in all the others. The proof then proceeds as before.

Remarks.

(1) It is possible to see directly that $\pi_1^{-1}(x)$ is homotopy equivalent to $\mathrm{Rat}_{k_1}(\mathbb{CP}(m))$ for every x as follows: There is a natural map $\pi_1^{-1}(x) \to SP^{k_1}(\mathbb{C})$ given by projecting onto the roots of p_{11}, the first entry in the first row of $y \in \pi_1^{-1}(x)$. But up to homotopy this is precisely the projection $\mathrm{Rat}_{k_1}(\mathbb{CP}(m)) \to SP^{k_1}(\mathbb{C})$ constructed in [6, §§3 and 5]. One observes that both total spaces are filtered over the same strata $F_r(SP^{k_1}(\mathbb{C}))$ with both fibers homotopy equivalent to $(S^{2m-1})^{k_1-r}$ over $F_r(SP^{k_1}(\mathbb{C})) - F_{r+1}(SP^{k_1}(\mathbb{C}))$ and by induction over the strata one can construct a homotopy equivalence $\pi_1^{-1}(x) \simeq \mathrm{Rat}_{k_1}(\mathbb{CP}(m))$. This fact can be used to give an alternate proof of Proposition (5.1) as π_1 is thus seen to be a smooth submersion with homotopy equivalent simply connected fibers.

(2) The proof of Theorem A given in the next section will also imply that the sequence of fibrations given in (5.2) is homologically a product.

6. The proof of Theorem A

Having decomposed $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ into smooth strata $X(K)$, each having a trivial normal bundle in the total space, we are ready to reassemble $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ and compute its homology. This we do by a spectral sequence comparison argument, using the known results for $H_*(\Omega_k^2(\mathbb{G}_{n,n+m}))$ and the known stable behavior of $H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}))$ given by Kirwan's result.

Recall the lexicographical order on the nonnegative partitions of k given by setting $(l_1, \cdots, l_n) < (k_1, \cdots, k_n)$ if $l_n < k_n$ or if $l_i = k_i$ for $i > j$ and $l_j < k_j$.

Definition 6.1.

$$Y(k_1, \cdots, k_n) = \bigcup_{(l_1, \cdots, l_n) \geq (k_1, \cdots, k_n)} X(l_1, \cdots, l_n).$$

Then the $Y(k_1, \cdots, k_n)$ yield an decreasing filtration of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ where

$$(6.2)\qquad Y_L/Y_{L+1} \cong \Sigma^{2t(L)}(Y_L - Y_{L+1}) \cong \Sigma^{2t(L)}X(L).$$

Here $L = (l_1, \cdots, l_n)$ runs over the partitions of k, $t(L) = (n-1)k - \sum_{j=2}^{n}(j-1)k_j$, $L+1$ is the smallest partition greater than L in the lexicographical order, and we have used the fact that the tubular neighborhood of $X(K)$ in $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ is trivial and lies in $Y(K)$ (recall (4.4)). It will be necessary to distinguish two types of partitions of k; those where $k_n = 0$ and those where $k_n > 0$, called type 1 and type 2, respectively.

The proof of Theorem A now proceeds by induction on n. The case when $n = 1$ is Theorem 1.5 of [6]. There is a spectral sequence associated to filtration (6.2) with E_2-term given by $H_*\left(\Sigma^{2t(L)}X(L)\right)$ which converges to $H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}))$. First notice that the union of strata of type 1 is precisely a copy of $\mathrm{Rat}_k(\mathbb{G}_{n-1,n-1+m})$. It embeds as a submanifold of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ with trivial normal bundle. Hence the E_2 term contains

$$(6.3)\qquad \bigoplus_K H_*(\Sigma^{2t(K)}(\mathrm{Rat}_{k_1}(\mathbb{CP}(m)) \times \ldots \times \mathrm{Rat}_{k_{n-1}}(\mathbb{CP}(m)))_+),$$

where the sum runs over partitions $K = (K', 0)$ of type 1.

Next, let $T\,\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ denote the subset $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ given by the union of the strata indexed by partitions of type 2. Using a collaring technique similar to that in [S] one can construct inclusions

$$(6.4)\qquad T\,\mathrm{Rat}_k(\mathbb{G}_{n,n+m}) \subset T\,\mathrm{Rat}_{k+1}(\mathbb{G}_{n,n+m}) \subset \mathrm{Rat}_{k+1}(\mathbb{G}_{n,n+m})$$

which preserve the filtration above. More precisely, for $x \in X(k_1, \ldots, k_{n-1}, k_n)$ given by

$$(6.5)\qquad x = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} & q_{11} & \cdots & q_{1m} \\ 0 & p_{22} & \cdots & p_{2n} & q_{21} & \cdots & q_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{nn} & q_{n1} & \cdots & q_{nm} \end{bmatrix},$$

then define $x*[1] \in X(k_1, \ldots, k_{n-1}, k_n+1)$ by multiplying the p_{nn} entry by $(z-N)$ and leaving all other entries alone. That is,

$$(6.6)\qquad x*[1] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} & q_{11} & \cdots & q_{1m} \\ 0 & p_{22} & \cdots & p_{2n} & q_{21} & \cdots & q_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (z-N)p_{nn} & q_{n1} & \cdots & q_{nm}. \end{bmatrix},$$

where we need to explain our choice of N. Recall from §5 that there is a compact manifold pair $(M(K), \partial M(K))$ contained, as a deformation retract, in each strata $X(K)$. Proposition 4.4 and the discussion preceding (5.5) imply that there is a compact manifold pair $(M, \partial M)$ contained, as a deformation retract, in $T\operatorname{Rat}_k(\mathbb{G}_{n,n+m})$. Next, consider the set B of $N \in \mathbb{C}$ such that for $x \in (M, \partial M)$ the row rank of $x * [1]$ drops for some point $z \in \mathbb{C}$. Since $(M, \partial M)$ is compact and the rank condition is governed by the vanishing of various determinants it follows that B is bounded. Hence, there is a choice of N outside of B for which the composition $T\operatorname{Rat}_k(\mathbb{G}_{n,n+m}) \to (M, \partial M) \to T\operatorname{Rat}_{k+1}(\mathbb{G}_{n,n+m})$ is the required inclusion.

Remarks.

(1) There is an embedding of $\operatorname{Rat}_k(\mathbb{G}_{n,n+m}) \subset \operatorname{Rat}_{k+1}(\mathbb{G}_{n,n+m})$ given in ([9, pp. 350–352]) which is distinct from ours and does not appear to respect the same filtration.[2]

(2) There is a similar inclusion on the strata of type 1 into strata of the form $K = (K', 1)$. However, the inclusions from the strata indexed by the two types of partitions do not glue together well and so we have used the inductive hypothesis to handle the classes in the image of this first inclusion.

In particular, this implies we have the following composite of inclusions:

$$\begin{array}{ccc} H_*(T\operatorname{Rat}_k(\mathbb{G}_{n,n+m})) & \xrightarrow{j(k,n,m)} & H_*(\operatorname{Rat}_{k+1}(\mathbb{G}_{n,n+m})) \\ & & \downarrow i(k+1,n,m) \\ & & H_*(\Omega^2_{k+1}(\mathbb{G}_{n,n+m})). \end{array} \tag{6.7}$$

By Kirwan's theorem the vertical map is an equivalence through a range that increases with k.

Lemma 6.8. *$j(k, n, m)$ is an inclusion.*

Proof. Proposition 5.1 shows that, in every dimension $*$, $E_2(H_*(T\operatorname{Rat}_k(\mathbb{G}_{n,n+m})))$ is bounded above by

$$\bigoplus_K H_*(\Sigma^{2t(K)}(\operatorname{Rat}_{k_1}(\mathbb{CP}(m)) \times \dots \times \operatorname{Rat}_{k_n}(\mathbb{CP}(m)))_+), \tag{6.9}$$

[2] We believe that Delchamps' embedding is closely connected with the loop space multiplication in $\Omega^2(\mathbb{G}_{n,n+m})$. However, our filtration is definitely distinct from the usual loop space type of splitting. We think that the two structures together, ours and the $\mathbb{C}(2)$ operad structure of a second loop space, homologically decompose the space $\operatorname{Rat}_k(\mathbb{G}_{n,n+m})$ into pieces which look like suspensions of products $\Sigma^{2L}D_{i_1} \wedge \cdots \wedge D_{i_n}$.

where the sum runs over all type 2 partitions. For any partition $K = (k_1, \dots, k_n)$ of k let $K+(1) = (k_1, \dots, k_n+1)$ be the associated partition of $k+1$ obtained by increasing the last coordinate by one. Then $t(K) = t(K+(1))$ and, since $H_*(\mathrm{Rat}_j(\mathbb{CP}(m)))$ injects in $H_*(\mathrm{Rat}_{j+1}(\mathbb{CP}(m)))$ for all j and m, it follows that the assignment $K \mapsto K+(1)$ indexes the inclusion of $E_2(H_*(T\,\mathrm{Rat}_k(\mathbb{G}_{n,n+m})))$ into a summand of $E_2(H_*(\mathrm{Rat}_{k+1}(\mathbb{G}_{n,n+m})))$. However, by inspection of (2.11), (2.14), (6.3) and (6.9) one sees that, for each k, the total dimension of $E_2(H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m}); \mathbb{Z}/p))$ is never larger, as a $\mathbb{Z}/p$ vector space, than the corresponding dimension for $H_*(\Omega^2_k(\mathbb{G}_{n,n+m}); \mathbb{Z}/p)$. Consequently, by letting k tend to ∞, naturality and Kirwan's theorem imply that the nontrivial elements given in (6.9) have nontrivial image in $H_*(\Omega^2_{k+l}(\mathbb{G}_{n,n+m}))$ for sufficiently large l. q.e.d.

Thus, we have shown that the E_2 term for the filtration given in (6.2) contains two summands: one, given by (6.9), which must, by Lemma (6.8), survive to E_∞ in its entirety; and the other, given by (6.3). Specifically, the union of the type 1 partitions is precisely the image of the inclusion $\mathrm{Rat}_k(\mathbb{G}_{n-1,n+m-1}) \hookrightarrow \mathrm{Rat}_k(\mathbb{G}_{n,n+m})$, and from the inductive hypothesis the spectral sequence for the filtered image collapses. However, the spectral sequence for $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ does not see the filtered image directly, but rather, the Thom space of the normal bundle to this image. This normal bundle is the trivial bundle, so the terms appearing in the spectral sequence for $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$ are actually the $2k$-fold suspension of the previous terms, and none of the previous differentials are changed. So, since they were previously trivial they continue to be trivial, and any nontrivial differentials on the classes in (6.3) must land on classes in the summand given by formula (6.9).

Finally, we have seen that these classes are all infinite cycles. Therefore, there cannot be any nontrivial differentials in the spectral sequence associated to filtration (6.2) of $\mathrm{Rat}_k(\mathbb{G}_{n,n+m})$. This also implies that the sequences of fibrations given in Proposition 5.1 are always homologically products and that

$$(6.10)\qquad \begin{aligned} &H_*(\mathrm{Rat}_k(\mathbb{G}_{n,n+m})) \\ &\quad \cong \bigoplus_K H_*(\Sigma^{2t(K)}(\mathrm{Rat}_{k_1}(\mathbb{CP}(m)) \times \dots \times \mathrm{Rat}_{k_n}(\mathbb{CP}(m)))_+) \end{aligned}$$

which establishes Theorem A.

Remark 6.11. For fixed n and k, consider the sequence of inclusions

$$(6.12)\qquad \mathrm{Rat}_k(\mathbb{G}_{n,n+m}) \to \Omega^2_k(\mathbb{G}_{n,n+m}) \to \Omega^2_k(BU(n)) \to \Omega(SU(n)).$$

As mentioned in the Introduction, Mitchell [20] has constructed a filtration $\mathbb{F}_{n,k}$ of $\Omega(SU(n))$, and we note that, for fixed n and k as $m \longrightarrow \infty$, it is precisely the pure suspension homology classes given in (6.10) which map via the composite (6.12) onto $H_*(\mathbb{F}_{n,k})$ to fill out the homology of the Mitchell filtration. It is also very interesting to note that, again, in homology, both the Mitchell filtration and the pure suspension classes given in (6.10) fill out the homology of $\Omega(SU(n))$ in precisely the same manner as in the original computation of $H_*(\Omega(SU(n)))$ given by Bott and Samelson [2].

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