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# AN EXPANSION OF CONVEX HYPERSURFACES

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**Abstract**

We study the motion of smooth, closed, uniformly convex hypersurfaces in a Euclidean $(n+1)$-space $\mathbf{R}^{n+1}$ expanding in the direction of their normal vectorfield with speed given by a suitable degree one homogeneous, positive, symmetric, concave function of the principal radii of curvature. We show that the hypersurfaces remain smooth and uniformly convex for all time and that asymptotically they become round.

## 1. Introduction

Let $M_0$ be a smooth, closed, uniformly convex hypersurface in a Euclidean $(n+1)$-space $\mathbf{R}^{n+1}$. Suppose that $M_0$ is given by a smooth embedding $X_0: S^n \to \mathbf{R}^{n+1}$. We consider the initial value problem

$$(1.1) \qquad \begin{aligned} &\frac{\partial X}{\partial t}(x, t) = k(x, t)\nu(x, t), \\ &X(\cdot, 0) = X_0, \end{aligned}$$

where $k(\cdot, t)$ is a suitable curvature function of the hypersurface $M_t$ parametrized by $X(\cdot, t): S^n \to \mathbf{R}^{n+1}$, and $\nu(\cdot, t)$ is the outer unit normal vectorfield to $M_t$.

Problems of this kind have been studied from several points of view. The motion of surfaces by their mean curvature was studied by Brakke [3] using the methods of geometric measure theory, while (1.1) with $k(\cdot, t) = -K(\cdot, t)$, where $K$ is the Gauss curvature, was proposed by Firey [7] as a model for the wearing of stones on a beach by water waves.

More recently, Huisken [11] considered the case $k(\cdot, t) = -H(\cdot, t)$, where $H$ is the mean curvature, and showed that in this case the initial value problem (1.1) has a unique smooth solution for some maximal time interval $[0, T)$, and as $t \to T$, the hypersurfaces $M_t$ converge to a point $P$. Moreover, the hypersurfaces $\widetilde{M}_t$, obtained from $M_t$ by a homothety

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about $P$ keeping the area of $\widetilde{M}_t$ constant, converge to a sphere as $t \to T$. The corresponding one-dimensional result was proved by Gage and Hamilton [8].

Using different methods, Tso [18] considered the case $k(\cdot, t) = -K(\cdot, t)$ and proved that (1.1) has a unique solution for a maximal time interval $[0, T)$ and $M_t$ converges to a point as $t \to T$.

By combining the methods of Huisken and Tso, Chow [5], [6] was able to prove analogous results for the cases $k(\cdot, t) = -K(\cdot, t)^{\beta}$ and $k(\cdot, t) = -R(\cdot, t)^{1/2}$, where $K$ and $R$ are the Gauss and scalar curvatures respectively, and $\beta$ is a positive constant. However, except for the case $k(\cdot, t) = -K(\cdot, t)^{1/n}$, his results were not as complete as those of Huisken for the mean curvature case in that the initial hypersurface $M_0$ had to satisfy a suitable pinching condition to obtain convergence to a sphere, while for the case $k(\cdot, t) = -R(\cdot, t)^{1/2}$, this was needed for the existence of a solution as well as the convergence to a sphere.

The problem for hypersurfaces contracting by other curvature functions is unsolved at present, but we expect results analogous to those of Huisken for the mean curvature case.

Here we consider the case of expanding convex hypersurfaces. We shall prove that a smooth solution of (1.1) exists for all time and that $M_t$ becomes spherical as $t \to \infty$ for a large class of curvature functions. Our approach is similar to that of Tso, and involves studying the evolution equation satisfied by the support function of the hypersurfaces $M_t$, rather than working directly with (1.1).

To formulate our results, we shall suppose that the curvature function $k$ can be expressed as

$$k(\cdot, t) = f(R_1, \cdots, R_n), \tag{1.2}$$

where $R_1, \cdots, R_n$ are the principal radii of curvature of the hypersurface $M_t$, and $f \in C^{\infty}(\Gamma^+)$ is a positive, symmetric function on the positive cone $\Gamma^+ = \{(\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n : \lambda_i > 0 \text{ for all } i\}$. The reason for expressing $k$ as a function of the principal radii of curvature rather than in terms of the principal curvatures will become apparent later. The function $f$ is assumed to satisfy the following conditions:

$$f \text{ is homogeneous of degree one on } \Gamma^+, \tag{1.3}$$

$$\frac{\partial f}{\partial \lambda_i} > 0 \text{ on } \Gamma^+, \tag{1.4}$$

$$f \text{ is concave on } \Gamma^+. \tag{1.5}$$

Our main result is the following:

**Theorem 1.1.** *Let $M_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbf{R}^{n+1}$ parametrized by a smooth embedding $X_0\colon S^n \to \mathbf{R}^{n+1}$. Let $f \in C^\infty(\Gamma^+)$ be a symmetric, positive function on the positive cone satisfying* (1.3), (1.4), *and* (1.5). *Suppose that one of the following three conditions is satisfied*:

(i) $f \in C^0(\overline{\Gamma^+})$ *and* $f \equiv 0$ *on* $\partial\Gamma^+$;

(ii) *the function* $g$ *defined by*

$$g(\lambda_1, \cdots, \lambda_n) = \frac{1}{f(1/\lambda_1, \cdots, 1/\lambda_n)}$$

*is concave on* $\Gamma^+$;

(iii) $n = 2$.

*Then the initial value problem* (1.1), (1.2) *has a unique smooth solution* $X$ *defined on the time interval* $[0, \infty)$. *For each* $t \in [0, \infty)$, $X(\cdot, t)$ *is a parametrization of a smooth, closed, uniformly convex hypersurface* $M_t$ *in* $\mathbf{R}^{n+1}$. *Furthermore, if* $\widetilde{M}_t$ *is the hypersurface parametrized by* $\widetilde{X}(\cdot, t) = e^{-\beta t} X(\cdot, t)$, *where*

$$\beta = f(1, \cdots, 1), \tag{1.6}$$

*then* $\widetilde{M}_t$ *converges to a sphere centered at the origin in the* $C^\infty$ *topology as* $t \to \infty$, *and there exists a positive constant* $H^*$, *depending only on* $n$, $f$, $\beta$, *and* $M_0$, *such that for any positive* $\gamma < 2$ *and any positive integer* $m$ *we have*

$$\|\tilde{g}_{il}(\cdot, t)\tilde{h}^{jl}(\cdot, t) - \delta_i^j H^*\|_{C^m(S^n)} \le C_m e^{-\gamma\beta t} \tag{1.7}$$

*and*

$$\|g_{il}(\cdot, t)h^{jl}(\cdot, t) - \delta_i^j H^* e^{\beta t}\|_{C^m(S^n)} \le C_m e^{-(\gamma-1)\beta t}, \tag{1.8}$$

*where* $\tilde{g}_{ij}$, $\tilde{h}^{ij}$, $g_{ij}$, $h^{ij}$ *are the metric and inverse of the second fundamental form of* $\widetilde{M}_t$, $M_t$ *respectively, and* $C_m$ *depends only on* $n$, $m$, $\beta$, $\gamma$, $f$, $M_0$, *and* $X_0$.

Let us make some remarks about our hypotheses. As we have already mentioned, we shall study the equation satisfied by the support function of the hypersurfaces $M_t$. Condition (1.4) ensures that this equation is parabolic. Condition (1.5) is used repeatedly in our proof and, together with (1.4), is essential for our method. We do not know whether the conclusion of Theorem 1.1 remains valid if these two conditions are weakened.

We believe that conditions (i) and (ii) are superfluous, but we have not been able to avoid these except in the case of two dimensions. As we shall see later, condition (ii) can be replaced by a weaker, but somewhat artificial hypothesis, which follows easily from the more natural condition (ii). Nevertheless, the cases of greatest geometric interest are not ruled out by hypotheses (i) and (ii).

Before proceeding to the proof of Theorem 1.1 we give some examples of functions $f$ satisfying the required hypotheses. For any integer $m$ such that $1 \le m \le n$, the $m$th elementary symmetric function $S_m$ is defined by

$$S_m(\lambda_1, \cdots, \lambda_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \cdots \lambda_{i_m}. \tag{1.9}$$

We define

$$\sigma_m(\lambda_1, \cdots, \lambda_n) = S_m(\lambda_1, \cdots, \lambda_n)^{1/m} \tag{1.10}$$

and

$$\tilde{\sigma}_m(\lambda_1, \cdots, \lambda_n) = S_m(1/\lambda_1, \cdots, 1/\lambda_n)^{-1/m}. \tag{1.11}$$

Then $\sigma_m$ and $\tilde{\sigma}_m$ are smooth, positive, symmetric functions on the positive cone, and both are homogeneous of degree one. It is easily checked that (1.4) holds for $\sigma_m$ and $\tilde{\sigma}_m$. To verify the concavity condition (1.5), we use a result proved in [16, §2.15], which asserts that for any integers $p$, $r$ with $1 \le p \le r \le n$, the function $(S_r/S_{r-p})^{1/p}$ is concave on $\Gamma^+$. The concavity of $\sigma_m$ then follows by choosing $r = p = m$ (we take $S_0 \equiv 1$), while that of $\tilde{\sigma}_m$ follows by choosing $r = n$ and $p = m$ and observing that $\tilde{\sigma}_m = (S_n/S_{n-m})^{1/m}$.

Condition (i) of Theorem 1.1 is satisfied by $\tilde{\sigma}_m$ for all $m = 1, \cdots, n$, but not by $\sigma_m$ unless $m = n$. However, $\sigma_m$ satisfies the alternative condition (ii) because $\sigma_m(1/\lambda_i, \cdots, 1/\lambda_n)^{-1} = \tilde{\sigma}_m(\lambda_1, \cdots, \lambda_n)$, and $\tilde{\sigma}_m$ is a concave function on $\Gamma^+$. Similarly, $\tilde{\sigma}_m$ satisfies (ii), but it is simpler to use (i) in this case.

The functions $\sigma_m$ and $\tilde{\sigma}_m$ are the main examples of geometric interest. The case $\tilde{\sigma}_m$ corresponds to the case where the curvature function $k$ is the reciprocal of the $m$th root of the $m$th mean curvature.

In the one-dimensional case, we may allow $M_0$ to have self-intersections. In this case we take

$$k(\cdot, t) = \rho(\cdot, t), \tag{1.12}$$

where $\rho(\cdot, t)$ is the radius of curvature of the curve $M_t$. We have the following result.

**Theorem 1.2.** *Let $M_0$ be a smooth, closed, immersed curve in $\mathbf{R}^2$ with positive curvature and rotation number $m > 0$, and suppose that $M_0$ is* *parametrized by an immersion* $X_0: S^1 \to \mathbf{R}^2$. *Then the initial value problem* (1.1), (1.12) *has a unique smooth solution* $X$ *defined on the time interval* $[0, \infty)$. *For each* $t \in [0, \infty)$, $X(\cdot, t)$ *is a parametrization of a smooth, closed, immersed curve* $M_t$ *in* $\mathbf{R}^2$ *having positive curvature and rotation number* $m$. *Furthermore, if* $\widetilde{M}_t$ *is the curve parametrized by* $\widetilde{X}(\cdot, t) = e^{-t}X(\cdot, t)$, *then* $\widetilde{M}_t$ *converges to an* $m$-*fold cover of a circle centered at the origin in the* $C^\infty$ *topology as* $t \to \infty$, *and there exists a positive number* $\rho^*$, *depending only on* $M_0$, *such that for any positive integer* $l$ *we have*

$$\|\tilde{\rho}(\cdot, t) - \rho^*\|_{C^l(S^1)} \le C_l e^{-2t/m^2}, \tag{1.13}$$

*where* $\tilde{\rho}(\cdot, t)$ *is the radius of curvature of* $\widetilde{M}_t$ *and* $C_l$ *depends only on* $l$, $m$, $M_0$, *and* $X_0$.

We shall prove Theorems 1.1 and 1.2 in the remaining sections of the paper. In §2 we shall derive the equation for the support function of the hypersurfaces $M_t$, assuming a solution of (1.1) exists, and show that the initial value problem (1.1) can be reformulated as an initial value problem for the support function. In §3 we derive the a priori estimates we need to prove the existence of a solution of this problem for all time, and show that after an appropriate rescaling the hypersurfaces $M_t$ become spherical as $t \to \infty$. In the final section we indicate the modifications which need to be made to our arguments if $f$ is replaced by $f^\alpha$ in (1.2) for some constant $\alpha \in [0, 1)$. We shall prove the following result.

**Theorem 1.3.** *Let* $M_0$, $X_0$, *and* $f$ *satisfy the hypotheses of Theorem* 1.1 *in cases* (i) *and* (ii), *and suppose that the curvature function* $k$ *in* (1.1) *is given by*

$$k(\cdot, t) = f(R_1, \cdots, R_n)^\alpha \tag{1.14}$$

*for some constant* $\alpha \in [0, 1)$, *where* $R_1, \cdots, R_n$ *are the principal radii of curvature of* $M_t$. *Then the initial value problem* (1.1), (1.14) *has a unique smooth solution* $X$ *defined on the time interval* $[0, \infty)$ *and for each* $t \in [0, \infty)$, $X(\cdot, t)$ *is a parametrization of a smooth, closed, uniformly convex hypersurface* $M_t$ *in* $\mathbf{R}^{n+1}$. *Furthermore, if* $\widetilde{M}_t$ *is the hypersurface parametrized by* $\widetilde{X}(\cdot, t) = (1 + \delta\beta t)^{1/\delta} X(\cdot, t)$, *where* $\beta$ *is given by* (1.6) *and* $\delta = 1 - \alpha$, *then* $\widetilde{M}_t$ *converges to a sphere centered at the origin in the* $C^\infty$ *topology as* $t \to \infty$.

A version of Theorem 1.3 for immersed curves $M_0$ in $\mathbf{R}^2$ similar to Theorem 1.2 also holds. We leave it to the reader to formulate this.

Gerhard Huisken [12] has proved a version of Theorems 1.1 and 1.2. His method avoids consideration of the support function and is similar to that used in his earlier work [11].

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## 2. The equation for the support function

In this section we show that the initial value problem (1.1) can be reduced to an initial value problem for the support function of the convex hypersurfaces parametrized by $X(\cdot, t)$. We shall recall briefly some facts about convex hypersurfaces.

Let $M$ be a smooth, closed, uniformly convex hypersurface in $\mathbf{R}^{n+1}$. We may assume that $M$ is parametrized by the inverse Gauss map $X: S^n \to M \subset \mathbf{R}^{n+1}$. Without loss of generality, we may assume that $M$ encloses the origin. The support function $H$ of $M$ is defined by

$$H(x) = \langle x, X(x)\rangle \quad \text{for all } x \in S^n, \tag{2.1}$$

where $\langle\cdot, \cdot\rangle$ denotes the standard inner product of $\mathbf{R}^{n+1}$. We extend $H$ and $X$ to be homogeneous functions on $\mathbf{R}^{n+1} - \{0\}$ of degrees one and zero respectively. Evidently we also have

$$H(x) = \sup_{y\in M}\langle x, y\rangle \quad \text{for all } x \in \mathbf{R}^{n+1},$$

so $H$ is convex, since it is a supremum of linear functions. Furthermore, $H$ is smooth, since $X$ is smooth. Since $H$ is homogeneous of degree one, we have

$$H(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} x_i D_i H,$$

where $D = (D_1, \cdots, D_{n+1})$ is the gradient on $\mathbf{R}^{n+1}$. If $M$ is a sphere of radius $R$ centered at the origin, then $H(x) = R|x|$.

Conversely, if $H$ is a convex function which is smooth and homogeneous of degree one on $\mathbf{R}^{n+1} - \{0\}$, then it can be shown that $H$ is the support function of a unique convex hypersurface $M = \partial B$, where $B$ is the convex body

$$B = \bigcap_{x\in S^n} \{y \in \mathbf{R}^{n+1}: \langle x, y\rangle \le H(x)\}.$$

$B$ is an intersection of halfspaces, and so is convex. Furthermore, the coordinates of the point of $M$ with outer unit normal $x$ are given by $X_i = D_i H$. Proofs of these assertions can be found in [2].

Next we compute the metric and the second fundamental form of $M$ in terms of the support function. As before, we assume that $M$ is given as an embedding of $S^n$ via the inverse Gauss map. Let $e_1, \cdots, e_n$ be a smooth local orthonormal frame field on $S^n$, and let $\nabla$ be the gradient on $S^n$. Differentiating (2.1) we obtain

$$\nabla_i H = \langle \nabla_i X, x\rangle + \langle X, \nabla_i x\rangle = \langle X, \nabla_i x\rangle,$$

since $\nabla_i X(x)$ is tangential to $M$ at $X(x)$, and $x$ is the normal to $M$ at $X(x)$. Differentiating once again, and writing $\nabla_{ij} = \nabla_i \nabla_j$, we obtain

$$\nabla_{ij} H = \langle X, \nabla_{ij} x\rangle + \langle \nabla_i X, \nabla_j x\rangle = \langle X, \nabla_{ij} x\rangle + h_{ij},$$

where $h_{ij}$ is the second fundamental form of $M$. To compute the term $\langle X, \nabla_{ij} x\rangle$ we differentiate the equation $\langle x, x\rangle \equiv 1$ and obtain

$$\langle x, \nabla_i x\rangle = 0 \tag{2.2}$$

and

$$\langle x, \nabla_{ij} x\rangle = -\langle \nabla_i x, \nabla_j x\rangle = -\delta_{ij}, \tag{2.3}$$

since $e_1, \cdots, e_n$ is an orthonormal frame field, and finally,

$$\langle \nabla_{ki} x, \nabla_j x\rangle + \langle \nabla_i x, \nabla_{kj} x\rangle = 0. \tag{2.4}$$

From (2.2) and (2.3), we see that $\nabla_1 x, \cdots, \nabla_n x$ form an orthonormal basis for $T_{X(x)}M$, and hence

$$\begin{aligned}
\langle X, \nabla_{ij} x\rangle &= \Big\langle \langle X, x\rangle x, \nabla_{ij} x\Big\rangle + \Big\langle \langle X, \nabla_k x\rangle \nabla_k x, \nabla_{ij} x\Big\rangle \\
&= \langle X, x\rangle\langle x, \nabla_{ij} x\rangle + \nabla_i \Big\langle \langle X, \nabla_k x\rangle \nabla_k x, \nabla_j x\Big\rangle \\
&\quad - \Big\langle \nabla_i(\langle X, \nabla_k x\rangle \nabla_k x), \nabla_j x\Big\rangle \\
&= -H\delta_{ij} + \nabla_i\langle X, \nabla_j x\rangle - \nabla_i\langle X, \nabla_k x\rangle\langle \nabla_k x, \nabla_j x\rangle \\
&\quad - \langle X, \nabla_k x\rangle\langle \nabla_{ik} x, \nabla_j x\rangle \\
&= -H\delta_{ij} - \langle X, \nabla_k x\rangle\langle \nabla_{ik} x, \nabla_j x\rangle
\end{aligned}$$

by virtue of (2.2) and (2.3). Similarly

$$\langle X, \nabla_{ji} x\rangle = -H\delta_{ij} - \langle X, \nabla_k x\rangle\langle \nabla_{jk} x, \nabla_i x\rangle,$$

so by adding together the above two expressions and using the symmetry of $\nabla_{ij} x$ together with (2.4), we obtain

$$h_{ij} = \nabla_{ij} H + \delta_{ij} H. \tag{2.5}$$

To compute the metric $g_{ij}$ of $M$ we use the Gauss-Weingarten relations

$$\nabla_i x = h_{ik} g^{kl} \nabla_l X, \tag{2.6}$$

from which we obtain

$$\delta_{ij} = \langle \nabla_i x, \nabla_j x \rangle = h_{ik} g^{kl} h_{jm} g^{ms} \langle \nabla_l X, \nabla_s X \rangle = h_{ik} h_{jl} g^{kl}.$$

Since $M$ is uniformly convex, $h_{ij}$ is invertible, and hence

$$g_{ij} = h_{ik} h_{jk}. \tag{2.7}$$

The principal radii of curvature are the eigenvalues of the matrix $b_{ij} = h^{ik} g_{jk}$, which, by virtue of (2.5) and (2.7), is given by

$$b_{ij} = h_{ij} = \nabla_{ij} H + \delta_{ij} H. \tag{2.8}$$

Next we reduce the initial value problem (1.1) to an initial value problem for the support function. This is carried out in [18] for the case of the Gauss curvature, and the proof in general is the same. We include it for completeness.

Let $X$ be a solution of (1.1), and suppose that for each $t \in [0, \infty)$, $X(\cdot, t)$ is a parametrization of a smooth, closed, uniformly convex hypersurface $M_t$. Let $H(\cdot, t)$ be the support function of $M_t$, and let $\nu_t\colon S^n \to S^n$ be the Gauss map of $X(\cdot, t)$. Note that now we do not assume that $X(\cdot, t)$ is the inverse Gauss map. We define a new parametrization $\overline{X}(\cdot, t)$ by

$$\overline{X}(x, t) = X(\nu_t^{-1}(x), t) \tag{2.9}$$

and

$$\bar{k}(x, t) = k(\nu_t^{-1}(x), t). \tag{2.10}$$

Then, assuming we have extended $X$ to be homogeneous of degree zero on $\mathbf{R}^{n+1} - \{0\}$, we have

$$\frac{\partial \overline{X}}{\partial t} = \frac{\partial X}{\partial s_i} \frac{\partial (\nu_t^{-1})_i}{\partial t} + \frac{\partial X}{\partial t} = \frac{\partial X}{\partial s_i} \frac{\partial (\nu_t^{-1})_i}{\partial t} + \bar{k} x$$

by (1.1). Thus

$$\frac{\partial H}{\partial t} = \left\langle \frac{\partial \overline{X}}{\partial t}(x, t), x \right\rangle = \bar{k}(x, t),$$

since $\partial X / \partial s_i$ is tangential. We see therefore that the support function satisfies the initial value problem

$$\begin{aligned} &\frac{\partial H}{\partial t} = F(\nabla^2 H + HI) \quad \text{on } S^n \times [0, \infty), \\ &H(\cdot, 0) = H_0, \end{aligned} \tag{2.11}$$

where $I$ is the identity matrix, $H_0$ is the support function of $M_0$,

$$F(a_{ij}) = f(\mu_1, \cdots, \mu_n), \tag{2.12}$$

where $\mu_1, \cdots, \mu_n$ are the eigenvalues of $[a_{ij}]$, and the curvature function $k$ is given by (1.2).

To obtain the higher order estimates needed, it will be convenient to express the equation for $H$ in a local coordinate chart. The mapping

$$(x_1, \cdots, x_n) \to \frac{(x_1, \cdots, x_n, -1)}{(1+|x|^2)^{1/2}} \tag{2.13}$$

maps $\mathbf{R}^n$ onto $S^n_-$, and gives a coordinate system for $S^n_-$. In this coordinate system the metric $e_{ij}$ on $S^n_-$ is given by

$$e_{ij} = (1+|x|^2)^{-1}\left(\delta_{ij} - \frac{x_i x_j}{1+|x|^2}\right). \tag{2.14}$$

Let $H$ be the support function of the convex hypersurface $M$, and set

$$u(x_1, \cdots, x_n) = (1+|x|^2)^{1/2} H\left(\frac{x_1, \cdots, x_n, -1}{(1+|x|^2)^{1/2}}\right). \tag{2.15}$$

By the degree one homogeneity of $H$, $u$ is just the restriction of $H$ to $x_{n+1} = -1$. Straightforward computations yield

$$h_{ij} = (1+|x|^2)^{-1/2} D_{ij} u. \tag{2.16}$$

From the Gauss-Weingarten relations (2.6) we obtain

$$e_{ij} = h_{ik} h_{jl} g^{kl} \tag{2.17}$$

and hence

$$g_{ij} = h_{ik} e^{kl} h_{jl}. \tag{2.18}$$

Thus by (2.14) and (2.16)

$$g_{ij} = (\delta_{kl} + x_k x_l) D_{ik} u D_{jl} u. \tag{2.19}$$

In the coordinate system given by (2.13), the matrix $[b_{ij}]$, the eigenvalues of which are the principal radii of curvature, is given by

$$b_{ij} = (1+|x|^2)^{1/2} (\delta_{il} + x_i x_l) D_{jl} u. \tag{2.20}$$

However, the matrix $(\delta_{ik} + x_i x_k) D_{jk} u$ is not symmetric in general and it is convenient to replace it by a symmetric matrix. It is not difficult to check that the matrix

$$\hat{b}_{ij} = \left(\delta_{ik} + \frac{x_i x_k}{1+(1+|x|^2)^{1/2}}\right)\left(\delta_{jl} + \frac{x_j x_l}{1+(1+|x|^2)^{1/2}}\right) D_{kl} u \tag{2.21}$$

is symmetric and has the same eigenvalues as $(\delta_{ik} + x_i x_k) D_{jk} u$.

Now let $u$ be given by (2.15). Since $H$ is homogeneous of degree one, so is $\partial H/\partial t$. Thus

$$\frac{\partial u}{\partial t}(x_1, \cdots, x_n) = (1+|x|^2)^{1/2}\frac{\partial H}{\partial t}\left(\frac{x_1, \cdots, x_n, -1}{(1+|x|^2)^{1/2}}\right).$$

We see therefore that $u = H|_{x_{n+1}=-1}$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= (1+|x|^2)F(\hat{b}_{ij}) \quad \text{on } \mathbf{R}^n \times [0, \infty), \\ u(\cdot, 0) &= u_0. \end{aligned} \tag{2.22}$$

Similar equations also hold for $H|_{x_j=\pm 1}$ for any $j = 1, \cdots, n+1$.

In §3 we shall also consider the initial value problem (2.11) in the case that $\nabla^2 H + HI$ is not necessarily positive definite. In view of this, we note that in this case, in the local coordinate system given by (2.13), the function $u$ defined by (2.15) still satisfies the initial value problem (2.22).

From (2.20) we see that at $(0, \cdots, 0, -1)$,

$$b_{ij} = D_{ij}u = D_{ij}H \quad \text{for } i, j \leq n.$$

By a rotation of the coordinates $x_1, \cdots, x_n$ we may diagonalize $[D_{ij}H]_{i,j\leq n}$ at $(0, \cdots, 0, -1)$. The eigenvalues of this matrix are the principal radii of curvature of $M$ at the point with outer unit normal $(0, \cdots, 0, -1)$. The remaining eigenvalue of $D^2 H$, corresponding to the radial direction, is of course zero, since $H$ is homogeneous of degree one.

Now suppose that we have a smooth solution of (2.11) such that the matrix $\nabla^2 H + HI$ is positive definite on $S^n \times [0, \infty)$. Then if we extend $H(\cdot, t)$ to be homogeneous of degree one on $\mathbf{R}^{n+1} - \{0\}$, $H(\cdot, t)$ is convex for each $t$. To see this, we observe that in the coordinate system given by (2.13), for $u = H(\cdot, t)|_{x_{n+1}=-1}$ we have

$$\nabla_{ij}H(0, -1, t) + \delta_{ij}H(0, -1, t) = D_{ij}u(0),$$

so $D^2u(0)$ is positive definite. Since similar assertions hold for $H(\cdot, t)$ restricted to any $n$-dimensional hyperplane in $\mathbf{R}^{n+1} - \{0\}$, we conclude that $H(\cdot, t)$ is convex. For each $t \in [0, \infty)$, $H(\cdot, t)$ is therefore the support function of a unique convex hypersurface $M_t$. Let $M(\cdot, t)$ be the parametrization of $M_t$ given by the inverse Gauss map. We want to show that there exist diffeomorphisms $\varphi(\cdot, t): S^n \to S^n$ such that

$$X(s, t) = M(\varphi(s, t), t) \tag{2.23}$$

satisfies (1.1). We have

$$\frac{\partial X}{\partial t} = \frac{\partial M}{\partial x_i}\frac{\partial \varphi_i}{\partial t} + \frac{\partial M}{\partial t} = \frac{\partial M}{\partial x_i}\frac{\partial \varphi_i}{\partial t} + \left(\frac{\partial M}{\partial t}\right)^T + k\varphi,$$

where $(\partial M/\partial t)^T$ denotes the tangential component of $\partial M/\partial t$, since

$$\left\langle \frac{\partial M}{\partial t}, x\right\rangle = \frac{\partial H}{\partial t} = k.$$

Therefore we require $\varphi$ to satisfy

$$\frac{\partial M}{\partial x_i}\frac{\partial \varphi_i}{\partial t} + \left(\frac{\partial M}{\partial t}\right)^T = 0. \tag{2.24}$$

Now let $s \in S^n$ and choose a coordinate system so that $\varphi(s, t) = (0, -1)$, and $\mathbf{R}^n \times \{-1\}$ is the tangent hyperplane to $S^n$ at $\varphi(s, t)$. The coordinate functions $M_i$ are given by $M_i = D_i H$, so we also have $D_i M_j = D_{ij} H$. Since $H$ is homogeneous of degree one, and therefore $D_{i,n+1} H = 0$ at $(0, -1)$ for all $i = 1, \cdots, n+1$, at $(s, t)$ we can write (2.24) as

$$\sum_{i=1}^n D_{ij} H(\varphi(s, t), t)\frac{\partial \varphi_i}{\partial t} + \left(\frac{\partial M_j}{\partial t}\right)^T = 0, \qquad j = 1, \cdots, n.$$

Since $[D_{ij}H]_{i,j\le n}$ is positive definite at $(0, -1)$, we deduce that $\varphi$ satisfies a system of the form

$$\frac{\partial}{\partial t}\varphi(s, t) = Y(\varphi(s, t), t) \quad \text{for all } s \in S^n, \tag{2.25}$$

where $Y(\cdot, t)$ is a smooth tangential vectorfield on $S^n$. Standard results on ordinary differential equations with a parameter imply that (2.25) has a unique smooth solution for any given initial condition, in particular for

$$\varphi(s, 0) = s. \tag{2.26}$$

We have proved that (1.1) is equivalent to (2.11) together with the condition

$$\nabla^2 H + HI > 0 \quad \text{on } S^n \times [0, \infty). \tag{2.27}$$

In the remainder of the paper we shall study the initial value problem (2.11), and we shall prove the existence of a solution satisfying (2.27).

We end this section by recalling the definitions of the various function spaces and norms which we shall use. Let $k$ be nonnegative integer and let $\alpha \in (0, 1]$. $C^k(S^n)$ is the Banach space of real valued functions on $S^n$ which are $k$-times continuously differentiable, equipped with the norm

$$\|u\|_{C^k(S^n)} = \sum_{|\beta|\le k} \sup_{S^n} |\nabla^\beta u|.$$

$C^{k,\alpha}(S^n)$ is the space of functions in $C^k(S^n)$ such that the norm

$$\|u\|_{C^{k,\alpha}(S^n)} = \|u\|_{C^k(S^n)} + \sup_{|\beta|=k} \sup_{\substack{x,y\in S^n \\ x\neq y}} \frac{|\nabla^\beta u(x) - \nabla^\beta u(y)|}{|x-y|^\alpha}$$

is finite. Here $|x-y|$ denotes the distance between $x$ and $y$ in $S^n$.

We shall also need norms defined on $S^n \times I$, where $I = [a, b] \subset \mathbf{R}$. We denote by $\tilde{C}^k(S^n \times I)$ the space of real valued functions on $S^n \times I$ which are $k$-times continuously differentiable with respect to $x$, and $[\frac{1}{2}k]$-times continuously differentiable with respect to $t$, such that the norm

$$\|u\|_{\tilde{C}^k(S^n\times I)} = \sum_{|\beta|+2r\leq k} \sup_{S^n\times I} |\nabla^\beta D_t^r u|$$

is finite. We denote by $\tilde{C}^{k,\alpha}(S^n \times I)$ the space of functions in $\tilde{C}^k(S^n \times I)$ such that the norm

$$\|u\|_{\tilde{C}^{k,\alpha}(S^n\times I)} = \|u\|_{\tilde{C}^k(S^n\times I)} + \sup_{|\beta|+2r=k} \sup_{\substack{(x,s),(y,t)\in S^n\times I \\ (x,s)\neq(y,t)}} \frac{|\nabla^\beta D_t^r u(x,s) - \nabla^\beta D_t^r u(y,t)|}{(|x-y|^2 + |s-t|)^{\alpha/2}}$$

is finite.

We shall also use the spaces $C^k(\overline{\Omega})$, $C^{k,\alpha}(\overline{\Omega})$, $\tilde{C}^k(\overline{\Omega} \times I)$, and $\tilde{C}^{k,\alpha}(\overline{\Omega} \times I)$, where $\Omega$ is a bounded domain in $\mathbf{R}^n$. These are defined in the same way as above.

## 3. Initial value problems on $S^n \times [0, \infty)$

In this section we shall study initial value problems of the form

$$\begin{aligned} &\frac{\partial H}{\partial t} = F(\nabla^2 H + HI) \quad \text{on } S^n \times [0, \infty), \\ &H(\cdot, 0) = H_0, \end{aligned} \tag{3.1}$$

where $F$ is a smooth positive function of the form (2.12). In the previous section we reduced the solvability of the initial value problem (1.1) to the solvability of a problem such as (3.1), subject to the additional restriction (2.27). It may be that $f$ is in fact defined on some larger cone $\Gamma$ than the positive cone and $f$ satisfies (1.3), (1.4), and (1.5) on $\Gamma$, but nevertheless, for application to the initial value problem (1.1), we should consider only solutions $H$ of (3.1) which satisfy (2.27). However, from the point of view of the theory of partial differential equations, such a restriction is generally neither natural nor necessary.

We shall consider therefore a somewhat more general situation. We shall assume that $F$ is given by (2.12) where $f$ is a smooth, positive, symmetric function on some open convex cone $\Gamma \subsetneq \mathbf{R}^n$ having vertex at the origin and containing the positive cone. $\Gamma$ is assumed to be invariant under the interchange of any two coordinates $\lambda_i$ and $\lambda_j$. It is not difficult to see that

$$\Gamma \subset \left\{\lambda \in \mathbf{R}^n : \sum_{i=1}^{n} \lambda_i > 0\right\}. \tag{3.2}$$

We shall assume furthermore that $f$ satisfies the following conditions:

$$f \text{ is homogeneous of degree one on } \Gamma, \tag{3.3}$$

$$\frac{\partial f}{\partial \lambda_i} > 0 \text{ on } \Gamma, \tag{3.4}$$

$$f \text{ is concave on } \Gamma. \tag{3.5}$$

The main examples of functions $f$ which satisfy the above hypotheses on some cone $\Gamma$ strictly larger than the positive cone are the functions $\sigma_m$ with $m = 1, \cdots, n-1$. It is shown in [4] that $\sigma_m$ satisfies (3.4) and (3.5) on the cone $\Gamma_m$ which is the connected component containing $\Gamma^+$ of the set where $\sigma_m$ is positive. It is clear then that $\sigma_m = 0$ on $\partial\Gamma_m$.

Any smooth function $\varphi$ on $S^n$ for which the eigenvalues of $\nabla^2\varphi(x) + \varphi(x)I$ belong to $\Gamma$ for all $x \in S^n$ is said to be *admissible with respect to* $\Gamma$, or $\Gamma$-*admissible.* Usually we will just write "admissible" when the cone $\Gamma$ is understood from the context. We also use the term "admissible" for functions $\varphi$ on $S^n \times [0, T)$ to mean that for each $t \in [0, T)$, $\varphi(\cdot, t)$ is admissible.

Our main result in this section is the following:

**Theorem 3.1.** *Let $F$, $\Gamma$, and $f$ be as above, with $f$ positive, smooth and symmetric and satisfying* (3.3), (3.4), *and* (3.5). *Suppose also that one of the following three conditions is satisfied*:

(i) $f \in C^0(\overline{\Gamma})$ *and* $f \equiv 0$ *on* $\partial\Gamma$;

(ii) $\Gamma = \Gamma^+$ *and the function* $g$ *defined by*

$$g(\lambda_1, \cdots, \lambda_n) = \frac{1}{f(1/\lambda_1, \cdots, 1/\lambda_n)}$$

*is concave on* $\Gamma$;

(iii) $n = 2$.

*Then for any smooth admissible function* $H_0$ *on* $S^n$, *there exists a unique admissible solution* $H \in C^\infty(S^n \times [0, \infty))$ *of the initial value problem* (3.1). *Furthermore, there exists a positive constant* $H^*$, *depending only on* $n$, $f$, $\Gamma$, $H_0$, *and* $\beta$, *where*

$$\beta = f(1, \cdots, 1), \tag{3.6}$$

*such that if* $\tilde{H} = e^{-\beta t} H$, *then* $\tilde{H}$ *converges to* $H^*$ *as* $t \to \infty$, *and for any positive* $\gamma < 1$ *and any positive integer* $k$ *we have*

$$\|\tilde{H}(\cdot, t) - H^*\|_{C^k(S^n)} \leq C_k e^{-\gamma\beta t}, \tag{3.7}$$

*where* $C_k$ *depends only on* $n$, $k$, $\beta$, $\gamma$, $f$, $\Gamma$, *and* $H_0$.

The existence and uniqueness assertions of Theorem 1.1 follow from Theorem 3.1 for the case $\Gamma = \Gamma^+$, by virtue of the results of §2. The assertions of Theorem 1.1 concerning the asymptotic convergence will be explained at the end of this section.

Before proceeding to the proof of Theorem 3.1, we make some remarks about (3.1). First, by introducing the new time variable $s = \beta t$, it suffices to consider the case $\beta = 1$. Henceforth we shall assume this. Next, since $f$ is homogeneous of degree one, we see that

$$F \text{ is homogeneous of degree one on } \mathscr{M}(\Gamma), \tag{3.8}$$

where $\mathscr{M}(\Gamma)$ is the cone of real symmetric $n \times n$ matrices $[a_{ij}]$ such that the eigenvalues of $[a_{ij}]$ belong to $\Gamma$. Clearly we have $\mathscr{M}(\Gamma^+) = S_+^{n\times n}$, the cone of real positive symmetric $n \times n$ matrices.

From (3.6) and the normalization $\beta = 1$, we see that

$$F(\delta_{ij}) = 1. \tag{3.9}$$

Next, it is not difficult to see that the eigenvalues of $[F_{ij}] = [\partial F/\partial a_{ij}]$ are $\partial f/\partial \lambda_1, \cdots, \partial f/\partial \lambda_n$. Thus from (3.4) we obtain

$$[F_{ij}] > 0 \quad \text{on } \mathscr{M}(\Gamma), \tag{3.10}$$

which yields that the equation in (3.1) is parabolic for admissible solutions.

In [4], it is proved that the concavity of $f$ on $\Gamma$ implies the concavity of $F$ on $\mathscr{M}(\Gamma)$. Thus

$$F_{ij,kl}(a)\eta_{ij}\eta_{kl} \leq 0 \tag{3.11}$$

for all $a = [a_{ij}] \in \mathscr{M}(\Gamma)$ and all real symmetric $n \times n$ matrices $[\eta_{ij}]$, where $F_{ij,kl} = \partial^2 F/\partial a_{kl}\partial a_{ij}$.

We shall need two inequalities concerning $F$ which follow from the hypotheses (3.3), (3.4), and (3.5).

**Lemma 3.2.** *We have*

$$\mathscr{T} \equiv \sum_{i=1}^{n} F_{ii} \geq 1 \quad on\ \mathscr{M}(\Gamma). \tag{3.12}$$

*Proof.* It is sufficient to prove the inequality

$$\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} \geq 1 \quad \text{on } \Gamma. \tag{3.13}$$

Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma$ and suppose without loss of generality that $\lambda_1 \leq \cdots \leq \lambda_n$. For any $\mu > 0$ we have $(\mu + \lambda_1, \cdots, \mu + \lambda_n) \in \Gamma$. Thus using (3.3), (3.4), (3.5), the convexity of $\Gamma$, and the fact that $\lambda_1 \leq \cdots \leq \lambda_n$, we obtain

$$\begin{aligned} \mu + \lambda_1 &\leq f(\mu + \lambda_1, \cdots, \mu + \lambda_n) \\ &\leq f(\lambda_1, \cdots, \lambda_n) + \mu \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(\lambda) \\ &\leq \lambda_n + \mu \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(\lambda). \end{aligned}$$

Dividing by $\mu$ and letting $\mu \to \infty$ yield (3.13).

**Lemma 3.3.** *Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma$ and suppose that $\lambda_1 \leq \cdots \leq \lambda_n$. Then*

$$\lambda_n - f(\lambda_1, \cdots, \lambda_n) \geq \frac{1}{n} \sum_{i=1}^{n-1} (\lambda_n - \lambda_i). \tag{3.14}$$

*Proof.* We have

$$1 = f(1, \cdots, 1) = \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(1, \cdots, 1).$$

Thus since $f$ is symmetric and (3.3) holds, we find

$$\frac{\partial f}{\partial \lambda_i} = \frac{1}{n} \quad \text{on the diagonal } \mathscr{D} = \{\lambda \in \Gamma : \lambda_1 = \cdots = \lambda_n\}. \tag{3.15}$$

Using this together with (3.3), (3.5), and the convexity of $\Gamma$, we obtain

$$\begin{aligned} f(\lambda_1, \cdots, \lambda_n) &\leq f(1, \cdots, 1) + \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i}(1, \cdots, 1)(\lambda_i - 1) \\ &= \frac{1}{n} \sum_{i=1}^{n} \lambda_i. \end{aligned} \tag{3.16}$$

(3.14) follows from this, so the lemma is proved.

**Remark.** Condition (3.4) with the strict inequality was not used in the proof of Lemmas 3.2 and 3.3. Thus they are valid in the degenerate case where

$$\frac{\partial f}{\partial \lambda_i} \geq 0 \quad \text{on } \Gamma. \tag{3.17}$$

Since $\Gamma \supset \Gamma^+$, this condition is implied by the positivity and concavity of $f$.

Lemma 3.2 is used to show that the lower bound on $F$, the upper bound on the maximum eigenvalue of $\nabla^2 H + HI$, and in case (ii) of Theorem 3.1, the lower bound on the minimum eigenvalue of $\nabla^2 H + HI$, are preserved, as well as in the proof of the asymptotic convergence. Lemma 3.3 is used only for the last of these and is not essential for the proof, for at that stage uniform parabolicity has already been established, and thus (3.14) holds with the constant $\frac{1}{n}$ replaced by a constant depending only on the parabolicity. However, much of our proof is valid in the degenerate case (3.17), and for this reason we prefer to use Lemma 3.3.

In establishing the necessary a priori estimates it will be convenient to work with the equation for $\widetilde{H} = e^{-t} H$ rather than that for $H$ itself. Using the degree one homogeneity of $F$, we see that $\widetilde{H}$ satisfies the initial value problem

$$\begin{aligned} &\frac{\partial \widetilde{H}}{\partial t} = F(\nabla^2 \widetilde{H} + \widetilde{H} I) - \widetilde{H} \quad \text{on } S^n \times [0, \infty), \\ &\widetilde{H}(\cdot, 0) = H_0, \end{aligned} \tag{3.18}$$

and $\widetilde{H}$ is admissible if $H$ is. We shall derive all our a priori estimates for $\widetilde{H}$; these are then readily translated into a priori estimates for $H$. Thus for the remainder of this section, $H$ will denote a solution of the normalized problem (3.18), rather than of (3.1).

We begin with the estimate for $H$.

**Lemma 3.4.** *Let $H$ be an admissible solution of* (3.18) *on* $S^n \times [0, T)$. *Then*

$$\min_{S^n} H_0 \leq H(\cdot, t) \leq \max_{S^n} H_0 \tag{3.19}$$

*for all* $t \in [0, T)$.

*Proof.* At a point where $H$ attains a maximum with respect to the spatial variables, we have $\nabla^2 H \leq 0$. Thus using (3.8), (3.9), and (3.10), we see that $\partial H / \partial t \leq 0$ at such a point. Now let $0 < t_1 < t_2 < T$ and set

$$H_{\max}(t) = \max_{x \in S^n} H(x, t).$$

Since $H$ is smooth, $H_{\max}$ is evidently a Lipschitz function of $t$. Suppose that $H_{\max}(t_1) = H(x_1, t_1)$ and $H_{\max}(t_2) = H(x_2, t_2)$. Then

$$\begin{aligned} H_{\max}(t_2) - H_{\max}(t_1) &= H(x_2, t_2) - H(x_1, t_1) \\ &= H(x_2, t_2) - H(x_2, t_1) + H(x_2, t_1) - H(x_1, t_1) \\ &\le H(x_2, t_2) - H(x_2, t_1). \end{aligned}$$

It follows that

$$\limsup_{h\to 0^+} \frac{H_{\max}(t+h) - H_{\max}(t)}{h} \le 0$$

for all $t \in (0, T)$. By a result of Hamilton [10] we conclude that

$$H_{\max}(t) \le H_{\max}(0).$$

A completely analogous argument yields

$$H_{\min}(t) \equiv \min_{x\in S^n} H(x, t) \ge H_{\min}(0),$$

so the lemma is proved.

We now prove an upper bound for the eigenvalues of $\nabla^2 H + HI$. Similar ideas were used by Pogorelov [17] in the elliptic case, and later by Tso [18] and Chow [5], [6] in the parabolic case.

**Lemma 3.5.** *Let* $H$ *be an admissible solution of* (3.18) *on* $S^n \times [0, T)$. *If at* $t = 0$ *we have*

$$\nabla^2 H + HI \le KI \tag{3.20}$$

*for a positive constant* $K$, *then this remains true for all* $t \in [0, T)$.

*Proof.* Let $t \in (0, T)$ and suppose that the maximum eigenvalue of the matrix $[h_{ij}(\cdot, t)] = \nabla^2 H(\cdot, t) + H(\cdot, t)I$ on $S^n$ is attained at $x_t \in S^n$ with unit eigenvector $\xi_t \in T_{x_t} S^n$. By a rotation of the frame $e_1, \cdots, e_n$ at $x_t$, we may assume that $\xi_t = e_1$ at $x_t$.

Let us now differentiate the equation

$$\frac{\partial H}{\partial t} = F(\nabla^2 H + HI) - H \tag{3.21}$$

twice. We obtain

$$\frac{\partial}{\partial t}\nabla_l H = F_{ij}\nabla_l(\nabla_{ij} H + \delta_{ij} H) - \nabla_l H, \tag{3.22}$$

$$\begin{aligned} \frac{\partial}{\partial t}\nabla_{kl} H = {} & F_{ij}\nabla_{kl}(\nabla_{ij} H + \delta_{ij} H) \\ & + F_{ij,rs}\nabla_l(\nabla_{ij} H + \delta_{ij} H)\nabla_k(\nabla_{rs} H + \delta_{rs} H) - \nabla_{kl} H. \end{aligned} \tag{3.23}$$

Using the standard formulas for interchanging the order of covariant differentiation, we have

$$\begin{aligned}\nabla_{klij}H = \nabla_{ijkl}H &+ (\nabla_k R_j{}^m{}_{li} + \nabla_i R_l{}^m{}_{kj})\nabla_m H \\ &+ R_l{}^m{}_{kj}\nabla_{im}H + R_l{}^m{}_{ki}\nabla_{jm}H \\ &+ R_j{}^m{}_{li}\nabla_{km}H + R_j{}^m{}_{ki}\nabla_{lm}H,\end{aligned} \tag{3.24}$$

where $R_i{}^j{}_{kl}$ is the Riemann curvature tensor of $S^n$. Since $e_1, \cdots, e_n$ is a local orthonormal frame on $S^n$, using the Gauss equations we have

$$R_i{}^j{}_{kl} = R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \tag{3.25}$$

which together with (3.24) gives

$$\begin{aligned}\nabla_{klij}H = \nabla_{ijkl}H &+ 2\delta_{kl}\nabla_{ij}H - 2\delta_{ij}\nabla_{kl}H \\ &+ \delta_{jk}\nabla_{il}H - \delta_{il}\nabla_{jk}H.\end{aligned} \tag{3.26}$$

Using (3.26) in (3.23), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_{kl}H = F_{ij}\nabla_{ijkl}H &+ 2\delta_{kl}F_{ij}\nabla_{ij}H - \mathscr{T}\nabla_{kl}H \\ &+ F_{ik}\nabla_{il}H - F_{il}\nabla_{ik}H \\ &+ F_{ij,rs}\nabla_l(\nabla_{ij}H + \delta_{ij}H)\nabla_k(\nabla_{rs}H + \delta_{rs}H) - \nabla_{kl}H.\end{aligned} \tag{3.27}$$

Since $F$ is homogeneous of degree one, from (3.21) it follows that

$$\delta_{kl}\frac{\partial H}{\partial t} = \delta_{kl}F_{ij}\nabla_{ij}H + \delta_{kl}(\mathscr{T} - 1)H. \tag{3.28}$$

Adding this to (3.27), and using the degree one homogeneity of $F$ once again, we see that $h_{kl} = \nabla_{kl}H + \delta_{kl}H$ satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}h_{kl} = F_{ij}\nabla_{ij}h_{kl} &+ 2\delta_{kl}F - (\mathscr{T} + 1)h_{kl} \\ &+ F_{ik}\nabla_{il}H - F_{il}\nabla_{ik}H + F_{ij,rs}\nabla_l h_{ij}\nabla_k h_{rs}.\end{aligned} \tag{3.29}$$

Let us now set $k = l = 1$ in (3.29). Using the concavity of $F$ to estimate the term involving $F_{ij,rs}$ we get

$$\frac{\partial}{\partial t}h_{11} \le F_{ij}\nabla_{ij}h_{11} - (\mathscr{T} + 1)h_{11} + 2F. \tag{3.30}$$

Then the fact that the maximum eigenvalue of $[h_{ij}(\cdot, t)]$ over $S^n$ is equal to $h_{11}(x_t, t)$, together with (3.8), (3.9), (3.10), and Lemma 3.2, leads to

$$\frac{\partial}{\partial t}h_{11} \le 0 \quad \text{at } (x_t, t). \tag{3.31}$$

An argument similar to that used in the proof of Lemma 3.4 now implies the conclusion of the lemma.

**Remark.** Since (3.16) holds, the above proof also shows that the upper bound for $\Delta H + nH$ is preserved. However, we do not get an upper bound for the eigenvalues of $\nabla^2 H + HI$ from this, unless we also have a suitable lower bound, for example $\nabla^2 H + HI \geq 0$.

A bound for $H$ in the $\widetilde{C}^2$ norm now follows easily.

**Lemma 3.6.** *Let $H$ be an admissible solution of* (3.18) *on* $S^n \times [0, T]$. *Then we have*

$$\|H\|_{\widetilde{C}^2(S^n \times [0, T])} \leq C, \tag{3.32}$$

*where $C$ depends only on $n$ and* $\|H_0\|_{C^2(S^n)}$.

*Proof.* We can obtain an upper bound for the eigenvalues of $\nabla^2 H$ from Lemmas 3.4 and 3.5, and a lower bound from the upper bound and the inequality $\Delta H + nH \geq 0$. Then we have a full second derivative bound, and the gradient bound for $H$ follows easily from this. To prove the bound for $\partial H/\partial t$, we use (3.21), together with the bounds we have already established and the fact that $F \geq 0$.

The next step is the derivation of a suitable lower bound on the eigenvalues of $\nabla^2 H + HI$ which, together with Lemma 3.5, will imply that (3.21) is uniformly parabolic. First we show that the lower bound on $F$ is preserved.

**Lemma 3.7.** *Let $H$ be an admissible solution of* (3.18) *on* $S^n \times [0, T)$. *Then for all* $t \in [0, T)$ *we have*

$$\min_{S^n} F(\nabla^2 H(\cdot, t) + H(\cdot, t)I) \geq \min_{S^n} F(\nabla^2 H_0 + H_0 I). \tag{3.33}$$

*Proof.* Differentiating equation (3.21) with respect to $t$ we get

$$\frac{\partial^2 H}{\partial t^2} = F_{ij}\nabla_{ij}\frac{\partial H}{\partial t} + (\mathscr{T} - 1)\frac{\partial H}{\partial t}. \tag{3.34}$$

Since $F$ is homogeneous of degree one we have

$$\frac{\partial H}{\partial t} = F_{ij}\nabla_{ij}H + (\mathscr{T} - 1)H. \tag{3.35}$$

Thus $F = \partial H/\partial t + H$ satisfies the equation

$$\frac{\partial F}{\partial t} = F_{ij}\nabla_{ij}F + (\mathscr{T} - 1)F. \tag{3.36}$$

Since $\mathscr{T} \geq 1$ by Lemma 3.2 and $F > 0$ at $t = 0$, the conclusion of the lemma follows.

**Lemma 3.8.** *Let $H$ be an admissible solution of* (3.18) *on $S^n \times [0, T)$. Then in each of the cases* (i) *and* (ii) *in Theorem* 3.1, *the eigenvalues of $\nabla^2 H + HI$ lie in a compact subset of $\Gamma$, and moreover, in case* (ii), *if for some constant $\varepsilon > 0$ we have*

$$\nabla^2 H + HI \geq \varepsilon I \tag{3.37}$$

*at $t = 0$, then this remains true for all $t \in [0, T)$.*

*Proof.* We need to consider each case separately.

(i) By Lemma 3.5 the eigenvalues of $\nabla^2 H + HI$ are bounded above by a positive constant $K$ independent of $t$. Since $f$ is uniformly continuous on $\overline{\Gamma}_K = \{\lambda \in \overline{\Gamma}: \lambda_i \leq K \text{ for all } i\}$, and $F(\nabla^2 H + HI)$ is bounded from below by a positive constant by Lemma 3.7, condition (i) implies that the eigenvalues of $\nabla^2 H + HI$ remain in a fixed compact subset of $\Gamma$, which is independent of $t$.

(ii) To show that the lower bound on the eigenvalues of $[h_{ij}]$ is preserved we would naturally like to use (3.29) directly. However, this does not seem to work, and instead we consider the equation satisfied by the inverse matrix $[h^{ij}]$ and try to bound its maximum eigenvalue. Let us compute the evolution equation for $h^{pq}$. We denote the partial derivatives $\partial h^{pq}/\partial h_{kl}$ and $\partial^2 h^{pq}/\partial h_{rs} \partial h_{kl}$ by $h^{pq}_{kl}$ and $h^{pq}_{kl,rs}$ respectively. We have

$$h^{pq}_{kl} = -h^{pk} h^{ql}, \tag{3.38}$$

$$h^{pq}_{kl,rs} = h^{pr} h^{ks} h^{ql} + h^{pk} h^{qr} h^{ls}, \tag{3.39}$$

$$\nabla_j h^{pq} = h^{pq}_{kl} \nabla_j h_{kl} \tag{3.40}$$

$$\nabla_{ij} h^{pq} = h^{pq}_{kl} \nabla_{ij} h_{kl} + h^{pq}_{kl,rs} \nabla_i h_{rs} \nabla_j h_{kl}. \tag{3.41}$$

Use of (3.29), (3.38), (3.39), and (3.41) gives

$$\begin{aligned}
\frac{\partial}{\partial t} h^{pq} &= h^{pq}_{kl} \frac{\partial}{\partial t} h_{kl} \\
&= h^{pq}_{kl} (F_{ij} \nabla_{ij} h_{kl} + 2\delta_{kl} F - (\mathscr{T} + 1) h_{kl} \\
&\qquad + F_{ik} \nabla_{il} H - F_{il} \nabla_{ik} H + F_{ij,rs} \nabla_l h_{ij} \nabla_k h_{rs}) \\
&= F_{ij} \nabla_{ij} h^{pq} - 2F h^{pk} h^{qk} + (\mathscr{T} + 1) h^{pq} - h^{pk} h^{ql} F_{ij,rs} \nabla_l h_{ij} \nabla_k h_{rs} \\
&\quad - h^{pk} h^{ql} (F_{ik} \nabla_{il} H - F_{il} \nabla_{ik} H) \\
&\quad - F_{ij} (h^{pr} h^{ks} h^{ql} + h^{pk} h^{qr} h^{ls}) \nabla_i h_{rs} \nabla_j h_{kl}.
\end{aligned}$$

To proceed further we use the fact that $\nabla_i h_{jk}$ is invariant under the interchange of any pair of indices. Since $h_{ij}$ is symmetric, it suffices to show that $\nabla_i h_{jk} = \nabla_j h_{ik}$. By means of the standard formula for commuting the order of covariant differentiation, together with (3.25), we obtain

$$\begin{aligned}\nabla_i h_{jk} &= \nabla_i(\nabla_{jk}H + \delta_{jk}H) \\ &= \nabla_{jik}H + \delta_{jk}\nabla_i H + R_{ijk}{}^l\nabla_l H \\ &= \nabla_{jik}H + \delta_{jk}\nabla_i H + \delta_{ik}\nabla_j H - \delta_{jk}\nabla_i H \\ &= \nabla_j h_{ik}.\end{aligned}$$

Notice that we have not used the positivity of $[h_{ij}]$ to derive this. Using this we find

$$\begin{aligned}&h^{pk}h^{ql}F_{ij,rs}\nabla_l h_{ij}\nabla_k h_{rs} + F_{ij}(h^{pr}h^{ks}h^{ql} + h^{pk}h^{qr}h^{ls})\nabla_i h_{rs}\nabla_j h_{kl} \\ &\quad= (h^{pk}h^{ql}F_{ij,rs} + F_{rj}h^{pk}h^{is}h^{ql} + F_{ir}h^{pl}h^{qk}h^{js})\nabla_k h_{rs}\nabla_l h_{ij} \\ &\quad= h^{pk}h^{ql}(F_{ij,rs} + 2F_{ir}h^{js})\nabla_k h_{rs}\nabla_l h_{ij}.\end{aligned}$$

Substituting the above equation into the equation for $h^{pq}$ gives

$$\begin{aligned}\frac{\partial}{\partial t}h^{pq} &= F_{ij}\nabla_{ij}h^{pq} - 2Fh^{pk}h^{ql} + (\mathscr{T}+1)h^{pq} \\ &\quad - h^{pk}h^{ql}(F_{ij,rs} + 2F_{ir}h^{js})\nabla_k h_{rs}\nabla_l h_{ij} \\ &\quad - h^{pk}h^{ql}(F_{ik}\nabla_{il}H - F_{il}\nabla_{ik}H).\end{aligned}\tag{3.42}$$

In particular, for $p = q$ we have

$$\begin{aligned}\frac{\partial}{\partial t}h^{pp} &= F_{ij}\nabla_{ij}h^{pp} - 2Fh^{pk}h^{pk} + (\mathscr{T}+1)h^{pp} \\ &\quad - h^{pk}h^{pl}(F_{ij,rs} + 2F_{ir}h^{js})\nabla_k h_{rs}\nabla_l h_{ij},\end{aligned}\tag{3.43}$$

and if we impose the condition

$$(F_{ij,rs} + 2F_{ir}h^{js})\eta_{ij}\eta_{rs} \geq 0\tag{3.44}$$

for all real symmetric $n \times n$ matrices $[\eta_{ij}]$, then

$$\frac{\partial}{\partial t}h^{pp} \leq F_{ij}\nabla_{ij}h^{pp} - 2Fh^{pk}h^{pk} + (\mathscr{T}+1)h^{pp}\tag{3.45}$$

for any $p = 1, \cdots, n$.

Now let us suppose that the maximum eigenvalue of $[h^{ij}]$ over $S^n$ at time $t$ is attained at a point $x_t \in S^n$ with unit eigenvector $\xi_t \in T_{x_t}S^n$.

By a rotation of the frame $e_1, \cdots, e_n$ at $x_t$, we may assume that at $x_t$ we have $\xi_t = e_1$. At $(x_t, t)$ we thus have

$$\begin{aligned}\frac{\partial}{\partial t}h^{11} &\le -2F(h^{11})^2 + (\mathcal{T}+1)h^{11} \le -2\mathcal{T}h^{11} + (\mathcal{T}+1)h^{11} \\ &= -(\mathcal{T}-1)h^{11} \le 0\end{aligned}$$

since $\mathcal{T} \ge 1$ and $h^{11} > 0$. By an argument similar to the one used in the proof of Lemma 3.4, we conclude that the maximum eigenvalue of $[h^{ij}]$ over $S^n$ at time $t$ is bounded by the maximum eigenvalue of $[h^{ij}]$ over $S^n$ at $t = 0$.

Case (ii) is proved, except for the verification of condition (3.44). This condition appears to be somewhat artificial, but as was pointed out to us by Gerhard Huisken, it follows easily from the more natural hypothesis (ii). For, by condition (ii), the function $\widetilde{F}$ defined by

$$F(h_{ij}) = \frac{1}{\widetilde{F}(h^{ij})} \tag{3.46}$$

is a concave function, and we have

$$F_{ij} = \widetilde{F}^{-2}\widetilde{F}_{mn}h^{mi}h^{nj}, \tag{3.47}$$

$$\begin{aligned}F_{ij,rs} = &-\widetilde{F}^{-2}\widetilde{F}_{mn}(h^{mr}h^{is}h^{nj} + h^{mi}h^{nr}h^{js}) \\ &-\widetilde{F}^{-2}\widetilde{F}_{mn,pq}h^{mi}h^{nj}h^{pr}h^{qs} + 2\widetilde{F}^{-3}\widetilde{F}_{mn}h^{mi}h^{nj}\widetilde{F}_{pq}h^{pr}h^{qs}.\end{aligned} \tag{3.48}$$

Using the concavity of $\widetilde{F}$, and the symmetry of $\widetilde{F}_{mn}$ to interchange some indices, we obtain

$$(F_{ij,rs} + 2F_{ir}h^{js})\eta_{ij}\eta_{rs} \ge 2F^{-1}(F_{ij}\eta_{ij})^2 \ge 0 \tag{3.49}$$

for any real symmetric $n \times n$ matrix $[\eta_{ij}]$. Thus (3.44) holds and the lemma is proved.

**Remarks.** (i) In case (i) of Lemma 3.8 it is sufficient to assume

$$\limsup_{\lambda \to \overline{\Gamma}_K \cap \partial\Gamma} f(\lambda) < \min_{S^n} F(\nabla^2 H_0 + H_0 I),$$

where $K$ is the maximum eigenvalue of $\nabla^2 H_0 + H_0 I$ over $S^n$ and $\Gamma_K = \{\lambda \in \Gamma : \lambda_i \le K \text{ for all } i\}$.

(ii) Lemmas 3.5, 3.7, and 3.8 are of special interest for the case $\Gamma = \Gamma^+$, for in the context of the initial value problem (1.1), they imply that the upper and lower bounds for the principal radii of curvature (and hence also for the principal curvatures) of the normalized hypersurfaces $\widetilde{M}_t$, and the lower bound for the normalized curvature function $\tilde{k}$, are preserved by the evolution. In particular, all three assertions are valid in the case that $f$ is given by $\sigma_m$ or $\tilde{\sigma}_m$.

Once we know that the eigenvalues of $\nabla^2 H + HI$ remain in a fixed compact subset of $\Gamma$, (3.21) is uniformly parabolic by virtue of (3.4) and the smoothness of $f$. Thus there exist positive constants $\lambda$ and $\Lambda$, depending only on $n$, $f$, $\Gamma$, and $H_0$, such that

$$\lambda|\xi|^2 \le F_{ij}(\nabla^2 H + HI)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{3.50}$$

for any $\xi \in \mathbf{R}^n$. Hölder continuity estimates for $\nabla^2 H$ and $\partial H/\partial t$ now follow from results of Krylov and Safonov [13], [14], and once we have these, estimates for higher derivatives follow from the standard theory of linear uniformly parabolic equations.

**Lemma 3.9.** *Let $H$ be an admissible solution of* (3.18) *on $S^n \times [0, T]$. Then for any $t \in (0, T]$ we have*

$$\left\|\frac{\partial H}{\partial t}\right\|_{\tilde{C}^{0,\alpha}(S^n\times[t,T])} + \|\nabla^2 H\|_{\tilde{C}^{0,\alpha}(S^n\times[t,T])} \le C, \tag{3.51}$$

*where $\alpha \in (0, 1)$ depends only on $n$, $\Lambda$, and $\lambda$, and $C$ depends in addition on $t^{-1}$ and* $\|H\|_{\tilde{C}^2(S^n\times[0,T])}$.

*Proof.* The results of Krylov and Safonov are proved for equations on domains in Euclidean space, so it is convenient to work in a local coordinate chart in order to apply these. Let $u$ be given by (2.15). In the local coordinate system given by (2.13) $u$ solves

$$\begin{aligned} \frac{\partial u}{\partial t} &= (1+|x|^2)F(\hat{b}_{ij}) - u \quad \text{on } \mathbf{R}^n \times [0, T], \\ u(\cdot, 0) &= u_0, \end{aligned} \tag{3.52}$$

where $\hat{b}_{ij}$ is given by (2.21), and $u_0$ by (2.15) with $u$, $H$ replaced by $u_0$, $H_0$. Since we have bounds for the spatial derivatives of $H$ up to order two and for the first time derivative of $H$ on $S^n \times [0, T]$, we also have similar bounds for $u$ on $\overline{B_2(0)} \times [0, T]$. Furthermore, (3.52) is uniformly parabolic on $\overline{B_2(0)} \times [0, T]$ with parabolicity constants $\tilde{\lambda}$ and $\tilde{\Lambda}$ depending only on $\lambda$ and $\Lambda$ in (3.50).

Differentiating (3.52) with respect to $t$ yields

$$\frac{\partial^2 u}{\partial t^2} = A_{ij} D_{ij} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t}, \tag{3.53}$$

where

$$A_{ij} = (1+|x|^2)F_{kl}\left(\delta_{ik} + \frac{x_i x_k}{1+(1+|x|^2)^{1/2}}\right)\left(\delta_{jl} + \frac{x_j x_l}{1+(1+|x|^2)^{1/2}}\right).$$

Thus $\partial u/\partial t$ satisfies a linear locally uniformly parabolic equation on $\mathbf{R}^n \times [0, T]$. A result of Krylov and Safonov [14] (see also [13, Theorem 4.2.7]) implies that for any $t \in (0, T)$ we have

$$(3.54) \qquad \left\| \frac{\partial u}{\partial t} \right\|_{\tilde{C}^{0,\alpha_1}(\overline{B_1(0)} \times [t, T])} \leq C_1,$$

where $\alpha_1 \in (0, 1)$ depends only on $n$, $\tilde{\lambda}$, and $\tilde{\Lambda}$, and $C_1$ depends in addition on $t^{-1}$ and $\|\partial u/\partial t\|_{\tilde{C}^0(\overline{B_2(0)} \times [0, T])}$.

Next, since $F$ is concave we may apply a result of Krylov [13, Theorem 5.5.2] to deduce that for any $t \in (0, T)$ we have

$$(3.55) \qquad \|D^2 u\|_{\tilde{C}^{0,\alpha_2}(\overline{B_1(0)} \times [t, T])} \leq C_2,$$

where $\alpha_2 \in (0, 1)$ depends on the same quantities as $\alpha_1$, and $C_2$ depends in addition on $t^{-1}$ and $\|u\|_{\tilde{C}^2(\overline{B_2(0)} \times [0, T])}$.

Notice that in the estimate (3.55) there is no explicit dependence on the derivatives of $F$. This is because the first derivatives $F_{ij}$ are controlled by the parabolicity constant $\tilde{\Lambda}$, and an examination of the proof of (3.55) shows that dependence on the second derivatives of $F$ does not occur, since the right-hand side of (3.52) is concave as a function of the matrix $\hat{b}_{ij}$, and not just of $D^2 u$.

Finally, since $H$ restricted to any hyperplane $x_j = \pm 1$, $j = 1, \cdots, n+1$, satisfies equations similar to (3.52), a covering argument yields the conclusion of the lemma.

Standard parabolic theory (see [15]) applied to (3.52) together with appropriate covering arguments yields higher order estimates.

**Lemma 3.10.** *Let $H$ be an admissible solution of* (3.18) *on $S^n \times [0, T]$. Then for any $t \in (0, T)$, any positive integer $k$, and any $\alpha \in (0, 1)$ we have*

$$(3.56) \qquad \|H\|_{\tilde{C}^{k,\alpha}(S^n \times [t, T])} \leq C,$$

*where $C$ depends only on $n$, $k$, $\alpha$, $\Lambda$, $\lambda$, $t^{-1}$, $F$, and $\|H\|_{\tilde{C}^2(S^n \times [0, T])}$.*

**Remark.** The estimates (3.51) and (3.56) blow up as $t \to 0$. They may be extended up to $t = 0$, but we do not need this.

We are now in a position to prove the existence assertion of Theorem 3.1 in cases (i) and (ii). Since $H_0$ is a smooth admissible function, a standard argument using the implicit function theorem yields the existence of a unique smooth admissible solution of (3.18) on $S^n \times [0, T)$ for some small positive $T$. Let $[0, T^*)$ be the maximal interval for which a smooth admissible solution exists, and suppose that $T^* < \infty$. The estimates obtained above show that this solution can be extended smoothly to $[0, T^*]$ and that $H(\cdot, T^*)$ is admissible. We may now use the implicit function theorem again to obtain a smooth admissible solution on an interval strictly larger than $[0, T^*]$. This contradicts the maximality of $T^*$, so we conclude that a smooth admissible solution of (3.18) exists for all time.

The uniqueness of smooth admissible solutions of (3.18) is easily established. If $H_1$ and $H_2$ are two such solutions, using the convexity of $\mathscr{M}(\Gamma)$ it is easy to see that $w = H_1 - H_2$ satisfies a parabolic differential equation of the form

$$\frac{\partial w}{\partial t} = a_{ij}\nabla_{ij}w + cw, \tag{3.57}$$

where $c \geq 0$. By arguing as in the proof of Lemma 3.4, we see that $w \leq 0$ on $S^n \times [0, \infty)$. The proof that $H_1 \geq H_2$ is similar.

Notice that this argument also shows that if $H_1$ and $H_2$ are two admissible solutions of (3.21) and $H_1 \leq H_2$ at $t = 0$, then $H_1 \leq H_2$ for all time.

Let us now prove the existence of an admissible solution in case (iii) of Theorem 3.1. Thus we assume $n = 2$. Since $H_0$ is admissible and smooth, the eigenvalues of $\nabla^2 H_0 + H_0 I$ lie in a compact subset $K$ of $\Gamma$. Thus we may find a symmetric, convex cone $\Gamma'$ with vertex at the origin such that $K \subset \Gamma'$ and $\overline{\Gamma'} - \{0\} \subset \Gamma$. We may suppose that $\Gamma'$ is given by

$$\Gamma' = \left\{\lambda \in \mathbf{R}^2 : \frac{\min\{\lambda_1, \lambda_2\}}{\max\{\lambda_1, \lambda_2\}} > \mu,\ \max\{\lambda_1, \lambda_2\} > 0\right\} \tag{3.58}$$

for some constant $\mu > -1$.

Let us now define a function $\tilde{f}$ by

$$\tilde{f}(\lambda) = \inf_{\lambda_0 \in \Gamma'} \{f(\lambda_0) + Df(\lambda_0) \cdot (\lambda - \lambda_0)\}. \tag{3.59}$$

Using the properties of $f$, it is not difficult to check that $\tilde{f}$ agrees with $f$ on $\overline{\Gamma'}$, the set $\tilde{\Gamma}$ where $\tilde{f}$ is positive is a convex, open, symmetric cone with vertex at the origin containing $\Gamma'$, and that conditions (3.3), (3.4), and (3.5) with $f$, $\Gamma$ replaced by $\tilde{f}$, $\tilde{\Gamma}$ are satisfied. Furthermore, since $f$ is smooth, we have $\tilde{f} \in C^{1,1}(\tilde{\Gamma}) \cap C^0(\overline{\tilde{\Gamma}})$, and $\tilde{f} \equiv 0$ on $\partial\tilde{\Gamma}$. Thus $\tilde{f}$ satisfies all the hypotheses of Theorem 3.1(i), except that $\tilde{f}$ is of class $C^{1,1}$ rather than $C^\infty$. By a straightforward approximation argument, we deduce the existence of a unique $\tilde{\Gamma}$-admissible solution

$\widetilde{H} \in \widetilde{C}^{2,\alpha}(S^2 \times [0, \infty))$ of the initial value problem

$$(3.60)\qquad \begin{aligned} \frac{\partial \widetilde{H}}{\partial t} &= \widetilde{F}(\nabla^2 \widetilde{H} + \widetilde{H} I) - \widetilde{H} \quad \text{on } S^2 \times [0, \infty), \\ \widetilde{H}(\cdot, 0) &= H_0, \end{aligned}$$

where $\widetilde{F}$ is defined in the same way as $F$, with $f$ replaced by $\tilde{f}$. Since the eigenvalues of $\nabla^2 H_0 + H_0 I$ lie in a compact subset of $\Gamma'$, the same is true for $\widetilde{H}$ on a sufficiently small time interval $[0, T)$. We assert that the eigenvalues of $\nabla^2 \widetilde{H} + \widetilde{H} I$ in fact remain in $\overline{\Gamma'}$ for all time. Once we have shown this, we see that $\widetilde{H}$ is a $\Gamma$-admissible solution of (3.18) on $S^2 \times [0, \infty)$, since $\tilde{f}$ agrees with $f$ on $\overline{\Gamma'}$. Thus Theorem 3.1 in case (iii) will be proved.

Let us denote the ratio of the minimum and maximum eigenvalues of $[h_{ij}] = [\nabla^2 \widetilde{H} + \widetilde{H} I]$ by $w$. Then we want to show that $w \geq \mu$ on $S^2 \times [0, \infty)$. Suppose that at time $t$, $w(\cdot, t)$ attains its minimum at a point $x_t \in S^2$. By a rotation of the frame $e_1$, $e_2$ at $x_t$, we may assume that $w = h_{11}/h_{22}$ at $(x_t, t)$ with $h_{11} < h_{22}$ and $h_{22} > 0$ there. Let us also assume that $w(x_t, t) < \mu$. Then at $(x_t, t)$ the eigenvalues of $[h_{ij}]$ lie in $\widetilde{\Gamma} - \overline{\Gamma'}$, and since $u$ is of class $\widetilde{C}^{2,\alpha}$, this is also true on a small neighborhood of $(x_t, t)$. Since $n = 2$, $\tilde{f}$ is linear on each of the two components of $\widetilde{\Gamma} - \overline{\Gamma'}$, and we see that $\widetilde{H}$ is of class $C^\infty$ in a neighborhood of $(x_t, t)$. We may also suppose that $h_{22} > 0$ on this neighborhood. Then we may differentiate $h_{11}/h_{22}$ twice near $(x_t, t)$ to obtain

$$(3.61)\qquad \nabla_j \left( \frac{h_{11}}{h_{22}} \right) = \frac{\nabla_j h_{11}}{h_{22}} - \frac{h_{11} \nabla_j h_{22}}{(h_{22})^2},$$

$$(3.62)\qquad \begin{aligned} \nabla_{ij} \left( \frac{h_{11}}{h_{22}} \right) = \frac{\nabla_{ij} h_{11}}{h_{22}} - \frac{\nabla_j h_{11} \nabla_i h_{22}}{(h_{22})^2} - \frac{\nabla_i h_{11} \nabla_j h_{22}}{(h_{22})^2} \\ + 2 h_{11} \frac{\nabla_i h_{22} \nabla_j h_{22}}{(h_{22})^3}. \end{aligned}$$

Using (3.29), we see that near $(x_t, t)$, $v = h_{11}/h_{22}$ satisfies the equation

$$(3.63)\qquad \begin{aligned} \frac{\partial}{\partial t} v = \widetilde{F}_{ij} \nabla_{ij} v + 2 \widetilde{F}_{ij} \frac{\nabla_j h_{22}}{h_{22}} \nabla_i v + \frac{2\widetilde{F}}{(h_{22})^2} (h_{22} - h_{11}) \\ + \frac{1}{h_{22}} \widetilde{F}_{ij,rs} \nabla_1 h_{ij} \nabla_1 h_{rs} - \frac{h_{11}}{(h_{22})^2} \widetilde{F}_{ij,rs} \nabla_2 h_{ij} \nabla_2 h_{rs}. \end{aligned}$$

Since $\tilde{f}$ is linear on each of the two components of $\tilde{\Gamma} - \Gamma'$ we have $\tilde{F}_{ij,rs} = 0$ at $(x_t, t)$, and so we deduce that

$$\frac{\partial v}{\partial t} \geq 0 \quad \text{at } (x_t, t). \tag{3.64}$$

Since $w > \mu$ at $t = 0$, a slight modification of the argument used in Lemma 3.4 now shows that $w \geq \mu$ for all time. The existence assertion of Theorem 3.1 in case (iii) is proved.

**Remark.** If $n \geq 3$, then $\tilde{f}$ is not linear in $\tilde{\Gamma} - \Gamma'$. All we obtain in this case is that two of the eigenvalues of $D^2\tilde{f}$ on $\tilde{\Gamma} - \Gamma'$ are zero, which is not sufficient for our argument. However, we expect that if $H$ is a smooth solution of (3.18) and $H(\cdot, 0)$ is $\Gamma$-admissible, then $H$ remains $\Gamma$-admissible for all time, without the hypotheses (i) or (ii) of Theorem 3.1. More precisely, we conjecture that if $\Gamma'$ is the smallest closed, convex, symmetric cone with vertex at the origin which contains the eigenvalues of $\nabla^2 H + HI$ at $t = 0$, then the eigenvalues of $\nabla^2 H + HI$ remain in $\Gamma'$ for all time.

The only assertions of Theorem 3.1 which are still to be proved are those concerning the asymptotic behavior of $H$. To do this we consider the matrix $[r_{ij}] = \nabla^2 H + (H - F)I$, and suppose that its maximum eigenvalue over $S^n$ at time $t$ is attained at a point $x_t \in S^n$ with unit eigenvector $\xi_t \in T_{x_t} S^n$. As usual we may assume that $\xi_t = e_1$ at $x_t$. From (3.30) and (3.36) it follows that $r_{11}$ satisfies the differential inequality

$$\frac{\partial}{\partial t} r_{11} \leq F_{ij} \nabla_{ij} r_{11} - (\mathscr{T} + 1) r_{11} - 2(\mathscr{T} - 1)F. \tag{3.65}$$

Since $r_{11}$ is the maximum eigenvalue of $[r_{ij}]$ at $(x_t, t)$ and $F$ satisfies (3.8), (3.9), and (3.10), we see that $r_{11}$ is nonnegative at $(x_t, t)$. Using this together with the fact that $F > 0$ and $\mathscr{T} \geq 1$, we obtain

$$\frac{\partial}{\partial t} r_{11} \leq -2 r_{11} \quad \text{at } (x_t, t). \tag{3.66}$$

An argument similar to that used in the proof of Lemma 3.4 then yields

$$\max_{x \in s^n} [\lambda_{\max}(x, t) - f(\lambda(x, t))] \leq C_1 e^{-2t}, \tag{3.67}$$

where $\lambda(x, t) = (\lambda_1(x, t), \cdots, \lambda_n(x, t))$ denotes the set of eigenvalues of $\nabla^2 H + HI$ at $(x, t)$, $\lambda_{\max}(x, t)$ is the maximum eigenvalue of $\nabla^2 H + HI$ at $(x, t)$, and $C_1$ is a positive constant depending only on $H_0$ and $f$. Using Lemma 3.3 we obtain

$$(\lambda_{\max}(x, t) - \lambda_{\min}(x, t)) \leq n C_1 e^{-2t}, \tag{3.68}$$

where $\lambda_{\min}(x, t)$ denotes the minimum eigenvalue of $\nabla^2 H + HI$ at $(x, t)$. Thus for each $x \in S^n$,

$$\operatorname{dist}(\lambda(x, t), \mathscr{D}) \le Ce^{-2t}, \tag{3.69}$$

where $\mathscr{D}$ is the diagonal of $\Gamma$. Since $f$ is smooth on $\Gamma$ and the eigenvalues of $\nabla^2 H + HI$ remain in a fixed compact convex subset $K$ of $\Gamma$, using (3.15) we deduce that

$$\left|\frac{\partial f}{\partial \lambda_i}(\lambda(x, t)) - \frac{1}{n}\right| \le \sup_K |D^2 f| \operatorname{dist}(\lambda(x, t), \mathscr{D}) \le Ce^{-2t}, \tag{3.70}$$

and hence, using (3.16), we obtain

$$\begin{aligned} \frac{1}{n}\Delta H + H - Ce^{-2t} &\le F_{ij}(\nabla_{ij} H + \delta_{ij} H) \\ &= F(\nabla^2 H + HI) \le \frac{1}{n}\Delta H + H, \end{aligned} \tag{3.71}$$

$$\frac{1}{n}\Delta H - Ce^{-2t} \le \frac{\partial H}{\partial t} \le \frac{1}{n}\Delta H. \tag{3.72}$$

Next we show that $\overline{H}(t) = (1/|S^n|)\int_{S^n} H(x, t)\,dx$ converges as $t \to \infty$. Integrating (3.72) over $S^n$ and using the divergence theorem give $-Ce^{-2t} \le \frac{d}{dt}\overline{H}(t) \le 0$, and then integrating this over any interval $[t_1, t_2] \subset [0, \infty)$ we obtain

$$0 \le \overline{H}(t_1) - \overline{H}(t_2) \le Ce^{-2t_1}.$$

Since $\overline{H}$ is a nonincreasing function and $\overline{H}$ is bounded below, we conclude that $H^* \equiv \lim_{t\to\infty} \overline{H}(t)$ exists, and furthermore, that

$$|\overline{H}(t) - H^*| \le Ce^{-2t}. \tag{3.73}$$

To show that $H$, and not just $\overline{H}$, converges to $H^*$ multiplying (3.72) by $H$, integrating over $S^n$, and using the Poincaré inequality on $S^n$ we obtain

$$\frac{d}{dt}\int_{S^n} H^2 \le -\frac{2}{n}\int_{S^n} |\nabla H|^2 + Ce^{-2t} \le -2\int_{S^n}(H - \overline{H})^2 + Ce^{-2t}.$$

Since $\overline{H} > 0$ and $-Ce^{-2t} \le \frac{d}{dt}\overline{H} \le 0$ by (3.16), and

$$\int_{S^n}(H^2 - \overline{H}^2) = \int_{S^n}(H - \overline{H})^2,$$

we have

$$\int_{S^n}(H - \overline{H})^2 \le C(\gamma)e^{-\gamma t}, \tag{3.74}$$

and hence

$$\int_{S^n} (H - H^*)^2 \le C(\gamma)e^{-\gamma t} \tag{3.75}$$

for any $\gamma < 2$.

To obtain the convergence of $H$ to $H^*$ in the $C^k$ norms, we use an interpolation inequality (see [9, Corollary 12.7]).

**Lemma 3.11.** *Let $T$ be a smooth tensor field on $S^n$. Then for any integers $k$, $m$ such that $0 \le k \le m$, we have*

$$\int_{S^n} |\nabla^k T|^2 \le C\left(\int_{S^n} |\nabla^m T|^2\right)^{k/m} \left(\int_{S^n} |T|^2\right)^{1-k/m}, \tag{3.76}$$

*where $C$ depends only on $m$ and $n$.*

Applying this to $H - H^*$ and using the fact that all derivatives of $H$ are bounded independently of $t$, we obtain

$$\int_{S^n} |\nabla^k H|^2 \le C_k(\gamma, \tilde{\gamma})e^{-\tilde{\gamma} t} \tag{3.77}$$

for any $\tilde{\gamma} < \gamma < 2$. By the Sobolev embedding theorem on $S^n$ (see [1, §2.7]) we have

$$\|H - H^*\|_{C^l(S^n)} \le C(k, l)\left(\int_{S^n} |\nabla^k H|^2 + |H - H^*|^2\right)^{1/2} \tag{3.78}$$

for any $k > l + n/2$. The estimates (3.7) now follow from (3.75), (3.77), and (3.78), and Theorem 3.1 is completely proved.

Next, we prove the assertions of Theorem 1.1 concerning the asymptotic convergence. In the case that the matrix $\nabla^2 H + HI$ is positive definite, which is true in any case for $t$ large enough, we saw in §2 that $H(\cdot, t)$ is the support function of a convex hypersurface $\widetilde{M}_t$, and from above, we know that $\widetilde{M}_t$ converges in the $C^\infty$ topology to a sphere $\widetilde{M}$ of radius $H^*$ centered at the origin. Thus $\lambda_{\max}(x, t)$ and $\lambda_{\min}(x, t)$ converge to $H^*$ as $t \to \infty$ for all $x \in S^n$, and from (3.68) it follows that

$$\sup_{x \in S^n} |\nabla^2 H(x, t) + H(x, t)I - H^* I| \le Ce^{-2t}. \tag{3.79}$$

By applying the interpolation inequality (3.76) to $\nabla^2 H(\cdot, t) + H(\cdot, t)I - H^* I$, and using the Sobolev inequality (3.78) as before, we deduce that (1.7) holds. The unnormalized inequality (1.8) then follows.

Finally, let us prove Theorem 1.2. This can be done working directly with the initial value problem (1.1). However, it can also be proved using our method. Since $M_0$ has positive curvature and rotation number $m > 0$, the Gauss map of $M_0$ covers $S^1$ $m$ times. Thus we may think of the Gauss map of $M_0$ as a one-to-one map of $M_0$ onto $S^1_m$, the $m$-fold cover of $S^1$. We may define the support function $H_0$ of $M_0$ on $S^1_m$ in the obvious way, and arguing as in §2 with only minor modifications, we see that after normalization, (1.1) can be reduced to the initial value problem

$$\begin{aligned} &\frac{\partial H}{\partial t} = H'' \quad \text{on } S^1_m \times [0, \infty), \\ &H(\cdot, 0) = H_0, \end{aligned} \tag{3.80}$$

together with the condition

$$H'' + H > 0, \tag{3.81}$$

where $H''$ denotes the second derivative of $H$ on $S^1_m$. The appropriate assertions concerning the existence, uniqueness, and asymptotic behavior of solutions of (3.80) and (3.81) are easily proved.

**Remark.** (i) An examination of the proofs of Lemmas 3.4 to 3.8 shows that they are still valid in the degenerate case (3.17), and the estimates obtained do not depend quantitatively on $f$ in any way, while in Lemma 3.9 the $\widetilde{C}^{2,\alpha}$ bound depends only on positive upper and lower bounds for $\partial f/\partial\lambda_i$ on a suitable compact subset of $\Gamma$, and in case (i) of Theorem 3.1, also on the modulus of continuity of $f$ on $\overline{\Gamma}\cap\overline{B_R}(0)$ for some $R > 0$ depending only on the initial data. Thus by using a suitable approximation argument we may deduce the existence of $\widetilde{C}^{2,\alpha}$ admissible solutions of (3.18) for nonsmooth functions $f$ satisfying the hypotheses of Theorem 3.1 in cases (i) and (iii). Thus, for example, we may consider functions $f$ of the form $f = \inf_{1\leq k\leq N} f_k$, where each $f_k$ is a smooth function satisfying the hypotheses of $f$ in Theorem 3.1 in cases (i) and (iii). We may also allow infima over a countably infinite number of such functions $f_k$, provided the hypotheses are satisfied uniformly with respect to $k$. In case (ii) an approximation argument is more delicate because condition (ii) needs to be preserved by the approximations to $f$. In this case however, we recall that condition (ii) implies (3.44), which in turn, together with the concavity of $F$, implies that $F$ is locally of class $C^{1,1}$. Parabolic regularity theory (see [15]) then yields that $\widetilde{C}^2$ admissible solutions of (3.18) are in fact smooth enough for us to apply the Bony maximum principle [13, Theorem 3.4.10] in our arguments, and the proofs of Lemmas 3.4 to 3.9, and the consequent existence assertion proceed as before.

(ii) In the proof of asymptotic convergence we can still prove that the eigenvalues of $\nabla^2 H + HI$ approach one another at each point of $S^n$ at an exponential rate, provided $f$ is of class $C^{1,1}$, or at least $C^{1,\alpha}$ for some $\alpha > 0$. The proof of (3.75) proceeds as before, but the proof of convergence in higher norms breaks down, because we do not have bounds for sufficiently high derivatives of $H$, if $f$ is not smooth.

(iii) If we drop the hypothesis (3.4), we still have the degenerate parabolicity condition (3.17) since $\Gamma \supset \Gamma^+$ and $f$ is positive and concave. By approximation by strictly parabolic problems, we can deduce the existence of $\widetilde{C}^{1,1}$ admissible solutions of (3.18) for $C^{1,1}$ admissible initial data, provided the remaining hypotheses of Theorem 3.1, together with (i) or (iii), are satisfied. The proof that the eigenvalues of $\nabla^2 H + HI$ approach one another at each point of $S^n$ at an exponential rate proceeds as before, as does the proof of (3.75). However, since $\partial f/\partial\lambda_i = 1/n$ on the diagonal of $\Gamma$, the smoothness of $f$ then implies that after a finite time $T$, the equation becomes uniformly parabolic, and higher order estimates for the derivatives of $H$ after time $T$ follow, as does the proof of asymptotic convergence in any $C^k$ norm. However, if $f$ is merely Lipschitz, all we can conclude is the existence of a $\widetilde{C}^{1,1}$ admissible solution. An example of such a function is given by

$$f(\lambda_1, \cdots, \lambda_n) = \min\{\lambda_1, \cdots, \lambda_n\}.$$

(iv) Similar assertions apply in the case of Theorem 1.1. We leave it to the reader to formulate these. We remark only that if $H$ is not sufficiently smooth, then solutions of the initial value problem (2.25), (2.26) may not be unique. However, the hypersurfaces $M_t$ will still be unique.

## 4. An extension

In this section we shall study the initial value problem (3.1), where now $F$ is given by

$$F(a_{ij}) = f(\mu_1, \cdots, \mu_n)^\alpha, \tag{4.1}$$

where $\mu_1, \cdots, \mu_n$ are the eigenvalues of $[a_{ij}]$, $\alpha \in [0, 1)$ is a constant, and $f$ satisfies the hypotheses of Theorem 3.1. Then $\tilde{f} = f^\alpha$ is homogeneous of degree $\alpha$, and it is easy to see that

$$\frac{\partial \tilde{f}}{\partial \lambda_i} > 0 \quad \text{on } \Gamma \tag{4.2}$$

and

$$\tilde{f} \text{ is concave on } \Gamma, \tag{4.3}$$

since $\alpha \in [0, 1)$, and $f$ is positive and satisfies (3.4) and (3.5). As in the previous section, we may reduce to the case where $f(1, \cdots, 1) = 1$.

It is convenient to consider a normalized problem rather than (3.1). To derive this, we set $\delta = 1 - \alpha > 0$ and

$$\widetilde{H} = (1+\delta t)^{1/\delta} H. \tag{4.4}$$

Then, using the degree $\alpha$ homogeneity of $\tilde{f}$, and hence also of $F$, we see that $\widetilde{H}$ satisfies

$$\begin{aligned}&(1+\delta t)\frac{\partial \widetilde{H}}{\partial t} = F(\nabla^2 \widetilde{H} + \widetilde{H} I) - \widetilde{H} \quad \text{on } S^n \times [0, \infty), \\ &\widetilde{H}(\cdot, 0) = H_0.\end{aligned} \tag{4.5}$$

By replacing $t$ by the new time variable $s = \frac{1}{\delta}\log(1+\delta t)$, we may instead consider the initial value problem

$$\begin{aligned}&\frac{\partial \widetilde{H}}{\partial t} = F(\nabla^2 \widetilde{H} + \widetilde{H} I) - \widetilde{H} \quad \text{on } S^n \times [0, \infty), \\ &\widetilde{H}(\cdot, 0) = H_0.\end{aligned} \tag{4.6}$$

We shall prove the following result.

**Theorem 4.1.** *Let $\Gamma$ be an open convex cone in $\mathbf{R}^n$ as in Theorem* 3.1 *and let $F$ be given by* (4.1), *where $f \in C^\infty(\Gamma)$ is a positive symmetric function satisfying* (3.3), (3.4), *and* (3.5). *Suppose also that $f$ satisfies one of the hypotheses* (i) *and* (ii) *of Theorem* 3.1. *Then for any smooth positive admissible function $H_0$ on $S^n$, there exists a unique positive admissible solution $H \in C^\infty(S^n \times [0, \infty))$ of the initial value problem* (4.6), *and for any positive $\gamma < 1$ and any positive integer $k$, we have*

$$\|H(\cdot, t)^\delta - 1\|_{C^k(S^n)} \le C_k e^{-\gamma\delta t}, \tag{4.7}$$

*where $C_k$ is a positive constant depending only on $n$, $k$, $\alpha$, $\gamma$, $f$, $\Gamma$, and $H_0$.*

**Remark.** For admissible solutions of (4.5) we get the asymptotic behavior

$$\|H(\cdot, t)^\delta - 1\|_{C^k(S^n)} \le C_k (1+\delta t)^{-\gamma t}. \tag{4.8}$$

The proof of Theorem 4.1 is very similar to that of Theorem 3.1, and we shall indicate only the modifications which need to be made. Assume $\alpha > 0$. We shall use the following modification of Lemma 3.2.

**Lemma 4.2.** *We have*

$$\mathcal{T} \equiv \sum_{i=1}^n F_{ii} \ge \alpha F^{1-1/\alpha}. \tag{4.9}$$

*Proof.* The function $\tilde{f}^{1/\alpha}$ satisfies the hypotheses of Lemma 3.2. Thus

$$\frac{1}{\alpha}\tilde{f}^{1/\alpha-1}\sum_{i=1}^{n}\frac{\partial\tilde{f}}{\partial\lambda_i}\geq 1,$$

which implies (4.9).

We can now derive the a priori estimates we need.

**Lemma 4.3.** *Let $H$ be an admissible solution of* (4.6) *on $S^n\times[0,T)$. Then the following hold.*

(a) $\min\{1, \min_{S^n}H_0\}\leq H\leq\max\{1, \max_{S^n}H_0\}$.

(b) $\min_{S^n\times[0,T)}F(\nabla^2H+HI)\geq\min\{1, \min_{S^n}F(\nabla^2H_0+H_0I)\}$.

(c) *If $\nabla^2H+HI\leq KI$ at $t=0$, where $K\geq 1$ is a constant, then this remains true for all $t\in[0,T)$.*

(d) *In each of the cases* (i) *and* (ii) *of Theorem* 4.1, *the eigenvalues of $\nabla^2H+HI$ lie in a compact subset of $\Gamma$.*

*Proof.* (a) At a nonnegative spatial minimum of $H$ we have $\nabla^2H\geq 0$, so using the degree $\alpha$ homogeneity of $F$, we obtain $\partial H/\partial t\geq H^\alpha-H$ at such a point. At a negative spatial minimum of $H$ we have $\partial H/\partial t\geq 0$, since $F\geq 0$. The first estimate of (a) follows, and the second is proved similarly.

(b) It is easily verified that $F$ satisfies the equation

$$\frac{\partial F}{\partial t}=F_{ij}\nabla_{ij}F+(\mathscr{T}-\alpha)F.$$

Since $\mathscr{T}\geq\alpha F^{1-1/\alpha}$ by Lemma 4.2, the result follows.

(c) From the proof of Lemma 3.5 we see that for any $k=1,\cdots,n$,

$$\frac{\partial}{\partial t}h_{kk}\leq F_{ij}\nabla_{ij}h_{kk}+(1+\alpha)F-(\mathscr{T}+1)h_{kk},$$

where $h_{kk}=\nabla_{kk}H+H$. At a point $(x_t, t)$ where the maximum eigenvalue of $\nabla^2H+HI$ over $S^n$ at time $t$ is attained, which we may assume has eigenvector $e_1$, we have

$$\frac{\partial}{\partial t}h_{11}\leq(1+\alpha)F-(\mathscr{T}+1)h_{11}\leq h_{11}^{\alpha}-h_{11},$$

after some manipulations using Lemma 4.2 and the degree $\alpha$ homogeneity of $F$. Thus assertion (c) follows.

(d) Case (i) is exactly as before. In case (ii) we see that for any $p=1,\cdots,n$, $h^{pp}$ satisfies

$$\frac{\partial}{\partial t}h^{pp}=F_{ij}\nabla_{ij}h^{pp}-(1+\alpha)Fh^{pk}h^{pk}+(\mathscr{T}+1)h^{pp}$$
$$-h^{pk}h^{pl}(F_{ij,rs}+2F_{ir}h^{js})\nabla_kh_{rs}\nabla_lh_{ij}.$$

To handle this last term, we observe that since the function $g$ given by

$$g(\lambda_1, \cdots, \lambda_n) = \frac{1}{f(1/\lambda_1, \cdots, 1/\lambda_n)}$$

is concave by hypothesis, so is $g^\alpha$, since $g$ is positive. Thus (3.44) holds, and proceeding as before we see that at a point $(x_t, t)$ where the maximum eigenvalue of $[h^{ij}]$ over $S^n$ at time $t$ is attained, which we may suppose has eigenvector $e_1$, we have

$$\frac{\partial}{\partial t} h^{11} \le -(1+\alpha) F (h^{11})^2 + (\mathscr{T} + 1) h^{11} \le h^{11} - (h^{11})^{2-\alpha}$$

after some manipulations using the degree $\alpha$ homogeneity of $F$. Thus the maximum eigenvalue of $[h^{ij}]$ remains bounded, and the conclusion of the lemma follows.

Higher order estimates now follow exactly as before, as does the existence of solutions in cases (i) and (ii).

To prove the assertion concerning the asymptotic behavior of $H$, we recall that if $H_1$ and $H_2$ are two admissible solutions of (4.6) and $H_1 \le H_2$ at $t = 0$, then $H_1 \le H_2$ for all time. It is easily checked that positive solutions of the differential equation $du/dt = u^\alpha - u$, which are not identically one, are given by $u(t) = (1 - (1 - u(0)^\delta) e^{-\delta t})^{1/\delta}$, where $\delta = 1 - \alpha$. If $a \le \min_{S^n} H_0$ and $b \ge \max_{S^n} H_0$, with $0 < a < 1 < b$, then

$$(a^\delta - 1) e^{-\delta t} \le H(\cdot, t)^\delta - 1 \le (b^\delta - 1) e^{-\delta t}.$$

Thus $H$ converges to one exponentially. To obtain the convergence to zero of the derivatives of $H$, we use the interpolation and Sobolev inequalities (3.76) and (3.78) as before.

Finally, Theorem 4.1 in the case $\alpha = 0$ follows trivially by solving the initial value problem

$$\frac{d}{dt} H(x, t) = 1 - H(x, t),$$
$$H(x, 0) = H_0(x),$$

and regarding $x$ as a parameter.

We leave it to the reader to work out the analogues of (1.7) and (1.8) for the problem (1.1), (1.14).

## Added in proof

The results of this paper have recently been extended to the case of starshaped initial hypersurfaces by the author (*On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, to appear in Math. Z.) and by Claus Gerhardt (*Flow of nonconvex hypersurfaces into spheres*, J. Differential Geometry **32** (1990) 299–314).

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