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GAUSS MAPS OF SPACELIKE CONSTANT MEAN CURVATURE HYPERSURFACES OF MINKOWSKI SPACE

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Abstract

The Gauss map of a spacelike constant mean curvature hypersurface of Minkowski space is a harmonic map to hyperbolic space. The properties of such hypersurfaces are interpreted in terms of the harmonic mapping. Given an arbitrary closed set in the ideal boundary at infinity of hyperbolic space, there exists a complete entire spacelike constant mean curvature hypersurface whose Gauss map is a diffeomorphism onto the interior of the hyperbolic space convex hull of the set. Identifying ideal infinity with the light cone, this set corresponds to the lightlike directions of the hypersurface. In terms of this extrinsic data we give conditions for the hyperbolicity or parabolicity of this hypersurface. For example, if the set of lightlike directions has nonempty interior in the unit sphere, then this hypersurface can be constructed so as to admit nontrivial bounded harmonic functions. This gives many new examples of harmonic maps of the disk and the complex plane to the hyperbolic plane, which are of full rank.

There are relatively few examples of harmonic maps between noncompact manifolds. A class of such maps arises as the Gauss maps of entire spacelike constant mean curvature hypersurfaces of Minkowski space. These are harmonic maps into hyperbolic space by a verson of the Ruh-Vilms theorem. We construct certain constant mean curvature hypersurfaces and analyse their geometric and function theoretic properties. Thus, we are able to answer a question of Eells and Lemaire [17] by constructing many harmonic maps from $\mathbf{R}^2$ to the hyperbolic plane, which have rank two everywhere. We also construct maps of the hyperbolic plane into itself. A completely elementary example of family of nonconformal harmonic diffeomorphisms of the hyperbolic plane to itself has been constructed using similar methods [14]. The relation between harmonic maps and constant mean curvature surfaces has been studied by T. K. Milnor

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[31]. Akutagawa and Nishikawa [1] and O. Kobayashi [25], [26] have used harmonic maps to construct constant mean curvature surfaces.

Building on the earlier work of Cheng-Yau [11] and Treibergs [38], we obtain a structure theory of the Gauss map. The entire spacelike hypersurfaces u of constant mean curvature are asymptotically null in certain directions called the lightlike directions, L_u. We prove that after splitting off a trivial factor, the Gauss map is a harmonic diffeomorphism onto the interior of the convex hull of L_u in hyperbolic space. The convex hull is taken in the following sense: the hyperbolic space is canonically identified with the hyperboloid consisting of unit future-pointing timelike vectors and then the lightlike rays are naturally identified with the points at infinity of hyperbolic space. The convex hull of L_u is thus the usual convex hull in hyperbolic space with respect to the hyperbolic metric. The proof is based on the observations that the entire constant mean curvature hypersurfaces are graphs of convex functions u, and that the Gauss map into the Klein model of hyperbolic space is the tangential mapping of u.

Conversely, any closed subset of the boundary at infinity is the set of lightlike directions of some entire spacelike constant mean curvature hypersurface. This is a direct interpretation of the solubility of the mean curvature equation with boundary values at infinity [38]. By sharpening the existence theory, we construct constant mean curvature hypersurfaces with given asymptotic behavior along the lightlike directions showing that there are many different surfaces with the same L_u. We can also construct solutions which have good asymptotic estimates in interior directions of L_u. This guarantees that there is an abundant supply of interesting harmonic maps to hyperbolic space. The existence of entire prescribed mean curvature hypersurfaces has also been studied in Minkowski space [8], [36] as well as in more general Lorentzian manifolds (e.g. [6], [7], [16], [18].).

To understand the mappings obtained as Gauss maps of constant mean curvature hypersurfaces, we are able to determine the function theory of the hypersurfaces from the lightlike set in some cases. We show that if the lightlike set contains a ball, then the hypersurface is hyperbolic in the sense that it admits a nontrivial bounded harmonic function. In this case the geometry is well controlled in the sector corresponding to the ball. It is quasi-isometric to a sector of the hyperbolic space, and the sectional curvature of the sector is pinched between two negative constants. This implies that there are nonconstant bounded harmonic functions on u. The proof of these facts depends on using the asymptotic behavior to estimate the metric and integral Gauss-Kronecker curvature. Then we prove a local Harnack-like mean value inequality and apply it to the Gauss-Kronecker curvature to conclude the result.

In §1, we set up the notation and state the known intrinsic results about constant mean curvature hypersurfaces. Theorem 1.2 provides a proof of a version of the Ruh-Vilms result. In §2 we state and prove the mean value inequality for the supersolutions we need. In §3, we show a splitting theorem used to simplify some future matters. It states that the constant mean curvature cuts decompose as the metric product of strictly convex cuts by hyperplanes. §4 is devoted to the extrinsic properties of constant mean curvature cuts and their implications to the Gauss map. In §5 we construct certain special constant mean curvature hypersurfaces by solving and estimating an ordinary differential equation. In §6 we use these new solutions to sharpen the existence theory and construct the cuts whose function theory will be determined. In §7 we use the special solutions to estimate the intrinsic properties of cuts. In §8, the existence of nontrivial bounded harmonic functions is established and in §9 theorems establishing the existence of certain harmonic mappings are assembled.

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1. Geometric preliminaries

Minkowski space $\mathbf{R}^{n,1}$ is $\mathbf{R}^n \times \mathbf{R}^1$ endowed with the metric $ds^2 = \sum_{i=1}^n (dx^i)^2 - (dx^{n+1})^2$, where $x = (x^1, \cdots, x^n)$ and x^{n+1} are the coordinates of $\mathbf{R}^n$ and $\mathbf{R}^1$. Let $\{\mathbf{e}_\alpha\}_{\alpha=1,\cdots,n+1}$ be an orthonormal frame, and let $\{w^\alpha\}_{\alpha=1,\cdots,n+1}$ be the dual coframe. Therefore, $w^\alpha(\mathbf{e}_\beta) = \delta^\alpha{}_\beta$. Let D be the usual Levi-Civita connection for $\mathbf{R}^{n,1}$. Then one can define the connection forms $\{w_\alpha{}^\beta\}$ by

$$D\mathbf{e}_\alpha = \sum_\beta w_\alpha{}^\beta \mathbf{e}_\beta.$$

We use the convention that all repeated Greek indices are summed from 1 to $n+1$, and all repeated Latin indices are summed from 1 to n. Hence

$$w_i{}^j + w_j{}^i = 0, \quad w_i{}^{n+1} = w_{n+1}{}^i \quad \text{for } i, j = 1, \cdots, n$$

which reflects the symmetry of the Maurer-Cartan forms of $\mathbf{SO}(n, 1)$. The torsion free condition of D is equivalent to the first structure equation

$$dw^\alpha = \sum_\beta w^\beta \wedge w_\beta{}^\alpha,$$

and the flatness of $\mathbf{R}^{n,1}$ is equivalent to the second structure equation

$$dw_\alpha{}^\beta = \sum_\gamma w_\alpha{}^\gamma \wedge w_\gamma{}^\beta.$$

Now suppose M is a spacelike hypersurface of $\mathbf{R}^{n,1}$, which means that M is a hypersurface whose induced metric is Riemannian. Locally M is given as a graph of a function $x^{n+1} = u(x^1, \cdots, x^n)$ satisfying the spacelike condition $|Du| < 1$. Throughout the paper we denote the hypersurface by M or u interchangeably. If the projection $M \ni (x^1, \cdots, x^n, x^{n+1}) \mapsto (x^1, \cdots, x^n)$ is onto $\mathbf{R}^n$ we say that M is entire. Suppose now that we have chosen an orthonormal frame adapted to M. In other words, $\mathbf{e}_1, \cdots, \mathbf{e}_n$ are tangent to M, and $\mathbf{e}_{n+1}$ is the future pointing unit timelike normal vector. Since $w^{n+1} = 0$ when restricted to M, we have

$$0 = dw^{n+1} = \sum_i w^i \wedge w_i{}^{n+1}.$$

By Cartan's lemma, there are h_{ij} such that $w_i{}^{n+1} = \sum_j h_{ij} w^j$ and $h_{ij} = h_{ji}$ for $i, j = 1, \cdots, n$. This gives the second fundamental form of M. The mean curvature H is defined by

$$H = \frac{1}{n} \sum_i h_{ii}.$$

Local geometry of M can be read off the structure equations. The second structure equation gives

$$dw_i{}^j - \sum_k w_i{}^k \wedge w_k{}^j = w_i{}^{n+1} \wedge w_{n+1}{}^j = \sum_{k,l} h_{ik} h_{jl} w^k \wedge w^l.$$

On the other hand,

$$dw_i{}^j - \sum_k w_i{}^k \wedge w_k{}^k = \Omega_i{}^j = -\frac{1}{2} \sum_{k,l} R_i{}^j{}_{kl} w^k \wedge w^l,$$

where $\Omega_i{}^j$ is the curvature 2-form, and $R_i{}^j{}_{kl}$ is the usual component of the Riemann curvature tensor. We lower the upper index to the second slot to get

$$R_{ijkl} = -h_{ik} h_{jl} + h_{il} h_{jk}. \tag{1.1}$$

Note that our sign convention is such that R_{1212} is the sectional curvature of the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. Therefore the Ricci tensor R_{ij} and the scalar curvature S are given by

$$R_{ij} = -nHh_{ij} + \sum_k h_{ik} h_{kj}, \qquad S = -n^2 H^2 + \sum_{i,j} h_{ij}^2. \tag{1.2}$$

Following [12], we define the covariant derivative h_{ijk} of h_{ij} by

$$\sum_k h_{ijk} w^k = dh_{ij} - \sum_k h_{ik} w_j{}^k - \sum_k h_{kj} w_i{}^k.$$

Similarly one defines h_{ijkl} by

$$\sum_l h_{ijkl} w^l = dh_{ijk} - \sum_l h_{ijl} w_k{}^l - \sum_l h_{ilk} w_j{}^l - \sum_l h_{ljk} w_i{}^l.$$

Commutation formulas hold [11], [38]:

$$\begin{gathered} h_{ij} = h_{ji}, \qquad h_{ijk} = h_{ikj}, \\ h_{ijkl} - h_{ijlk} = \sum_p (h_{ip} R_{pjkl} + h_{pj} R_{pikl}). \end{gathered} \tag{1.3}$$

Define $A = \sum_{i,j} h_{ij}^2$ and $B = \sum_{i,j,k} h_{ijk}^2$. Then the following Bochner type formula (Simons identity) holds [38]:

$$\frac{1}{2}\Delta A = B + n\sum_{i,j} H_{ij} h_{ij} + A^2 - nH \sum_{i,j,k} h_{ij} h_{jk} h_{ki}.$$

Therefore, if H is a constant, we have

$$\Delta A \geq 2A^2 - 2nHA^{3/2}. \tag{1.4}$$

Before we proceed, let us list some important earlier results.

Proposition 1.1 [11], [38]. *Let M be an entire spacelike hypersurface of Minkowski space $\mathbf{R}^{n,1}$ which is closed with respect to the Euclidean topology of $\mathbf{R}^{n+1}$. Suppose the mean curvature H is constant. Then the following hold:*

(1) *M can be represented as a graph of a function $x^{n+1} = u(x) = u(x^1, \cdots, x^n)$ such that u is defined for all $x \in \mathbf{R}^n$, and u satisfies the equation*

$$\mathscr{M} u = \frac{1}{n} \sum_i \frac{\partial}{\partial x_i} \left(\frac{\partial u / \partial x^i}{\sqrt{1 - |Du|^2}} \right) = H.$$

(2) *M is complete with respect to the induced Riemannian metric.*

(3) *M has nonpositive sectional curvature. If $H \geq 0$, all principal curvatures are nonnegative, and therefore u is a convex function.*

(4) $nH^2 \leq A \leq n^2 H^2$.

(5) $B \leq 36 n^5 H^4$.

We say M is a constant mean curvature cut if M is an entire spacelike hypersurface of $\mathbf{R}^{n,1}$ with constant mean curvature H, and which is closed with respect to the Euclidean topology. Cheng and Yau [11] proved that if M is a cut and $H \equiv 0$, then M is a hyperplane. For constant mean curvature cuts M, Proposition 1.1 implies that Ricci curvature R_{ij} and the scalar curvature $S = -n^2H^2 + A$ are nonpositive. Therefore $F = |S| = n^2H^2 - A$ is a smooth nonnegative function. Formula (1.4) can be now rewritten as

$$\Delta F = -\Delta A \leq \frac{2A^{3/2}(n^2H^2 - A)}{nH + \sqrt{A}} \leq 2n^2H^2F. \tag{1.5}$$

We end this section by checking that the Gauss map of a constant mean curvature spacelike hypersurface is a harmonic map into hyperbolic space. This was proved by Ruh-Vilms [34] in Euclidean space, by T. K. Milnor [31] for surfaces in Minkowski space and by Ishihara [24] in general. First, recall the harmonic map equation in moving frames [39]. Suppose M is a Riemannian manifold with local orthonormal coframe $\{w^i\}$ and connection form $\{w_i{}^j\}$, and let N be another Riemannian manifold with local orthonormal coframe $\{\theta^a\}$ and connection form $\{\theta_a{}^b\}$. Suppose $f\colon M \to N$ is a map. We define $f^a{}_i$ by $f^*\theta^a = \sum_i f^a{}_i w^i$. The covariant derivative $f^a{}_{ij}$ is defined by

$$\sum_j f^a{}_{ij} w^j = df^a{}_i - \sum_j f^a{}_j w_i{}^j + \sum_b f^b{}_i f^* \theta_b{}^a.$$

The map f is harmonic if and only if $\sum_i f^a{}_{ii} = 0$ for all a.

Let M be a spacelike hyperspace, and let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ be an orthonormal frame adapted to M. The Gauss map $\mathscr{G}$ is a map defined by $\mathscr{G} = \mathbf{e}_{n+1}\colon M \to H^n$, where H^n is the set of future pointing unit timelike vectors. But H^n with induced metric is the usual model of hyperbolic space of constant curvature -1 [27]. Let $\{\theta^a\}$ be an orthonormal coframe for H^n, and $\{\theta_a{}^b\}$ be its connection form. Differentiating $\mathbf{e}_{n+1}$, we have

$$d\mathbf{e}_{n+1} = \sum_{a=1}^{n} w_{n+1}{}^a \mathbf{e}_a.$$

Since $\mathbf{e}_{n+1}$ is the position vector of H^n, $\{\mathbf{e}_a\}$ can be thought as an orthonormal frame for H^n at $\mathbf{e}_{n+1}$. Therefore we have

$$\mathscr{G}^*\theta^a = w_{n+1}{}^a = h_{ai} w^i,$$

which implies that

$$\mathscr{G}^a{}_i = h_{ai}. \tag{1.6}$$

The second structure equation gives

$$dw_{n+1}{}^{a} = \sum_{b=1}^{n} w_{n+1}{}^{b} \wedge w_{b}{}^{a},$$

which implies that $\mathcal{G}^{*}\theta_{b}{}^{a} = w_{b}{}^{a}$. Differentiating both sides of (1.6),

$$\begin{aligned} d\mathcal{G}^{a}{}_{i} &= \sum_{j} \mathcal{G}^{a}{}_{ij} w^{j} + \sum_{j} \mathcal{G}^{a}{}_{j} w_{i}{}^{j} - \sum_{b} \mathcal{G}^{b}{}_{i} \mathcal{G}^{*}\theta_{b}{}^{a} \\ &= dh_{ai} = \sum_{j} h_{aij} w^{j} + \sum_{j} h_{aj} w_{i}{}^{j} + \sum_{b} h_{bi} w_{a}{}^{b}. \end{aligned}$$

Therefore we can conclude that $\mathcal{G}^{a}{}_{ij} = h_{aij}$. Thus

$$\sum_{i} \mathcal{G}^{a}{}_{ii} = \sum_{i} h_{aii} = nH_{a},$$

by the symmetry of h_{ijk}. This proves a version of Ruh-Vilms Theorem in our context:

Theorem 1.2. *Let M^{n} be a spacelike hypersurface of $\mathbf{R}^{n,1}$. The Gauss map is a harmonic map into H^{n} if and only if M^{n} has constant mean curvature.*

2. A mean value inequality for supersolutions

The purpose of this section is to describe a local mean value inequality for supersolutions of a linear equation. Applied to the scalar and Gauss-Kronecker curvature equations of a constant mean curvature hypersurface, this will give uniform curvature bounds on sectors for some cuts whose lightlike set has nonempty interior.

Let u be an entire spacelike hypersurface of constant mean curvature $H > 0$ of $\mathbf{R}^{n,1}$. For $x \in u$ and $r > 0$ let $B(x, r) \subset u$ be an intrinsic geodesic ball about x of radius r. By Proposition 1.1, the Ricci and sectional curvatures of u satisfy $-n^{2}H^{2}/4 \leq \mathbf{Ric}$ and $-n^{2}H^{2} \leq \mathbf{K} \leq 0$ so there is a lower bound $\operatorname{vol} B(x, r) \geq c_{1}(n) r^{n}$ as well as an upper bound $\operatorname{vol} B(x, r) \leq c_{2}(n, R_{0}, H) r^{n}$ for all $r \leq R_{0}$. Since there is an estimate for the first Neumann eigenvalue of compact convex domains in terms of the lower bound of Ricci curvature, radius of the largest inscribed ball, and the lower bound of volume [30], [40], we have the *Poincaré inequality* for $r \leq R_{0}$:

$$\int_{B(x,r)} |f - \bar{f}|^{2}\, dV \leq c_{p} r^{2} \int_{B(x,r)} |Df|^{2}\, dV \tag{2.1}$$

for all $f \in W^{1,2}(B(x, r))$, where

$$\bar{f} = \fint_{B(x,r)} f\, dV.$$

There is an estimate of the isoperimetric constant involving the same quantities [15], [29]. Hence there holds for some $c_s(H, n, r)$ the *Sobolev inequality*:

$$\left(\int_{B(x,r)} f^{n/(n-1)}\, dV\right)^{(n-1)/n} \le c_s \int_{B(x,r)} |Df|\, dV \tag{2.2}$$

for all $f \in W_0^{1,1}(B(x, r))$. The mean value inequality for supersolutions is deduced from the mean value inequality for subsolutions using the well-known argument involving the Moser iteration scheme and the John-Nirenberg Lemma. For completeness, we present an argument modifying Schoen [35]. See also [19].

Lemma 2.1. *Assume that the Sobolev inequality* (2.2) *holds with constant c_s for functions supported in $B(x_0, R_0)$. Then for all $p \ge 2$ and $c \ge 2$ there is a constant $c_4(c, c_s, n, p, R_0) > 0$ so that given $0 \le u \in W^{1,2}(B(x, R))$ satisfying*

$$\Delta u \ge -cu \tag{2.3}$$

for some $B(x, R) \subset B(x_0, R_0)$, there holds

$$\sup_{B(x,\theta R)} u \le c_4 \left(\frac{1}{(1-\theta)^n R^n} \int_{B(x,R)} u^p\right)^{1/p} \tag{2.4}$$

for all $\theta \in (0, 1)$.

Proof. For $\rho, \sigma > 0$ such that $\sigma + \rho \le R$ we let $\eta(\mathrm{dist}(\cdot, x))$ be a cut off function such that $\eta(r) = 1$ for $r \le \rho$, $\eta(r) = (\sigma + \rho - r)/\sigma$ for $\rho \le r \le \rho + \sigma$, and $\eta(r) = 0$ otherwise. Let $q \ge 2$. By multiplying (2.3) by $\eta^2 u^{q-1}$ and integrating we have

$$(q-1)\int \eta^2 u^{q-2}|Du|^2 \le 2\int \eta u^{q-1}|Du|\,|D\eta| + c\int \eta^2 u^q.$$

Estimate the second term by

$$2\eta u^{q-1}|Du|\,|D\eta| \le \frac{1}{2}(q-1)u^{q-2}\eta^2|Du|^2 + \frac{2}{q-1}|D\eta|^2 u^q.$$

Hence

$$\frac{2(q-1)}{q^2}\int \eta^2 |Du^{q/2}|^2 \le c\int \eta^2 u^q + \frac{2}{q-1}\int u^q |D\eta|^2. \tag{2.5}$$

But

$$|D\eta^2 u^q| \le 2\eta u^q |D\eta| + q\eta^2 u^{q-1}|Du|$$
$$= 2\eta u^q|D\eta| + 2\eta^2 u^{q/2}|Du^{q/2}|.$$

Hence using the inequality $2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$,

$$|D\eta^2 u^q| \le \frac{2\eta^2 u^q}{\sigma} + \sigma u^q |D\eta|^2 + \sigma\eta^2 |Du^{q/2}|^2.$$

Substituting this in (2.5) and applying the Sobolev inequality we obtain for $\kappa = n/(n-1)$,

$$\left(\int \eta^{2\kappa} u^{q\kappa}\right)^{1/\kappa} \le c_s\left(\frac{cq^2\sigma}{2(q-1)} + \frac{2}{\sigma}\right)\int \eta^2 u^q$$
$$+ c_s\sigma\left(\frac{q^2}{(q-1)^2} + 1\right)\int u^q |D\eta|^2.$$

Using $|D\eta| \le 1/\sigma$ and $q \ge 2$,

$$\text{(2.6)}\qquad \left(\int_{B(x,\rho)} u^{q\kappa}\right)^{1/\kappa} \le c_s\left(\frac{cq^2\sigma}{2(q-1)} + \frac{7}{\sigma}\right)\int_{B(x,\rho+\sigma)} u^q$$
$$\le \frac{c_3 q}{\sigma}\int_{B(x,\rho+\sigma)} u^q,$$

where $c_3 = c_s(cR_0^2 + 4)$. Now iterate this inequality by setting $q_i = p\kappa^i$, $\rho_0 + \sigma_0 = R$, $\sigma_i = (1-\theta)Re^{-i-1}$, and $\rho_i + \sigma_i = \rho_{i-1}$. Hence $\rho_\infty = \theta R$. Put

$$I_i = \left(\frac{1}{(1-\theta)^n R^n}\int_{B(x,\rho_i+\sigma_i)} u^{q_i}\right)^{1/q_i}.$$

Then (2.6) implies $I_{i+1} \le (c_3 q_i 2^{i+1})^{1/q_i} I_i$. So we obtain (2.4) as a finite infinite product

$$\text{(2.7)}\qquad \sup_{B(x,\theta R)} u = \lim_{i\to\infty} I_i \le I_0 \prod_{i=0}^{\infty}(c_3 p\kappa^i 2^{i+1})^{1/p\kappa_i} = c_4 I_0.$$

Lemma 2.2. *Let A, β and p be constants such that $0 < p < 2$, $0 < A$ and $1 < \beta$, and suppose the sequence $\{J_i\} \subset \mathbf{R}$ satisfies*

$$\text{(2.8)}\qquad \log J_i = \mathbf{o}(\lambda^{-1}) \quad \textit{and} \quad J_i \le A\beta^i J_{i+1}^\lambda \quad \textit{for } i = 1, 2, 3, \cdots,$$

where $\lambda = 1 - p/2$. Then $J_0 \le A^{2/p}\beta^{4\lambda/p^2}$.

Proof. By taking the log of (2.8),

$$\log J_i \le \log A + i \log \beta + \lambda \log J_{i+1}.$$

Repeating the application of this inequality gives

$$\log J_0 \le \lambda^k \log J_k + \log A \sum_{i=0}^{k-1} \lambda^i + \log \beta \sum_{i=0}^{k-1} i\lambda^i ,$$

where k is any positive integer. Letting $k \to \infty$ and making use of $\log J_i = \mathbf{o}(\lambda^{-i})$, we obtain

$$\log J_0 \le \frac{\log A}{1-\lambda} + \frac{\lambda \log \beta}{(1-\lambda)^2}.$$

The proof is completed by taking the exponential. q.e.d.

These lemmas imply a *mean value inequality for subsolutions.*

Lemma 2.3. *Assume that the Sobolev inequality* (2.2) *holds with constant* c_s *for functions supported in some* $B(x_0, R_0)$. *Let* $c \ge 0$, $p > 0$. *Then there is a constant* $c_5(c, c_s, n, p, R_0) > 0$ *such that given any* $0 \le u \in W^{1,2}$ *satisfying*

$$\Delta u \ge -cu \tag{2.9}$$

in $B(x, R) \subset B(x_0, R_0)$, *and any* $\theta \in (0, 1)$, *there holds*

$$\sup_{B(x, \theta R)} u \le c_5 \left(\frac{1}{(1-\theta)^n R^n} \int_{B(x, R)} u^p \right)^{1/p} . \tag{2.10}$$

Proof. By Lemma 2.1, the result holds for $p \ge 2$, so suppose $0 < p < 2$ and let $\lambda = 1 - p/2$ and $0 < \rho < \rho + \sigma \le R$. From (2.4) with squares we estimate

$$\sup_{B(x, \rho)} u \le \frac{c_6}{\sigma^{n/2}} \sup_{B(x, \rho+\sigma)} u^\lambda \left(\int_{B(x, \rho+\sigma)} u^p \right)^{1/2} ,$$

where $c_6 = c_4(c, c_s, n, 2, R_0)$. We may suppose that $0 < \|u\|_{p, B(x, R)}$ or else both sides are zero. By substituting

$$v = \frac{u}{(\int_{B(x, R)} u^p)^{1/p}} , \tag{2.11}$$

we obtain

$$\sup_{B(x, \rho)} v \le \frac{c_6}{\sigma^{n/2}} \sup_{B(x, \rho+\sigma)} v^\lambda .$$

Now iterate using $\rho_0 = \theta R$, $\sigma_i = (1-\theta)R2^{-i-1}$ and $\rho_i + \sigma_i = \rho_{i+1}$ so that $\rho_\infty = R$. By letting $J_i = \sup_{B(x,\rho_i)} v$, from (2.11) we obtain the recursion

$$J_i \le c_6 \left(\frac{2}{(1-\theta)R} \cdot 2^i \right)^{n/2} J_{i+1}^\lambda .$$

On the other hand, by Lemma 2.1,

$$\sup_{B(x,\rho_i)} u \le c_4 \left(\frac{1}{(R-\rho_i)^n} \int_{B(x,R)} u^2 \right)^{1/2} \le \frac{c_4 2^{ni/2} \|u\|_{2,B(x,R)}}{(1-\theta)^{n/2} R^{n/2}} .$$

For i large enough, $\|u\|_{p,B(x,\rho_i)} > \frac{1}{2}\|u\|_{p,B(x,R)}$ so $\log J_i = \mathbf{O}(i)$. Now we may apply Lemma 2.2 to obtain the result.

Lemma 2.4. *Let $\kappa = n/(n-1)$, and let α, $\beta \ge 0$ be constants. Suppose the sequence $\{I_i\} \subset \mathbf{R}$ satisfies*

$$I_{i+1}^{1/\kappa} \le 2^i (\alpha I_i + \beta (4\kappa^i)^{2\kappa^i}) \tag{2.12}$$

for $i = 0, 1, 2, \ldots$. Then there is a constant $c_7(n) > 0$ so that

$$I_i \le [c_7 \alpha^{n/2} I_0^{1/2} + c_7 (1 + \alpha^{(n-1)/2} + \beta^{1/2}) \kappa^i]^{2\kappa^i} . \tag{2.13}$$

Proof. Substitute $J_i = I_i^{1/2\kappa^i}$. Taking $2\kappa^i$ roots of both sides of (2.12) and then using the calculus inequality $(x+y)^\theta \le x^\theta + y^\theta$ for $x, y \ge 0$ and $0 \le \theta \le 1$ gives for $i \ge 0$

$$J_{i+1} \le 2^{i/2\kappa^i} (\alpha^{1/2\kappa^i} J_i + 4\beta^{1/2\kappa^i} \kappa^i).$$

Iterate this inequality to obtain the estimate

$$\begin{aligned} J_{j+1} &\le 2^{(\frac{1}{2}\sum_{i=0}^j i\kappa^{-i})} \alpha^{(\frac{1}{2}\sum_{i=0}^j \kappa^{-i})} J_0 \\ &\quad + 4 \cdot \sum_{l=1}^{j-1} 2^{(\frac{1}{2}\sum_{i=l}^j i\kappa^{-i})} \kappa^{l-1} \beta^{1/2\kappa^{l-1}} \alpha^{\frac{1}{2}(\sum_{i=l}^j \kappa^{-i})} \\ &\quad + 4 \cdot 2^{j/2\kappa^j} \kappa^j \beta^{1/2\kappa^j} . \end{aligned} \tag{2.14}$$

Observing $\sum_0^\infty \kappa^{-j} = n$ and $\sum_0^\infty j\kappa^{-j} = n(n-1)$, and that the total powers of α and β are less than $\frac{1}{2}n$ we conclude (2.13). q.e.d.

There is also a *mean value inequality for supersolutions.*

Lemma 2.5. *Let $c \ge 0$ and $0 < \theta < 1$. Assume that the Poincaré and Sobolev inequalities* (2.1) *and* (2.2) *hold with constants c_p and c_s*

for functions supported in some $B(x_0, R_0)$. *Suppose* M *satisfies the volume condition* $\operatorname{vol} B(x, r) \le c_2 r^n$ *for* $r \le \theta^{-1}R$ *and* $B(x, \theta^{-1}R) \subset B(x_0, R_0)$. *Then there are constants* $p_0(c, c_2, c_p, c_s, n, R_0, \theta) > 0$ *and* $c_8(c, c_2, c_p, c_s, n, p, R_0, \theta) > 0$ *such that given any* $W^{1,2}$ *supersolution* $u \ge 0$ *satisfying*

$$\Delta u \le cu \tag{2.15}$$

in $B(x, \theta^{-1}, R)$ *and* $0 < p < p_0$, *there holds*

$$\inf_{B(x, \theta R)} u \ge c_8 \left(\frac{1}{R^n} \int_{B(x, R)} u^p \right)^{1/p}. \tag{2.16}$$

Proof. From (2.15) we obtain

$$\begin{aligned}\Delta u^{-p} &= -pu^{-p-1}\Delta u + p(p+1)u^{p-2}|Du|^2 \\ &\ge -pcu^{-p} + p(p+1)u^{p-2}|Du|^2 \ge -pcu^{-p}.\end{aligned}$$

Lemma 2.3 implies

$$\sup_{B(x, \theta R)} u^{-p} \le \frac{c_5}{(1-\theta)^n R^n} \int_{B(x, R)} u^{-p}.$$

Putting $c_9 = c_5^{-1/p}(pc, c_s, n, 1, R_0)(1-\theta)^{n/p} > 0$, we have

$$\begin{aligned}\inf_{B(x, \theta R)} u &\ge c_9 \left(\frac{1}{R^n} \int_{B(x, R)} u^{-p} \right)^{-1/p} \\ &\ge \frac{c_9 (\frac{1}{R^n} \int_{B(x, R)} u^p)^{1/p}}{(\frac{1}{R^n} \int_{B(x, R)} u^{-p})^{1/p} (\frac{1}{R^n} \int_{B(x, R)} u^p)^{1/p}}.\end{aligned}$$

The result will follow when we show

$$\left(\int_{B(x, R)} u^{-p} \right) \left(\int_{B(x, R)} u^p \right) \le c_{10} R^{2n}. \tag{2.17}$$

For this purpose, consider the function $w = \beta - \log u$. (2.17) follows if we can show that for some $0 < p_0$ and any β,

$$\int_{B(x, R)} e^{p|w|} \le c_{11} R^n \quad \text{for all } 0 < p \le p_0. \tag{2.18}$$

We have the weak inequality

$$\Delta w = \frac{-\Delta u}{u} + \frac{|Du|^2}{u^2} \geq -c + |Dw|^2. \tag{2.19}$$

Multiplying (2.19) by η^2, where η is the cutoff defined in the proof of Lemma 2.1, we obtain

$$\begin{aligned}\int \eta^2|Dw|^2 &\leq \int c\eta^2 + \eta^2\Delta w = \int c\eta^2 - 2\int \eta D\eta \cdot Dw \\ &\leq \int c\eta^2 + 2|D\eta|^2 + \tfrac{1}{2}\int \eta^2|Dw|^2,\end{aligned}$$

$$\int \eta^2|Dw|^2 \leq 2c\int \eta^2 + 4\int |D\eta|^2 \leq \left(2c + \frac{4}{\sigma^2}\right)\operatorname{vol} B(x, \rho+\sigma). \tag{2.20}$$

Letting $\rho = R$ and $\rho + \sigma = \theta^{-1}R$, by (2.20) and the Poincaré inequality (2.1), we find

$$\begin{aligned}\int_{B(x,R)} |w|^2 &= \int_{B(x,R)} |\beta - \log u|^2 \leq c_p R^2 \int_{B(x,R)} |Dw|^2 \\ &\leq c_p R^2 \int_{B(x,\theta^{-1}R)} \eta^2 |Dw|^2 \\ &\leq c_p \operatorname{vol} B(x, \theta^{-1}R) R^2 \left(2c + \frac{4}{(\theta^{-1}-1)^2R^2}\right),\end{aligned}$$

where we take $\beta = ⨍_{B(x,R)} \log u$. Hence, by the Schwarz inequality,

$$\left(\int_{B(x,R)} |w|\right)^2 \leq c_p \operatorname{vol}^2 B(x, \theta^{-1}R) R^2 \left(2c + \frac{4\theta^2}{(1-\theta)^2R^2}\right). \tag{2.21}$$

Choose $q \geq 1$. Multiplying (2.19) by $\eta^2|w|^{2q}$ and integrating give

$$\int \eta^2|w|^{2q}|Dw|^2 \leq \int 2q\eta^2|w|^{2q-1}|Dw|^2 - \int 2\eta|w|^{2q} D\eta \cdot Dw + \int c\eta^2|w|^{2q}.$$

Using Young's inequality $ab \leq \varepsilon a^p/p + \varepsilon^{-p'/p} b^{p'}/p'$, where $1/p + 1/p' = 1$ with $a = |w|^{2q-1}$, $\varepsilon = p/2$ and $p = 2q(2q-1)^{-1}$, and Schwarz's inequality, we obtain

$$\begin{gathered}2q|w|^{2q-1} \leq \tfrac{1}{2}|w|^{2q} + (4q)^{2q-1}, \\ 2|Dw|\,|D\eta|\eta \leq \tfrac{1}{4}\eta^2|Dw|^2 + 4|D\eta|^2.\end{gathered}$$

Thus

$$\int \eta^2|w|^{2q}|Dw|^2 \leq \int (4q)^{2q}\eta^2|Dw|^2 + \int (16|D\eta|^2 + 4c\eta^2)|w|^{2q}. \tag{2.22}$$

However,

$$|D(\eta^2|w|^{2q})| \le 2\eta|w|^{2q}|D\eta| + 2q|w|^{2q-1}\eta^2|Dw|$$
$$\le 2\eta|w|^{2q}|D\eta| + |w|^{2q-2}\eta^2\left(\sigma w^2|Dw|^2 + \frac{q^2}{\sigma}\right).$$

By means of $|D\eta| \le 1/\sigma$ and $q^2|w|^{2q-2} \le |w|^{2q} + q^{2q-1}$ we find

$$|D(\eta^2|w|^{2q}) \le \frac{2\eta+\eta^2}{\sigma}|w|^{2q} + \sigma\eta^2|w|^{2q}|Dw|^2 + \frac{q^{2q-1}\eta^2}{\sigma}.$$

Integrating this inequality, using (2.20), (2.22) and collecting terms yield

$$\int |D(\eta^2|w|^{2q})| \le \int\left(\frac{19}{\sigma} + 4c\sigma\right)|w|^{2q} + \int\left(\frac{5(4q)^{2q}}{\sigma} + 2c\sigma\right).$$

Applying the Sobolev inequality (2.2), and setting $\kappa = n/(n-1)$, and $c_{12} = c_s(19 + 4cR_0^2)$, we arrive at

$$(2.23)\qquad \left(\int_{B(x,\rho)} |w|^{2q\kappa}\right)^{1/\kappa} \le \frac{c_{12}}{\sigma}\int_{B(x,\rho+\sigma)}|w|^{2q} + \frac{c_{12}(4q)^{2q}}{\sigma}\operatorname{vol} B(x,\theta^{-1}R).$$

Now this inequality is iterated. Let $q_i = \kappa^i$, $\rho_0 = \theta^{-1}R$, $\sigma_{i+1}+\rho_{i+1} = \rho_i$, and $\sigma_i = (\theta^{-1}-1)R2^{-i}$. Then $\rho_\infty = R$. Denote

$$I_i = \frac{1}{R^n}\int_{B(x,\rho_i)}|w|^{2\kappa^i}.$$

Upon substituting $q = q_i$, $\rho = \rho_{i+1}$ and $\sigma = \sigma_{i+1}$ in (2.23) and using the volume bound we obtain the recursion

$$I_{i+1}^{1/\kappa} \le 2^i(c_{13}I_i + c_{14}(4\kappa^i)^{2\kappa^i}),$$

where $c_{13} = 2c_{12}\theta(1-\theta)^{-1}$ and $c_{14} = 2c_2\theta^{1-n}(1-\theta)^{-1}$. Applying Lemma 2.4 gives

$$I_i^{1/2\kappa^i} \le c_{15}I_0^{1/2} + c_{16}\kappa^i,$$

where $c_{15}, c_{16} > 0$ depend on c, c_2, c_s, n, R_0 and θ. To estimate the exponential, for each $j \in \mathbf{Z}^+$ we take i such that $2\kappa^{i-1} < j \le 2\kappa^i$. Then by the Hölder inequality we have

$$\frac{1}{R^n}\int_{B(x,R)}|w|^j \le I_i^{j/2\kappa^i}c_2^{1-j/2\kappa^i} \le c_{17}c_{18}^j(j+\sqrt{I_0})^j \le c_{17}c_{18}^j j^j\exp(\sqrt{I_0}).$$

Thus (2.18) follows: using Stirling's inequality $j^j < e^j j!$ and (2.21) we obtain

$$\frac{1}{R^n}\int_{B(x,R)} e^{p|w|} \le c_2 + c_{19}p + \sum_2^\infty p^j c_{17} c_{18}^j e^j \exp(\sqrt{I_0})$$
$$\le c_2 + c_{19}p_0 + c_{17}\exp(\sqrt{I_0}) = c_{11},$$

where $c_{11}, c_{17}, c_{18}, c_{19} > 0$ depend on c, c_2, c_p, c_s, n, R_0 and θ and provided that

$$0 \le p \le p_0 \le 1/(2ec_{18}).$$

3. A splitting theorem for constant mean curvature cuts

In this section we prove a splitting theorem. It was first observed for $n = 2$ by T. K. Milnor [31]. The mean value inequality of §2 for the scalar curvature and (1.5) may be used to show that if M is a constant mean curvature spacelike hypersurface of Minkowski space, and if there is a point $p \in M$ where the second fundamental form has rank one, then M splits as $H^1 \times \mathbf{R}^{n-1}$. Similarly, as in Lemma 7.3, the Gauss-Kronecker curvature can be used to show the result for full rank. We present the proof which works for all ranks due to N. Korevaar [28].

Theorem 3.1. *Let M^n be an entire spacelike hypersurface of constant mean curvature $H > 0$ (< 0) in $\mathbf{R}^{n,1}$. Then, after a $\mathbf{R}^{n,1}$ rigid motion, $\mathbf{R}^{n,1}$ splits as a product $\mathbf{R}^{k,1} \times \mathbf{R}^{n-k}$ such that M^n also splits as a product $M^k \times \mathbf{R}^{n-k}$, where $M^k = M^n \cap \mathbf{R}^{k,1}$ is a strictly convex (concave) hypersurface of $\mathbf{R}^{k,1}$ with constant mean curvature nH/k. In particular, if M^n is represented as a graph of an entire function u, the Hessian u_{ij} has constant rank k everywhere.*

Proof. Suppose that for some unit vector v and at some point $x \in \mathbf{R}^n$, $u_{vv}(x) = 0$. By applying an isometry of $\mathbf{R}^{n,1}$, we may arrange that $x = 0$, $u(0) = 0$, $v = \partial/\partial x^1$, $Du(0) = 0$, $u_{11}(0) = 0$ and $u_{ij}(0)$ is positive semidefinite. The constant mean curvature equation can be rewritten as

$$\Delta u = \frac{-1}{1-|Du|^2} u_i u_j u_{ij} + nH\sqrt{1-|Du|^2}.$$

To analyse this equation, we consider it as

$$\Delta u = a(|Du|^2) u_i u_j u_{ij} + b(|Du|^2),$$

where a and b are analytic functions in $|Du|^2$. We claim that $u_{11} \equiv 0$.

Differentiating twice with respect to $\partial/\partial x^1$, we have

$$
\begin{aligned}
\Delta u_{11} &= 4a'' u_p u_{p1} u_m u_{m1} u_i u_j u_{ij} + 2a' u_{m1} u_{m1} u_i u_j u_{ij} \\
&\quad + 2a' u_m u_{m11} u_i u_j u_{ij} + 8a' u_m u_{m1} u_{i1} u_j u_{ij} \\
&\quad + 4a' u_m u_{m1} u_i u_j u_{ij1} + 2a u_{i11} u_j u_{ij} \\
&\quad + 2a u_{i1} u_{j1} u_{ij} + 4a u_{i1} u_j u_{ij1} \\
&\quad + a u_i u_j u_{ij11} + 4b'' u_p u_{p1} u_m u_{m1} \\
&\quad + 2b' u_{m1} u_{m1} + 2b' u_m u_{m11}.
\end{aligned} \tag{3.1}
$$

Assuming that u_{11} is not identically zero, since u is analytic, we may take a power series expansion of u_{11} at 0, $u_{11} = h+p$, where the lowest order term h is a nonzero homogeneous polynomial of degree $m \geq 2$ and the rest of the series is p. We observe that the convexity of u_{ij} implies the nonnegativity of all 2×2 minors so that for any i,

$$u_{ii} u_{11} - u_{i1}^2 \geq 0.$$

Summing over i gives

$$(\Delta u) u_{11} \geq \sum_i u_{i1}^2 \geq u_{i1}^2.$$

Since $\Delta u(0) = nH$, each term u_{i1} is of order at least $m/2$. Hence u_{ij1} is of order at least $m/2 - 1$. u_i is of order at least 1, and since u_{11} is of order m, u_{11i} and u_{11ij} are of order at least $m-1$ and $m-2$ respectively. Using these we can check that the right-hand side of equation (3.1) is of order at least m. On one hand, Δh is either identically zero, or of order $m-2$ which cannot occur because of order consideration. Therefore $\Delta h \equiv 0$. On the other hand, by convexity, $u_{11} \geq 0$, which implies that $h \geq 0$. But $h(0) = 0$, so by the strong maximum principle $h \equiv 0$, which contradicts the assumption that the power series expansion of u_{11} starts at an mth order term. Therefore $u_{11} \equiv 0$. By analyticity, $u_{11} \equiv 0$ everywhere, therefore M is ruled by lines in the x^1 direction. Since u was arranged to be nonnegative, $u(0) = 0$, and u is convex, we conclude that M is ruled by lines parallel to the x^1-axis. Therefore $M^n = M^{n-1} \times \mathbf{R}^1$ and also $\mathbf{R}^{n,1} = \mathbf{R}^{n-1,1} \times \mathbf{R}^1$. Repeating this procedure completes the proof.

4. Extrinsic properties and convexity

In this section we show that the image of an entire constant mean curvature hypersurface under the Gauss map is convex. In fact, the image is the convex hull of a set of points of ideal infinity of hyperbolic space, H^n. By applying known existence theory [38], for any subset $L \subset H(\infty)$, there is a constant mean curvature cut of $\mathbf{R}^{n,1}$ whose Gauss map image is the convex hull $\mathbf{Conv}(L) \subset H^n$. We begin by describing and sharpening some known [38] extrinsic properties of constant mean curvature cuts of Minkowski space.

Many of the results of this section are elementary so we often only sketch the argument. We usually assume $H > 0$ so u is convex. $H < 0$ and u concave is handled similarly. Associated to M is a cone, a positively homogeneous of degree one function V_M, gotten by blowing down.

Lemma 4.1. *Let u be a convex spacelike function on $\mathbf{R}^n$. Then*

$$V_u(x) = \lim_{\substack{r\to\infty \\ r>0}} \frac{u(rx)}{r} \tag{4.1}$$

exists for all x, and V_u is an achronal (nontimelike) positively homogeneous function.

Proof. Choose $p \in \mathbf{R}^n$ and let $f_r(x) = (u(rx) - u(0))/r$ for $r > 0$. Because of convexity, for $\theta > 1$ we have $f_r(x) \le f_{\theta r}(x)$ so f_r is an increasing sequence. Because f_r is spacelike, it is also bounded, $|f_r(x)| \le |x|$, so the limit converges pointwise. One checks that it is a convex achronal function with homogeneity of degree one, i.e., if $\theta > 0$ then $V_u(\theta x) = \theta V_u(x)$. q.e.d.

The constant mean curvature cuts are asymptotically lightlike. We give a slight improvement of [38, p. 52].

Lemma 4.2. *Suppose M_u is an entire spacelike constant mean curvature hypersurface of $\mathbf{R}^{n,1}$ which is the graph of u. Then V_u is null in the sense that for all $x \in \mathbf{R}^n$ and $\delta > 0$, there is a $y \in \mathbf{R}^n$ so that $|x - y| = \delta$ and $|V_u(x) - V_u(y)| = \delta$.*

Proof. We may assume $H > 0$ so u is convex. If the conclusion is false, then there exist an $x \in \mathbf{R}^n$ and $\varepsilon > 0$ so that for all $\xi \in \partial B(x, \delta) \subset \mathbf{R}^n$ we have

$$V_u(\xi) \le V_u(x) + (1 + 2\varepsilon)\delta.$$

Since $u_r(x) := u(rx)/r \to V_u(x)$ uniformly on compacta, we may choose an r_0 large so that for all $r > r_0$ and all $\xi \in \partial B(x, \delta)$,

$$u(r\xi)/r \le V_u(x) + (1 - \varepsilon)\delta.$$

But the mean curvature of u_r satisfies $\mathscr{M} u_r = rH$. By the maximum principle, u_r is less than another mean curvature rH surface with larger boundary values on $\partial B(x, \delta)$,

$$u_r(y) \le V_u(x) + (1 - \varepsilon)\delta + \sqrt{r^{-2}H^{-2} + |x - y|^2} - \sqrt{r^{-2}H^{-2} + \delta^2}$$

for all $y \in B(x, \delta)$. In particular this holds at $y = x$. Letting $r \to \infty$ leads to a contradiction. q.e.d.

Denote the class of all null achronal positively homogeneous degree one convex functions on $\mathbf{R}^n$ by $\mathscr{Q}$. It is known [38] that every $w \in \mathscr{Q}$ is the blowdown $w = V_u$ of some constant mean curvature cut u of $\mathbf{R}^{n,1}$. The class $\mathscr{Q}$ is in one-to-one correspondence with $\mathscr{F}$, the set of closed subsets of $\mathbf{S}^{n-1}$.

Lemma 4.3. *Let E be a closed subset of $\mathbf{S}^{n-1}$. Then the function on $\mathbf{R}^n$ given by*

$$V_E(x) = \sup_{\xi \in E} \xi \cdot x, \tag{4.2}$$

where the inner product is the usual one from $\mathbf{R}^n$, is convex, homogeneous and null. In fact, the mapping $\mathscr{F} \to \mathscr{Q}$ given by $E \mapsto V_E$ is a one-to-one correspondence.

Proof. Convexity and positive homogeneity are immediate. To see that V_E is null, suppose that it is not at some $x \in \mathbf{R}^n$ and $\delta > 0$. Then there is an $\varepsilon > 0$ so that $V_E(\xi + x) \leq V_E(x) + (1 - 2\varepsilon)\delta$ for all $|\xi| = \delta$. By definition there is a $z \in E$ so that $V_E(x) \leq x \cdot z + \varepsilon\delta$. Now, taking $\xi = \delta z$ and combining yields $z \cdot (x + \delta z) \leq V_E(x + \xi) \leq x \cdot z + (1 - \varepsilon)\delta$, which is a contradiction.

To show that $E \mapsto V_E$ is injective, suppose that for two sets $E_1 \neq E_2$ we have $V_{E_1} = V_{E_2}$. By switching if necessary, there is a point $z \in E_1 - E_2$. By closedness, there exists an $\varepsilon > 0$ so $B(z, \varepsilon) \cap E_2 = \varnothing$. Hence $|\xi - z|^2 > \varepsilon^2$ for all $\xi \in E_2$ which implies $2 - \varepsilon^2 > 2z \cdot \xi$, so $V_{E_2}(z) \leq 1 - \frac{1}{2}\varepsilon^2$. However this leads to a contradiction since $V_{E_1}(z) = 1$.

To show that the mapping $E \mapsto V_E$ is surjective, we construct an inverse using the tangential map of a convex function. Following Bakelman [4], for an arbitrary convex function ζ defined on a convex set $G \subset \mathbf{R}^n$, and $x_0 \in G$, the tangential mapping of a point is defined to be the set of supporting directions

$$\chi_\zeta(x_0) = \{p = (p^1, \cdots, p^n) \in \mathbf{R}^n : \zeta(x) \geq p \cdot (x - x_0) + \zeta(x_0) \text{ for all } x \in \mathbf{R}^n\}.$$

If u is differentiable at x_0, then the tangential map is $\chi_u(x_0) = Du(x_0)$. Now, for a set $E \subset G$, define

$$\chi_\zeta(E) = \cup_{x_0 \in E} \chi_\zeta(x_0).$$

For $v \in \mathscr{Q}$ let

$$E_v = \chi_v(0) \cap \mathbf{S}^{n-1}. \tag{4.3}$$

This is a closed subset since $\chi_v(0)$ is closed. Now let $v' = V_{E_v}$. To show that this is an inverse it suffices to check $v' = v$ on unit vectors. If there is a $|z| = 1$ so that $v'(z) > v(z)$, by the definition of v', there is a $y \in E_v$ so that $z \cdot y > v(z)$ which contradicts E_v are supporting directions of v, namely, $v(\xi) \geq \xi \cdot y$ for all $\xi \in \mathbf{R}^n$. On the other hand suppose for some $|z| = 1$ we had $v'(z) < v(z)$. Because v is null, there is a $|y| = 1$ so that $v(z+y) - v(z) = 1$. Since v is achronal and convex, this is extremal, so $v(z+\xi) - v(z) \geq \xi \cdot y$ for all $\xi \in \mathbf{R}^n$, in particular, $v(z) \leq z \cdot y$. Replacing ξ by $s\xi$, dividing by s, using the homogeneity and letting $s \to \infty$ shows that $v(\xi) \geq y \cdot \xi$ for all $\xi \in \mathbf{R}^n$, so that $y \in E_v$ is a unit supporting vector at 0. But this means $v'(z) \geq y \cdot z$, contradicting the assumption. q.e.d.

If M_u is an entire convex achronal hypersurface of $\mathbf{R}^{n,1}$ which is the graph of u, we define

$$L_u = \chi_{V_u}(0) \cap \mathbf{S}^{n-1} = E_{V_u} \tag{4.4}$$

to be the set of lightlike directions of u. In case u is also a constant mean curvature, it is smooth with null blowdown. Hence there is another interpretation of the lightlike vectors.

Lemma 4.4. *Suppose that u is a constant mean curvature $H > 0$ cut of $\mathbf{R}^{n,1}$. Then*

$$\overline{Du(\mathbf{R}^n)} = \overline{\chi_{V_u}(\mathbf{R}^n)} = \chi_{V_u}(0).$$

Proof. To show that $\overline{Du(\mathbf{R}^n)} \subset \overline{\chi_{V_u}(\mathbf{R}^n)}$ choose $\xi \in \overline{Du(\mathbf{R}^n)}$ and $x_i \in \mathbf{R}^n$ such that $Du(x_i) = \xi_i \to \xi$. The supporting linear functions $l_i(x) = \xi_i \cdot (x - x_i) + u(x_i)$ satisfy $l_i(y) \leq u(y)$ for all $y \in \mathbf{R}^n$. This relation is preserved in blowing down, so that $x \cdot \xi_i = V_{l_i}(x) \leq V_u(x)$. Since both sides are homogeneous, it suffices to check equality on $\overline{B}(0, 1)$. There, $x \cdot \xi_i \to x \cdot \xi$ uniformly.

To show $\overline{\chi_{V_u}(\mathbf{R}^n)} \subset \overline{\chi_{V_u}(0)}$ let $\xi \in \overline{\chi_{V_u}(\mathbf{R}^n)}$ and $\xi_i \in \chi_{V_u}(\mathbf{R}^n)$ so that $\xi_i \to \xi$. Then choose $x_i \in \mathbf{R}^n$ so that $\xi_i \in \chi_{V_u}(x_i)$. This means that for all y,

$$V_u(y) \geq \xi_i \cdot (y - x_i) + V_u(x_i). \tag{4.5}$$

By replacing y with sy for $s > 0$, dividing by s, using the homogeneity and letting $s \to \infty$ we get $V_u(y) \geq \xi_i \cdot y$ for all y. Hence $\xi_i \in \chi_{V_u}(0)$ which is a closed set (e.g. [33, p. 10]) so $\xi \in \chi_{V_u}(0)$.

Finally, to show that $\overline{\chi_{V_u}(0)} = \chi_{V_u}(0) \subset \overline{Du(\mathbf{R}^n)}$ we may assume that u is strictly convex, since by the Splitting Theorem 3.1, if in some isometric image when u splits off an $\mathbf{R}^k$ then so does V_u. We may also assume, after a rigid motion of $\mathbf{R}^{n,1}$, that u has its minimum at 0. Therefore, zero is an interior point of the convex set $\chi_{V_u}(0)$. First suppose that ξ is an interior point of $\chi_{V_u}(0)$. Let $c = \min\{V_u(y) - \xi \cdot y : |y| = \delta\}$, where $\delta > 0$ is fixed. $c > 0$ since ξ is an interior point of $\chi_{V_u}(0)$. Because u_r converges uniformly to V_u on $B(0, \delta)$ as $r \to \infty$, for r sufficiently large we have $u_r(y) - \xi \cdot y > c/2$ whenever $|y| = \delta$ and $u_r(0) < c/3$. Therefore the function $f(x) = u_r(x) - \xi \cdot x$ has its minimum at some point x with $|x| < \delta$. Hence $\xi = Du_r(x) = Du(rx)$. If ξ is in the boundary of $\chi_{V_u}(0)$, then there exists a sequence of interior points $\xi_i \to \xi$. Thus there are $x_i \in \mathbf{R}^n$ so that $Du(x_i) = \xi_i$ and the proof is completed by letting $i \to \infty$. q.e.d.

We are now in the position to relate the lightlike set of a constant mean curvature cut u to the image of the Gauss map $\mathscr{G}(u)$. For this purpose, we observe that we can interpret the preceding Euclidean convexity considerations in terms of the hyperbolic space.

Lemma 4.5. *The projective map $\wp$ taking the set of future pointing unit timelike vectors $H^n \subset \mathbf{R}^{n,1}$ to the unit disk*

$$\wp\colon (x^1, \cdots, x^n, x^{n+1}) \mapsto \left(\frac{x^1}{x^{n+1}}, \cdots, \frac{x^n}{x^{n+1}}\right)$$

is an isometry of hyperbolic space taking the hyperboloid model H^n to the Klein model K^n with metric

$$ds_K^2 = \frac{1}{1-|p|^2}\left(\delta_{ij} + \frac{p^i p^j}{1-|p|^2}\right) dp^i \otimes dp^j. \tag{4.6}$$

The Gauss map $\mathscr{G}\colon M_u \to H^n$ from the graph coordinates $\varphi\colon \mathbf{R}^n \to M_u$ to the Klein model $(B(0, 1), ds_K^2)$ takes the form

$$\wp \circ \mathscr{G} \circ \varphi = Du(x).$$

Proof. In graph coordinates, the induced metric of the hyperboloid is

$$(g_H(x))_{ij} = \delta_{ij} - \frac{x^i x^j}{1+|x|^2}.$$

Let $p^i = \wp(x^i) = x^i/x^{n+1} = x^i/\sqrt{1+|x|^2}$ be the coordinate in the disk. Hence $x^i = p^i/\sqrt{1-|p|^2}$ so changing variables gives (4.6). In graph coordinates, the Gauss map is

$$\mathscr{G}(x) = \mathbf{e}_{n+1} = \left(\frac{Du(x)}{\sqrt{1-|Du|^2}}, \frac{1}{\sqrt{1-|Du|^2}} \right),$$

so $\wp(\mathscr{G}(x)) = Du(x)$. q.e.d.

The previous lemma can also be seen geometrically. A totally geodesic hypersurface of H^n is an ambient rotation $\gamma \in \mathrm{SO}(n, 1)$ of the intersection of a vertical hyperplane of $\mathbf{R}^{n,1}$ with H^n. The projection $\wp$ maps the rotated hyperplane to itself, so in particular, the image of a totally geodesic hyperplane is a Euclidean straight hyperplane of the disk. But the metric of the disk for which the Euclidean hyperplanes are totally geodesic is the Klein metric.

We may also identify the compatification or geometric boundary $H(\infty)$, the set of ideal points of hyperbolic space. $H(\infty)$ is defined to be the asymptotic classes of geodesic rays of H^n where we say that two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to H^n$ are asymptotic if $\mathrm{dist}_{H^n}(\gamma_1(t), \gamma_2(t))$ remains bounded as $t \to \infty$. This set is naturally identified to the boundary sphere of the Klein model of hyperbolic space $H(\infty) \leftrightarrow \partial B(0, 1)$. Hence **Conv** means both the convex hull in the Euclidean sense or intrinsically in the hyperbolic sense, since the affine structure of $\overline{K}^n$ and $\overline{B}(0, 1)$ agree. Then $\mathbf{Conv}(L)$, where $L \subset H(\infty)$, means the hull of ideal points, as usual.

Lemma 4.6. *Let u be a constant mean curvature $H > 0$ cut of $\mathbf{R}^{n,1}$. Then*

$$\chi_{V_u}(0) = \mathbf{Conv}(L_u).$$

Proof. Because $\chi_{V_u}(0)$ is a closed convex set we have $\mathbf{Conv}(L_u) \subset \chi_{V_u}(0)$. To show the other inclusion we have from Lemma 4.3 that

$$V_u(x) = \sup_{\zeta \in L_u} x \cdot \zeta.$$

Now suppose there is a $z \in \chi_{V_u}(0) - \mathbf{Conv}(L_u)$ such that $|z| < 1$. Because z is a supporting direction, we have $V_u(y) \geq z \cdot y$ for all y. However, since every point not in a compact convex set $\mathbf{Conv}(L_u)$ can be separated from it by a linear functional, there is $|\zeta| = 1$ and $\varepsilon > 0$ so that $\zeta \cdot \mathbf{Conv}(L_u) \leq \zeta \cdot z - \varepsilon$. Hence $V_u(\zeta) \leq \zeta \cdot z - \varepsilon$. A contradiction ensues for $y = \zeta$. q.e.d.

We may now combine everything.

Theorem 4.7. *Let $L \subset H^n(\infty)$ be an arbitrary closed set. Then there exists a harmonic map $\mathscr{G}$ from an entire spacelike constant mean curvature hypersurface M of $\mathbf{R}^{n,1}$ to hyperbolic space H^n such that*

$$\overline{\mathscr{G}(M)} = \mathbf{Conv}(L). \tag{4.7}$$

Proof. L can be realized as a subset of $\mathbf{S}^{n-1}$. Let V_L be the unique corresponding homogeneous, achronal, convex, null function defined by Lemma 4.3. By [38] there is a constant mean curvature cut u of $\mathbf{R}^{n,1}$ where $V_u = V_L$. The Gauss map $\mathscr{G}$ is harmonic by Theorem 1.2. By Lemma 4.5, the image of the Gauss map $\overline{\mathscr{G}(u)} = \overline{Du(\mathbf{R}^n)}$. This, in turn, by Lemma 4.4 is $\chi_{V_u}(0)$, which by Lemma 4.6 gives (4.7). q.e.d.

Theorem 4.8. *Suppose M of $\mathbf{R}^{n,1}$ is an entire spacelike constant mean curvature $H > 0$ hypersurface which is the graph of u. Then the Gauss map $\mathscr{G}$ is a harmonic map to H^n so that*

$$\overline{\mathscr{G}(M)} = \mathbf{Conv}(L_u),$$

where L_u is the lightlike set of u. If u is strictly convex, then $\mathscr{G}$ is a diffeomorphism onto the interior of $\mathbf{Conv}(L_u)$. More generally, if k is the largest integer for which $\mathbf{Conv}(L_u) \cap A_k$ has nonempty interior in A_k, for some A_k, a totally geodesic k-plane of $H^n \equiv$ a flat of $\mathbf{R}^n$, then the cut splits, up to ambient isometry, as $M = M^k \times \mathbf{R}^{n-k}$ intrinsically and extrinsically as in the splitting Theorem 3.1, where M^k is strictly convex in $\mathbf{R}^{k,1}$ and the restriction $\mathscr{G}\colon M^k \to (\mathbf{Conv}(L_u) \cap A_k)^\circ$ is a diffeomorphism. In particular, if L_u is affinely full, i.e., is contained in no A_k, $k < n$, then u is strictly convex.

Proof. Lemmas 4.4 and 4.5 and Theorem 1.2 work as before. The rest follows from the Splitting Theorem 3.1.

5. Comparison surfaces

In this section we describe hypersurfaces of revolution which will be important in the construction of barriers for the boundary value problem as well as in transferring extrinsic to intrinsic information. Various special constant mean curvature surfaces in Minkowski space have been studied ([8], [14], [21], [38]). These are the analogs of the Delaunay surfaces of revolution in higher dimensions. One such surface, called the semitrough, is asymptotic to the product of the $(n-1)$-dimensional hyperboloid and $\mathbf{R}^1$ along a ray and asymptotic to the hyperboloid in other directions. We need to be able to estimate the gap between the semitrough and hyperboloid carefully for our purpose. In $\mathbf{R}^{n,1}$ the spacelike surfaces invariant under the $\mathrm{SO}(n-1, 1)$ action are described by a single function $f(x_1)$. Denoting $x = x_1$ and $\overline{x} = (x_2, \cdots, x_n)$, the hypersurface is thus

$$u(x_1, x_2, \cdots, x_n) = \sqrt{f(x_1)^2 + |\overline{x}|^2}.$$

Substituting this into the mean curvature equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = nH$$

we find, taking $H = 1$,

$$\left(\frac{f'}{\sqrt{1-(f')^2}}\right)' + \frac{n-1}{f\sqrt{1-(f')^2}} = n. \tag{5.1}$$

A solution of this equation is given by the hyperboloid

$$h(x) = \sqrt{1+x^2}.$$

Another is the hyperboloid of one lower dimension crossed with a line,

$$l(x) = \frac{n-1}{n}.$$

Corresponding to the classical Delaunay surfaces [14], [22] there is a family interpolating these two. There is another complete surface, the semitrough, characterized by $f' > 0$, which was first observed by Calabi [10] and which has the property that $f \to h$ as $x \to \infty$ and $f \to l$ as $x \to -\infty$.

Lemma 5.1. *There is a solution $f(x)$ of (5.1) defined for all x with the following properties*:

(a) $0 < f' < 1$ *and* $f'' > 0$ *for all* x.

(b) $\lim_{x\to-\infty} f(x) = (n-1)/n = l$.

(c) $\max(l, x) < f(x) < h(x)$ *for all* x.

(d) *There is a constant $c > 0$ depending only on n so that* $|f(x) - h(x)| < cx^{-n-1}$ *whenever* $x \geq 3$.

Proof. First consider the initial value problem (5.1) for a function $y = f(x)$ such that $f(0) = y_0$ and $f'(0) = 0$ for some $l < y_0 < 1$. Thus, by (5.1) we have $f''(0) > 0$ so f and f' are increasing functions near $x \geq 0$. We find a first integral for (5.1) to solve the initial value problem. Differentiating out the first term of (5.1), changing variables $u(y) = f'(x)$, where $y = f(x)$ is the new independent variable, multiplying by y^{n-1} and integrating we find that

$$\frac{1}{\sqrt{1-u^2}} = y + \frac{y_0^{n-1}(1-y_0)}{y^{n-1}}, \tag{5.2}$$

with initial condition $u(y_0) = 0$. Since $y_0 > l$, (5.2) shows that $0 \leq u < 1$ and u is an increasing function which exists for all $y \geq y_0$. Hence f is convex and because $0 \leq f' < 1$, $f(x)$ exists for all $x \geq 0$.

Since the solutions of (5.1) are invariant under translation along the x-axis, we can construct a sequence of functions approximating the one desired in the lemma by translating solutions of the initial value problem. Choose a decreasing sequence $l < y_i < 1$ such that $y_i \to l$. Let $f_{[\alpha, i]}$ denote the solution of the initial value problem for (5.1) with $f_{[\alpha, i]}(\alpha) = y_i$ and $f'_{[\alpha. i]}(\alpha) = 0$ defined on (α, ∞). Because $f' < 1$ we have $f_{[1, i]}(x) < h(x)$ for all $1 \leq x$. Thus we may define

$$a_i = \inf\{\alpha \leq 1 : h(x) > f_{[\alpha, i]}(x) \quad \forall x \geq \alpha\}.$$

Let $\{f_i\}$ be the sequence of solutions $f_i(x) = f_{[a_i, i]}(x)$.

We claim that $a_i \to -\infty$. To see this, observe that $f_i(1) \geq 1$ and that if $a_i \leq 0$ then $f_i(0) < 1$. Estimating (5.2) on the left and right, we find

$$1 + \tfrac{1}{2}u^2 \leq 1 + \alpha z + \tfrac{1}{2}\beta z^2 \quad \text{for } y_i \leq y \leq 1, \tag{5.3}$$

where

$$y = y_i + z, \quad \alpha = 1 - \frac{(n-1)(1-y_i)}{y_i}, \quad \beta = \frac{2n(n-1)(1-y_i)}{y_i^2}.$$

Taking $0 < c_i \leq 1$ defined by $f_i(c_i) = 1$, using (5.3) we compute $b_i(y) = f_i^{-1}(y)$ for $y_i \leq y \leq 1$ by

$$\begin{aligned} c_i - b_i(y) = \int_y^1 \frac{dy}{u(y)} &\geq \int_{y-y_i}^{1-y_i} \frac{dz}{\sqrt{2\alpha z + \beta z^2}} \\ &\geq \frac{1}{\sqrt{\beta}} \log\left\{\frac{\alpha + 2\beta(1-y_i)}{2\alpha + 2\beta(y - y_i)}\right\}. \end{aligned} \tag{5.4}$$

Now $a_i = b_i(y_i)$. Note that as $y_i \to l$, $\alpha \to 0$ and $\beta \to 2n^2/(n-1)$. Hence $c_i - a_i \to \infty$. By the maximum principle, f_i and h do not make interior contact, and are therefore asymptotic. Since both are asymptotically lightlike, $h \to |x|$ and f', $h' \to 1$ as $x \to \infty$. Since $0 < f' < 1$ we also have $f_i(x) > \max\{l, x\}$. In fact, for any fixed y, using (5.2) we see that $f_i'(f_i^{-1}(y))$ is an increasing sequence so that by the maximum principle, f_i is a decreasing sequence of functions. But for fixed y by (5.4), $c_i - b_i(y)$ has a uniform lower bound as $i \to \infty$, which increases as $y \to l$. Therefore, for any compact interval $[b, d] \subset \mathbf{R}$ there is a uniform bound $0 < k_1(b) \leq f_i \leq k_2(d) < \infty$, and by (5.4) there is a bound of f' away from 0 and 1. Thus a subsequence converges to a solution f required in the lemma satisfying properties (a, b, c).

Finally we will compare f to the hyperbola. When $y_0 = 1$ we get the hyperbola solution $h(x)$, with corresponding $v(y) = h'(x)$ such that

$$\frac{1}{\sqrt{1-v^2}} = y. \tag{5.5}$$

Since f satisfies (5.2) with $y_0 = l$, we now estimate, using (5.5),

$$\frac{d}{dy}(h^{-1}(y) - f^{-1}(y)) = \frac{1}{v} - \frac{1}{u} \leq \frac{1}{\sqrt{1-y^{-2}}} - \frac{1}{\sqrt{1-(y+\varepsilon)^{-2}}},$$

where $\varepsilon = l^{n-1}(1-l)y^{1-n}$. By putting the expression over a common denominator we find when $y \geq 2$,

(5.6)

$$\leq \frac{2y\varepsilon + \varepsilon^2}{\sqrt{1-y^{-2}}\sqrt{1-(y+\varepsilon)^{-2}}(\sqrt{1-y^{-2}} + \sqrt{1-(y+\varepsilon)^{-2}})y^2(y+\varepsilon)^2} \leq \frac{c}{y^{n+2}}$$

for some $c = c(n)$. Because $\lim_{y\to\infty}(h^{-1}(y) - f^{-1}(y)) = 0$, we conclude from (5.6),

$$\begin{aligned} 0 < h(z) - f(z) &\leq f^{-1}(y) - h^{-1}(y) \\ &\leq \int_y^\infty \left(\frac{1}{v} - \frac{1}{u}\right) dy \leq \frac{c}{y^{n+1}} \leq \frac{c}{(z-1)^{n+1}}, \end{aligned}$$

where $z = h^{-1}(y)$, and so (d) follows.

6. Existence of constant mean curvature cuts

In this section we sharpen known results [38, Theorem 2] for prescribing constant mean curvature hypersurfaces in $\mathbf{R}^{n,1}$ by their asymptotic behavior. It was unknown how large the classes of solutions determined by their projective boundary values were [38]. In fact, by Theorem 6.2 we show a much finer structure and construct many different solutions within each projective class. The existence of these hypersurfaces reduces to construction of global barriers.

Proposition 6.1 [8] [38]. *Suppose there exist functions $v(x) \leq w(x) \in \mathbf{C}^{0,1}(\mathbf{R}^n)$ which are (weak) sub- and super-solutions to the prescribed* *mean curvature equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=nH, \tag{6.1}$$
$$|Du(x)|<1 \quad \textit{for all } x\in\mathbf{R}^n,$$

for $H=\text{const}>0$. *Then there exists a smooth solution* $u(x)$ *of* (6.1) *whose graph is an entire spacelike constant mean curvature hypersurface of* $\mathbf{R}^{n,1}$ *which is complete in the induced metric. Moreover,* $u(x)$ *satisfies*

$$v(x)\le u(x)\le w(x) \quad \textit{for all } x\in\mathbf{R}^n.$$

By using Lemma 5.1 we are able to construct barriers which sharpen the existence result to

Theorem 6.2. *Let* $L\subset\mathbf{S}^{n-1}$ *be a closed subset containing more than one point. Let* $f_0(\theta)$ *be any* $\mathbf{C}^0(L)$ *function. Then there exists an entire spacelike constant mean curvature hypersurface* $u(x)$ *with the property that the lightlike set* $L_u=L$ *and*

$$\lim_{r\to\infty}(u(r\theta)-r)=f_0(\theta) \quad \textit{for all } \theta\in L, \tag{6.2}$$

where we have identified $\mathbf{S}^{n-1}$ *with the set of unit vectors of* $\mathbf{R}^n$.

With the notation and method used previously [38], the idea is to replace the hyperboloid by the semitrough in the barrier construction. The three constant mean curvature $H=1$ hypersurfaces we use are the hyperboloid h, trough τ, and semitrough σ along the x_1-axis defined by

$$h(x)=\sqrt{1+|x|^2},\quad \tau(x)=\sqrt{\frac{n-1}{n}+|\bar{x}|^2},\quad \sigma(x)=\sqrt{f(x_1)^2+|\bar{x}|^2},$$

where $x=(x_1,\bar{x})$ and f is defined in Lemma 5.1. The theorem, we believe, can be proved for a wider class of f_0's.

Proof. By scaling $\mathbf{R}^{n,1}$, without loss of generality we may assume $H=1$. First assume that $f_0=f\in C^2(G)$ for some open $G\supset L$. Choose an open $G'\subset\mathbf{S}^{n-1}$ such that $L\subset G'\subset\overline{G'}\subset G$, and extend $f|_{G'}$ to a $\mathbf{C}^2$-function on $\mathbf{S}^{n-1}$ and then to $\mathbf{R}^n-\{0\}$ by $f(x)=f(x/|x|)$ in such a way that for all $x, y\in\mathbf{S}^{n-1}$,

$$|f(x)-f(y)-Df(y)(x-y)|\le M|x-y|^2=-2My\cdot(x-y), \tag{6.3}$$

where M depends only on L and f. To construct the supersolution, define $p(y)=-Df(y)+ky$ for $y\in\mathbf{S}^{n-1}$, where k is a large constant to be chosen later. Now define

$$z_2(x;y,f)=f(y)+k+\sqrt{1+|x-p(y)|^2}.$$

Let $\theta \in \mathbf{S}^{n-1}$ be arbitrary. Define the limit operator Λ on convex achronal hypersurfaces w by

$$\Lambda[w](\theta) = \lim_{r\to\infty}(w(r\theta) - r)$$

so that $y \cdot Df(y) = 0$ implies

$$\Lambda[z_2(\cdot\,; y, f)](\theta) = f(y) + k - \theta \cdot p(y), \qquad \Lambda[z_2(\cdot\,; y, f)](y) = f(y).$$

We compute angular and radial derivatives using the Schwarz inequality to show

$$\text{(6.4)} \qquad \left.\frac{\partial}{\partial \psi}\right|_{\psi=y} (f(y) + k - \psi \cdot p(y)) = Df(y), \qquad \frac{\partial}{\partial r}(z_2(r\theta\,; y, f) - r) \le 0.$$

Hence it follows by choosing $k = 2M$ that for all fixed $y \in \mathbf{S}^{n-1}$,

$$z_2(z\,; y, f) \ge |x| + f(x) \quad \text{for all } x \in \mathbf{R}^n.$$

For any closed set $F \subset \mathbf{S}^{n-1}$, a supersolution [38] which has F as its lightlike set can be defined by

$$q_2(x\,; F, f) = \inf_{y\in\mathbf{R}^n}\left(V_F(y) + \sqrt{1 + |x-y|^2}\right) + \sup_{\theta\in\mathbf{S}^{n-1}} |f(\theta)|,$$

where $V_F(x) = \sup\{x \cdot y : y \in F\}$ is the homogeneous function of degree one whose lightlike set is F. We can now define the supersolution

$$w(x\,; L, f) = \min\left(q_2(x\,; L, f), \inf_{\theta\in L} z_2(x\,; \theta, f)\right).$$

We have $w > V_L(x) + f(x)$ and $\Lambda[w](\theta) = f(\theta)$ for all θ in L.

Let $\overline{B}(\theta, \rho) \subset \mathbf{S}^{n-1}$ be a closed ball about θ of radius ρ in the sphere. Since the set of lightlike directions, L_σ, corresponding to $\sigma(x)$ is the half ball $B^+ = \mathbf{S}^{n-1} \cap \{x : x_1 \ge 0\}$, we can find an isometry β of $\mathbf{R}^{n,1}$ that takes σ into a hypersurface whose lightlike directions are exactly $\overline{B}(\theta, \rho)$. Denote the corresponding function by $\sigma_{\theta,\rho}(x)$. For constructing the subsolution, for $y \in \mathbf{S}^{n-1}$ let

$$\zeta(x\,; y, \rho, f) = f(y) - k + \sigma_{y,\rho}(x + p(y)).$$

Let z_1 be defined like ζ, but with h in place of σ. Then $\zeta < z_1 \le z_2$ follows from the fact that $\sigma < h$. Arguing similarly to (6.4), using the fact that $\sigma(r\theta) - h(r\theta) \to 0$ as $r \to \infty$, for all $\theta = (\theta_1, \cdots, \theta_n) \in \mathbf{S}^{n-1}$ such that $\theta_1 \geq 0$, we have

$$
\tag{6.5}
\begin{gathered}
\Lambda[\zeta(\cdot\,; y, \rho, f)](\theta) = f(y) - k + p(y) \cdot \theta, \\
\Lambda[\zeta(\cdot\,; y, \rho, f)](y) = f(y), \\
\left.\frac{\partial}{\partial \psi}\right|_{\psi=y} (f(y) - k + \psi \cdot p(y))) = Df(y).
\end{gathered}
$$

By compactness, we can fix an $\eta > 0$ such that $\overline{B}(\theta, \eta) \subset G'$ for all $\theta \in L$. Now take a sequence $0 < \rho_j < \eta$ such that $\rho_j \to 0$, and let $L_j = \{\theta \in \mathbf{S}^{n-1}\colon \operatorname{dist}(\theta, L) < \rho_j\}$ be the ρ_j parallel set to L. Take increasing and decreasing $f_j^- < f_j^+ \in C^2(\mathbf{S}^{n-1})$ approximating the boundary values $f_j^- < f_0 < f_j^+$ on L and $f_j^+ - f_j^- < \rho_j$ on L_j. Define the sequences

$$
v_j(x) = \sup_{\theta \in L} \zeta(z\,; \theta, \rho_j, f_j^-), \qquad w_j(x) = w(x\,; L_j, f_j^+);
$$

v_j is a subsolution and w_j is a supersolution by construction, and both have L_j as lightlike set. To check that $v_j < w_j$ it suffices to check

$$
\tag{6.6}
\zeta(x\,; \xi, \rho_j, f_j^-) < q_2(x\,; L_j, f_j^+), \qquad \zeta(x\,; \xi, \rho_j, f_j^-) < z_2(x\,; \mu, f_j^+)
$$

for all $x \in \mathbf{R}^n$, $\xi \in L$, $\mu \in L_j$. By isometry, we may assume without loss of generality that $L_\zeta = B^+$. The first inequality of (6.6) holds by construction, since $q_2 > h(x)$ for $x_1 \geq 0$ and also $q_2 \geq \tau(x)$ for $x_1 \leq 0$. By comparing (6.4) to (6.5) we see that $\Lambda[z_2(\cdot\,; y, f_j^+)](\theta) \geq f_j^+(\theta)$ for all $y, \theta \in \mathbf{S}^{n-1}$. But $\Lambda[\zeta(\cdot\,; y, \rho_j, f_j^-)](\theta) \leq f_j^-(\theta)$ for all $y \in L$ and $\theta \in \mathbf{S}^{n-1}$ so that asymptotically, the two unit mean curvature hypersurfaces satisfy

$$
\lim_{r \to \infty} (z_2(r\theta\,; y_2, f_j^+) - \zeta(r\theta\,; y_1, \rho, f_j^-)) > 0.
$$

Hence, by the maximum principle, we obtain the second inequality of (6.6).

Using Proposition 6.1, we conclude that there is a sequence of solutions u_j which satisfy $|\Lambda[u_j](\theta) - f_0(\theta)| < \rho_j$ for $\theta \in L$ but have $L_{u_j} = L_j$. By construction $q_2(x\,; L_j, f_j^+)$ is a decreasing sequence in j. The second term is bounded by $\inf_{\theta \in L} z_2(x, \theta, f_1^+)$ so there is a uniform supersolution independent of j. Similarly there is a global subsolution [38] which has lightlike set L defined by

$$
q_1(x\,; L) = \sup\{\Upsilon_{\theta, \psi}\colon \theta \neq \psi,\ \theta, \psi \in L\} - \sup\{|f_1^-(\theta)|\colon \theta \in \mathbf{S}^{n-1}\},
$$

where $\Upsilon_{\theta,\psi}$ is the isometric image of $\Upsilon(x_1, \bar{x}) = \sqrt{n^{-2} + x_1^2}$ whose lightlike set is $\{\theta, \psi\}$. Hence, exactly as in the proof of [38, Theorem 1] we have uniform a priori $\mathbf{C}^3$ bounds on compact subsets of $\mathbf{R}^n$, and thus a subsequence converges to a hypersurface (see also [8]). Moreover, the functions v_j, w_j provide local barriers at L as in [19, p. 105]. Hence the limiting surface also satisfies (6.2). q.e.d.

A simpler procedure yields the existence of $\mathbf{Z}$ invariant constant mean curvature hypersurface.

Theorem 6.3. *Let γ be an isometry of $\mathbf{R}^{2,1}$ which induces a hyperbolic motion of the Poincaré plane realized as the hyperboloid h with induced motion. Let $\{E, W\}$ denote the source and sink on $H(\infty) = \mathbf{S}^1$, and $\{N, S\}$ be two points in each of the arcs of $\mathbf{S}^1$ separated by E and W. Let*

$$L = \overline{\bigcup_{i \in \mathbf{Z}} \gamma^i\{N, S\}}.$$

Then there is a γ invariant entire spacelike constant mean curvature surface, $m(x)$, whose lightlike set $L_m = L$.

Proof. By Theorem 6.2 we know that a surface $m(x)$ exists whose $L_m = L$ and $f(\theta) = \text{const}$ for $\theta \in L$. To show the invariance, we will construct invariant barriers and prove that the hypersurface between them is uniquely determined—hence invariant itself. Let $\sigma(x) > \tau(x)$ be a semitrough and trough gotten by the same isometric image of $f(x) > l(x)$ in such a way that $L_\tau = \{N, \gamma N\}$ and $L_\sigma = [N, \gamma N] = \{\theta \colon N \le \theta \le \gamma N\}$, the interval of $\mathbf{S}^1$ containing S. For all $\gamma N < \theta < N$ (viewed as angles on $\mathbf{S}^1$) we have $\lim_{r\to\infty}(\sigma(r\theta) - \tau(r\theta)) = 0$. Let $\bar{\sigma} > \bar{\tau}$ be the corresponding construction (with possible addition of a constant) corresponding to the interval $[\gamma S, S]$. Thus there are barriers

$$v(x) = \sup_{i\in\mathbf{Z}} \max(\gamma^i\tau, \gamma^i\bar{\tau}), \qquad w(x) = \inf_{i\in\mathbf{Z}} \min(\gamma^i\sigma, \gamma^i\bar{\sigma}).$$

By construction, $v < w$ are invariant barriers with the property that $\lim_{r\to\infty}(\sigma(r\theta) - \tau(r\theta)) = 0$ for all $\theta \in \mathbf{S}^1$. By Proposition 6.1, there is a solution $m(x)$ between these two barriers. By the maximum principle, the solution is unique. q.e.d.

In constructing harmonic maps, we need a hypersurface whose asymptotic behavior is better understood. As in Theorem 6.3, the barrier construction is simpler than in Theorem 6.2.

Theorem 6.4. *Let $L \subset \mathbf{S}^{n-1}$ be any closed subset. Then there exists an entire spacelike constant mean curvature hypersurface $u(x)$ of $\mathbf{R}^{n,1}$ whose* *lightlike set* $L_u = L$ *and such that*

$$\lim_{r\to\infty}(u(r\theta) - r) = 0 \quad \textit{for all } \theta \in L. \tag{6.7}$$

Moreover, for any ball $\overline{B} \subset \mathbf{S}^{n-1}$ *such that* $\overline{B} \subset L$,

$$\sigma_{\overline{B}}(x) \le u(x) \le h(x) \quad \textit{for all } x \in \mathbf{R}^n,$$

where $\sigma_{\overline{B}}(x)$ *is the rotation of the semitrough solution whose lightlike directions are exactly* $\overline{B}$.

Proof. The existence of such a surface again follows from Theorem 6.2. To obtain the asymptotics, we shall reconstruct the barriers in this special case. Let $\{\theta_i\} \subset \mathbf{S}^{n-1} - L$ be a countable sequence and $\rho_i > 0$ be chosen so that

$$\mathbf{S}^{n-1} - L = \bigcup_{i=1}^{\infty} B(\theta_i, \rho_i).$$

For each i we construct a subsolution of (6.1) by taking an isometric image of the semitrough of Lemma 5.1, but restrict the isometries β_i to boosts that fix $\{x\colon x_1 = x_{n+1} = 0\}$ and rotation about the x_{n+1}-axis. This time let $\sigma_i = \beta_i\sigma$ such that the lightlike set of σ_i is $\mathbf{S}^{n-1} - B(\theta_i, \rho_i)$. Let $v(x) = \sup_{i=1,\cdots,\infty}\sigma_i(x)$. By construction, this barrier satisfies (6.7). The supersolution, similar to $q_2(x)$, is

$$w(x) = \inf_{y\in\mathbf{R}^n}\left(V_L(y) + \sqrt{\frac{1}{H^2} + |x-y|^2}\right).$$

Now we see that $v < w$ and (6.7) holds for both. The resulting solution obtained from applying Proposition 6.1 has the desired properties.

7. Intrinsic estimates from extrinsic data

In this section we demonstrate how the existence of very sharp upper and lower barriers enable transferring information from the lightlike structure of the hypersurface to the intrinsic geometry. We will compare the constant mean curvature hypersurface with a nearby hyperboloid at corresponding values of x. The hyperboloid has the induced metric of hyperbolic space with sectional curvature -1.

Lemma 7.1. *Let* u *be an entire spacelike hypersurface of constant mean curvature* $H = 1$. *Suppose the lightlike set* L_u *contains a ball* $\mathscr{B}(\theta, \rho)$ *for some* $\theta \in L_u \subset \mathbf{S}^{n-1}$ *and* $\rho > 0$. *Suppose that* u *is sandwiched between the semitrough and the hyperboloid*

$$\sigma_{\theta,\rho}(x) \le u(x) \le h(x) \quad \textit{for all } x \in \mathbf{R}^n,$$

where $\sigma_{\theta,p}$ is as in Theorem 6.2. Then h contains an intrinsic sector

$$\overline{\Sigma}(P, v; R, \alpha) = \overline{\exp_P}\{r\theta : R < r, |\theta - v| < \alpha\}$$

for some $P \in h$, $v \in \overline{\mathbf{S}_P^{n-1}} \subset T_P h$, $R(\rho) < \infty$, $\alpha(\rho) > 0$ such that on $\overline{\Sigma}$, u is quasi-isometric to hyperbolic space. That is, the metrics g of u and $\bar{g}$ of h are uniformly equivalent,

$$\left(1 - \frac{c}{\sqrt{|x|}}\right) \bar{g}_{ij}(x) \le g_{ij}(x) \le \left(1 + \frac{c}{\sqrt{|x|}}\right) \bar{g}_{ij}(x) \quad \textit{for all } x \in \overline{\Sigma},$$

where $|x|$ is the distance in $\mathbf{R}^n$.

Proof. The idea of the proof is to show that the sharp bounds obtained from Lemma 5.1 comparing the hyperboloid and semitrough solutions yield information about u. We abuse notation and denote by x or y any of the corresponding points of $\mathbf{R}^n$, σ, u or h.

By applying an isometry of $\mathbf{R}^{n,1}$ we may assume without loss of generality that the lightlike set $L_u \supseteq \mathscr{B}(\mathbf{e}_1, 3\pi/4)$, where $\mathbf{e}_1 = (1, 0, \cdots, 0)$, is the unit vector along the x^1-axis. The tangent space of u is spanned by the vectors

$$E_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial x^{n+1}},$$

so that the metric is

$$g_{ij} = \delta_{ij} - u_i u_j.$$

The hyperboloid $h(x) = \sqrt{1 + |x|^2}$ has

$$\bar{g}_{ij} = \delta_{ij} - \frac{x^i x^j}{1 + |x|^2}.$$

The gradient estimate is based on the observation that the tangent plane $T_x u \subset \mathbf{R}^{n+1}$ is a supporting hyperplane to the convex hull $\mathbf{Conv}(h \cup \{(x, u(x))\})$. In order to estimate Du at a point x in the cone $\{(x_1, \bar{x}) \in \mathbf{R} \times \mathbf{R}^{n-1} : |\bar{x}| < x_1\}$ we rotate $\mathbf{R}^{n,1}$ about the x^{n+1} axis so the x direction moves to $y = r\mathbf{e}_1$, where $r = |x|$, and so $\sigma_{\mathbf{e}_1, 3\pi/4}(x) \ge \sigma(y)$. We write $\sigma(y) = \sigma_{\mathbf{e}_1, \pi/2}(y)$. Call the rotated function u also. We estimate separately $u_1 = D_{\mathbf{e}_1} u$ and $u_\gamma = D_{\mathbf{e}_\gamma} u$, where $\gamma > 1$. Let $\varepsilon = h(y) - \sigma(y)$. The line in $\mathbf{R}^{n,1}$ passing through $(y, u(y))$ and $(y + a\mathbf{e}_\gamma, h(y + a\mathbf{e}_\gamma))$ becomes tangent to h if it satisfies

$$h(y + a\mathbf{e}_\gamma) - h_\gamma(y + a\mathbf{e}_\gamma)a = h(y) - \varepsilon.$$

Substituting $\sqrt{h(y)^2 + a^2}$ for $h(y + a\mathbf{e}_\gamma)$ gives expressions for $h(y)^2 + a^2$ and a^2 which are in terms of ε and $h(y)$ and whose quotient is the slope. Using Lemma 5.1, for r large enough we obtain

$$|u_\gamma(y)| \leq \sqrt{\frac{2\varepsilon(r)}{h(y)}} \leq \frac{c}{r^{n/2+1}}.$$

To estimate $u_1(r, 0, \cdots, 0)$, we consider $r, s \in \mathbf{R}^1$ so that the steepest supporting line has slope $m = h'(s)$ where

$$h(r) - \varepsilon = h(s) + h'(s)(r - s).$$

Inserting the hyperbola $h(r)$,

$$\sqrt{1 + r^2} - \varepsilon = rm + \sqrt{1 - m^2}.$$

Substituting

$$m = h'(r) + a = \frac{r}{\sqrt{1 + r^2}} + a$$

and solving for a yield

$$\frac{2\varepsilon}{(1 + r^2)^{3/2}} - \frac{\varepsilon^2}{(1 + r^2)^2} = \left(a + \frac{\varepsilon r}{1 + r^2}\right)^2 .$$

By absorbing the lower order terms and using Lemma 5.1, we find for r large enough that there is a $c > 0$ so that

$$\left| u_1(r, 0, \cdots, 0) - \frac{r}{\sqrt{1 + r^2}} \right| \leq \frac{c}{r^{n/2+2}}. \tag{7.2}$$

To show the uniform equivalence of the metrics, in these coordinates we have

$$\bar{g}_{ij} = \operatorname{diag}\left(\frac{r}{\sqrt{1 + r^2}}, 1, \cdots, 1 \right).$$

Therefore, for some constant $\xi \neq 0$,

$$(1 + \xi)\bar{g}_{ij} - g_{ij} = \begin{pmatrix} \frac{\xi}{1+r^2} - \mathbf{O}(r^{-n/2-2}) & \mathbf{O}(r^{-n/2-1}) & \cdots \\ \mathbf{O}(r^{-n/2-1}) & \xi + \mathbf{O}(r^{-n-2}) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

When $n > 2$ the matrix is $\xi \bar{g}_{ij}$ plus terms of order $\mathbf{O}(r^{-5/2})$ or less, so for sufficiently large r, it is dominated by $\xi \bar{g}_{ij}$. For $n = 2$,

$$\det((1 + \xi)\bar{g}_{ij} - g_{ij}) = \frac{\xi^2}{1 + r^2} + \mathbf{O}(r^{-3}) > 0,$$

and so is again positive or negative depending on the sign of ξ. In either case, there is a c so that when $\xi^2 \geq c/r$ the matrix is positive or negative depending on the sign of ξ, and the quasi-isometry follows.

Lemma 7.2. *Let u be an entire spacelike constant mean curvature hypersurface of $\mathbf{R}^{n,1}$ satisfying the hypotheses of Lemma 7.1. Then for some $c_1, c_2 > 0$ there is a sector $\overline{\Sigma}(P, v; R, \alpha) \subset h$ for some $P \in h$, $v \in \overline{\mathbf{S}_P^{n-1}}$, $R > 0$ and $\alpha > 0$, so that for all $y \in \overline{\Sigma}$, the integral Gauss-Kronecker curvature over $B(y, c_1) \subset u$ satisfies*

$$\int_{B(y, c_1)} \kappa \, d\,\mathrm{vol} \geq c_2,$$

where the Gauss-Kronecker curvature of u has the expression

$$\kappa(z) = \frac{\det(u_{ij}(z))}{(1 - |Du(z)|^2)^{n/2+1}}.$$

Proof. The idea of the proof is to estimate the integral curvature forced on a surface molded between two convex surfaces. Let $B(y, r)$ denote a metric ball in u and $\overline{B}(y, r)$ a metric ball in h.

Denote the projection $\pi: (x^1, \cdots, x^{n+1}) \mapsto (x^1, \cdots, x^n)$. Let $\overline{\Sigma}_1 \subset h$ be the sector on which the metrics of u and h are uniformly equivalent. Fix the radius $c_1 > 0$ appropriately small, as to be described. By quasi-isometry take $\overline{\Sigma}_2 \subset h$ to be a smaller sector such that $\pi B(y, c_1) \subset \pi\overline{\Sigma}_2$. For any $y \in \overline{\Sigma}_2$ we may rotate $\mathbf{R}^{n,1}$, as in Lemma 7.1, so that $y = (r, 0, \cdots, 0)$ and $\sigma(x) < u(x) < h(x)$ for all $x \in \mathbf{R}^n$. By quasi-isometry, we may choose $l > 0$ so that $\pi\overline{B}(x, l) \subset \pi B(x, c_1)$ for all $x \in \overline{\Sigma}_2$. The integral Gauss curvature can be estimated as follows:

$$\int_{B(y, c_1)} \kappa \, d\,\mathrm{vol} \geq \int_{\pi\overline{B}(y, l)} \frac{\det(u_{ij})}{(1 - |Du|^2)^{(n+1)/2}} \, dx^1 \ldots dx^n. \tag{7.3}$$

First we describe $\pi\overline{B}(y, l)$. Let $U(y, \rho)$ designate the Euclidean ball with center y and radius ρ in $\mathbf{R}^n$. Then by computing in the metric of h we see that $\pi\overline{B}(0, l) = U(0, \operatorname{sh} l)$. By applying the boost rotation of Minkowski angle ν so that $r = \operatorname{sh}\nu$,

$$\beta_\nu = \begin{pmatrix} \operatorname{ch}\nu & 0 & \ldots & 0 & \operatorname{sh}\nu \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \operatorname{sh}\nu & 0 & \ldots & 0 & \operatorname{ch}\nu \end{pmatrix},$$

to the equation of the ball $\overline{B}(0, l) = \{(x^1, \cdots, x^{n+1}) \in u: x^{n+1} < \operatorname{ch} l\}$ we get

$$\sqrt{1+|x|^2} < \frac{\operatorname{ch} l + x^1 \operatorname{sh} \nu}{\operatorname{ch} \nu} \tag{7.4}$$

if $(x^1, \cdots, x^n) \in \overline{B}(y, l)$ and $y = (\operatorname{sh}\nu, 0, \cdots, 0, \operatorname{ch}\nu)$. Thus we see that

$$\pi\overline{B}(y, l) = \pi\beta_\nu\overline{B}(0, l)$$
$$= \left\{(x^1, \cdots, x^n): \left(\frac{x^1}{\operatorname{ch}\nu} - \operatorname{ch} l \frac{\operatorname{sh}\nu}{\operatorname{ch}\nu}\right)^2 + (x^2)^2 + \cdots + (x^n)^2 < \operatorname{sh}^2 l\right\}.$$

We have $\beta_\nu(0) = y$, and $\pi\overline{B}(y, l)$ is an ellipsoid whose Euclidean center is $(\operatorname{sh}\nu \operatorname{ch} l, 0, \cdots, 0)$, whose major axis is in the x^1-direction with major radius $\operatorname{ch}\nu \operatorname{sh} l$ and whose $n-1$ minor radii are $\operatorname{sh} l$. Hence, by taking c_1 small enough that $\operatorname{sh} l < 1/2$ and a smaller $\overline{\Sigma}_3$ so that $\operatorname{ch}\nu < 2\operatorname{sh}\nu$, we have

$$\pi\overline{B}(y, l) \subset Q = \{(x^1, \cdots, x^n): (1 - 2\operatorname{sh} l)r \le x^1 \le 3r\operatorname{ch} l;$$
$$|x^\alpha| \le \operatorname{sh} l, \ \alpha = 2, 3, \cdots, n\}.$$

By taking larger R_4, thus even smaller $\overline{\Sigma}_4$, and using (7.1) and (7.2) from Lemma 7.1, we can estimate for all $y \in \overline{\Sigma}_4$ and $x \in \overline{B}(y, l)$

$$\left| |Du(x)|^2 - \frac{r^2}{1+r^2} \right| \le \frac{c_3}{r^{n/2+2}}$$

for some $c_3 > 0$. Hence for all $y \in \overline{\Sigma}_4$ and $x \in \overline{B}(y, l)$,

$$\frac{1}{(1 - |Du(x)|^2)^{(n+1)/2}} \ge c_4 r^{n+1}, \tag{7.5}$$

where $c_4 > 0$ depends on n and l.

From (7.4) and Lemma 5.1 it follows that

$$\begin{aligned} \frac{\operatorname{ch} l}{\operatorname{ch}\nu} + x^1 \tanh\nu - \frac{c_5}{r^{n+1}} &\le u(x) \quad \text{for all } x \in \partial\overline{B}(y, l), \\ u(x) &\le \sqrt{1+|x|^2} \quad \text{for all } x \in \overline{B}(y, l). \end{aligned} \tag{7.6}$$

Thus $u(y) \le \operatorname{ch}\nu$.

To estimate the integral of the Hessian determinant, we interpret it as the area of the tangential mapping (§4). The set function

$$\omega(\zeta, E) = \mathscr{L}^n \chi_\zeta(E)$$

on $E \subset G$ is nonnegative completely additive on the Borel sets of G where $\mathscr{L}$ is Lebesgue measure and ζ is a convex function. An important property of this set function is that

$$\omega(\zeta, G) \geq \omega(K, G),$$

where K is the function whose graph is the convex cone with vertex at $(x_0, \zeta(x_0))$ and base $\partial G \times \{c\}$ such that

$$\zeta(x_0) \leq c \leq \inf_{z \in \partial G} \zeta(z).$$

Applying to the present situation, take $\zeta = u - x^1 \tanh \nu$, $x_0 = y$ and $G = \overline{B}(y, l)$, where ν is fixed such that $\operatorname{sh} \nu = |y|$. By (7.6), the fact that $x^1 = \operatorname{sh} \nu \operatorname{ch} l$ at y, $\zeta(y) \leq \operatorname{sech} \nu$ and, possibly choosing a smaller $\overline{\Sigma}_5 \ni y$, for $x \in \partial \overline{B}(y, l)$, we obtain

$$\zeta(x) \geq \frac{1 + \operatorname{ch} l}{2 \operatorname{ch} \nu}.$$

Thus

$$\begin{aligned} \int_{\overline{B}(y, l)} \det(u_{ij})\, dx^1 \ldots dx^n &= \int_G \det(\zeta_{ij})\, dx^1 \ldots dx^n \\ &= \omega(\zeta, G) \geq \omega(K, G) \geq \omega(K', Q), \end{aligned}$$

where K' is the shallower conic function with the same vertex constructed using the cube Q. The estimates on ζ show that the height of K' over Q is at least $c_6(l)/r$. Hence, for some $c_7(n)$, $c_8(n, l) > 0$,

$$\omega(K', Q) \geq c_7 \left(\frac{1}{3r^2 \operatorname{ch} l} \right) \left(\frac{c_6}{r \operatorname{sh} l} \right)^{n-1} = \frac{c_8}{r^{n+1}}.$$

Combining this inequality with (7.5) yields for some $c_9(n, l) > 0$,

$$\begin{aligned} \int_{B(y, c_1)} \kappa \, d\mathrm{vol} &\geq \inf_{\pi \overline{B}(y, l)} \frac{1}{(1 - |Du|^2)^{(n+1)/2}} \int_{\pi \overline{B}(y, l)} \det(u_{ij})\, dx^1 \ldots dx^n \\ &\geq c_4 r^{n+1} \times \frac{c_7}{r^{n+1}} \geq c_2 > 0. \end{aligned}$$

Lemma 7.3. *Let u be an entire spacelike constant mean curvature hypersurface of $\mathbf{R}^{n,1}$ satisfying the hypotheses of Lemma 7.1. Then there is a sector $\overline{\Sigma}(P, v; R, \alpha) \subset h$ for some $P \in h$, $v \in \overline{\mathbf{S}_P^{n-1}}$, $R > 0$ and $\alpha > 0$ and a constant $c_{10} > 0$ so that the sectional curvature K_u of u is pinched:*

$$-\tfrac{1}{2} n^2 H^2 \leq K_u(x) \leq -c_{10} < 0 \quad \textit{for all } x \in \overline{\Sigma}.$$

Proof. From §1 the sectional curvature in the $\mathbf{e}_1$, $\mathbf{e}_j$ directions is

$$K_u(\mathbf{e}_i, \mathbf{e}_j) = R_i{}^j{}_{ij} = h_{ij}^2 - h_{ii} h_{jj}$$

and if $i \neq j$, the lower bound follows from the inequality

$$h_{ii}h_{jj} \leq \frac{1}{2}\sum_{ij} h_{ij}^2 \leq \frac{1}{2}n^2H^2.$$

By the Splitting Theorem 3.1, we have $h_{ij} > 0$. Let h^{ij} denote the inverse matrix. The upper bound follows from a uniform positive bound on the principal curvatures, which in turn follows from a uniform bound on the Gauss-Kronecker curvature. Take the sector $\overline{\Sigma}$ the same as in Lemma 7.2. By diagonalizing $h_{ij} = \operatorname{diag}(\kappa_1, \cdots, \kappa_n)$, we have

$$\kappa_i \geq \frac{c_{11}(n)}{\psi_{n-1}} \inf_{y\in\overline{\Sigma}} \kappa(y) \geq \frac{c_{12}(n)}{H^{n-1}} \inf_{y\in\overline{\Sigma}} \kappa(y),$$

where ψ_k is the kth elementary symmetric function of $\{\kappa_1, \cdots, \kappa_n\}$, and we have used Newton's inequalities $\psi_{k+1}^k \leq \psi_k^{k+1}$. On the other hand, computing with $\varphi = \kappa^{1/n}$ gives

$$n^2\Delta\varphi = \varphi h^{ij}h_{ijp}h^{st}h_{stp} - n\varphi h^{is}h_{stp}h^{tj}h_{ijp} + n\varphi h^{ij}h_{ijpp}.$$

Diagonalizing at a point and using the Schwarz inequality we obtain

$$(h^{ij}h_{ijp})^2 = \left(\sum_{ij} \frac{\delta_{ij}h_{ijp}}{\sqrt{\kappa_i\kappa_j}}\right)^2 \leq n\sum_{ij}\frac{h_{ijp}^2}{\kappa_i\kappa_j} = nh^{is}h_{stp}h^{tj}h_{ijp}.$$

Combining this with the commutation formula for h_{ijpp} and using facts from §1 yield

$$\Delta\varphi \leq \varphi\left(\sum_{ij} h_{ij}^2 - nH^2\right) \leq n(n-1)H^2\varphi.$$

So by the mean value inequality for supersolutions, Lemma 2.5, for appropriate $0 < p < n$, we find

$$\begin{aligned}\varphi(y) &\geq \inf_{B(y,c_1/2)} \varphi \geq c_{13}H^{(p-n)/p} \sup_{B(y,c_1)} \kappa^{(n-p)/(np)} \left(\fint_{B(y,c_1)} \varphi^p\right)^{1/p} \\ &\geq c_{13}H^{(p-n)/p}\left(\fint_{B(y,c_1)} \kappa\right)^{1/p}.\end{aligned}$$

The last quantity is uniformly positive by Lemmas 7.1 and 7.2.

8. Existence of harmonic functions on some cuts

In this section we consider the function theory of constant mean curvature hypersurfaces. A noncompact manifold which admits nonconstant bounded harmonic functions is called *hyperbolic*; one that does not is called *parabolic*. A complete manifold whose sectional curvature is pinched, $-b^2 \le K_M \le -a^2 < 0$, is hyperbolic since one can find harmonic functions taking arbitrary boundary values at infinity [2], [3], [13], [37]. However $\mathbf{R}^n$ is parabolic. The entire spacelike constant mean curvature hypersurfaces lie somewhere between $\mathbf{R}^n$ and the space form with constant curvature $-\frac{1}{2}n^2H^2$. By §7, certain constant mean curvature hypersurfaces whose lightlike sets have nonempty interior contain large subsets with nonnegatively pinched sectional curvature; these are shown hyperbolic. Global sub- and super-solutions are found by combining our knowledge of the intrinsic geometry of cuts and the availability of extrinsically defined functions with estimates.

Theorem 8.1. *Let u be an entire constant mean curvature hypersurface of $\mathbf{R}^{n,1}$ with mean curvature $H = 1$. Suppose that the set of lightlike directions L_u of u contains a ball $\mathscr{B}(\theta, \rho) \subset L_u \subset \mathbf{S}^{n-1}$ for some $\rho > 0$. Suppose that u is sandwiched between the semitrough and the hyperboloid*

$$\sigma_{\theta,\rho}(x) \le u(x) \le h(x) \quad \textit{for all } x \in \mathbf{R}^n,$$

where $\sigma_{\theta,\rho}$ is the solution constructed in Lemma 5.1 such that the lightlike set $L_{\sigma_{\theta,\rho}} = \overline{\mathscr{B}}(\theta, \rho)$. Then u is hyperbolic.

Proof. Let π denote the identification of points of the hyperboloid h to u with the same x coordinate. By Lemmas 7.1 and 7.2, there is a sector $\overline{\Sigma} \subset h$ which is quasi-isometric to $\Sigma = \pi\overline{\Sigma}$ and such that the sectional curvature satisfies

$$-b^2 \le K_u(x) \le -a^2 < 0 \tag{8.1}$$

for all $x \in \Sigma$ with constants $a, b > 0$ depending on n, H and ρ. By taking a further ambient isometry of $\mathbf{R}^{n,1}$, we may assume that $u(x) \to \infty$ as $|x| \to \infty$ on Σ and that the sector contains an intrinsic cone $C = \pi\overline{C} = \pi\overline{\exp_o}\{(\rho, \vartheta) \colon \rho \ge 0,\ \vartheta \in \mathscr{B}(\vartheta_0, \alpha)\}$ whose vertex is $x = 0$ and a neighborhood $\{y \in u \colon \operatorname{dist}_u(C, y) \le 2\}$ satisfies (8.1).

Our construction involves a mollified extrinsically defined function $\phi \in \mathbf{Lip}(h)$ (cf. [3]). Let $0 \le \zeta \le 1$ in $C^\infty(\mathbf{R})$ be a cutoff function such that $\zeta(r) = 1$ for $r \le 1$, $\zeta(r) = 0$ for $r \ge 2$ and $|D\zeta| + |D^2\zeta| \le c_3$. Define the smoothing operator on u by

$$\mathscr{S}\phi(y) = \frac{\int_u \phi(z)\zeta(\operatorname{dist}_u^2(y, z))\, dz}{\int_u \zeta(\operatorname{dist}_u^2(y, z))\, dz},$$

where $\operatorname{dist}_u(x, y)$ is distance taken in u. By the Hessian comparison theorem [20], in view of (8.1), it follows that the function $r(x) = \operatorname{dist}_u(x, y)$

satisfies

$$2r_ir_j + 2ar\coth(ar)(\delta_{ij} - r_ir_j) \le (r^2(x))_{ij}$$
$$\le 2r_ir_j + 2br\coth(br)(\delta_{ij} - r_ir_j).$$

Hence one obtains the estimate for $x_0 \in C$,

$$|\mathscr{S}(\phi - \phi(x_0))| + |D(\mathscr{S}\phi)| + |D^2(\mathscr{S}\phi)| \le c_4 \sup_{B(x_0, 2)} |\phi(x) - \phi(x_0)|,$$
$$\le c_4 \sup_{\overline{B}(x_0, 2c_1)} |\phi(x) - \phi(x_0)|,$$

where c_4 depends on a, b, c_3 and n. Suppose now that $\phi(\rho, \vartheta) = R(\rho)\Theta(\vartheta)$ where $\vartheta \in \mathbf{S}_0^{n-1}h$ and $\rho > 0$. Suppose that $R(\rho)$ is chosen to be an increasing smooth function which is zero in $\overline{B}(0, c_7)$ and 1 off $\overline{B}(0, 2c_7)$ such that $R(\rho)' \le 2/c_7$, and Θ is chosen in $\mathbf{Lip}(\mathbf{S}_o^{n-1})$. Then for $x_0 \in \overline{C}$,

$$|\phi(x) - \phi(y)| \le \frac{4c_1}{c_7}\mathbf{I}_{B(0, c_1+2c_7)}(x_0) \cdot \sup_{\mathbf{S}_0^{n-1}} \Theta + \angle x\#y \cdot \mathbf{Lip}_{\mathbf{S}_o^{n-1}}\Theta$$

for all $x, y \in B(x_0, 2c_1)$, where $\mathbf{I}_E$ is the characteristic function for E. But for $x, y \in \overline{B}(x_0, 2c_1)$ the law of cosines in h gives

$$\mathrm{ch}(\mathrm{dist}_h(x, y)) = \mathrm{ch}(\mathrm{dist}_h(x, 0))\,\mathrm{ch}(\mathrm{dist}_h(y, 0))$$
$$- \cos(\angle_h x\#y)\,\mathrm{sh}(\mathrm{dist}_h(x, 0))\,\mathrm{sh}(\mathrm{dist}_h(y, 0)),$$

where $\mathrm{dist}_h(x, y)$ is taken in h. Hence

$$\angle x\#y \le c_8 e^{-t} \le \frac{c_9}{1 + |x_0|},$$

where $t = \mathrm{dist}_h(0, x_0) = \mathrm{sh}\,|x|$ because $\mathrm{dist}_h(x, y) \le 2c_1$ and $\mathrm{dist}_h(\#, x)$, $\mathrm{dist}_h(\#, y) \ge t - c_1$.

To construct the barriers, consider now two functions globally defined on u,

$$\mu = -\langle P, T\rangle, \qquad \nu = -\langle \mathbf{e}_{n+1}, T\rangle,$$

where P is the position vector of the hypersurface u, and $T = (0, \cdots, 0, 1) \in \mathbf{R}^{n,1}$ is a fixed unit timelike vector. From §1 we see that these functions satisfy

$$|D\mu|^2 = -1 + \nu^2, \qquad \Delta\mu = nH\nu,$$

$$|D\nu|^2 = \sum_i \Big(\sum_j -h_{ij}\langle \mathbf{e}_j, T\rangle\Big)^2, \qquad \Delta\nu = \nu\sum_{ij} h_{ij}^2 = \nu A.$$

On the other hand, the functions are comparable. On C, by Lemma 7.1 we find that

$$\nu = \frac{1}{\sqrt{1-|Du|^2}} \geq c_{10}\sqrt{1+|x|^2} \geq c_{10}\mu.$$

For δ, $c_{11} > 0$ consider

$$\begin{aligned}\Delta(\nu + c_{11})^{-\delta} \\ &= \frac{\delta}{(\nu + c_{11})^{\delta+2}}\left((\delta+1)\sum_i\left(\sum_j -h_{ij}\langle \mathbf{e}_j, T\rangle\right)^2 - \nu(\nu+c_{11})A\right).\end{aligned}$$

Since all the principal curvatures have a lower bound by Lemma 7.3, the eigenvalues of $h_{ij}(x)$ exceed $\sqrt{c_{12}A(x)} > 0$ for all $x \in \overline{C}$ so $h_{11}^2 \geq c_{12}A$. By a rotation of axes, we may pick $\mathbf{e}_1$ orthogonal to T. Thus in $\overline{C}$,

$$\sum_{i=1}^{n}\left(\sum_{\beta=2}^{n} -h_{i\beta}\langle \mathbf{e}_\beta, T\rangle\right)^2 \leq \sum_{i\beta} h_{i\beta}^2(-1+\nu^2) \leq (1-c_{12})A(-1+\nu^2).$$

By choosing δ sufficiently small so that $(1+\delta)(1-c_{12}) < 1$ and large constants c_{11}, c_{13} we have shown

$$\Delta(\mathscr{S}\phi \pm c_{13}(\nu + c_{11})^{-\delta}) \lesseqgtr 0. \tag{8.2}$$

Choose an open $G \subset \mathscr{B}(\vartheta_0, \alpha) \subset \mathbf{S}_0^{n-1}$. Then choose a smooth function Θ on $\mathbf{S}_0^{n-1}$ which is nonconstant on G, and such that $\phi = R\Theta$ has support in $\overline{C}$ so that $\mathscr{S}\phi$ is zero off $\overline{C}$. Let $\infty > c_{14} > \sup|\Theta|$. Now choose smooth functions $\Theta_1 \leq \Theta \leq \Theta_2$ so that $\Theta_1 = \Theta = \Theta_2$ on G but $\Theta_1 < -c_{14}$ and $\Theta_2 > c_{14}$ off $\mathscr{B}(\vartheta_0, \alpha)$. Let $\phi_i(\rho, \vartheta) = R(\rho)\Theta_i(\vartheta)$, $i = 1, 2$, be the extensions to $\overline{C}$. As in (8.2) we may choose constants so that

$$\begin{aligned}\phi_- &= \max(-c_{14}, \mathscr{S}\phi - c_{13}(\nu + c_{11})^{-\delta}) \\ &\leq \min(c_{14}, \mathscr{S}\phi + c_{13}(\nu + c_{11})^{-\delta}) = \phi_+\end{aligned}$$

are sub- and super-harmonic functions on C. By taking c_{13} larger, if necessary, $\phi_\pm$ extend as constants beyond C, and are, in fact, globally defined barriers on u. Finally, consider a harmonic function f which is pinched between $\phi_\pm$ found, for example, by the Perron process. It is a bounded and nontrivial harmonic function of u since it takes nontrivial boundary values on G.

9. Examples of harmonic maps and the conformal structure of cuts

We now summarize the relation between the harmonic Gauss maps of entire spacelike constant mean curvature hypersurfaces of $\mathbf{R}^{n,1}$ and the conformal structure of these hypersurfaces.

Theorem 9.1. *Let $L \subset \mathbf{S}^1$ be a finite set with cardinality $\#L$. Then there is a harmonic map φ from the complex plane $\mathbf{C}$ into the hyperbolic plane H^2 such that*

$$\overline{\varphi(\mathbf{C})} = \mathbf{Conv}(L).$$

If $\#L \geq 3$, then φ is a diffeomorphism from $\mathbf{C}$ to $\mathbf{Conv}(L)^\circ$.

Proof. By Theorem 4.7 or 6.2 there exists a constant mean curvature $H > 0$ cut u whose Gauss map satisfies $\mathscr{G}(u) = \mathbf{Conv}(L)^\circ$. The Gauss map is a diffeomorphism for $\#L \geq 3$ by strict convexity. The total curvature of u is the area of the Gauss image so

$$\int_u \kappa\, dA = -\mathrm{Area}(\mathscr{G}(u)) = \pi(2 - \#L),$$

which is finite. By the Blanc-Fiala-Huber Theorem ([9], [23]), M is conformal to $\mathbf{C}$. Hence there is a conformal diffeomorphism $\zeta\colon \mathbf{C} \to u$, and the desired harmonic map is $\varphi = \mathscr{G} \circ \zeta$.

Theorem 9.2. *Suppose $L \subset \mathbf{S}^{n-1}$ is a closed set with nonempty interior. Then there is a harmonic map $\varphi\colon M \to H^n$ from a hyperbolic spacelike entire constant mean curvature hypersurface M to hyperbolic space which satisfies*

$$\overline{\varphi(M)} = \mathbf{Conv}(L)$$

and which is a diffeomorphism from M to $\mathbf{Conv}(L)^\circ$.

Proof. By Theorem 6.4 there is an entire spacelike constant mean curvature hypersurface u whose lightlike set is L, so its Gauss map $\mathscr{G}$ has the desired harmonic mapping properties, as in §4. Moreover, the solution u is pinched between a semitrough and hyperboloid. By Theorem 8.1, such u is hyperbolic.

Corollary 9.3. *Let $L \subset \mathbf{S}^1$ be a closed set containing an interval. Then there is a harmonic map φ from the Poincaré disk $\mathbf{D}$ into the hyperbolic plane H^2 such that*

$$\overline{\varphi(\mathbf{D})} = \mathbf{Conv}(L)$$

so that φ is a diffeomorphism from $\mathbf{D}$ to $\mathbf{Conv}(L)^\circ$.

Proof. By Theorem 9.2 there is a constant mean curvature $H > 0$ cut u of $\mathbf{R}^{2,1}$ satisfying the mapping conditions and hyperbolicity. Hence u is conformally diffeomorphic to the disk.

Theorem 9.4. *Let γ be an isometry of $\mathbf{R}^{2,1}$, which induces a hyperbolic motion of the Poincaré plane realized as the hyperboloid h with induced motion. Let $\{E, W\}$ denote the source and sink on $H(\infty) = \mathbf{S}^1$, and $\{N, S\}$ be two points in each of the arcs of $\mathbf{S}^1$ separated by E and W. Let*

$$L = \overline{\bigcup_{i\in\mathbf{Z}} \gamma^i\{N, S\}}.$$

Then there is an entire spacelike constant mean curvature $H > 0$ surface u such that its Gauss map

$$\overline{\mathscr{G}(u)} = \mathbf{Conv}(L),$$

so it has infinite total curvature, but is parabolic.

Proof. By Theorem 6.3, there is a γ invariant entire spacelike constant mean curvature surface, $u(x)$, whose lightlike set $L_u = L$. The Gauss map satisfies the desired properties. Since u is invariant, we let q be the quotient of u by the infinite cyclic group of isometries generated by γ. q is a cylinder with finite total curvature, since the Gauss image of the fundamental domain is contained in an ideal geodesic polygon with finite area. By uniformization, q is conformally diffeomorphic to the flat cylinder and by lifting, u is conformally $\mathbf{C}$. q.e.d.

We end with some questions. For surfaces, we have shown that if the lightlike set is finite, then there exists a parabolic constant mean curvature surface, and if the lightlike set has nonempty interior, there exists a hyperbolic surface which has a harmonic diffeomorphism to the convex hull in the hyperbolic plane. Theorem 9.4 gives another example not covered by these cases. Does the lightlike set of a constant mean curvature surface determine its conformal type? Which sets correspond to each type? For arbitrary dimension we have shown that given the lightlike set there is a constant mean curvature cut whose Gauss map establishes a harmonic map to the convex hull of the set and for which the conformal properties may be deduced. By Theorem 6.2, however, there are many cuts having the same lightlike directions. Can one deduce the conformal properties for these as well? More generally, does the structure of the extreme points of the harmonic maps image restrict the domain? For example, must every harmonic map to the convex hull of ideal points factor through a constant mean curvature cut as in the case, locally, for surfaces [1], [25], [26]?

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