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$L^{n/2}$-CURVATURE PINCHING

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The famous sphere theroem states that a complete, simply connected 1/4-pinched manifold is homeomorphic to the standard sphere [2], [22], [26]. It is also known that the homeomorphism theorem can be sharpened to a diffeomorphism theorem, if a more restrictive pinching condition is imposed [12], [27], [28], [20].

On the other hand, Gromov proved the negative pinching theorem provide the pinching constant also depending on the diameter of the manifold [14], [17].

In this paper we prove pinching theorems for $L^{n/2}$-curvature bounded Riemannian manifolds. We denote the norm of the curvature tensor $\mathrm{Rm}(g)$ of the metric g by $|\mathrm{Rm}(g)|$. Our main results may now be stated as follows.

Theorem A. *For any $i_0 > 0$, $H > 0$, and integer $n \geq 4$, there exists a constant $\mu = \mu(H, i_0, n) > 0$, such that if (M, g) is a complete Riemannian manifold with* $\operatorname{diam} M = n \geq 4$, *and*

(a) $\mathrm{Ric}(g) \geq -Hg$,

(b) $\mathrm{inj}(g) \geq i_0$,

(c)

$$\max_{x\in M} \int_{B_{i_0}(x)} |\mathrm{Rm}(g)|^{n/2}\, dg \leq H,$$

(d)

$$\max_{x\in M} \int_{B_{i_0}(x)} |R(g)_{ijkl} - (g_{ik}g_{jl} - g_{il}g_{jk})|^2\, dg \leq \mu,$$

then M is homotopic to a Riemannian manifold $\overline{M}$ of positive constant sectional curvature, in particular, M is compact. Furthermore, M is covered by a topological sphere.

Theorem B. *For each $H > 0$, $i_0 > 0$, $d > 0$, and integer $n \geq 4$, there exists a small constant $\mu = \mu(H, i_0, d, n) > 0$, such that if (M, g) is a*

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compact Riemannian manifold with $\dim M = n \geq 4$, *and*

(a) $\mathrm{Ric}(g) \geq -Hg$,

(b) $\mathrm{inj}(g) \geq i_0 > 0$,

(c)

$$\max_{x\in M} \int_{B_{i_0}(x)} |\mathrm{Rm}(g)|^{n/2}\, dg \leq H,$$

(d)

$$\max_{x\in M} \int_{B_{i_0}(x)} |R(g)_{ijkl} - \Delta(g_{ik}g_{jl} - g_{il}g_{jk})|^2\, dg \leq \mu,$$

where $\Delta = -1$ *or* 0.

(e) $\mathrm{diam}(g) \leq d$,

then M *is homotopic to a manifold with constant sectional curvature* $\Delta = -1$ *or* 0.

Using the results of Kervaire and Milnor in [21], we have

Corollary C. *If* (M, g) *is as in Theorem* A, *and if* $n \leq 6$, *then* M *is covered by a diffeomorphism sphere.*

If $\mathrm{inj}(M, g) \geq i_0 > 0$, we can consider the metric on a geodesic ball $B_{i_0}(x)$ in polar coordinate $\{r, \theta\}$ on $B_{i_0}(x)$,

$$g = dr^2 + g(r),$$

where $g(r)$ are metrics on the sphere $S^{n-1} = \{x \in R^n,\ |x| = 1\}$.

As a consequence of the proof, we have the following interesting result (see §2).

Theorem D. *Let* (M, g) *be a complete Riemannian manifold with* $\dim M = n \geq 2$, *such that*

(a) $\mathrm{Ric}(g) \geq -Hg$,

(b) $\mathrm{inj}(g) \geq i_0 > 0$,

and for any $x \in M$,

(c)

$$\int_{B_{i_0}(x)} |\mathrm{Rm}(g)|^{n/2}\, dg \leq K.$$

Then there exist constants

$$C_1 = C_1(H, K, i_0, n) > 0, \qquad C_2 = C_2(H, K, i_0, n) > 0,$$

and a diffeomorphism $\phi\colon S^{n-1} \to S^{n-1}$, *such that*

$$0 < e^{-C_1/r}\, d\theta^2 \leq \phi^* g(r) \leq e^{C_1}\, d\theta^2,$$

where $d\theta^2$ *is the standard metric on* S^{n-1}.

For the proofs of the main theorems it is worth noticing that there is no evolution equation method available as in the case of

$$\int_M |\mathrm{Rm}|^{p/2} \le K < \infty, \qquad p > n.$$

as given in 36. Instead, we develop a series of estimates on the metric which are based on integral bounds on curvature, but are not consequences of the Rauch comparison theorem. $L^{n/2}$ curvature pinching is much more interesting and far more difficult than $L^{p/2}$ curvature pinching, $p > n$. The difficulty is caused by the facts that the power $n/2$ of $L^{n/2}$ curvature pinching is the same as the critical power of Sobolev inequality and that the $L^{n/2}$-norm of curvature is a scale invariant.

Remark 0.1. In some sense, the bound on the scale invariant $L^{n/2}$-norm of curvature is necessary. In fact, M. Gromov pointed out that any compact manifold M carries Riemannian metrics of volume 1 with L^p-norm of curvature as small as you like for $p < n/2$.

Remark 0.2. The constant $\mu(H, i_0, n)$ is not estimated explicitly here, although an estimation may be possible, but would be very complicated. We use a noneffective argument.

The proofs of the theorems will be given in the remaining sections which are organized as follows.

1. First order estimate on geodesic balls.
2. Zero order estimate on geodesic balls.
3. The diameter estimate of small geodesic sphere.
4. $L^{n/2}$-curvature pinching estimates.
5. $L^{n/2}$-curvature pinching theorems.
6. Miscellaneous results.

In §1, we first prove a metric comparison result of concentrated geodesic spheres by using the lower bounds of Ricci curvature and injectivity radius. We then use it to derive the L^n estimate of the second fundamental form of small geodesic spheres. Thus, combined with the Gauss equations, this result implies an estimate on the $L^{n/2}$-norm of curvature tensor of $(n-1)$-dimensional geodesic spheres with controlled radius.

In §2, we study the geodesic spheres with $L^{n/2}$-norm of curvature bounded. We are able to use the evolution equation [19] to deform the metrics on such spheres. Since the exponent $n/2$ of the $L^{n/2}$-norm of curvature is greater than the critical exponent $(n-1)/2$ of the scale invariant $L^{(n-1)/2}$-norm of curvature of geodesic spheres, the evolution equation is well behaved. Here, the lower bound of the injectivity radius is used to obtain a Sobolev inequality on small geodesic balls ([7], [6] and [5]), which then implies a weaker but more suitable version of the Sobolev inequality on geodesic spheres.

In §3, we use the precompactness theorem of Gromov to give a diameter estimate of small geodesic spheres, which is needed in §2 to estimate the metric of the geodesic sphere of fixed radius.

In §4, we estimate the second fundamental form of small geodesic spheres. We do this by bounding the L^2 distance of the second fundamental form and $b(r)g(r)$ of the geodesic sphere with metric $g(r)$ and radius r by using integral estimate on Jacobi fields (see §4 for the definition of $b(r)$). This is used to estimate the L^1-norm of scalar curvature free part of curvature tensor. Due to the nature of L^p estimates, the estimates are somewhat complicated.

In §5, we prove the theorems by contradiction. We start with a sequence of Riemannian manifolds with the bounds as in Theorems A and B. We show that the geodesic spheres of fixed radius converge to the standard geodesic sphere of space form of the same radius, and the metric tensor converges to the standard metric tensor of space form in $L^{n/2}$-norm. These are used to show that the limit manifold is a space form. We then prove that many manifolds of the given sequence are homotopic to the limit manifold; this part of the argument is taken from [18].

We refer to [13], [4], [17], [19], and [20] for basic tools and results in Riemannian geometry, which will be used freely.

1. First order estimate on geodesic balls

Let us consider an n-dimensional ($n \geq 4$) compact Riemannian manifold M with metric g, and denote by $\mathrm{Rm}(g)$ and $\mathrm{Ric}(g)$ the Riemann curvature tensor and Ricci curvature tensor of the metric g respectively. We denote the injectivity radius of g by $\mathrm{inj}(g)$. Throughout this section we shall make the following hypothesis.

Assumption 1.1. There exist constants $H > 0$ and $I_0 > 0$, such that

(a) $\mathrm{Ric}(g) \geq -Hg$.
(b) $\mathrm{inj}(g) \geq i_0 > 0$, $i_0 \leq \pi/2$.

We fix a point $x_0 \in M$. Let $B_\rho(x_0) = \{x \in M \colon d(x_0, x) \leq \rho\}$ be the geodesic ball of M at x_0 with radius $\rho < i_0$; here d is the induced distance function on M. We consider any geodesic polar coordinate $\{r, x^1, \cdots, x^{n-1}\}$ on $B_\rho(x_0)$. By identifying $B_\rho(x_0)$ with the Euclidean ball $B_\rho = \{v \in R^n ;\ |v| \le \rho\}$, we have

$$g = dr^2 + \sum g_{ij}(r, x)\, dx^i\, dx^j, \tag{1.2}$$

$$R_{irrj} = \frac{1}{2}\frac{\partial^2}{\partial r^2} g_{ij} + \frac{1}{4}\sum g^{kl}\frac{\partial}{\partial r} g_{ik}\frac{\partial}{\partial r} g_{jl}, \tag{1.3}$$

where $f(x) = d(x, x_0)$, and $\{x^1, \cdots, x^{n-1}\}$ is any coordinate on the unit sphere $S^{n-1} = \{v \in R^n,\ |v| = 1\}$. (1.3) implies that

$$R_{rr} = -\frac{\partial^2}{\partial r^2}\ln\sqrt{g} - \frac{1}{4}\left|\frac{\partial}{\partial r} g\right|_g^2, \tag{1.4}$$

where

$$\sqrt{g}\, d\omega = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^{n-1},$$

for the standard volume form $d\omega$ of $S_1 = S^{n-1} \subseteq R^n$, and

$$\left|\frac{\partial g}{\partial r}\right|_g^2 = \sum g^{ij} g^{kl}\frac{\partial}{\partial r} g_{ij}\frac{\partial}{\partial r} g_{jl}.$$

It is an interesting and important fact that $\frac{\partial}{\partial r} g_{ij}$ can be estimated by the constants in Assumption 1.1. The next few paragraphs are devoted to the proof of such estimates.

Proposition 1.5. *For $\rho \le \frac{1}{2} i_0$, we have*

$$\int_0^\rho r^2\left|\frac{\partial}{\partial r} g\right|^2 dr \le C_1(H, n)\rho.$$

Proof. Taking ϕ to be a piecewise smooth function of r with $\phi(\rho) = 0$, noting that

$$\lim_{r\to 0} r^2\frac{\partial}{\partial r}\ln\sqrt{g} = 0,$$

using (1.4), and integrating by parts, we have

$$\begin{aligned}\int_0^\rho r^2\phi^2 R_{rr}\, dr &= -\frac{1}{2}\int_0^\rho r^2\phi^2\frac{\partial^2}{\partial r^2}\ln g\, dr - \frac{1}{4}\int_0^\rho r^2\phi^2\left|\frac{\partial}{\partial r} g\right|^2 dr\\ &= \frac{1}{2}\int_0^\rho \frac{\partial}{\partial r}(r^2\phi^2)\frac{\partial}{\partial r}\ln g\, dr - \frac{1}{4}\int_0^\rho r^2\phi^2\left|\frac{\partial}{\partial r} g\right|^2 dr\\ &= \frac{1}{2}\int_0^\rho (2r\phi^2 + 2r^2\phi\phi')\frac{\partial}{\partial r}\ln g\, dr\\ &\quad - \frac{1}{4}\int_0^\rho r^2\phi^2\left|\frac{\partial}{\partial r} g\right|^2 dr.\end{aligned} \tag{1.6}$$

The Cauchy inequality gives

$$\left|\frac{\partial}{\partial r}\ln g\right|^2 \leq (n-1)\left|\frac{\partial}{\partial r}g\right|_g^2,$$

and, therefore,

$$\int_0^\rho r^2\phi^2 R_{rr}\,dr \leq \int_0^\rho \left\{r\phi^2\sqrt{n-1}\left|\frac{\partial}{\partial r}g\right| + r^2\phi|\phi'|\sqrt{n-1}\left|\frac{\partial}{\partial r}g\right|\right\} dr - \frac{1}{4}\int_0^\rho r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2 dr.$$

Applying the Cauchy inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ to the first and second terms on the right-hand side, we obtain

(1.7)

$$\begin{aligned}\int_0^\rho r^2\phi^2 R_{rr}\,dr \leq \frac{1}{2}\int_0^\rho \Bigg(\frac{1}{\varepsilon}r^2\phi^2(n-1) + \varepsilon r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2 \\ + \frac{1}{\varepsilon}(n-1)\phi^2 + \varepsilon r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2\Bigg) dr \\ - \frac{1}{4}\int_0^\rho r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2 dr \\ \leq -\left(\frac{1}{4}-\varepsilon\right)\int_0^\rho r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2 dr + \frac{(n-1)}{2\varepsilon}\int_0^\rho (r^2\phi'^2+\phi^2)\,dr\end{aligned}$$

and hence

$$\left(\frac{1}{4}-\varepsilon\right)\int_0^\rho r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2 dr \leq \frac{n-1}{2\varepsilon}\int_0^\rho (r^2\phi'^2+\phi^2)\,dr - \int_0^\rho r^2\phi^2 R_{rr}. \tag{1.8}$$

Taking $\varepsilon = 1/8$ and $\phi = \rho - r$ in (1.8), we get

$$\begin{aligned}\int_0^\rho r^2(\rho-r)^2\left|\frac{\partial}{\partial r}g\right|^2 dr &\leq 32(n-1)\int_0^\rho (r^2+(\rho-r)^2)\,dr \\ &\quad + H\int_0^\rho r^2(\rho-r)^2\,dr \\ &\leq C(H, n)\rho^3)\end{aligned}$$

and, therefore,

$$\int_0^{\frac{\rho}{2}} r^2\left|\frac{\partial}{\partial r}g\right|^2 dr \leq \frac{1}{(\frac{\rho}{2})^2}\int_0^\rho r^2(\rho-r)^2\left|\frac{\partial}{\partial r}g\right|^2 dr \leq \frac{1}{2}C_1(H, n)\rho.$$

This completes the proof of Proposition 1.5.

Proposition 1.9. *For $r \le \frac{1}{2} i_0$, we have*

$$r\left|\frac{\partial}{\partial r}\ln\sqrt{g}\right| \le C_2(H, i_0, n).$$

Proof. Using (1.4) again, and integrating by parts, we have

(1.10)

$$\begin{aligned}\int_0^r r^2 R_{rr}\,dr &= -\frac{1}{2}\int_0^r r^2\frac{\partial^2}{\partial r^2}\ln g\,dr - \frac{1}{4}\int_0^r r^2\left|\frac{\partial}{\partial r}g\right|^2 dr\\ &= -\frac{1}{2}r^2\frac{\partial}{\partial r}\ln g + \frac{1}{2}\int_0^r 2r\frac{\partial}{\partial r}\ln g - \frac{1}{4}\int_0^r r^2\left|\frac{\partial}{\partial r}g\right|^2 dr.\end{aligned}$$

Combining (1.10) and Proposition 1.5 with the Cauchy inequality yields

$$\begin{aligned}r^2\frac{\partial}{\partial r}\ln\sqrt{g} &\le H\int_0^r r^2\,dr + \frac{1}{4}C_1(H, n)r + \left(\int_o^r r^2\left|\frac{\partial}{\partial r}\ln g\right|^2\right)^{1/2} r^{1/2}\\ &\le \frac{1}{3}Hr^3 + \frac{1}{4}C_1(H, n)r + \left((n-1)\int_0^r r^2\left|\frac{\partial}{\partial r}g\right|_g^2 dr\right)^{1/2} r^{1/2}\\ &\le C(H, i_0, n)r + \sqrt{n-1}\,C_1(H, n)^{1/2} r \le C_2(H, i_0, n)r,\end{aligned}$$

which is just Proposition 1.9.

Remark. One should note that Proposition 1.9 does not imply that $|\frac{\partial}{\partial r}\sqrt{g}| \le C$. In the case of bounded sectional curvature, the Rauch comparison theorem gives that $|\frac{\partial}{\partial r}\sqrt{g}| \le C$.

As a consequence of Proposition 1.5, we can compare the induced metrics $\bar{g}(r) = \Sigma g_{ij}(r, x)\,dx^i\,dx^j$ on the geodesic spheres $S_r(x_0) = \{x \in M, d(x, x_0) = r\}$ for $r \le \frac{1}{2} i_0$.

Proposition 1.11. *There exists a constant $C_2(H, n) > 0$, such that*

$$e^{-C_3 r_2/r_1}\bar{g}(r_1) \le \bar{g}(r_2) \le e^{C_3 r_2/r_1}\bar{g}(r_1)$$

for $0 < r_1 \le r_2 \le \frac{1}{2} i_0$.

Proof. From Proposition 1.5 it follows that

$$\int_0^r r\left|\frac{\partial}{\partial r}g\right| dr \le \left(\int_0^r r^2\left|\frac{\partial}{\partial r}g\right|^2 dr\right)^{1/2} r^{1/2} \le C_1^{1/2} r \tag{1.12}$$

for $r \le \frac{1}{2} i_0$. Taking a fixed vector $v = (v_i) \in TS_1$, and letting $h(r) = \bar{g}(r)(v, v) = \Sigma g_{ij}(r, x)v^i v^j$, we have

$$\left|\frac{\partial}{\partial r}h\right| = \left|\frac{\partial}{\partial r}g_{ij}v^i v^j\right| \le \left|\frac{\partial}{\partial r}g\right|_g h(r),$$

which gives

$$\left|\frac{\partial}{\partial r}\ln h(r)\right| \le \left|\frac{\partial}{\partial r}g\right|_g .$$

Thus, combining this with (1.12), we obtain

$$\begin{aligned}\left|\ln\frac{h(r_2)}{h(r_1)}\right| &\le \int_{r_1}^{r_2}\left|\frac{\partial}{\partial r}\ln h\right| dr \le \frac{1}{r_1}\int_0^{r_2} r\left|\frac{\partial}{\partial r}g\right| dr \\ &\le C_1^{1/2}\frac{r_2}{r_1} = C_3(H, n)\frac{r_2}{r_1},\end{aligned}$$

and hence

$$e^{-C_3 r_2/r_1} \le \frac{h(r_2)}{h(r_1)} \le e^{C_3 r_2/r_1}.$$

Since v is any vector of TS_1, this implies Proposition 1.11.

Proposition 1.11 gives the ratio estimate of the metrics $\bar{g}(r)$ on $S_r(x_0)$. Our first main goal is to estimate $\bar{g}(r)$. To this end, we need to control the $L^{n/2}$-norm of the Riemann curvature tensor $\overline{\mathrm{Rm}}(r) = \mathrm{Rm}(\bar{g}(r))$ of $g(r)$ on $S_r(x_0)$. The next few paragraphs are devoted to such estimates.

We make the following hypothesis.

Assumption 1.13. Let $\rho = i_0 > 0$. There exists a constant $K > 0$, such that for any $x_0 \in M$

$$\int_{B_\rho(x_0)} |\mathrm{Rm}|^{n/2}\, dg \le K.$$

Theorem 1.14. *For any $\rho \le i_0/4$, there exists $r_\rho > 0$ such that $\rho/2 \le r_\rho \le \rho$, and*

$$\int_{S_{r_\rho}(x_0)} |\overline{\mathrm{Rm}}(r_\rho)|_{\bar{g}}^{n/2}\, d\bar{g} \le C_4\frac{1}{r_\rho}$$

for a constant $C_4 = C_4(H, K, i_0, n) > 0$.

Remark 1.14(a). In general, for any $0 < \bar{r} < i_0/4$, we have

$$\int_{B_{i_0/4}(x_0)-B_{\bar{r}}(x_0)} |\overline{\mathrm{Rm}}(\bar{g})|_{\bar{g}}^{n/2}\, dg < C(H, K, i_0, n, \bar{r}).$$

First we recall a well-known volume estimate of Bishop [3].

Lemma 1.15. *For $r \le i_0$, there is a constant $C_5 = C_5(H, i_0, n)$, such that $\sqrt{g} \le C_5 r^{n-1}$.*

We start with several lemmas.

Lemma 1.16. *Given a geodesic γ with length $l \le i_0/2$, and a Jacobi field Y on γ, such that $Y(\gamma(0)) = 0$ and $\langle Y(\gamma(l)), \dot\gamma(l)\rangle = 0$, we have*

$$|Y(\gamma(r_1))|^2 \le e^{C_3 r_3/r_1}|Y(\gamma(r_2))|^2$$

for $0 < r_1 < r_2 \le l$, and C_3 in Proposition 1.11.

Proof. Since $l \le i_0/2$ and γ is a minimal geodesic, we take $x_0 = \gamma(o)$ and choose the polar geodesic coordinate $\{r, x^1, \cdots, x^{n-1}\}$ on $B_l(x_0)$ such that $\partial/\partial x^1 = Y(\gamma(l))$ at $\gamma(l)$. Thus $\partial/\partial x^1 = Y$ on γ, and Proposition 1.11 implies the lemma.

Lemma 1.17. *Given a geodesic γ with length $l \le i_0/4$, and a Jacobi vector field Y on γ such that $Y(\gamma(0)) = 0$ and $\langle Y(\gamma(l)), \gamma'(l)\rangle = 0$, there exists a constant $C_6 = C_6(H, i_0, n) > 0$, such that*

$$|Y(\gamma(t))| \le C_6 |Y(\gamma(l))|$$

for $0 \le t \le l$.

Proof. Let $\bar\gamma$ be an extension of γ defined as follows:

$$\bar\gamma(t) = \begin{cases} \gamma(t), & 0 \le t \le l, \\ \exp_{\gamma(0)}(t\gamma'(0)), & -l \le t \le 0. \end{cases}$$

Then $\bar\gamma$ has length $2l \le i_0/2$, and $\bar\gamma$ is a minimal geodesic. There exists a unique Jacobi vector field $\overline{Y}$ along $\bar\gamma$, such that $\overline{Y}(\bar\gamma(l)) = Y(\gamma(l))$, and $\overline{Y}(\bar\gamma(-l)) = 0$. Applying Lemma 1.16 to $\overline{Y}$ and Y in turn, we obtain

$$|\overline{Y}(\gamma(t))|^2 \le e^{2C_3} |\overline{Y}(\gamma(l))|^2 = e^{2C_3}|Y(\gamma(l))|^2 \quad \text{for } t \ge 0, \tag{1.18}$$

$$|Y(\gamma(t))|^2 \le e^{2C_3}|Y(\gamma(l))|^2 \quad \text{for } t \ge l/2. \tag{1.19}$$

On the other hand, letting $Z = \overline{Y} - Y$ on γ, we have $\langle Z, \gamma'\rangle = 0$, and $Z(\gamma(l)) = 0$. Applying Lemma 1.16 again gives

$$|Z(\gamma(t))| \le e^{2C_3}|Z(\gamma(0))|^2 = e^{2C_3}|\overline{Y}(\gamma(0))|^2 \quad \text{for } t \le l/2.$$

Combining this with (1.18) and (1.19), we deduce

$$|Y(\gamma(t))| \le |\overline{Y}(\gamma(t))| + |Z(\gamma(t))| \le 2e^{2C_3}|Y(\gamma(l))| \quad \text{for } t \le l/2,$$
$$|Y(\gamma(t))| \le e^{C_2}|Y(\gamma(l))| \quad \text{for } t \ge l/2.$$

We finish the proof of Lemma 1.17 by taking $C_6 = 2e^{2C_3}$.

Lemma 1.20. *There exists a constant $\overline{C}_6 = \overline{C}_6(H, i_0, n) > 0$, such that for each $\rho \le i_0/4$, we have*

$$\int_{B_\rho(x_0) - B_{\rho/2}(x_0)} \left|\frac{\partial}{\partial r} g\right|^n dg < \overline{C}_6 \int_{B_\rho(x_0)} |\mathrm{Rm}(g)|^{n/2} dg + \frac{\overline{C}_6}{\rho}.$$

Proof. Let B be the second fundamental form of $S_r(x_0)$. Then we have

$$B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{1}{2}\frac{\partial}{\partial r} g_{ij},$$

and $|\frac{\partial}{\partial r}g|^2 = 4|B|^2$ is independent of the choice of coordinate $\{x^i\}$. For a fixed point $y \in S_\rho(x_0)$, we choose a coordinate $\{x^i\}$ on $S_\rho(x_0)$, such that

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \delta_{ij}$$

at y. Let us denote $\partial/\partial x^i$ by Y_i on $B_\rho(x_0)$. Then Y_i is a Jacobi vector field along the geodesic $\gamma(r) = \{r, y\} \in B_\rho(x_0)$, and we have $Y_i'' + R(Y_i, T)T = 0$, where $T = \gamma'(r)$ is the tangent vector field of γ. We take the parallel vector fields E_i and $\overline{E}_i$ along γ, such that $E_i(\rho) = Y_i(\rho)$ and $\overline{E}_i(\rho/2) = Y_i(\rho/2)$. Thus by Lemma 1.17, $|Y_i(r)| \le C_6$, and $|E_i(\rho)| \le C_6$, $|\overline{E}_i| \le C_6$ for $r \le \rho$. Defining the vector field A_i on γ by

$$A_i(r) = Y_i(r) - \frac{(r-\rho/2)}{(\rho/2)}E_i(r) - \frac{(\rho - r)}{(\rho/2)}\overline{E}_i(r), \qquad \frac{\rho}{2} \le r \le \rho, \tag{1.21}$$

we then have

$$|A_i(r)| \le 3C_6, \quad A_i(\rho/2) = 0, \quad A_i(\rho) = 0,$$
$$A_i'' + R(Y_i, T)T = 0.$$

We now integrate by parts,

$$\int_{\rho/2}^{\rho} |A_i'|^n\, dr = \int_{\rho/2}^{\rho} \langle A_i', A_i'\rangle |A_i'|^{n-2}\, dr,$$

which gives, in consequence of equation $A_i'' + R(Y_i, T)T = 0$,

$$\begin{aligned}
\int_{\rho/2}^{\rho} |A_i'|^n\, dr &= -\int_{\rho/2}^{\rho} \langle A_i'', A_i\rangle |A_i'|^{n-2}\, dr \\
&\quad - (n-2)\int_{\rho/2}^{\rho} \langle A_i', A_i\rangle |A_i'|^{n-4} \langle A_i'', A_i'\rangle\, dr \\
&= + \int_{\rho/2}^{\rho} \langle R(Y_i, T)T, A_i\rangle |A_i'|^{n-2}\, dr \\
&\quad + (n-2)\int_{\rho/2}^{\rho} \langle A_i', A_i\rangle |A_i'|^{n-4} \langle R(Y_i, T)T, A_i'\rangle\, dr \\
&\le 3C_6^2 \int_{\rho/2}^{\rho} |\mathrm{Rm}|\, |A_i'|^{n-2}\, dr + (n-2)3C_6^2 \int_{\rho/2}^{\rho} |\mathrm{Rm}|\, |A_i'|^{n-2}\, dr \\
&\le C(H, i_0, n) \int_{\rho/2}^{\rho} |\mathrm{Rm}|\, |A_i'|^{n-2}\, dr.
\end{aligned}$$

Then the Hölder inequality implies

$$\int_{\rho/2}^{\rho} |A_i'|^n\, dr \le C\left(\int_{\rho/2}^{\rho} |\mathrm{Rm}|^{n/2}\, dr\right)^{2/n} \left(\int_{\rho/2}^{\rho} |A_i'|^n\, dr\right)^{(n-2)/n},$$

that is,

$$\int_{\rho/2}^{\rho} |A_i'|^n \, dr \le C \int_{\rho/2}^{\rho} |\mathrm{Rm}|^{n/2} \, dr,$$

or

$$\sum_{i=1}^{n-1} \int_{\rho/2}^{\rho} |A_i'|^n \, dr \le C \int_{\rho/2}^{\rho} |\mathrm{Rm}|^{n/2} \, dr. \tag{1.22}$$

Using (1.21), we have

$$A_i' = Y_i' - \frac{2}{\rho} E_i + \frac{2}{\rho} \overline{E}_i,$$

which combines with (1.22) to imply

$$\sum_{i=1}^{n-1} \int_{\rho/2}^{\rho} |Y_i'|^n \, dr \le \frac{C}{\rho^n} + C \int_{\rho/2}^{\rho} |\mathrm{Rm}|^{n/2} \, dr. \tag{1.23}$$

We now consider

$$\left| \frac{\partial}{\partial r} g \right|^2 \le 4 \sum g^{ij} g^{jl} \langle Y_i', Y_j \rangle \langle Y_k', Y_l \rangle,$$

which, together with $|Y_i| \le C_6$, gives

$$\left| \frac{\partial}{\partial r} g \right|^2 \le C \sum g^{il} g^{jl} |Y_i'| \, |Y_k'|.$$

Note that $g_{ij}(\rho) = \delta_{ij}$ at $y \in S_\rho(x_0)$. Applying Proposition 1.11, we then have $|g^{ij}(r)| \le C$ on γ for $\rho/2 \le r \le \rho$, and hence

$$\left| \frac{\partial}{\partial r} g \right|^2 \le C \sum |Y_i'|^2,$$

which clearly implies that

$$\left| \frac{\partial}{\partial r} g \right|^n \le C(H, i_0, n) \sum |Y_i'|^n.$$

This with the help of (1.23) yields

$$\int_{\rho/2}^{\rho} \left| \frac{\partial}{\partial r} g \right|^n dr \le C(H, i_0, n) \left(\frac{1}{\rho^n} + \int_{\rho/2}^{\rho} |\mathrm{Rm}|^{n/2} \, dr \right).$$

We now note that $|\frac{\partial}{\partial r} g|^n$ and $|\mathrm{Rm}|^n$ are independent of the choice of $\{x^i\}$ on $S_\rho(x_0)$. For any such $\{x_i\}$ we have by Proposition 1.11,

$$C(H, i_0, n)^{-1} \sqrt{g}(\rho) \le \sqrt{g}(r) \le C(H, i_0, n) \sqrt{g}(\rho)$$

on $\gamma(r)$, $\rho/2 \le r \le \rho$. This implies

$$\int_{\rho/2}^{\rho} \left|\frac{\partial}{\partial r} g\right|^n \sqrt{g}\, dr \le C\left(\frac{1}{\rho^n}\sqrt{g}(\rho) + \int_{\rho/2}^{\rho} |\operatorname{Rm}|^{n/2}\sqrt{g}\, dr\right).$$

Integrating over $S_\rho(x_0)$, we then obtain

$$\int_{B_\rho(x_0)-B_{\rho/2}(x_0)} \left|\frac{\partial}{\partial r} g\right|^n dg \le C\frac{1}{\rho^n}\int_{S_\rho(x_0)} dg + C\int_{B_\rho(x_0)} |\operatorname{Rm}|^{n/2} dg.$$

Using Lemma 1.15, we have

$$\int_{B_\rho(x_0)-B_{\rho/2}(x_0)} \left|\frac{\partial}{\partial r} g\right|^n dg \le \frac{C}{\rho} + C\int_{B_\rho(x_0)} |\operatorname{Rm}|^{n/2} dg.$$

We take $\rho = i_0/4$; then

$$\int_{B_{i_0/4}(x_0)-B_{i_0/8}(x_0)} |g|^n dg \le \overline{C}_6(H, i_0, n) + \overline{C}_6\int_{B_{i_0/4}(x_0)} \cdot |\operatorname{Rm}|^{n/2} dg. \tag{1.24}$$

For any $\rho \le \frac{1}{4}i_0$, applying (1.24) to the metric $g' = \tau^{-2}g$, where $\tau = 4\rho/i_0$, and noting that $\operatorname{Ric}(g') \ge -Hg$; $\operatorname{inj}(g) \ge i_0$, by the scale invariance of (1.24), we then finally obtain

$$\begin{aligned}\int_{B_\rho(x_0)-B_{\rho/2}(x_0)} \left|\frac{\partial}{\partial r} g\right|^n dg &= \int_{B_{i_0/4}-B_{i_0/8}} \left|\frac{\partial}{\partial r} g\right|^n dg' \\ &\le \overline{C}_6 + \overline{C}_6\int_{B_{i_0/4}(x_0)} |\operatorname{Rm}(g')|^{n/2} dg' \\ &= \overline{C}_6 + \overline{C}_6\int_{B_\rho(x_0)} |\operatorname{Rm}(g)|^{n/2} dg.\end{aligned}$$

Now we are ready to prove Theorem 1.14.

Proof of Theorem 1.14. From Lemma 1.20, we have

$$\int_{\rho/2}^{\rho}\int_{S_1} \left|\frac{\partial}{\partial r} g\right|^n d\bar{g}\, dr \le C(H, K, i_0, n). \tag{1.25}$$

Let us recall the Gauss formula on $S_r(x_0)$,

$$R_{ijkl} = \overline{R}_{ijkl} + \frac{1}{4}\left(\frac{\partial}{\partial r}g_{ik}\frac{\partial}{\partial r}g_{jl} - \frac{\partial}{\partial r}g_{jk}\frac{\partial}{\partial r}g_{il}\right),$$

which, together with (1.25), implies that

$$\int_{\rho/2}^{\rho}\int_{S_r} |\overline{\operatorname{R}}\mathrm{m}(\bar{g})|^{n/2} d\bar{g}\, dr \le C\int_{\rho/2}^{\rho}\int_{S_r} |\operatorname{Rm}|_g^{n/2} d\bar{g}\, dr + C. \tag{1.26}$$

(1.26) means that there exists an $r_\rho > 0$ with $\rho/2 \le r_\rho \le \rho$, such that

$$\int_{S_{r_\rho}(x_0)} |\overline{\mathrm{R}}\mathrm{m}(r_\rho)|^{n/2}\, d\bar{g} \le C\frac{1}{r_\rho}.$$

This completes the proof of Theorem 1.14.

Remark 1.14(a) follows from above; the only change is that we have to replace every constant by a constant which also depends on $\bar{r}$.

2. Zero order estimate on geodesic balls

In this section we state and prove the central estimates of the paper, a compactness estimate of the metric g on geodesic balls.

Let (M, g) be a Riemannian manifold as in §1.

Theorem 2.1. *For any $\rho \le i_0/4$, there exist $r_\rho > 0$ as in Theorem 1.14, a constant $C_7 = C_7(H, K, i_0, n) > 0$ and a smooth Riemannian metric $h(r_\rho)$ on the geodesic sphere $S_{r_\rho}(x_0)$, such that*

$$C_7^{-1}\bar{g}(r_\rho) \le r_\rho^2(r_\rho) \le C_7\bar{g}(r_\rho)$$

and $|\mathrm{Rm}(h)| \le C_7$.

First, let us recall a well-known result ([7], [6], [5]).

Theorem 2.2. *If $\rho < i_0/2$, then there is a constant $C_8 = C_8(n) > 0$ such that for any $f \in C_0^\infty(U_\rho(x_0))$, we have*

$$\left(\int_{B_\rho(x_0)} |f|^{n/(n-1)}\right)^{(n-1)/n} \le C_8 \int_{B_\rho(x_0)} |\nabla f|\, dg,$$

where $U_\rho(x_0) = \{x \in M, d(x, x_0) < \rho\}$ is an open geodesic ball of g.

Using Theorem 2.2 and the Hölder inequality, and replacing f in Theorem 2.2 by a power $|f|^{23(n-1)/n-2)}$ of $|f|$, we can prove the following.

Theorem 2.3. *For $\rho < i_0/2$ and $f \in C_0^\infty(U_\rho(x_0))$, we have*

$$\left(\int_{B_\rho(x_0)} |f|^{2n/(n-2)}\, dg\right)^{(n-2)/n} \le \left(2\frac{(n-1)}{n-2}\right)^2 C_8^2 \int_{B_\rho(x_0)} |\nabla f|^2\, dg.$$

Let us now consider the metric $\bar{g}(r)$ on $S_r(x_0)$, and define a new metric $\bar{h}(r)$ on $S_r(x_0)$ by

$$\bar{h}(r) = \frac{1}{r^2}\bar{g}(r).$$

We need to prove a Sobolev inequality for the metric $\bar{h}(r)$ on $S_r(x_0)$.

To start, from Proposition 1.11, we have

$$e^{-C_3 r_2/r_1}\left(\frac{r_1}{r_2}\right)^2 \bar{h}(r_1) \le \bar{h}(r_2) \le e^{C_3 r_2/r_1}\bar{h}(r_1) \tag{2.4}$$

for $0 < r_1 \le r_2 \le i_0/2$.

Theorem 2.5. *For* $0 < r \le i_0/4$, *there exists a constant* $C_9 = C_9(H, i_0, n) > 0$, *such that, for any* $0 < l \le r$ *and* $f \in C^\infty(S_r(x_0))$, *we have*

$$\frac{1}{l^{2/n}}\left(\int_{S_r(x_0)} |f|^{2n/(n-2)}\, d\bar{g}\right)^{(n-2)/n} \le C_9\left(\int_{S_r} |\nabla f|^2\, d\bar{g} + \frac{1}{l^2}\int_{S_r} f^2\, d\bar{g}\right).$$

Proof. We take a cut-off function $\phi\colon [r, r\, l] \to [0, 1]$, such that $\phi(r) = 0$, $\phi(r+l) = 0$, $\phi(t) = 1$ for $r + l/4 \le t \le r + 3l/4$, and ϕ is linear on $[r, r+l/4]$ and $[r+3l/4, r+l]$. We have

$$|\nabla\phi| = |\phi'| \le \frac{4}{l}.$$

We can consider ϕf as a function on $B_\rho(x_0)$ with support in $B_{r+l}(x_0) - U_r(x_0)$. Applying Theorem 2.3 to ϕf, we obtain

$$\begin{aligned}\left(\int_r^{r+l}\int_{S_t(x_0)} |\phi f|^{2n/(n-2)}\, d\bar{g}\, dt\right)^{(n-2)/n} &\le C\int_r^{r+l}\int_{S_t(x_0)} |\nabla(\phi f)|_g^2\, d\bar{g}\, dt\\ &\le C\left\{\int_r^{r+l}\int_{S_t(x_0)} |\nabla f|_g^2\, d\bar{g}\, dt + \frac{1}{l2}\int_r^{r+l}\int_{S_t(x_0)} f^2\, d\bar{g}\, dt\right\}.\end{aligned}$$

Since $l \le r$ and all the metrics $\bar{g}(t)$ for $t \in [r, r+l]$ are equivalent by Proposition 1.11, we can replace the metrics $\bar{g}(t)$ for $t \in [r, r+l]$ by $\bar{g}(r)$. Thus we have

$$\begin{aligned}&\left(\frac{1}{2}lC\int_{S_r(x_0)} |f|^{2n/(n-2)}\, d\bar{g}\right)^{(n-2)/n}\\ &\quad\le \left(\int_r^{r+l}\int_{S_t(x_0)} |\phi f|^{2n/(n-2)}\, d\bar{g}\, dt\right)^{(n-2)/n}\\ &\quad\le C\left(l\int_{S_r(x_0)} |\nabla f|^2\, d\bar{g} + \frac{1}{l}\int_{S_r(x_0)} f^2\, d\bar{g}\right),\end{aligned}$$

which clearly implies Theorem 2.5.

We identify $S_r(x_0)$ with $S_1 \subset R^n$, and consider $\bar{g}(r)$ and $\bar{h}(r)$ to be metrics on S_1. From Theorem 2.5, we obtain

Corollary 2.6. *For* $0 < r \le i_0/4$, $0 < l \le r$, *and* $f \in C^\infty(S_1)$, *we have*

$$\frac{1}{L^{2/n}}\left(\int_{S_1} |f|^{2n/(n-2)}\, d\bar{h}\right)^{(n-2)/2} \le C_9\left\{\int_{S_1} |\nabla f|^2\, d\bar{h} + \frac{1}{L^2}\int_{S_1} f^2\, d\bar{h}\right\};$$

here $L = l/r$ *and* $0 < L \le 1$.

For the metric $\bar{h}(r)$ on S, from Theorem 1.1.4, we also obtain

Corollary 2.7. *For any* $\rho \le i_0/4$, *there exists* $r_\rho \ge 0$, *such that* $\rho \le r \le 2\rho$ *and*

$$\int_{S_1} |\operatorname{Rm}(\bar{h}(r_\rho))|^{n/2}\, d\bar{h}(r_\rho) \le C_4. \tag{2.8}$$

We now use the evolution equation of Hamilton [19] to deform the metric $\bar{h}(r_\rho)$. We fix $\rho \le i_0/4$ for the next few paragraphs, and consider the evolution equation

$$\frac{\partial}{\partial t} h(t) = -2\operatorname{Ric}(h), \tag{2.9}$$

where $h(0) = \bar{h}(r_\rho)$. From [19], we have

Theorem 2.10. *The evolution equation*

$$\frac{\partial}{\partial t} h_{ij} = -2\operatorname{Ric}(h)_{ij}$$

has a unique solution on a maximal time interval $0 \le t < t \le \infty$. *If* $T < \infty$, *then* $\max_{S_1} |\operatorname{Rm}(h)| \to \infty$ *as* $t \to T$.

We shall estimate the T for $\bar{h}(r_\rho)$ from below by a constant which depends only on H, K, i_0, and n, and estimate the uniform norm of $\operatorname{Rm}(h)$. We start with the following.

Theorem 2.11. *There exist*

$$T = T(H, K, i_0, n) > 0 \quad \text{and} \quad C_{10} = C_{10}(H, K, i_0, n) > 0,$$

such that (2.9) *has a solution on* $[0, T]$, *and for* $t \in [0, T]$, $0 < L \le 1$, $f \in C^\infty(S_1)$, *we have the following*:

(a)

$$\int_{S_1} |\operatorname{Rm}(h)|^{n/2}\, dh \le 2C_4,$$

(b)

$$\frac{1}{L^{2/n}}\left(\int_{S_1} |f|^{2n/(n-2)}\, dh\right)^{(n-2)/n} \le 2C_9\left\{\int_{S_1} |\nabla f|^2\, dh + \frac{1}{L^2}\int_{S_1} f^2\, dh\right\},$$

(c)

$$\max_{S_1} |\operatorname{Rm}(h)|_h(\tau) \le \frac{C_{10}}{\tau^{1-1/n}} \left(\int_{S_1} |\operatorname{Rm}(h)|^{n/2}\, dh \right)^{2/n} \le \frac{C_{10}}{\tau^{1-1/n}} (C_4)^{2/n}.$$

Proof. Note that (a) and (b) are satisfied at $t = 0$. We take $\eta > 0$ to be the maximal number such that (2.9) has a solution on $[0, \eta)$, and (a), (b) hold on $[0, \eta)$. We may assume that $\eta \le 1$. First we prove

Lemma 2.12. (c) *is satisfied on* $[0, \eta)$.

Proof. Letting $\tau[0, \eta)$, we have

$$\frac{\partial}{\partial t} h(t) = -2\operatorname{Ric}(h),$$

which is invariant under the transformation $t \to \tau t$ and $h_{ij}(t) \to (1/\tau) h_{ij}(\tau t) = h'_{ij}(t)$. Thus we obtain

$$\frac{\partial}{\partial t} h'(t) = -2\operatorname{Ric}(h')$$

for h', and this equation has a solution on $[0, 1]$. For this metric h', we have

(a_1)

$$\int_{S_1} |\operatorname{Rm}(h')|^{n/2}\, dh' = \tau^{1/2} \int_{S_1} |\operatorname{Rm}(h)|^{n/2}\, dh' \le 2C_4 \tau^{1/2},$$

(b_1)

$$\frac{\tau^{n/4}}{L^{2/n}} \left(\int_{S_1} |f| \frac{2n}{n}\, dh' \right)^{(n-2)/2} \le 2C_9 \left\{ \int_{S_1} |\nabla f|^2\, dh' + \frac{\tau}{L^2} \int_{S_1} f^2\, dh' \right\}$$

for any $0 < L \le 1$ and $f \in C^\infty(S_1)$. Let $\overline{L} = L/\tau^{1/2}$. Then $\overline{L}$ can take any value on $(0, 1]$, and (b_1) can be rewritten as

(b_1)

$$\frac{1}{\overline{L}^{2/n}} \left(\int_{S_1} |f|^{2n/(n-2)}\, dh' \right)^{(n-2)/n} \le 2C_9 \left\{ \int_{S_1} |\nabla f|^2\, dh' + \frac{1}{\overline{L}^2} \int_{S_1} f^2\, dh' \right\}.$$

From [18] it follows that

$$\frac{\partial}{\partial t} |\operatorname{Rm}|^2 \le \Delta |\operatorname{Rm}|^2 - 2|\nabla \operatorname{Rm}|^2 + C(n) |\operatorname{Rm}|^3. \tag{2.13}$$

For $p \geq 1$, by integrating and using (2.13), we obtain
(2.14)

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{S_1} |\operatorname{Rm}|^{2p}\,dh' \\
&\quad\leq 2p\int_{S_1} |\operatorname{Rm}|^{2(p-1)}\langle \Delta \operatorname{Rm}, \operatorname{Rm}\rangle\,dh' + Cp\int_{S_1} |\operatorname{Rm}|^{2p+1}\,dh' \\
&\quad\leq -2p\int_{S_1} |\operatorname{Rm}|^{2(p-1)}|\nabla \operatorname{Rm}|^2\,dh' - 4p(p-1) \\
&\qquad\times\int_{S_1} |\operatorname{Rm}|^{2(p-2)}\langle \nabla \operatorname{Rm}, \operatorname{Rm}\rangle^2\,dh' + Cp\int_{S_1} |\operatorname{Rm}|^{2p+1}\,dh' \\
&\quad\leq -\left(\frac{(p-1)}{p} + \frac{2}{p}\right)\int_{S_1} |\nabla|\operatorname{Rm}|p|^2\,dh' + Cp\int_{S_1} |\operatorname{Rm}|^{2p+1}\,dh',
\end{aligned}$$

so that

$$\frac{\partial}{\partial t}\int_{S_1} |\operatorname{Rm}|^{2p}\,dh' + \int_{S_1} |\nabla|\operatorname{Rm}|^p|^2\,dh \leq Cp\int_{S_1} |\operatorname{Rm}|^{2p+1}\,dh'. \tag{2.15}$$

Similarly, for any nonnegative function ϕ of t, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\phi\int_{S_1} |\operatorname{Rm}|^{2p}\,dh'\right) + \phi\int_{S_1} |\nabla|\operatorname{Rm}|p|^2|\,dh' \\
&\quad\leq Cp\phi\int_{S_1} |\operatorname{Rm}|^{2p+1}\,dh' + \left(\frac{d}{dt}\phi\right)\int_{S_1} |\operatorname{Rm}|^{2p}\,dh'.
\end{aligned} \tag{2.16}$$

Now we use the standard Di Geogi-Nash-Moser iteration. We take $\phi(t) = 1$ for $t > \delta'$, $\phi(t) = 0$ for $t \leq \delta$, and ϕ is linear on $[\delta, \delta']$. For such ϕ, we obtain

$$|\phi'| = \left|\frac{d}{dt}\phi\right| \leq \frac{1}{(\delta' - \delta)}. \tag{2.17}$$

For each $p \geq 1$, we take $\overline{L}$ small, such that

$$\frac{1}{\overline{L}^{2/n}} = 4c_9 c(n) p (2c_4)^{2/n} + 4c_9 \geq 1.$$

Applying (b_1) and (2.16) yields

$$\frac{\partial}{\partial t}\left(\phi\int_{S_1}|\mathrm{Rm}|^{2p}\,dh'\right)+\frac{1}{2}\phi\int_{S_1}|\nabla|\mathrm{Rm}|^p|^2\,dh'$$
$$+\frac{1}{2}\phi\frac{1}{(2C_9\overline{L}^{2/n})}\left(\int_{S_1}|\mathrm{Rm}|^{2np/(n-2)}\,dh\right)^{(n-2)/n}$$
$$\le Cp\phi\int_{S_1}|\mathrm{Rm}|^{2p+1}\,dh'+\frac{1}{2}\frac{1}{\overline{L}^2}\int_{S_1}|\mathrm{Rm}|^{2p}\,dh'$$
$$+\phi\int_{S_1}|\mathrm{Rm}|^{2p}\,dh'. \tag{2.18}$$

Let $\kappa=n/(n-2)>1$. By the Hölder inequality we have

$$\frac{\partial}{\partial t}\left(\phi\int_{S_1}|\mathrm{Rm}|^{2p}\,dh'\right)+\frac{1}{2}\phi\int|\nabla|\mathrm{Rm}|^p|^2\,dh'$$
$$+\phi\left(\int_{S_1}|\mathrm{Rm}|^{2p\kappa}\,dh'\right)^{1/\kappa}\frac{1}{4C_9\overline{L}^{2/n}}$$
$$\le cp\phi\left(\int|\mathrm{Rm}|^{2p\kappa}\,dh'\right)^{1/\kappa}\left(\int|\mathrm{Rm}|^{n/2}\,dh'\right)^{2/n}$$
$$+\frac{1}{2}\frac{\phi}{\overline{L}^2}\int|\mathrm{Rm}|^{2p}\,dh'+\frac{1}{\delta'-\delta}\int|\mathrm{Rm}|^{2p}\,dh',$$

which implies

$$\frac{\partial}{\partial t}\left(\phi\int|\mathrm{Rm}|^{2p}\,dh'\right)+\frac{1}{2}\phi\int|\nabla|\mathrm{Rm}|^p|^2\,dh'+\phi\left(\int|\mathrm{Rm}|^{2p\kappa}\right)^{1/\kappa}$$
$$\le\frac{\phi}{2\overline{L}^2}\int|\mathrm{Rm}|^{2p}\,dh'+\frac{1}{\delta'-\delta}\int|\mathrm{Rm}|^{2p}\,dh',$$

so that

$$\frac{\partial}{\partial t}\left(\phi\int|\mathrm{Rm}|^{2p}\,dh'\right)+\phi\left(\int|\mathrm{Rm}|^{2p\kappa}\,dh'\right)^{1/\kappa}+\frac{1}{2}\phi\int|\nabla|\mathrm{Rm}|^p|^2\,dh'$$
$$\le C(H,K,i_0,n)p^n\int|\mathrm{Rm}|^{2p}\,dh'+\frac{1}{\delta'-\delta}\int|\mathrm{Rm}|^{2p}\,dh'.$$

Let

$$H_{2p}(\delta)=\int_\delta^1\left(\int_{S_1}|\mathrm{Rm}|^{2p}\,dh'\right)dt,$$
$$D_{2p}(\delta)=\int_\delta^1\left(\int_{S_1}|\nabla|\mathrm{Rm}|^p|\,dh'\right)dt,$$

$$M_{2p}(\delta) = \max_{\delta \le t \le 1} \int_{S_1} |\mathrm{Rm}|^{2p}\, dh'.$$

From (2.18) by integrating from δ to 1, we obtain

$$M_{2p}(\delta') \le Cp^n \frac{1}{\delta' - \delta} H_{2p}(\delta), \tag{2.19}$$

$$D_{2p}(\delta') \le Cp^n \frac{1}{\delta' - \delta} H_{2p}(\delta). \tag{2.20}$$

Lemma 2.21. *For $\bar{\kappa} = (n+2)/n > 1$,*

$$H_{2p\bar{\kappa}}(\delta) \le CM_{2p}(\delta)^{2/n}(H_{2p}(\delta) + D_{2p}(\delta)).$$

Proof. By the Hölder inequality,

$$\begin{aligned} H_{2p\bar{\kappa}}(\delta) &= \int_\delta^1 \left(\int_{S_1} |\mathrm{Rm}|^{2p\bar{\kappa}}\, dh'\right) dt \le \int_\delta^1 \left(\int |\mathrm{Rm}|^{2p+4p/n}\, dh'\right) \\ &\le \int_\delta^1 \left(\int |\mathrm{Rm}|^{2p\kappa}\, dh'\right)^{1/\kappa} \left(\int |\mathrm{Rm}|^{2p}\, dh'\right)^{2/n} \\ &\le M_{2p}(\delta)^{2/n} \int_\delta^1 \left(\int |\mathrm{Rm}|^{2p\kappa}\, dh'\right)^{1/\kappa}. \end{aligned} \tag{2.22}$$

Taking $\overline{L} = 1$ in (b_1), and applying (b_1) to the right-hand side of (2.22), we obtain

$$H_{2p\bar{\kappa}}(\delta) \le CM_{2p}(\delta)^{2/n}(H_{2p}(\delta) + D_{2p}(\delta)).$$

Using (2.19), (2.20), and Lemma 2.21, we then have

$$H_{2p\bar{\kappa}}(\delta') \le Cp^{n\bar{\kappa}} \left(\frac{1}{\delta' - \delta}\right)^{\bar{\kappa}} H_{2p}(\delta)^{\bar{\kappa}}.$$

Hence, for $q = 2p \ge 1$,

$$H_{\bar{\kappa}q}(\delta') \le C_{11}(H, K, i_0, n) q^{\bar{\kappa}n} \left(\frac{1}{\delta' - \delta}\right)^{\bar{\kappa}} H_q(\delta)^{\bar{\kappa}}. \tag{2.23}$$

Let $q_m = \bar{\kappa}^m q_0$, $\delta_m = \frac{1}{2} - \frac{1}{2}\cdot 2^{-m}$, $\delta_m - \delta_{m-1} = 2^{-(m+1)}$, and let $\Phi(q, \delta) = H_q(\delta)^{1/q}$. Then (2.23) implies that

$$\begin{aligned} \Phi(q_m, \delta_m) &\le C_{11}^{1/q_m}(q_{m-1})^{\bar{\kappa}n/q_m}(2^{m+1})^{\bar{\kappa}/q_m}\Phi(q_{m-1}, \delta_{m-1}) \\ &\le C_{11}^{\frac{1}{q_0\kappa^m}} (q_0)^{\frac{\bar{\kappa}n}{q_0\kappa^m}} (\bar{\kappa})^{\frac{m\bar{\kappa}n}{q_0\kappa^m}} (2)^{\frac{(m+1)\bar{\kappa}}{q_0\kappa^m}} \Phi(q_{m-1}, \delta_{m-1}) \\ &\le C_{11}^{\frac{1}{q_0}\sum\frac{1}{\kappa^m}} (q_0)^{\frac{\bar{\kappa}n}{q_0}\sum\frac{1}{\kappa^m}} (\bar{\kappa})^{\frac{\bar{\kappa}n}{q_0}\sum\frac{m}{\kappa^m}} (2)^{\frac{\bar{\kappa}}{q_0}\sum\frac{m+1}{\kappa^m}} \Phi(q_0, \delta_0) \\ &\le C_{12}(H, K, i_0, n, q_0)\Phi(q_0, 0). \end{aligned}$$

Letting $m \to \infty$, and taking limits, we obtain

$$\Phi(\infty, \tfrac{1}{2}) \le C_{12}\Phi(q_0, 0).$$

Taking $q_0 = n/2$ and $C_{13} = C_{12}(H, K, i_0, n)$ yields

$$\max_{1/2\le t\le 1} |\operatorname{Rm}(h')| = C_{13}\left(\int_0^1 \int_{S_1} |\operatorname{Rm}|^{n/2}\, dh'\right)^{2/n}.$$

Consequently

$$\max_{S_1} |\operatorname{Rm}(h')|(1) \le C_{13}\left(\int_0^1 \int_{S_1} |\operatorname{Rm}|^{n/2}\, dh\right)^{2/n}. \tag{2.24}$$

Now from (a_1) it follows that

$$\int_0^1 \int_{S_1} |\operatorname{Rm}|^{n/2}\, dh' \le 2C_4\tau^{1/2}.$$

Combining this with (2.24) gives

$$\max_{S_1} |\operatorname{Rm}(h')|(1) \le C_{13}(2C_4\tau^{1/2})^{2/n} \le C_{10}(C_4)^{2/n}\tau^{1/n}.$$

Changing back to the metric h, we therefore obtain the estimate (c).

To finish the proof of Theorem 2.11, we need only show that there exists $C_{14} = C_{14}(H, K, i_0, n) > 0$, such that $\eta \ge C_{14} > 0$. To this end, we consider the evolution equation for h,

$$\frac{\partial}{\partial t} h = -2\operatorname{Ric}(h).$$

We have [18] as in (2.15),

$$\frac{\partial}{\partial t}\int |\operatorname{Rm}|^{n/2}\, dh + \int \left|\nabla |\operatorname{Rm}|^{n/4}\right|^2 dh \le C(n)\int |\operatorname{Rm}|^{n/2+1}\, dh.$$

Hence, by the Hölder inequality,

$$\frac{\partial}{\partial t}\int |\operatorname{Rm}|^{n/2}\, dh + \int \left|\nabla |\operatorname{Rm}|^{n/4}\right|^2 dh \le C\left(\int |\operatorname{Rm}|^{n\kappa/2}\, dh\right)^{1/\kappa} \left(|\operatorname{Rm}|^{n/2}\, dh\right)^{2/n}.$$

Using (a) and (b), by taking $1/L^{2/n} = 2C_9C(n)(2C_4)^{2/n}$, we obtain

$$\frac{\partial}{\partial t}\int |\operatorname{Rm}|^{n/2}\, dh \le C_{15}(H, K, i_0, n)\int |\operatorname{Rm}|^{n/2}\, dh.$$

Integrating gives

$$\int |\mathrm{Rm}|^{n/2}\,dh \le e^{C_{15}t}\int |\mathrm{Rm}(\bar h)|\,d\bar h \le C_4 e^{C_{15}t}$$

and therefore, if $t \le (C_{15})^{-1}\ln\frac{3}{2}$, then

$$\text{(a}_2\text{)}\qquad \int |\mathrm{Rm}|^{n/2}\,dh \le \tfrac{3}{2}C_4 < 2C_4.$$

We now shall improve the constant in (b). For any fixed $f \in C^\infty(S_1)$, we have

$$\begin{aligned}\frac{\partial}{\partial t}&\left(\int_{S_1}|\nabla f|^2\,dh + \frac{1}{L^2}\int f^2\,dh\right)\\ &= \int_{S_1} 2\,\mathrm{Ric}(\nabla f, \nabla f)\,dh + \frac{1}{L^2}\int f^2(-R(h))\,dh + \int |\nabla f|^2(-R)\,dh\\ &\le 6\max_{S_1}|\mathrm{Ric}(h)|\left(\int_{S_1}|\nabla f|^2\,dh + \frac{1}{L^2}\int_{S_1} f^2\,dh\right).\end{aligned}$$

Using (c), we obtain

$$\frac{\partial}{\partial t}\left(\int_{S_1}|\nabla f|^2\,dh + \frac{1}{L^2}\int_{S_1} f^2\,dh\right) \le \frac{C_{16}}{t^{1-1/n}}\left(\int_{S_1}|\nabla f|^2\,dh + \frac{1}{L^2}\int_{S_1} f^2\,dh\right)$$

and therefore, by integration,

$$\text{(2.25)}\qquad \begin{aligned}&\int_{S_1}|\nabla f|^2\,dh + \frac{1}{L^2}\int_{S_1} f^2\,dh\\ &\qquad\le e^{nC_{16}t^{1/n}}\left(\int_{S_1}|\nabla f|^2\,dh + \frac{1}{L^2}\int_{S_1} f^2\,dh\right)_{t=0}.\end{aligned}$$

Similarly,

$$\text{(2.26)}\qquad \left(\int_{S_1}|f|^{\frac{2n}{n-2}}\,dh\right)^{\frac{n-2}{n}} \ge e^{-nC_{16}t}\left(\int_{S_1}|f|^{\frac{2n}{n-2}}\,dh\right)^{\frac{n-2}{n}}\Big|_{t=0}.$$

Combining (2.25) and (2.26) with Corollary 2.6, for $t \le (\frac{1}{2n}C_{16}\ln\frac{3}{2})^n$, we then obtain

$$\text{(2.27)}\qquad \frac{1}{L^{2/n}}\left(\int_{S_1}|f|^{\frac{2n}{n-2}}\,dh\right)^{\frac{n-2}{n}} \le \frac{3}{2}C_9\left[\int_{S_1}|f|^2\,dh + \frac{1}{L^2}\int_{S_1} f^2\,dh\right].$$

Now, if

$$\eta \le \min\left\{\frac{1}{C_{15}}\ln\frac{3}{2},\ \left(\frac{(\ln\frac{3}{2})}{2n}C_{16}\right)^n\right\} = C_{14},$$

then (c) implies that $|\operatorname{Rm}(h)|$ is bounded when $t \to \eta$. By Theorem 2.10, (2.9) has a solution on $[0, \bar{\eta}]$ for some $\bar{\eta} > \eta$, and (2.26) and (2.27) imply that (a) and (b) are true on $[0, \bar{\eta}]$. Therefore η cannot be maximal. This contradiction proves that $\eta \geq C_{14} > 0$, and we can take

$$T = T(H, K, i_0, n) = C_{14} > 0.$$

This finishes the proof of Theorem 2.11.

Proof of Theorem 2.1. We take $h(r_\rho) = h(T)$ as in Theorem 2.11. Then Theorem 2.11(c) implies that

$$|\operatorname{Rm}(h)| \leq C_7 \quad \text{for } C_7 \geq \left(\frac{C_{10}}{T^{(1-1/n)}}\right)(C_4)^{2/n} > 0.$$

We now prove that $\bar{g}(r_\rho)$ and $r_\rho^2 h(r_\rho)$ are equivalent. Recalling that $\bar{h}(r_\rho) = \bar{g}(r_\rho)/r_\rho^2$ and $h(0) = \bar{h}(r_\rho)$, we have from Theorem 2.11(c),

$$|\operatorname{Ric}(h)| \leq \frac{C(H, K, i_0, n)}{t^{1-1/n}},$$

and hence for any vector $v \in TS_1$

$$\left|\frac{\partial}{\partial t} h(v, v)\right| \leq 2|\operatorname{Ric}(h)| h(v, v) \leq \left(\frac{c}{t^{-1/1n}}\right) h(v, v).$$

Integrating both sides gives

$$\left|\ln \frac{h(T)(v, v)}{h(0)(v, v)}\right| \leq nCT^{1/n}.$$

Since v is arbitrary, by taking $C_7 = e^{nCT^{1/n}}$ we have

$$C_7^{-1}\bar{h}(r_\rho) \leq h \leq C_7 \bar{h}(r_\rho),$$

which clearly implies that

$$C_7^{-1}\bar{g}(r_\rho) \leq r_\rho^2 h(r_\rho) \leq C_7 \bar{g}(r_\rho).$$

This finishes the proof of Theorem 2.1.

We need to estimate $\bar{g}(r_\rho)$; by Theorem 2.1, we need only to estimate $h(r_\rho)$. We like to use the Gromov Convergence Theorem ([11], [16], [25]); for this, we have to control the volume and diameter of $h(r_\rho)$. For the volume, we have the results of Croke [7].

Theorem 2.28. *For $r \leq i_0/2$, there exists a constant $C_{17} = C_{17}(n)$, such that*

$$\operatorname{Vol}(\bar{g}(r)) = \operatorname{Vol}(S_r(x_0)) \geq C_{17} r^{n-1}.$$

Combining this with Theorem 2.1, we obtain the following.

Corollary 2.29. *For $\rho \le \frac{i_0}{4}$, there exists a constant $C_{18} = C_{18}(H, K, i_0, n) > 0$ such that*

$$\mathrm{Vol}(h(r_\rho)) \ge C_{18} > 0.$$

For the estimate of diameter, we have

Theorem 2.30. *There exists a constant $C_{19} = C_{19}(H, i_0, n) > 0$, such that for $r \le i_0/2$, we have*

$$\mathrm{diam}(\bar{g}(r)) \le C_{19} r.$$

We shall postpone the proof of Theorem 2.30 to the next section. For now, we assume Theorem 2.30. Then from Theorem 2.1, Corollary 2.29, and Theorem 2.30, we obtain

$$(2.31)\quad |\mathrm{Rm}(h(r_\rho))| \le C_7, \quad \mathrm{Vol}(h(r_\rho)) \ge C_{18} > 0, \quad \mathrm{diam}(h(r_\rho)) \le C_{19}.$$

We now can use the Gromov Compactness Theorem ([11], [25]) to estimate $h(r_\rho)$, and so $\bar{g}(r_\rho)$. These, together with Proposition 1.11 and Lemma 1.16, clearly imply Theorem D.

3. The diameter estimate of small geodesic sphere

In this section we will prove Theorem 2.30. Most of the proofs are straightforward, but for completeness we write them here. The main references of this section are [14]–[16]. We start with some lemmas.

We define the set of Riemannian manifolds

$$\mathscr{M}(i_0, H, n) = \{(M, g) | \mathrm{inj}(g) \ge i_0, \ \mathrm{Ric}(g) \ge -Hg, \ \dim M = n\}.$$

Lemma 3.1. *Let $(M_i, g_i) \in \mathscr{M}(i_0, H, n)$, and take $x_i \in M_i$, a point in M_i for each i. Then there exists a compact metric space X such that $\overline{M}_i = B_D^{M_i}(x_i) \subset X$ with $D = 4i_0$, and the distance functions on compact subspace $\overline{M}_i$ induced from X and the distance function of (M_i, g_i) are the same* [14], [15].

Lemma 3.2. *There exist a compact subspace M^0 of X, and a subsequence of $\{\overline{M}_i\}$ (say $\{\overline{M}_i\}$), such that $\overline{M}_i$ converges to M^0 in Hausdorff distance, which is denoted by $\overline{M}_i \xrightarrow{H} M^0$ in X.*

By passing to a subsequence if necessary, we may assume that $x_i \to x_0 \in M^0$ in X.

Lemma 3.3. *Let $B_{i_0}(x_0) = \{x \in M^0, d(x, x_0) \le i_0\}$. Then for any two points $x, y \in B_{i_0}(x_0)$, there exists a minimal geodesic γ from x to y in M, i.e., $L(\gamma) = d(x, y)$.*

Proof. By [16], we need only to prove that for any $\varepsilon > 0$, there is a point $z \in M^0$, such that

$$\max\{d(x, z), d(y, z)\} < \tfrac{1}{2} d(x, y) + 4\varepsilon.$$

Since $\overline{M}_i \xrightarrow{H} M^0$, for any $\varepsilon > 0$ we have $M^0 \subset U_\varepsilon(\overline{M}_i) = \{x \in X, d(x, \overline{M}_i) < \varepsilon\}$ for large i, and there exist $\bar{x}, \bar{y} \in \overline{M}_i$ for large i, such that $d(x, \bar{x}) < \varepsilon$ and $d(y, \bar{y}) < \varepsilon$. The triangle inequality implies

$$d(x, y) < d(\bar{x}, \bar{y}) + 2\varepsilon, \qquad d(\bar{x}, \bar{y}) < d(x, y) + 2\varepsilon.$$

Since M_i is a length space, clearly $d(x, y) \leq 2i_0$, and there exists $\bar{z} \in \overline{M}_i$ such that

$$\max\{d(\bar{x}, \bar{z}), d(\bar{y}, \bar{z})\} < \tfrac{1}{2} d(\bar{x}, \bar{y}) + \varepsilon \leq \tfrac{1}{2} d(x, y) + 2\varepsilon.$$

We also have $\overline{M}_i \subset U_\varepsilon(M^0)$ for large i, which implies that there is a $z \in M^0$ such that $d(z, \bar{z}) < \varepsilon$, and that

$$d(x, z) < d(\bar{x}, \bar{z}) + 2\varepsilon < \tfrac{1}{2} d(x, y) + 4\varepsilon,$$
$$d(y, z) < d(\bar{y}, \bar{z}) + 2\varepsilon < \tfrac{1}{2} d(x, y) + 4\varepsilon.$$

This completes the proof of Lemma 3.3.

Lemma 3.4. *For $r \leq i_0/2$, we have*

$$B_r(x_i) \xrightarrow{H} B_r(x_0),$$

where $B_r(x_i)$ and $B_r(x_0)$ are geodesic balls in $\overline{M}_i$ and M^0, respectively.

Proof. For $\varepsilon > 0$, we have $B_r(x_0) \subset U_\varepsilon(\overline{M}_i)$ for large i. For any $y_0 \in B_r(x_0)$, there exists $y \in \overline{M}_i$, such that $D(y_0, y) < \varepsilon$, and hence

$$\begin{aligned} d(y, x_i) &\leq d(y, y_0) + d(y_0, x_0) + d(x_0, x_i) \\ &\leq \varepsilon + r + \varepsilon \leq r + 2\varepsilon \end{aligned}$$

for large i. This implies that $y \in B_{r+2\varepsilon}(x_0)$. Since M_i is a length space, there is a $\bar{y} \in B_r(x_i)$, such that $d(y, \bar{y}) \leq 2\varepsilon$, and therefore, $d(y_0, \bar{y}) \leq 3\varepsilon$ and $B_r(x_0) \subset U_{3\varepsilon}(B_r(x_i))$.

Now for any $y \in B_r(x_i)$, since $M^0 \subset U_\varepsilon(\overline{M}_i)$ for large i, there exists a $\bar{y}_0 \in M^0$, such that $d(y, \bar{y}_0) < \varepsilon$, and

$$d(\bar{y}_0, x_0) \leq d(x_0, x_i) + d(x_i, y) + d(y, \bar{y}_0) \leq \varepsilon + r + \varepsilon \leq r + 2\varepsilon$$

for large i, which implies that $\bar{y}_0 \in B_{r+2\varepsilon}(x_0)$. If $\bar{y}_0 \notin B_r(x_0)$, then $d(x_0, \bar{y}_0) > r$, and by Lemma 3.3, there is a $y_0 \in B_r(x_0)$, such that $d(y_0, \bar{y}_0) \leq 2\varepsilon$, and $d(y, y_0) \leq 3\varepsilon$. If $\bar{y}_0 \in B_r(x_0)$, we take $y_0 = \bar{y}_0$

and $d(y, y_0) < \varepsilon$, and hence $y \in U_{3\varepsilon}(B_r(x_0))$ for large i, and $B_r(x_i) \subset U_{3\varepsilon}(B_r(x_0))$. This proves that $B_r(x_i) \xrightarrow{H} B_r(x_0)$.

Lemma 3.5. *Let* $S_r(x_i) = \{x \in \overline{M}_i, d(x, x_i) = r\}$ *and* $S_r(x_0) = \{x \in M^0, d(x, x_0) = r\}$, *for* $r \leq i_0/2$. *Then*

$$S_r(x_i) \xrightarrow{H} S_r(x_0) \qquad in\ X.$$

Proof. (1) We shall show that for any $\varepsilon > 0$,

$$S_r(x_0) \subset U_{3\varepsilon}(S_r(x_i)).$$

Taking any $y_0 \in S_r(x_0)$, we have $d(y_0, x_0) = r$. Lemma 3.4 implies that $B_r(x_0) \subset U_\varepsilon(B_r(x_i))$ for large i, so there is a $y \in B_r(x_i)$ such that $d(y_0, y) < \varepsilon$, and hence

$$r = d(x_0, y_0) \leq d(x_0, x_i) + d(x_i, y) + d(y, y_0) \leq 2\varepsilon + d(x_i, y)$$

for large i. This gives $d(x_i, y) \geq r - 2\varepsilon$, and $d(x_i, y) \leq r$. Since $r < \mathrm{inj}(M_i)$, there exists a $\bar{y} \in S_r(x_i)$, such that $d(y, \bar{y}) \leq 2\varepsilon$, and hence $d(y_0, \bar{y}) \leq 3\varepsilon$. Consequently, $S_r(x_0) \subset U_{3\varepsilon}(S_r(x_i))$.

(2) We now are going to show that $S_r(x_i) \subset U_{3\varepsilon}(S_r(x_0))$ for large i. We start with the following.

(a) For any $y \in B_r(x_0)$ and $d(x_0, y) \geq r - 2\delta$, there exists a $y_0 \in S_r(x_0)$, such that $d(y, y_0) \leq 2\delta$. In fact, for any $y \in B_r(x_0)$ and $\varepsilon > 0$, there is a $z \in B_r(x_i)$ for large i, such that $d(y, z) < \varepsilon$, and therefore

$$\begin{aligned} r - 2\delta &\leq d(x_0, y) \leq d(x_0, z) + d(z, y) \\ &\leq d(x_0, x_i) + d(x_i, z) + d(y, z) \\ &\leq 2\varepsilon + d(x_i, z) \end{aligned}$$

for large i. This implies that $r - 2\delta - 2\varepsilon \leq d(x_i, z) \leq r \leq i_0/2$, so that there is a $\bar{z}_\varepsilon \in S_r(x_i)$, such that $d(z, \bar{z}_\varepsilon) \leq 2\delta + 2\varepsilon$. For such $\bar{z}_\varepsilon$, there exists a $\bar{y}_0^\varepsilon \in B_r(x_0)$ such that $d(\bar{z}_\varepsilon, \bar{y}_0^\varepsilon) < \varepsilon$. We may assume that $\bar{y}_0^\varepsilon \to y_0 \in B_r(x_0)$ as $\varepsilon \to 0$, and $\bar{z}_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$. Then $d(\bar{z}_\varepsilon, \bar{y}_0^\varepsilon) < \varepsilon$ implies that $y_0 = \bar{z}$, and since $x_i \to x_0$, we have $d(x_0, \bar{z}) = \lim d(x_i, \bar{z})\varepsilon) = r$, and hence $d(x_0, y_0) = r$, $y_0 \in S_r(x_0)$. We have

$$d(y, y_0) \leq d(y, z) + d(z, \bar{z}) \leq \varepsilon + \lim_{\varepsilon \to 0} d(z, \bar{z}_\varepsilon) \leq \varepsilon + 2\delta + 2\varepsilon.$$

Letting $\varepsilon \to 0$, we then obtain $d(y, y_0) \leq 2\delta$.

(b) For any $y \in S_r(x_i)$, $d(y, x_i) = r$, for large i, there is a $\bar{y}_0 \in B_r(x_0)$ with $d(y, \bar{y}_0) < \varepsilon$, such that

$$\begin{aligned} r = d(x_i, y) &\leq d(x_i, x_0) + d(x_0, \bar{y}_0) + d(\bar{y}_0, y) \\ &\leq 2\varepsilon + d(x_0, \bar{y}_0) \end{aligned}$$

for large i. This implies that $d(x_0, \bar{y}_0) \geq r - 2\varepsilon$, and that $d(x_0, \bar{y}_0) \leq r < i_0$. By (a), there exists a $y_0 \in S_r(x_0)$, such that $d(y_0, \bar{y}_0) \leq 2\varepsilon$, and hence

$$d(y, y_0) \leq d(y, \bar{y}_0) + d(\bar{y}_0, y_0) \leq 3\varepsilon.$$

Consequently, $S_r(x_i) \subset U_{3\varepsilon}(S_r(x_0))$. This finishes the proof of Lemma 3.5.

Lemma 3.6. *$S_r(x_0)$ is connected for $r \leq i_0/2$.*

Proof. Since $S_r(x_i)$ is connected and compact, so is $S_r(x_0)$.

Define $PC_\delta = \{\alpha|\alpha|[t_i, t_{i+1}]$ is a minimal curve with length $\leq \delta$ in M^0, and $\alpha(t_i) \in S_r(x_0)$, $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$, α is C^0 in $M^0\}$.

Lemma 3.7. *For any $\delta > 0$ and $x, y \in S_r(x_0)$, there exists an $\alpha \in PC_\delta$ such that $\alpha(0) = x$, $\alpha(1) = y$.*

Proof. Let us fix x, and let

$$A = \{\alpha(1)|\alpha \in PC_\delta, \alpha(0) = x\}.$$

By Lemma 3.3, A is open and closed in $S_r(x_0)$, and hence $A = S_r(x_0)$.

Theorem 3.8. *For $r \leq i_0/4$, there is a constant $C_{20} = C_{20}(H, K, i_0, n, r) > 0$, such that* $\operatorname{diam}(S_r(x_0)) \leq C_{20}$.

Proof. Taking $\delta = r/8$, we then have for any $\alpha \in PC_\delta$,

$$\alpha \subset U_r^{M^0}(x_0)) \subset M^0.$$

Let $y_i, z_i \in S_r^i = S_r(x_i)$, such that

$$\operatorname{diam}(S_r^i) = \bar{d}(y_i, z_i) = \inf_{\gamma \subset S_r}\{L(\gamma)|\gamma(0) = y_i, \gamma(1) = z_i\}.$$

Since X is compact, and $S_r^i \xrightarrow{H} S_r = S_r(x_0)$ in X, we may assume that $y_i \to y$, $z_i \to z$ and $y, z \in S_r$.

Lemma 3.7 implies that there exists an $\alpha \in PC_\delta$ such that $L(\alpha) < \infty$, $\alpha(0) = y$ and $\alpha(1) = z$. Thus we have α: $[0, 1] \to M^0$ and $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$ such that $\alpha|[t_j, t_{j+1}]$ is minimal in $B_{2r}(x_0)$ and $\alpha(t_j) \in S_r(x_0)$. By Lemma 3.5, there exists an $\bar{\alpha}_j^i \in S_r^i$, such that for small $\varepsilon < \delta$ and i large, $d(\alpha(t_j), \bar{\alpha}_j^i) < \varepsilon$ $\bar{\alpha}_0^i = y_i$, and $\bar{\alpha}_{m+1}^i = z_i$. Therefore

$$\begin{aligned} d(\bar{\alpha}_j^i, \bar{\alpha}_{j+1}^i) &\leq d(\alpha(t_j), \bar{\alpha}_j^i) + d(\alpha(t_j), \alpha(t_{j+1})) + d(\alpha(t_{j+1}), \bar{\alpha}_{j+1}^i) \\ &\leq 2\varepsilon + \delta < 2\varepsilon \leq r/2, \end{aligned}$$

which implies that there is a minimal geodesic $\bar{\beta}_j^i$, from $\bar{\alpha}_j^i$ to $\bar{\alpha}_{j+1}^i$ in $\overline{M}_i$, such that

$$\bar{\beta}_j^i \subset U_{2\delta}(S_r^i) \subset B_{2r}(x_i).$$

Let $\bar\beta^i = \bigcup_{j=0}^m \bar\beta_j^i$. Then $\bar\beta(0) = y_i$, $\bar\beta^i(1) = z_i$, and $\bar\beta^i$ is a piecewise continuous minimal geodesic in $\overline{M}_i$. Thus

$$L(\bar\beta^i) = \sum_{j=0}^{m} d(\bar\alpha_j^i, \bar\alpha_{j+1}^i) \leq 2m\varepsilon + L(\alpha) < \infty. \tag{3.9}$$

Now note that $\bar\beta_j^i \subset B_{2r}(x_i) - B_{r/2}(x_i) \subset \overline{M}_i$. From Proposition 1.11, we have

$$g_i = g = dr^2 + \bar g_{kl}(r, x)\, dx^i\, dx^1 \quad \text{on } B_{2r}(x_i),$$

$$e^{4c_3}\bar g(r_1) \leq \bar g(r_2) \leq e^{4C_3}\bar g(r_1) \tag{3.10}$$

for $r/2 \leq r_1 \leq r_2 \leq 2r$. Let $\bar\beta_j^i(t) = (r(t), x(t))$. Then $r(t_j) = r(t_{j+1}) = r$.

Define $\beta_j(t) = (r, x(t))$ to be the radial projection of $\bar\beta_j^i$ on $S_r(x_i)$. Then from (3.10) it follows that

$$\begin{aligned} L(\bar\beta_j^i) &= \int_{t_j}^{t_{j+1}} \left|\frac{d}{dt}\beta_j\right|_{r(t)} dt \geq \int_{t_j}^{t_{j+1}} \left|\frac{d}{dt}x(t)\right|_{r(t)} dt \\ &\geq e^{-2C_3} \int_{t_j}^{t_{j+1}} \left|\frac{d}{dt}\beta_j^i\right|_{r} dt \geq e^{2C_3} L(\beta_j^i). \end{aligned}$$

Taking $\beta^i = \bigcup_{j=0}^m \beta_j^i$, where β^i is a C^0-piecewise C^1 curve in S_r^i, and $L(\beta^i) \leq e^{2C_3} L(\bar\beta^i)$, we have $\beta^i(0) = y_i$ and $\beta^i(1) = z_i$. By (3.9), we obtain

$$L(\beta^i) \leq e^{2C_3}(2m\varepsilon + L(\alpha)) < \infty.$$

Consequently

$$\operatorname{diam}(S_r^i) \leq e^{2c_3}(2m\varepsilon + L(\alpha)) < \infty$$

for large i. This completes the proof of Theorem 3.8.

Proof of Theorem 2.30. First we claim that

$$\operatorname{diam}\left(\bar g_i\left(\tfrac{i_0}{4}\right)\right) \leq C(H, i_0, n). \tag{3.11}$$

If (3.11) is false, then there exists a sequence

$$\{(M_i, g_i)\} \subset \mathscr{M}(H, i_0, n),$$

such that $\operatorname{diam}(\bar g_i(i_0/4)) \to \infty$, and thus $\operatorname{diam}(S_{i_0/4}^i) \to \infty$. But by Theorem 3.8, there exists a subsequence of $\{(M_i, g_i)\}$ (say $\{(M_i, g_i)\}$) such that $\operatorname{diam}(S_{i_0/4}^i) \leq C_{20}$. This contradiction proves (3.11).

Let $g_i' = (1/\tau^2)g_i$ for any $r \le i_0/4$ and $\tau = 4r/i_0$. Then since $\tau \le 1$, we have

$$\mathrm{Ric}(g_i') \ge -Hg_i', \qquad \mathrm{inj}(g_i') \ge i_0,$$

that is, $(M_i, g_i') \in \mathscr{M}(H, i_0, n)$. Applying (3.11) to (M_i, g_i'), we obtain

$$\mathrm{diam}\left(\bar{g}_i'\left(\frac{i_0}{4}\right)\right) \le C(H, i_0, n)$$

which implies

$$\mathrm{diam}(\bar{g}_i(r)) \le C(H, i_0, n)\frac{4r}{i_0}.$$

Hence the theorem is proved.

4. L^2-curvature pinching estimates

Let M be a compact Riemannian manifold with metric g as in the above sections, and assume that

$$\begin{aligned} &\mathrm{Ric}(g) \ge -Hg, \qquad \mathrm{inj}(g) \ge i_0 > 0, \\ &\max_{x_0 \in M} \int_{B_{i_0}(x_0)} |\mathrm{Rm}(g)|^{n/2}\, dg \ge K, \end{aligned} \tag{4.1}$$

where $H > 0$. In this section, we make the following additional hypothesis.

Assumption 4.2. Let

$$Tm = T_{ijkl} = R_{ijkl} - \Delta(g_{ij}g_{jl} - g_{il}g_{jk}),$$

with $\Delta = 1, 0$, or -1, and assume that

$$\max_{x_0 \in M} \int_{B_{i_0}(x_0)} |Tm|^2\, dg \le \mu$$

for a small $\mu > 0$.

As in earlier sections, we consider the metric g in polar geodesic coordinates on the geodesic ball $B_{i_0}(x_0)$,

$$g = dr + \sum g_{ij}(r, x)\, dx^i\, dx^j.$$

Let $\bar{g}(t)$ be the induced metric on the geodesic sphere $S_r(x_0)$. Denote the scalar curvature free curvature tensor of $\bar{g}(r)$ by $\mathring{\mathrm{R}}\mathrm{m}(r) = \mathring{\mathrm{R}}\mathrm{m}(\bar{g}(r))$, and the second fundamental form of $S_r(x_0)$ by $B(X, Y) = \langle \nabla_X Y, \partial/\partial r\rangle$ for X, Y vector fields on $S_r(x_0)$.

The main theorem of this section is:

Theorem 4.3. *Let* $\sigma(\mu) = \mu^{1/(n+1)}$. *Then for any* $x_0 \in M$ *and* $\rho = i_0/4$, *there is an* $x \in B_\sigma(x_0) \subset M$, *such that*

$$\int_{B_\rho(x)-B_{\rho/2}(x)} |\overset{\circ}{\mathrm{R}}\mathrm{m}(\bar{g})|\, d\bar{g} \le \sigma(\mu)^{1/2} C(H, K, i_0, n).$$

Remark 4.3(a). Similarly, if $0 < \bar{r} < i_0/4$, there exists an $x \in B_\sigma(x_0) \subset M$ such that

$$\int_{B_{i_0/4}(x)-B_{\bar{r}}(x)} |\overset{\circ}{\mathrm{R}}\mathrm{m}(\bar{g})|\, dg \le \sigma(\mu)^{1/2} C(H, K, i_0, n, \bar{r}).$$

We start with several lemmas.

Lemma 4.4. *Let* γ *be a geodesic of length* $l \le \pi/2$, *and* J *a vector field along* γ, *such that* $J(\gamma(0)) = 0$ *and* $J(\gamma(l)) = 0$. *Then for* $\Delta = 1, 0$, *or* -1, *we have*

$$\int_0^l |J|^2\, dt \le \int_0^l |J'|^2\, dt - \Delta \int_0^l |J|^2\, dt.$$

Proof. Since the Dirichlet eigenvalue problem of the Laplace operator $-d^2/dt^2$ on $[0, l]$ has the first eigenvalue $(\pi/l)^2 \ge 4$, we have

$$\int_0^l |J'|62 - \Delta \int_0^l |J|^2 \ge 4 \int_0^l |J|^2 - \Delta \int_0^l |J|^2 \ge \int_0^l |J|^2.$$

Lemma 4.5. *Let* γ *and* J *be as in Lemma* 4.4. *Then*

$$\int_0^l |J'|^2\, dt \le 2\left(\int_0^l |J'|^2\, dt - \Delta \int_0^l |J|^2\, dt\right).$$

Proof. We have

$$\begin{aligned}\int_0^l |J'|^2\, dt &= \int_0^l |J'|^2\, dt - \Delta \int_0^l |J|^2\, dt + \Delta \int_0^l |J|^2\, dt \\ &\le 2\left(\int_0^l |J'|^2\, dt - \Delta \int_0^l |J|^2\, dt\right).\end{aligned}$$

Lemma 4.6. *Let* γ *be a geodesic of length* $l \le i_0/4 \le \pi/2$, *and* Y *a Jacobi field along* γ *such that* $Y(\gamma(0)) = 0$, $|Y(\gamma(l))| = 1$, *and* Y *is perpendicular to* γ. *Let* E *be the parallel vector field along* γ *with* $E(\gamma(l)) = Y(\gamma(l))$, *and define*

$$A = \begin{cases} Y - (\sin t/\sin l)E & \text{if } \Delta = 1, \\ Y - (t/l)E & \text{if } \Delta = 0, \\ Y - (\sinh t/\sinh l)E & \text{if } \Delta = -1. \end{cases}$$

Then

$$\int_0^l |A'|^2\,dt \le C_{20}\int_\gamma |T_m|.$$

Proof. We have $Y'' + R(Y, \gamma')\gamma' = 0$, which can be rewritten as

$$Y'' + \Delta Y = -(R(Y, \gamma')\gamma' - \Delta Y),$$

so that

$$A'' + \Delta A = Y + \Delta Y = -(R(Y, \gamma')\gamma' - \Delta Y).$$

Noting that $(R(Y, \gamma')\gamma' - \Delta Y) \le |T_m|\,|Y|$, we deduce

$$|A'' + \Delta A| \le |T_m|\,|Y|. \tag{4.7}$$

Since $|Y(l)| = 1$, Lemma 1.17 implies that $|Y| \le C_6(H, i_0, n)$ so that

$$|A| \le |Y| + |E| \le C_6 + 1.$$

Combining this with (4.7) gives

$$\begin{aligned}\int_0^l |A'|^2 - \Delta\int_0^l |A|^2 &= -\int_0^l \langle A'' + \Delta A, A\rangle\,dt\\ &\le \int_0^l |T_m|\,|Y|\,|A| \le (C_6+1)^2\int_\gamma |T_m|.\end{aligned}$$

Thus using Lemmas 4.4 and 4.5, we thus prove Lemma 4.6.

Lemma 4.8. *Let γ, Y, and A be as in Lemma 4.6, and assume $i_0/100 \le l \le i_0/4$. Then*

$$|A'|^2(\gamma(l)) \le C_{21}\int_\gamma |T_m|^2$$

for a constant $C_{21} = C_{21}(H, K, i_0, n) > 0$.

Proof. Let ψ be a function on γ, such that $\psi(t) = 0$ for $t < l/8$, $\psi(t) = 1$ for $t \ge 7l/8$, $0 \le \psi \le 1$, and $|\psi'| \le 10/l$.

Integrating by parts yields

$$\begin{aligned}|\psi A'| &= \int_0^l |(\psi A')'|\,dt = \int_0^l |\psi A'' + \psi' A'|\,dt\\ &\le \int_0^l |-\Delta A - (R(Y, \gamma')\gamma' - \Delta Y)| + \frac{10}{l}\int_0^l |A'|\,dt\\ &\le \frac{10}{l}\int_0^l (|A| + |A'|)\,dt + \int_\gamma |T_m|\,|Y|.\end{aligned}$$

By the Hölder inequality and Lemma 4.6, we obtain

$$|A'|(\gamma(l)) \le \left(\frac{10}{l}\right) l^{1/2}\left(\int_0^l |A'|^2 + |A|^2\right)^{1/2} + C\left(\int_\gamma |T_n|^2\right)\frac{1}{2},$$

so that

$$|A'|^2 \cdot (\gamma(l)) \le C_{21} \int_\gamma |T_m|^2.$$

Remark 4.8(a). By taking $\psi = 1$ on $[\bar{r}, i_0/4]$ in the proof, we can show that, for any $0 < \bar{r} < i_0/4$, we have

$$\max_{\bar{r} \le r \le i_0/4} |A'|^2(\gamma(r)) \le C(H, K, i_0, n, \bar{r}) \int_\gamma |T_m|^2.$$

Lemma 4.9. *For each $x \in M$, let γ be the geodesic in M, and $\gamma(0) = x$ with the length l of γ satisfying $i_0/8 \le l \le i_0/4$. Then at $\gamma(l)$,*

$$|B - b(l)\bar{g}|^2(\gamma(l)) \le C_{22} \int_\gamma |T_m|^2$$

for a constant $c_{22} = c_{22}(H, K, i_0, n) > 0$, where

$$b(l) = \begin{cases} -\cos l/\sin l & \text{if } \Delta = 1, \\ -1/l & \text{if } \Delta = 0, \\ -\cos hl/\sin hl & \text{if } \Delta = -1, \end{cases}$$

and B and $\bar{g}$ are the second fundamental form and induced metric on $S_r(x)$, respectively.

Proof. Let X, Y be vector fields on $S_l(x)$, such that $|X(\gamma(l))| = 1$ and $|Y(\gamma(l))| = 1$, and let E, $\overline{E}$ be parallel vector fields on γ, with $E(\gamma(l)) = X(\gamma(l))$ and $\overline{E}(\gamma(l)) = Y(\gamma(l))$. We can extend X, Y to vector fields on $B_l(x)$, such that X, Y are Jacobi fields on each radial geodesic from x. Then clearly, we have

$$B(X, Y)(\gamma(l)) = -\langle \nabla_{\gamma'} X, Y \rangle(\gamma(l)).$$

From Lemma 4.8, it follows that

$$\begin{aligned} |B(X, Y) - b(l)\langle X, Y\rangle|^2(\gamma(l)) &= |\langle X', Y\rangle + b(l)\langle E, Y\rangle|^2(\gamma(l)) \\ &= |\langle (X' + b(l)E), Y\rangle|^2 \le C_{21}|Y| \int_\gamma |T_m|^2 \\ &\le C_6 C_{21} \int_\gamma |T_m|^2. \end{aligned}$$

Since this holds for any such vector field X, Y on $S_l(x)$, clearly this implies that

$$|B - b(l)\bar{g}|^2(\gamma(l)) \le C_{22} \int_\gamma |T_m|^2,$$

which finishes the proof of Lemma 4.9.

We now define a function f on $M \times M \cap \{(x, y) : d(x, y) \le i_0/2\}$. For each $(x, y) \in M \times M$ with $d(x, y) \le i_0/2$, there exists a unique geodesic γ from x to y in M with length $2l$, and $l < i_0/4$, $\gamma(0) = x$, $\gamma(2l) = y$. Let $B_x(l)$ and $\bar{g}_x(l)$ be the second fundamental form and induced metric on $S_l(x)$, respectively. Then we define

$$f(x, y) = |B_x(l) - b(l)\bar{g}_x(l)|^2(\gamma(l)).$$

Lemma 4.10. *Let $x_0 \in M$, and for small $1 > \delta > 0$, $B_\delta(x_0) = \{x \in M : d(x_0, x) < \delta\}$. Then there exists a constant $C_{23} = C_{23}(H, K, i_0, n) > 0$, such that for $i_0/8 \le l \le i_0/4$, we have*

$$\int_{B_\delta(x_0)} \left(\int_{S_{2l}(x)} f(x, y)\, d\bar{g}_x(y) \right) dg(x) \le C_{23}\mu.$$

Proof. Let $\Omega = \bigcup_{x \in B_\delta(x_0)} S_{2l}(x) \subset M$. We consider the distance function d on $B_\delta(x_0) \times M$. Since $2l \le i_0/2$, $2l$ is a regular value of d,

$$\sum = d^{-1}(2l) = \bigcup_{x \in B_\delta(x_0)} (x, S_{2l}(x))$$

is a smooth submanifold of $M \times M$ with $\dim \Sigma = 2n - 1$. Let us denote the $(2n-1)$-dimensional Haussdorff measure of $M \times M$ by $d\nu$. We use the coarea formula [8] to compute $\int\int_\Sigma f(x, y)\, d\nu$. On the one hand, we have

$$\int\int_{\Sigma} f(x, y)\, d\nu = \int_{B_\delta(x_0)} \left(\int_{S_{2l}(x)} f(x, y)\, d\bar{g}_x(y) \right) dg(x).$$

On the other hand, for each $y \in \Omega$, we set $\Omega_y = B_\delta(x_0) \cap S_{2l}(y) = \{x \in B_\delta(x_0), d(x, y) = 2l\}$. Then $\Omega_y \subset S_{2l}(y)$, and

$$\begin{aligned} \int\int_{\Sigma} f(x, y)\, d\nu &= \int_\Omega \left(\int_{\Omega y} f(x, y)\, d\bar{g}_y(x) \right) dg(y) \\ &\le \int_\Omega \left(\int_{S_{2l}(y)} f(x, y)\, d\bar{g}_y(x) \right) dg(y). \end{aligned} \tag{4.11}$$

Now for each $y \in \Omega$ and $x \in S_{2l}(y)$, we denote the geodesic from x to $\gamma(l)$ by $\bar{\gamma}$, i.e., $\bar{\gamma}(t) = \gamma(t)$ for $t \le l$. Using Lemma 4.9, we obtain

$$\int_{S_{2l}(y)} f(x, y)\, d\bar{g}_y(x) \le C_{22} \int_{S_{2l}(y)} \left(\int_{\bar{\gamma}} |Tm|^2 \right) d\bar{g}_y(x),$$

where $\int_{\bar{\gamma}} |Tm|^2$ is considered as a function of x and y with $d(x, y) = 2l$, and hence

$$\int_{S_{2l}(y)} f(x, y)\, d\bar{g}_y(x) \le C_{11} \int_l^{2l} \left(\int_{S_{2l}(y)} |Tm|^2(\gamma(2l - t))\, d\bar{g}_y(x) \right) dt.$$

From Proposition 1.11, we have $d\bar{g}_y(\gamma(2l-t)) \geq C(Y, i_0, n)\, d\bar{g}_y(x)$, $x \in S_{2l}(y)$, which implies

$$\int_{S_{2l}(y)} f(x, y)\, d\bar{g}_y(x) \leq C(H, K, i_0, n) \int_{B_{2l}(y)-B_l(y)} |Tm|^2\, dg$$
$$\leq C(H, K, i_0, n) \int_{B_{2l}(y)} |Tm|^2\, dg \leq C\mu.$$

Combining this with (4.11), we obtain

$$\iint_{\Sigma} f(x, y)\, d\nu \leq C\mu \operatorname{Vol}(\Omega).$$

Note that $\Omega \subset B_{\delta+2l}(x_0) \subset B_{\delta+i_0}(x_0)$ and $\operatorname{Ric} \geq -H$. From Lemma 1.15, it then follows that

$$\int_{B_\delta(x_0)} \left(\int_{S_{2l}(x)} f(x, y)\, d\bar{g}_x(y) \right) dg(x) \leq C_{23}\mu.$$

Lemma 4.12. *For* $i_0/8 \leq l \leq i_0/4$, $1 > \delta > 0$, *we have*

$$\int_{B_\delta(x_0)} \left(\int_{S_l(x)} |B_x(l) - b(l)\bar{g}_x(l)|^2\, d\bar{g}_x(y) \right) dg(x) \leq C_{24}\mu$$

for a constant $C_{24} = C_{24}(H, K, i_0, n) > 0$.

Proof. From Lemma 4.10, it follows that

$$\int_{B_\delta(x_0)} \left(\int_{S_{2l}(x)} |B_x(l) - b(l)\bar{g}_x(l)|^2\, d\bar{g}_x(y) \right) dg(x) \leq C_{23}\mu.$$

By Proposition 1.11, the metrics $\bar{g}_x(l)$ and $\bar{g}_x(2l)$ on $S_l(x)$ and $S_{2l}(x)$ respectively are equivalent. We thus obtain Lemma 4.12.

Remark 4.12(a). If we replace 4.8 by 4.8(a) in the proof above, we can show that for any $0 < \bar{r} < i_0/4$,

$$\int_{B_\delta(x_0)} \left(\max_{\bar{r} \leq r \leq i_0/4} \int_{S_r(x)} |B_x(r) - b(r)\bar{g}_x(r)|^2\, d\bar{g}_x(y) \right) dg(x)$$
$$\leq C(H, K, i_0, n, \bar{r})\mu.$$

We now can prove Theorem 4.3.

Proof of Theorem 4.3. For $\delta > 0$ small, by Theorem 2.28 we obtain $\operatorname{Vol}(B_\delta(x_0)) \geq C(H, i_0, n)\delta^n$. Then Lemma 4.12 implies that there is an $x \in B_\delta(x_0)$, such that

$$\int_{S_l(x)} |B_x(l) - b(l)\bar{g}_x(l)|^2\, d\bar{g} \leq C(H, K, i_0, n)\mu/\delta^n.$$

Taking $\delta^{n+1} = \mu > 0$, and $\sigma(\mu) = \mu^{1/(n+1)} = \delta > 0$, we have

$$\int_{S_l(x)} |B_x(l) - b(l)\bar{g}_x(l)|^2 \, d\bar{g} \le C\sigma(\mu), \quad \text{for } x \in B_\sigma(x_0), \tag{4.13}$$

and $\lim_{\mu\to 0}\sigma(\mu) = 0$.

Now let us recall the Gauss formula on $S_l(x)$,

$$R(g)_{ijkl} = R(\bar{g})_{ijkl} + (B_{ik}B_{jl} - B_{il}B_{jk}).$$

We will estimate

$$\int_{S_l(x)} |\operatorname{Rm}(\bar{g}) - (\Delta + b(l)^2)(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})| \, d\bar{g}.$$

First, noting that $g(\partial/\partial x^i, \partial/\partial x^j) = g_{ij} = \bar{g}_{ij}$ and $g(\partial/\partial r, \partial/\partial x^i) = 0$, we have

$$|R(g)_{ijkl} - \Delta(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})| \le |Tm| \quad \text{on } S_r(x),$$

and hence

$$\int_{B_\rho(x)-B_{\rho/2}(x)} |R(g)_{ijkl} - \Delta(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})|^2 \, dg \le \mu. \tag{4.14}$$

Secondly, (4.13) implies that

$$\int_{S_l(x)} |B_x(l)|^2 \, d\bar{g} \le C(H, K, i_0, n),$$

so that

$$\begin{aligned}
&\int_{S_l(x)} |B_{ik}B_{jl} - b(l)^2 \bar{g}_{ik}\bar{g}_{jl}| \, d\bar{g} \\
&\quad = \int_{S_l(x)} |B_{ik}(B_{jl} - b(l)\bar{g}_{jl}) + b(l)\bar{g}_{jl}(B_{ik} - b(l)\bar{g}_{ik})| \, d\bar{g} \\
&\quad = C\int_{S_l(x)} |B|\,|B - b(l)\bar{g}| \, dg + C\int_{S_l(x)} |B - b(l)\bar{g}| \, d\bar{g}.
\end{aligned}$$

By the Hölder inequality, we thus obtain

$$\begin{aligned}
&\int_{S_l(x)} |B_{ik}B_{jl} - b(l)^2 \bar{g}_{ik}\bar{g}_{jl}| \, d\bar{g} \\
&\quad \le C(H, K, i_0, n)\left(\int_{S_l(x)} |B - b(l)\bar{g}|^2 \, d\bar{g}\right)^{1/2} \le C\sigma(\mu)^{1/2},
\end{aligned}$$

and hence

$$\int_{S_l(x)} |(B_{ik}B_{jl} - B_{il}B_{jk}) - b(l)^2(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})| \, d\bar{g} \le C(H, K, i_0, n)\sigma(\mu)^{1/2}.$$

Combining this with the Hölder inequality and (4.14) yields

$$\int_{B_\rho(x)-B_{\rho/2}(x)} |R(\bar{g})_{ijkl} - (\Delta + b(\tau)^2)(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})|\, dg \tag{4.15}$$
$$\leq C\sigma(\mu)^{1/2} + C\mu^{1/2} \leq C(H, K, i_0, n)\sigma(\mu)^{1/2}$$

for $\mu \leq 1$. If $\overline{R}$ denotes the scalar curvature of $\bar{g}(r)$, then

$$\int_{B_\rho(x)-B_{\rho/2}(x)} |\bar{r} - (n-1)(n-2)(\Delta + b(r)^2)|\, dg \leq C\sigma(\mu)^{1/2}.$$

Combining this with (4.15), we thus obtain

$$\int_{B_\rho(x)-B_{\rho/2}(x)} |\mathring{\mathrm{R}}\mathrm{m}(\bar{g})|\, dg \tag{4.16}$$
$$= \int_{B_\rho(x)-B_{\rho/2}(x)} \left| R(\bar{g})_{ijkl} - \frac{\overline{R}}{(n-1)(n-2)}(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}) \right| dg$$
$$\leq C(H, K, i_0, n)\sigma(\mu)^{1/2},$$

which completes the proof of Theorem 4.3.

As a corollary of Theorem 4.3, we have

Corollary 4.17. *For the metric g on M, if*

(a) $\mathrm{Ric}(g) \geq -Hg$,

(b) $\mathrm{inj}(g) \geq i_0 > 0$,

(c)
$$\max_{x\in M} \int_{B_{i_0}(x)} |\,\mathrm{Rm}\,|^{n/2}\, dg \leq H,$$

(d)
$$\max_{x\in M} \int_{B_{i_0}(x)} |Tm|^2\, dg \leq \mu \leq 1,$$

then there exist a $\sigma(\mu) > 0$, $\lim_{\mu\to 0}\sigma(\mu) = 0$, and a constant $C_{25} = C_{25}(H, i_0, n) > 0$, such that for any $x_0 \in M$ and $\rho = i_0/4$, there is an $x \in B_\sigma(x_0) \in M$, and an $r_\rho > 0$, $\rho/2 \leq r_\rho \leq \rho$, and

$$\int_{S_{r_\rho}(x)} |\,\mathring{\mathrm{R}}\,\mathrm{m}(\bar{g})|\, d\bar{g} \leq C_{25}\sigma(\mu).$$

Proof. First (c) and Lemma 1.15 imply that

$$\max_{x\in M} \int_{B_{i_0}(x)} |\,\mathrm{Rm}(g)|^{n/2}\, dg \leq C(H, i_0, n),$$

so that $K \leq C(H, i_0, n)$. Theorem 2.28 yields that $\mathrm{Vol}(B_\rho - B_{\rho/2}) \geq C(H, i_0, n) > 0$. These and Theorem 4.3 thus immediately prove Corollary 4.17. Remark 4.3(a) follows from Remark 4.8(a) above.

5. $L^{n/2}$-curvature pinching theorems

In this section we shall prove the main theorems of this paper. We refer to the $L^{n/2}$-curvature pinching theorems.

Theorem 5.1. *For each $H > 0$ and $0 < i_0 \leq \pi/2$, there exists a small $\mu = \mu(H, i_0, n) > 0$ which depends only on H, i_0, and n, such that if (M, g) is a complete Riemannian manifold with* $\dim M = n \geq 4$, *and*

(a) $\mathrm{Ric}(g) \geq -Hg$,
(b) $\mathrm{inj}(g) \geq i_0$,
(c)
$$\max_{x\in M} \int_{B_{i_0}(x)} |\,\mathrm{Rm}(g)|^{n/2}\, dg \leq H,$$
(d)
$$\max_{x\in M} \int_{B_{i_0}(x)} |R(g)_{ijkl} - (g_{ik}g_{jl} - g_{il}g_{jk})|^2\, dg \leq \mu,$$

then M is homotopic to a Riemannian manifold $\overline{M}$ with positive constant sectional curvature, in particular, M is compact. Furthermore, M is covered by a homeomorphism sphere.

Theorem 5.2. *For each $H > 0$, $d > 0$, and $0 > i_0 \leq \pi/2$, there exists a small $\mu = \mu(H, i_0, n, d) > 0$ which depends only on H, i_0, n and d, such that if (M, g) is a compact Riemannian manifold with* $\dim M = n \geq 4$, *and*

(a) $\mathrm{Ric}(g) \geq -Hg$,
(b) $\mathrm{inj}(g) \geq i_0 > 0$,
(c)
$$\max_{x\in M} \int_{B_{i_0}(x)} |\,\mathrm{Rm}(g)|^{n/2}\, dg \leq H,$$
(d)
$$\max_{x\in M} \int_{B_{i_0}(x)} |R(g)_{ijkl} - \Delta(g_{ik}g_{jl} - g_{il}g_{jk})|^2\, dg \leq \mu,$$
(e) $\mathrm{diam}(g) \leq d$,

then M is homotopic to a manifold with constant sectional curvature $\Delta = -1$ or 0.

We shall prove Theorems 5.1 and 5.2 together. Condition (c) can be rewritten as

$$\max_{x\in M}\int_{B_{i_0}(x)} |Tm|^2\, dg \le \mu. \tag{5.3}$$

We first start with the following.

Theorem 5.4. *Let (M_k, g_k) be a sequence of Riemannian manifolds, such that*

(a) $\mathrm{Ric}(g_k) \ge -Hg_k$,

(b) $\mathrm{inj}(g_k) \ge i_0 > 0$,

(c)
$$\max_{x\in M}\int_{B_{i_0}(x)} |\mathrm{Rm}(g_k)|^{n/2}\, dg_k \le H,$$

(d)
$$\max_{x\in M}\int_{B_{i_0}(x)} |Tm(g_k)|^2\, dg_k \le \tfrac{1}{k} = \mu_k.$$

Then there is a subsequence of (M_k, g_k) which converges to a complete Riemannian manifold $\overline{M}$ of constant sectional curvature Δ in Haussdorff distance.

Proof. We take a point $x_k \in M_k$, for each k. Then the precompactness theorem of Gromov [16] implies that there exists a subsequence of (M_k, g_k) (for simplicity, say this subsequence is (M_k, g_k)) which converges to a length space $\overline{M}$. We claim that $\overline{M}$ is a Riemannian manifold of constant curvature Δ. Let us denote the $(n-1)$-dimensional standard Euclidean unit sphere by S_1.

Let
$$(M_k, x_k) \xrightarrow{H} (\overline{M}, \bar{x}).$$

We denote the ball of radius r in M_k at x_k by $B_r(x_k)$, and the sphere of radius r in M_k at x_k by $S_r(x_k)$. Similarly, we have the ball $B_r(\bar{x})$ and sphere $S_r(\bar{x})$ in $\overline{M}$. For any large $D > 0$, there is a compact metric space X, such that $B_D(x_k)$ is a subspace of X with induced metric, and also $B_D(\bar{x}) \subset X$. We have $x_k \to \bar{x}$ in X. From §3, for $r \le i_0/2$ $\bar{y} \in \overline{M}$, and $d(\bar{x}, \bar{y} \ll D$, we have $S_r(\bar{x}_k) \xrightarrow{H} S_r(\bar{y})$ in X for a sequence $\bar{x}_k \in B_D(x_k)$, and $\bar{x}_k \to \bar{y}$ in X.

Let $d\theta^2$ denote the standard metric of constant sectional curvature on S_1, and

$$g(r) = s(r)^2\, d\theta^2 = \begin{cases} \sin^2 r\, d\theta^2 & \text{if } \Delta = 1, \\ r^2\, d\theta^2 & \text{if } \Delta = 0, \\ \sin h^2 r\, d\theta^2 & \text{if } \Delta = -1. \end{cases}$$

Lemma 5.5. *For $\bar{y} \in \overline{M}$ and any $0 < \bar{r} \leq i_0/4$, there exists a $y_k \in M_k$ with $d(\bar{x}_k, y_k) < \sigma(1/k)$, such that for each $r \in [\bar{r}, i_0/4]$, and for $g_k = dr^2 + \bar{g}_k(r)$ on $B_{i_0/4}(y_k)$ there is a diffeomorphism $\phi_k \colon S_1 \to S_1$ such that (ϕ_k may depend on r)*

$$|\phi_k^* \bar{g}_k(r) - g(r)|_{g(r)} \to 0 \quad \text{on } S_1.$$

First we need the following.

Sublemma 5.6. *There exists $y_k \in M_k$ with $d(\bar{x}_k, y_k) < \sigma(1/k)$ such that for any $r \in [\bar{r}, i_0/4]$ and any $\delta > 0$, there is an $r_k > 0$, $r \leq r_k \leq \delta r + r$, and a diffeomorphism $\phi_k \colon S_1 \to S_1$ for each k, such that*

$$|\phi_k^* \bar{g}_k(r_k) - h_\delta|_{g(r)} \to 0,$$

$h_\delta = (1/\kappa_\delta)\, d\theta^2$ for some $\kappa_\delta > 0$.

Proof. Let us recall from Theorem 1.14, Remark 1.14(a) for any $x \in M_k$, and from Theorem 4.3, Remark 4.3(a) respectively:

$$\int_{\bar{r}}^{i_0/4} \int_{S_r(x)} |\mathrm{Rm}(\bar{g}_k)|^{n/2}\, d\bar{g})k\, dr \leq C(H, i_0, n, \bar{r});$$

$$\int_{\bar{r}}^{i_0/4} \int_{S_r(y_k)} \left| \mathring{\mathrm{R}}\mathrm{m}(\bar{g}_k) \right| d\bar{g}_k\, dr \leq C(H, i_0, n, \bar{r}) \sigma(\mu_k),$$

or

$$\int_{\bar{r}}^{i_0/4} \int_{S_r(y_k)} |\mathrm{Rm}(\bar{g}_k) - (\Delta + b(r)^2)(\bar{g}_{il}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})|\, d\bar{g}_k\, dr$$
$$\leq C(H, i_0, n, r)\sigma(\mu_k).$$

Since $y_k \to \bar{y}$ in X, we also have $S_r(y_k) \to S_r(\bar{y})$ in X for each $r \leq i_0/4$.

Now taking $r \in [\bar{r}, i_0/4]$ and $\delta > 0$ small, we have

$$\text{(5.7)} \qquad \int_r^{(\delta+1)r} \int_{S_r(y_k)} |\mathrm{Rm}(\bar{g}_k)|^{n/2}\, d\bar{g}_k\, dr \leq C(H, i_0, n, \bar{r}),$$

$$\text{(5.8)(a)} \qquad \int_r^{(\delta+1)r} \int_{S_r(y_k)} \left| \mathring{\mathrm{R}}\mathrm{m}(\bar{g}_k) \right| d\bar{g}_k\, dr \leq C(H, i_0, n, \bar{r}) \sigma(1/k),$$

$$\text{(5.8)(b)} \qquad \int_r^{(\delta+1)r} \int_{S_r(y_k)} |\mathrm{Rm}(\bar{g}_k) - (\Delta + b(r)^2)(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk})|\, d\bar{g}_k\, dr$$
$$\leq C(H, i_0, n, r)\sigma(1/k).$$

Note that (5.8)(a) is a consequence of (5.8)(b).

We define the subsets A_1^k and A_2^k of $[r, (\delta+1)r]$ by

$$A_1^k = \left\{ \rho \in [r, (\delta+1)r] ; \int_{S_\rho(y_k)} |\operatorname{Rm}(\bar{g}_k)|^{n/2} \, d\bar{g}_k \le C(H, i_0, n, \bar{r}) \frac{4}{\delta r} \right\},$$

$$A_2^k = \left\{ \rho \in [r, (\delta+1)r] : \int_{S_\rho(y_k)} |\operatorname{Rm}(\bar{g}_k) - (4 + b(\rho)^2)(\bar{g}_{ij}\bar{g}_{jl} - \bar{g}_{jl}\bar{g}_{jh})| \, d\bar{g}_k \right.$$
$$\left. \le 4C(H, i_0, n, \bar{r}) \frac{\sigma(1/k)}{\delta r} \right\}.$$

Then from (5.7) and (5.8), it follows that for each k,

$$m(A_1^k) > \tfrac{1}{2}\delta r \quad \text{and} \quad m(A_2^k) > \tfrac{1}{2}\delta r,$$

where $m(A_1^k)$ is the Lebesgue measure of A_1^k. These clearly imply that there exists an $r_k \in A_1^k \cap A_2^k$, for each k, such that $r \le r_k \le (\delta+1)r$ and

$$\int_{S_{r_k}(y_k)} |\operatorname{Rm}(\bar{g}_k)|^{n/2} \, d\bar{g}_k \le C(H, i_0, n, \bar{r}, \delta), \tag{5.9}$$

$$\int_{S_{r_k}(y_k)} \left| \mathring{\operatorname{R}}\operatorname{m}(\bar{g}_k) \right| \, d\bar{g}_k \le C(H, i_0, n, \bar{r}, \delta)\sigma\left(\frac{1}{k}\right), \tag{5.10)(a}$$

$$\begin{aligned} &\int_{S_{r_k}(y_k)} |\operatorname{Rm}(\bar{g}_k) - (\Delta + b(k)^2)(\bar{g}_{ih}\bar{g}_{jl} - \bar{j}_{il}\bar{g}_{jh})| \, d\bar{g}_k \\ &\qquad \le C(H, i_0, n, \bar{r}, \delta)\sigma\left(\frac{1}{k}\right). \end{aligned} \tag{5.10)(b}$$

Notation. We shall write $S(k) = S_{r_k}(y_k)$ (also identify $S(k)$ with S_1), and simply write S as $S_1 = S(k)$. For $r \le i_0/2$, we also identify $S_r(y_k)$ with S.

We now consider the evolution equation on $S_{r_k}(y_k) = S$:

$$\frac{\partial}{\partial t} h_k = -2\operatorname{Ric}(h_k) \tag{5.11}$$

such that $h_k(0) = \bar{g}_k(r_k)$. We apply Theorem 2.11 to $\bar{g}_k(r_k)$, and note from (5.9) and Theorem 5.4(c) that there exists a $T = T(H, i_0, n, \bar{r}, \delta) > 0$ such that (5.11) has a solution on $[0, T]$, and

$$\max_S |R_m(h_k)|(t) \le \frac{C(H, i, n, \bar{r})}{\delta^{2/n} t^{1-1/n}}.$$

From [19] and [20] it is a straightforward but long calculation to show

$$(5.12)\qquad \frac{\partial}{\partial t}\left|\overset{\circ}{R}_m(h_k)\right| \le \Delta\left|\overset{\circ}{R}_m(h_k)\right| + C(n)\left|\overset{\circ}{R}_m(h_k)\right|\,|R_m(h_k)|.$$

Then for each fixed τ, $0<\tau\le T$, and all k we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_S \left|\overset{\circ}{R}_m(h_k)\right| dh_k &\le C(n)\int_S \left|\overset{\circ}{R}_m(h_k)\right|\,|R_m(h_k)|\,dh_k\\ &\le \frac{C(H, i_0, n, \bar{r}, \delta)}{t^{1-1/n}}\int_S \left|\overset{\circ}{R}_m(h_k)\right| dh_k.\end{aligned}$$

Integrating both side yields

$$\int_S \left|\overset{\circ}{R}_m(h_k(\tau))\right| dh_k(\tau) \le C(H, i_0, n, \bar{r}, \delta)t^{1/n}\int_S \left|\overset{\circ}{R}_m(\bar{g}_k)\right| d\bar{g}_k.$$

Combining this with (5.10), and noting that $n\ge 4$, we obtain

$$(5.13)\qquad \int_S \left|\overset{\circ}{R}_m(h_k(\tau))\right| dh)k(\tau) \le C(H, i_0, n, \bar{r}, \delta)\tau^{1/n}\sigma(1/k)^{4/n}.$$

From §2, we also have

$$(5.14)\qquad \begin{aligned}&\mathrm{Vol}(h_k(\tau)) \ge C(H, i_0, n, \bar{r}) > 0 \quad \text{for } \tau\le\delta^2,\\ &\mathrm{diam}(h_k(\tau)) \le C(H, i_0, n) \quad \text{for } \tau \le \delta^2.\end{aligned}$$

Since $|R_m(h_k(\tau/2))| \le C/\tau^{1-1/n}$, using a smoothing theorem in [1] we find

$$|\nabla R_m(h_k(\tau))| \le \max\left|R_m\left(h_k\left(\frac{\tau}{2}\right)\right)\right|\frac{C(n)}{\tau} \le \frac{C(H, i_0, n, \bar{r}, \delta)}{\tau^{2-1/n}}.$$

For each fixed $0<\tau<\min\{T, \delta^2\}$, by the Gromov Convergence Theorem ([15], [25]), a subsequence of $(S(k), h_k(\tau))$ converges to a C^2 Riemannian manifold $(S, h(\tau))$; we still call this subsequence $(S(k), h_k(\tau))$. (5.13) implies that $\overset{\circ}{\mathrm{R}}\,\mathrm{m}(h(\tau)) = 0$, and hence $h(\tau)$ is a constant curvature metric on S.

From §2 again, we have

$$(5.15)\qquad |\bar{g}_k(r_k) - h_k(\tau)|_{h(\tau)} \le C(H, i_0, n, \bar{r}, \delta)\tau^{1/n}.$$

Recalling the Gromov Convergence Theorem ([11], [25]), we see that there exists a diffeomorphism $\phi_k\colon S\to S$ for each k, such that $\phi_k^* h_k(\tau)$ converges to a constant curvature metric $h(\tau)$ on S in C^2-topology, so that for large k,

$$(5.16)\qquad |\phi_k^*\bar{g}_k(r_k) - h(\tau)| \le C(H, i_0, n, \bar{r}, \delta)\tau^{1/n}.$$

From (5.14), the constant curvature $K(\tau)$ of $h(\tau)$ on S satisfies

$$0 < C(H, i_0, n, \bar{r}) \leq K(\tau) \leq C(H, i_0, n, \bar{r}) \quad \text{for } \tau \leq \delta^2$$

which shows that there exists a subsequence $\{\tau_\alpha\}$ such that $\tau_\alpha \to 0$, the sectional curvature $K(\tau_\alpha) \to \kappa_\delta$, and $h(\tau_\alpha) \xrightarrow{C^2} h_\delta$, where h_δ is a constant curvature metric on S with $K(h_\delta) = \kappa_\delta$. Then (5.16) implies that

$$(5.17) \qquad |\phi_k^* \bar{g}_k(r_k) - h_\delta|_{h_\delta} \to 0,$$

and that $\phi_k^*(\bar{g}_k(r_k)$ converges uniformly to h_δ on S.

We also have $0 < C(H, i_0, n, \bar{r}) \leq \kappa_\delta \leq C(H, i_0, n, r)$ and $h_\delta = (1/\kappa_\delta)\, d\theta^2$. This completes the proof of Sublemma 5.6.

We may take a subsequence $\{\delta_\alpha\}$ of $\{\delta\}$, such that $\delta_\alpha \to 0$ and $\kappa_{\delta_\alpha} \to \kappa > 0$. We have

$$0 < C(H, i_0, n, \bar{r})\, d\theta^2 \leq h_\delta \leq C(H, i_0, n, \bar{r})\, d\theta^2.$$

Sublemma 5.18. *There exists a subsequence of* $\{\phi_k^* \bar{g}_k(r)\}$ *and* $h(r) = (1/\kappa)\, d\theta^2$, *such that for this subsequence*

$$|\phi_k^* \bar{g}_k(r) - h(r)|_{h(r)} \to 0 \quad on\ S,$$

i.e., such a subsequence converges uniformly to $h(r)$ *on* S.

Proof. Note that $h_{\delta_\alpha} \to h$ uniformly on S. For any given $\varepsilon > 0$, we have for a small $\delta_\alpha > 0$,

$$(5.19) \qquad |h_{\delta_\alpha} - h|_h < \varepsilon.$$

For simplicity, we rewrite $\bar{g}_k(r)$ as $\phi_k^* \bar{g}_k(r)$, consider that ϕ_k is fixed, and use ϕ_k to define a new polar coordinate on $S(k)$, i.e., we compose ϕ_k with x's of polar coordinate $\{r, x^i\}$. Then in the new coordinate, $\phi_k^* \bar{g}_k(s)$ changes to $g_k(s)$ for all $s \leq i_0/2$. From Proposition 1.11, we have

$$\int_0^{i_0/4} r^2 \left|\frac{\partial}{\partial r} \bar{g}_k\right|^2_{\bar{g}_k} dr \leq C(H, i_0, n),$$

which implies

$$\begin{aligned}\int_r^{r_k} \left|\frac{\partial}{\partial r} \bar{g}_k\right|_{\bar{g}_k} dr &\leq \frac{1}{r} \int_r^{r_k} r \left|\frac{\partial}{\partial r} \bar{g}_k\right| dr \\ &\leq \frac{1}{\bar{r}} \left(int_r^{r_k} r^2 \left|\frac{\partial}{\partial r} \bar{g}_k\right|^2 dr\right)^{1/2} |r_k - r|^{1/2} \\ &\leq C(H, i_0, n, \bar{r}) |r_k - r|^{1/2}.\end{aligned}$$

We also take $\delta_\alpha > 0$ small such that

$$C(H, i_0, n, \bar{r})|r_k - r|^{1/2} < \varepsilon,$$

and fix such δ_α. For k large, from (5.17) we have

$$|\bar{g}_k(r_k) - h_{\delta_\alpha}|_h \leq C|\bar{g}_k(r_k) - h_{\delta_\alpha}|_{h_{\delta_\alpha}} \leq \varepsilon.$$

This implies that $|\bar{g}_k(r_k) - h|_h < 2\varepsilon$ for large k, and that the $\bar{g}_k(r_k)$ are equivalent to h for large k, so that

$$\int_r^{r_k} \left|\frac{\partial}{\partial r}\bar{g}_k\right|_h dr \leq C(H, i_0, n, \bar{r}) \int_r^{r_k} \left|\frac{\partial}{\partial r}\bar{g}_k\right|_{\bar{g}_k} dr$$
$$\leq C|r_k - r|^{1/2} < \varepsilon$$

and therefore

$$|\bar{g}_k(r_k) - \bar{g}_k(r)|_h \leq \int_r^{r_k} \left|\frac{\partial}{\partial r}\bar{g}_k\right|_h < \varepsilon.$$

From this and (5.19), one has

$$|\bar{g}_k(r) - h|_h < 3\varepsilon,$$

which proves Sublemma 5.18.

Remark 5.18(a). For each $r \in [\bar{r}, i_0/4]$, we have

$$0 < C(H, i_0, n, \bar{r})\, d\theta^2 \leq h(r) \leq C(H, i_0, n, \bar{r})\, d\theta^2.$$

Sublemma 5.20. *We can choose $\phi_k(r)$, such that*

$$h(r) = \left(\frac{S(r)^2}{S(\rho)^2}\right) h(\rho)$$

for $r \in [\bar{r}, i_0/4]$ and $\rho = i_0/8$.

Proof. Now we take $\rho = i_0/8$; then

$$|\phi_k^* \bar{g}_k(\rho) - h(\rho)|_{h(\rho)} \to 0,$$

where each ϕ_k depends on $\bar{g}_k(\rho)$. For each k, we use ϕ_k to define a polar coordinate on $B_{2\rho}(y_k)$, and write $\bar{g}_k(r)$ as $\phi_k^* \bar{g}_k(r)$ for all $r \leq i_0/4$. First we have

$$0 < C(H, i_0, n, \bar{r})\, d\theta^2 \leq \bar{g}_k(r) \leq C(H, i_0, n, \bar{r})\, d\theta^2.$$

Secondly, from Lemma 4.12 and Remark 4.12(a), noting that $B_k(r) = -\frac{1}{2}\frac{\partial}{\partial r}\bar{g}_k(r)$, we obtain

$$\max_{\bar{r} \leq r \leq i_0/4} \int_{S_r(y_k)} \left|\frac{\partial}{\partial r}\bar{g}_k(r) + 2b(r)\bar{g}_k(r)\right|_{\bar{g}_k}^2 d\bar{g}_k \leq C(H, i_0, n, \bar{r})\sigma\left(\frac{1}{k}\right).$$

(Clearly, we can choose y_k, such that this and (5.8) are satisfied.) From this it follows that

$$\int_{S_r(y_k)} \left|\frac{\partial}{\partial r}\bar{g}_k(r) + 2b(r)\bar{g}_k(r)\right|^2_{d\theta^2} d\theta \le C(H, i_0, n, \bar{r})\sigma\left(\frac{1}{k}\right),$$

where $d\theta$ is the volume element of $(S_1, d\theta^2)$. Now let

$$h_k(r) = \bar{g}_k(r) - \left(\frac{S(r)^2}{S(\rho)^2}\right) h(\rho),$$

where

$$S(r) = \begin{cases} \sin r & \text{if } \Delta = 1, \\ r & \text{if } \Delta = 0, \\ \sin hr & \text{if } \Delta = -1. \end{cases}$$

We have

$$\frac{\partial}{\partial r}\left(e^{-2\int_r^\rho b(r)\,dr} h_k(r)\right) = e^{-2\int_r^\rho b(r)\,dr}\left(\frac{\partial}{\partial r}\bar{g}_k + 2b(r)\bar{g}_k\right).$$

If we now integrate over $r \in [\bar{r}, i_0/4]$, we then obtain

$$\begin{aligned}\left|e_r^{-2\int_r^\rho b(r)\,dr} h_k(r) - h_k(\rho)\right| &\le C\int_r^{i_0/4}\left|\frac{\partial}{\partial r}\bar{g}_k(r) + 2b(r)\bar{g}_k(r)\right| dr \\ &\le C\left(\int_r^{i_0/4}\left|\frac{\partial}{\partial r}\bar{g}_k + 2b(r)\bar{g}_k\right|^2 dr\right)^{1/2},\end{aligned}$$

that is,

$$\left|e^{-2\int_r^\rho b(r)\,dr} h_k(r) - h_k(\rho)\right|^2 \le C\int_{\bar{r}}^{i_0/4}\left|\frac{\partial}{\partial r}\bar{g}_k + 2b(r)\bar{g}_k\right|^2 dr.$$

By integrating over S_1, we deduce

$$\begin{aligned}\int_S\left|e^{-2\int_r^\rho b(r)\,dr} h_k(r) - h_k(\rho)\right|^2 d\theta &\le C\int_{\bar{r}}^{i_0/4}\int_S\left|\frac{\partial}{\partial r}\bar{g}_k + 2b(r)\bar{g}_k\right|^2 d\theta\,dr \\ &\le C(H, i_0, n, \bar{r})\sigma\left(\frac{1}{k}\right),\end{aligned}$$

and thus

$$\int_S |h_k(r)|^2_{d\theta^2}\,d\theta \le C\int_S |h)k(\rho)|^2\,d\theta + C\sigma\left(\frac{1}{k}\right).$$

Noting that $|h_k(\rho)| = |\bar{g}_k(\rho) - h(\rho)| \to 0$, we then obtain

(5.19)(a) $$\int_S |h_k(r)|^2\,d\theta \to 0$$

and for each $r \in [\bar{r}, i_0/4]$, $\bar{g}_k(r)$ converges to $(S(r)^2/S(\rho)^2)h(\rho)$ almost everywhere. On the other hand, $\bar{g}_k(r)$ converges uniformly to $h(r)$ (up to diffeomorphisms). We have

$$\mathrm{Vol}(h(r)) = \mathrm{Vol}\left(\left(\frac{S(r)^2}{S(\rho)^2}\right) h(\rho)\right).$$

This clearly implies

$$h(r) = \left(\frac{S(r)^2}{(S(\rho)^2}\right) h(\rho),$$

and proves Sublemma 5.20.

Sublemma 5.21. *$h(r) = S(r)^2\, d\theta^2$ for $r \in [\bar{r}, i_0/4]$.*

Proof. All we need is to show that for one $r \in [\bar{r}, i_0/4]$, $h(r) = S(r)^2\, d\theta^2$.

To this end, we consider the evolution equation again:

$$\frac{\partial}{\partial r} h_k = -2\,\mathrm{Ric}(h_k),$$

where $h_k(0) = \bar{g}_k(r_k)$, and where we take $r = i_0/8$ and r_k as in Sublemma 5.6 for $\delta = 1$. We have

$$|\phi_k^* \bar{g}_k(r_k) - h_1| \to 0.$$

Now recalling (5.10)(b), we find

$$\int_S \left| R(\bar{g}_k(r_k)) - (n-1)(n-2)\left(\frac{1}{S(r_k)^2}\right)\right| d\bar{g}_k \le C(H, i_0, n, \bar{r})\sigma\left(\frac{1}{k}\right).$$

If we denote the scalar curvature of h_k by $R(h_k)$, we have

$$\int_S \left| R(h_k)(0) - \frac{(n-1)(n-2)}{S(r_k)^2}\right| dh_k(0) \le C\sigma\left(\frac{1}{k}\right). \tag{5.22}$$

We also have from (5.10)(a) and Theorem 2.11,

$$\int_S |R_m(h_k)|^{n/2}\, dh)k \le C(H, i_0, n, \bar{r})$$

for $0 \le t \le T = T(H, i_0, n, \bar{r})$.

Note that

$$\frac{\partial}{\partial t} R(h_k) = \Delta R(h_k) + 2|\,\mathrm{Ric}(h_k)|^2,$$

and that

$$\frac{\partial}{\partial t}\int R(h_k)\, dh_k = \int (2|\,\mathrm{Ric}(h_k)|^2 - R(h_k)^2)\, dh_k.$$

Then

$$\left|\frac{\partial}{\partial t}\int R(h_k)\,dh_k\right| \le C(H, i_0, n, \bar{r}),$$

which implies that

$$\left|\int R(h_k)\,dh)k - \int R(\bar{g}_k)(r_k)\right| \le C(H, i_0, n, \bar{r})t.$$

Combining this with (5.22) gives

$$\left|\int R(h_k)\,dh_k - \frac{(n-1)(n-2)}{s(r_k)^2}\operatorname{Vol}(\bar{g}_k(r_k)\right| \le C(H, i_0, n, \bar{r})t + C\sigma\left(\frac{1}{k}\right).$$

Passing to a subsequence if necessary, we may assume that $r_k \to R_0$, $i_0/8 \le r_0 \le i_0/4$. Noting that

$$|\operatorname{Vol}(\bar{g}_k(r_k) - \operatorname{Vol}(h_k)| \le C(H, i_0, n, \bar{r})t^{1/n},$$

we then have

$$\left|\int R(h_k)\,dh_k - \frac{(n-1)(n-2)}{s(r_k)}\operatorname{Vol}(h_k)\right| \le Ct^{1/n} + C\sigma\left(\frac{1}{k}\right).$$

Taking a subsequence, and then letting $k \to \infty$, keeping in mind that such a subsequence converges to h_1 in C^2-topology, we obtain

$$\left|\int R(h_1)\,dh_1 - \frac{(n-1)(n-2)}{s(r_k)^2}\operatorname{Vol}(h_1)\right| \le Ct^{1/n}.$$

We observe that $h-1$ is a constant sectional curvature metric on S, and $h_1(t) = (1/\kappa_1(t))\,d\theta^2$, so that

$$\left|\kappa_1(t) - \frac{1}{s(r_0)^2}\right| \le Ct^{1/n}.$$

Letting $t \to 0$, we have $h_1(t) \to s(r_0)^2\,d\theta^2$. As in the proof of Sublemma 5.6, this implies that

$$|\phi_k^* \bar{g}_k(r_0) - S(r_0)^2\,d\theta^2| \to 0,$$

and therefore $h(r_0) = s(r_0)^2\,d\theta^2$. Hence Sublemma 5.21 is proved.

Now we are ready to prove Lemma 5.5.

Proof of Lemma 5.5. From Sublemma 5.6 through Sublemma 5.21, there exists a subsequence of $\{\bar{g}_k\}$ which converges to $g(r) = h(r) = s(r)^2\,d\theta^2$ for all $r \in [\bar{r}, i_0/4]$. We can apply this to any subsequence of a subsequence of $\{\bar{g}_k\}$, which all have the same limit. This clearly implies that the sequence $\{\bar{g}_k\}$ itself converges to $g(r) = s(r)^2\, d\theta^2$ for each $r \in [\bar{r}, i_0/4]$, i.e.,

$$|\phi_k^* \bar{g}_k(r) - g(r)|_{g(r)} \to 0 \quad \text{on } S, \tag{5.23}$$

which proves Lemma 5.5.

Since we can take any $\bar{r}$ $(0 < \bar{r} < i_0/4)$, as an immediate consequence, we have the following.

Corollary 5.24. *For any $\bar{y} \in \overline{M}$, we have*

$$S_r(\bar{y}) = (S_1, g(r)) = (S_1, s(r)^2\, d\theta^2),$$

$0 < r \leq i_0/4$.

Estimate (5.19)(a) also gives the following.

Corollary 5.25. *Let $\bar{y} \in \overline{M}$, and $y_k \in M_k$ as above. For a proper choice of polar coordinate on $B_{i_0}(y_k)$, we have*

$$\int_S |\bar{g}_k(r) - g(r)|^2_{g(r)}\, dg(r) \to 0$$

uniformly for $r \in [\bar{r}, i_0/4]$. In particular $\bar{g}_k(r)$ converges to $g(r)$ almost everywhere for each $r \in [\bar{r}, i_0/4]$.

Proof. Corollary 5.25 follows from (5.19)(a) and Sublemma 5.21.

Corollary 5.24 says that any metric sphere $S_r(\bar{y})$ of $\overline{M}$ for $0 < r \leq i_0/4$ is the Euclidean sphere $(S_1, g(r))$, which almost implies that $B_{i_0/4}(\bar{y})$ is isometric to $(B_{i_0/4}(0), g)$, where $B_r(0) = \{x \in R^n,\ |x| \leq r\}$ and $g = dr^2 + s(r)^2\, d\theta^2$, i.e., $B_{i_0/4}(\bar{y})$ is isometric to the geodesic ball of radius $i_0/4$ in the constant sectional curvature space form. The next few paragraphs are devoted to the proof of this fact.

Lemma 5.26. *Let $\bar{y} \in \overline{M}$ and $y_k \in M_k$ be as above. Let $\phi_k : S_1 \to S_1$ be as in Lemma 5.5 for each fixed $r \in [\bar{r}, i_0/4]$. Then we have*

$$|\phi_k^* \bar{g}_k(r) - g(r)|_{g(r)} \to 0 \quad \text{on } S_1.$$

If $\bar{d}$ denotes the distance function of $(S_1, g(r))$, then for any $p, q \in S_1$

$$\bar{d}(\phi_k(p), \phi_k(q)) \to \bar{d}(p, q) \quad \text{and} \quad \bar{d}(\phi_k^{-1}(p), \phi_k^{-1}(q)) \to \bar{d}(p, q).$$

Here we agree to take the proper polar coordinate on $B_{i_0}(y_k)$, such that

$$|\bar{g}_k(\rho) - g(\rho)|_{g\rho)} \to 0 \quad \text{on } S, \quad \rho = i_0/8.$$

Proof. From Corollary 5.25,

$$\int_S |\bar{g}_k(r) - g(r)|^{n/2}\, dg(r) \to 0 \tag{5.27}$$

uniformly for $r \in [\bar{r}, i_0/4]$. Note that

$$0 < C(H, i_0, n) g(\rho) \le \bar{g}_k(\rho) \le C(H, i_0, n) g(\rho).$$

This and Proposition 1.11 imply

$$0 < C(H, i_0, n, \bar{r})\, d\theta^2 \le \bar{g}_k(r) \le C(H, i_0, n, r)\, d\theta^2. \tag{5.28}$$

From this, we have

$$|D\phi_k| + |D\phi_k^{-1}| \le C(H, i_0, n, \bar{r}) \tag{5.29}$$

where $D\phi_k$ is the tangential map of ϕ_k, the pointwise norm $|D\phi_k|$ is taken with respect to any one of $d\theta^2$, $g(r)$ or $\bar{g}_k(r)$.

Now (5.27), (5.28), and (5.29) imply

$$\begin{aligned}
&\int_S |\phi_k^* g(r) - g(r)|^2\, dg(r) \\
&\quad \le C \int_S |\phi_k^* g(r) - \phi_k^* \bar{g}_k|^2\, dg(r) + C \int_S |\phi_k^* \bar{g}_k(r) - g(r)|^2\, dg(r) \\
&\quad \le C(H, i_0, n, \bar{r}) \int_S |\bar{g}_k(r) - g(r)|^2\, dg + C \int_S |g(r) - \phi_k^* g_k(r)|^2\, dg,
\end{aligned}$$

and therefore

$$\int_S |\phi_k^* g(r) - g(r)|^2\, dg(r) \to 0. \tag{5.30}$$

Similarly

$$\begin{aligned}
&\int_S |\phi_k^{-1*} g(r) - g(r)|^2\, dg(r) \\
&\quad \le C \int_S |\phi_k^{-1*} g(r) - \bar{g}_k(r)|^2\, dg(r) + C \int_S |\bar{g}_k(r) - g(r)|^2\, dg \\
&\quad \le C(H, i_0, n, \bar{r}) \int_S |g(r) - \phi_k^* \bar{g}_k(r)|^2\, dg + C \int_S |\bar{g}_k(r) - g_k(r)|^2\, dg
\end{aligned}$$

and thus

$$\int_S |(\phi_k^{-1})^* g(r) - g(r)|^2\, dg \to 0. \tag{5.31}$$

Clearly, Lemma 5.26 will follow from the following.

Sublemma 5.32. *Let $\phi_k \colon S_1 \to S_1$ be a diffeomorphism, such that*

$$\int_S |\phi_k^* g(r) - g(r)|^2\, dg \to 0 \quad \text{on } S_1,$$

$$|D\phi_k| + |D\phi_k^{-1}| \le C(H, i_0, n, \bar{r}).$$

If $\phi_k \xrightarrow{C^0} \phi$ *on* S_1, *then for any* $p, q \in S_1$, *we have*

$$\bar{d}(\phi(p), \phi(q)) \leq \bar{d}(p, q).$$

Proof. Let $h_k = \phi_k^* g(r)$. Then h_k converges to $g(r)$ almost everywhere on S_1.

Let γ be the minimal geodesic from p to q in $(S, g(r))$. We have the formula for the length $L(\gamma)$ of γ:

$$L(\gamma) = \bar{d}(p, q).$$

Given $\varepsilon > 0$, we take two totally geodesic discs D and D' in the sphere $(S, g(r))$ through p and q with center p and q and the same radius $\leq \varepsilon$, which are perpendicular to γ. It is obvious that there are parallel geodesics connecting each point of D to the corresponding point of D'. These geodesics define a cylinder Σ in S_1 with axis γ. Clearly, $\mathrm{Vol}(\Sigma) > 0$, and $h_k \to g(r)$ almost everywhere on Σ, and the Fubini Theorem implies that there is a parallel geodesic γ' in Σ, such that $h_k \to g(r)$, almost everywhere on γ'. In fact, we have

$$\int_{\gamma'} |h_k - g|^{n/2} \to 0.$$

Note that

$$L(\gamma') \leq L(\gamma) + \varepsilon \leq \bar{d}(p, q) + \varepsilon.$$

Now $h_i \to g(r)$ on γ' implies that for large k,

$$L_k(\gamma') \leq L(\gamma) + \varepsilon \leq d(p, q) + 2\varepsilon,$$

where $L_k(\gamma)$ is the length of γ' in (S_1, h_k), and hence

$$L(\phi_k(\gamma')) \leq \bar{d}(p, q) + 2\varepsilon,$$

$$\bar{d}(\phi_k(\gamma'(0)), \phi_k(\gamma'(1))) \leq \bar{d}(p, q) + 2\varepsilon.$$

Since $\bar{d}(p, \gamma'(0) < \varepsilon$ and $\bar{d}(q, \gamma'(1)) < \varepsilon$, we have

$$\bar{d}(\phi_k(p), \phi_k(\gamma'(0))) < C\varepsilon, \qquad \bar{d}(\phi_k(q), \phi_k(\gamma'(1))) < C\varepsilon.$$

Therefore, the triangle inequality gives

$$\bar{d}(\phi_k(p), \phi_k(q)) \leq \bar{d}(p, q) + 2\varepsilon + 2C\varepsilon.$$

Letting $k \to \infty$, we thus obtain

$$\bar{d}(\phi(p), \phi(q)) \leq \bar{d}(p, q) + 2\varepsilon + 2C\varepsilon.$$

Since ε is arbitrary and C is independent of ε, this proves Sublemma 5.32.

Now we can finish the proof of Lemma 5.26. We apply Sublemma 5.32 to any subsequence of $\{\phi_k\}$ and $\{\phi_k^{-1}\}$.

For if ϕ is a limit of any subsequence of $\{\phi_k\}$, then ϕ^{-1} is also a limit of a subsequence of $\{\phi_k^{-1}\}$. We have

$$\bar{d}(\phi(p), \phi(q)) \leq \bar{d}(p, q),$$

$$\bar{d}(p, q) = \bar{d}(\phi^{-1}\phi(p), \phi^{-1}\phi(q)) \leq \bar{d}(\phi(p), \phi(q)).$$

Hence

$$\bar{d}(p, q) = \bar{d}(\phi(p), \phi(q)).$$

Since all of them have the same limit $\bar{d}(p, q)$, this clearly implies Lemma 5.26.

Remark 5.33. If $d(r)$ denotes the distance function for $g(r)$, then we clearly have

$$d(r)(\phi_k(p), \phi_k(q)) \to d(r)(p, q) \quad \text{for all } 0 < r < i_0/4$$

(since $g(r)$ are constant multiples of each other). Note that ϕ_k depends on $r \in [\bar{r}, i_0/4]$.

Corollary 5.34. *For a fixed $r \in [\bar{r}, i_0/4]$, let $\bar{d}_k$ denote the distance function of $(S, \bar{g}_k(r))$. Then for any $p, q \in S$,*

$$\lim_{k\to\infty} \bar{d}_k(p, q) = \bar{d}(p, q).$$

Proof. First we have $|\phi_k^* \bar{g}_k(r) - g(r)|_g \to 0$ on S_1 and hence $|\bar{g}_k(r) - \phi_k^{-1*} g(r)|_g \to 0$ on S_1. Therefore,

$$\begin{aligned}|\bar{d}_k(p, q) - d(p, q)| \leq |\bar{d}_k(p, q) - \bar{d}(\phi_k^{-1}(p), \phi_k^{-1}(q))| \\ + |\bar{d}(\phi_k^{-1}(p), \phi_k^{-1}(q)) - \bar{d}(p, q)| \to 0,\end{aligned}$$

which finishes the proof of Corollary 5.34.

Now we want to prove a similar fact for the metrics $g_k = dr^2 + \bar{g}_k(r)$ on $B_{i_0/4}(y_k) - B_{2\bar{r}}(y_k)$.

Lemma 5.35. *If d_k and d denote the distance function of g_k and $g = dr^2 + g(r)$, then for any $p, q \in B_{i_0/4}(y_k) - B_{2\bar{r}}(y_k) = \Omega_k$ and $d(p, q) \leq \bar{r}$,*

$$\lim_{k\to\infty} d_k(p, q) = d(p, q).$$

First, for any $\phi_k: S \to S$ as above and $r \in [\bar{r}, i_0/4]$, we have

$$|\phi_k^* \bar{g}_k(r) - g(r)|_{g(r)} \to 0 \quad \text{on } S_1.$$

We define diffeomorphisms $\psi_k(r)\colon \Omega_k \to \Omega_k$ by $\psi_k(r)(t, p) = (t, \phi_k(r))$; here we identify Ω_k with $[2\bar{r}, i_0/4] \times S$, and r is considered as a fixed number. It is easy to see that from Lemma 5.26 we obtain

Sublemma 5.36. *For $p, q \in \Omega_k$,*

$$\lim_{k\to\infty} d(\psi_k^{-1}(r)(p), \psi_k^{-1}(r)(q)) = d(p, q).$$

Sublemma 5.37. *For $p, q \in \Omega_k$ and $d(p, q) \le \bar{r}$,*

$$d(p, q) \le \inf \operatorname{Lim} d_k(p, q).$$

Proof. First we have

$$\int_0^{i_0/4} r^2 \left|\frac{\partial}{\partial r}\bar{g}_k\right|^2 dr \le C(H, i_0, n),$$

and hence for r, $r_1 \in [\bar{r}, o_0/4]$,

$$\begin{aligned}
|\bar{g}_k(r_1) - \bar{g}_k(r)|_{\bar{g}_k(r)} &\le \frac{1}{r}\int_r^{r_1} r\left|\frac{\partial}{\partial r}\bar{g}_k\right|_{\bar{g}_k(r)} dr \\
&\le C(H, i_0, n, \bar{r})\left(\int_r^{r_1} r^2\left|\frac{\partial}{\partial r}\bar{g}_k\right|^2 dr\right)^{2/2} |r_1 - r|^{1/2} \\
&\le C(H, i_0, n, \bar{r})|r_1 - r|^{1/2}.
\end{aligned}$$

This implies that for $\phi_k\colon S \to S$ as above,

$$|\phi_k^*\bar{g}_k(r_1) - \phi_k^*\bar{g}_k(r)|_{g(r)} \le C(H, i_0, n, \bar{r})|r_1 - r|^{1/2}.$$

Given any $\varepsilon > 0$, for large k we have

$$|\phi_k^*\bar{g}_k(r) - g(r)|_{g(r)} < \varepsilon,$$

and therefore

$$|\phi_k^*\bar{g}_k(r_1) - g(r)|_{g(r)|} < \varepsilon + C|r_1 - r|^{1/2}.$$

Taking $|r_1 - r|^{1/2} < \varepsilon$, we thus obtain

$$|\psi_k(r)^* g_k - g|_g < \varepsilon + C\varepsilon$$

on $[r, \varepsilon^2 + r] \times S$. Dividing $[\bar{r}, i_0/4]$ to finite number intervals $\bar{r} = r_0 < r_1 < \cdots < r_m = i_0/4$, with $r_{i+1} = r_i = \varepsilon^2$, we have

$$|\psi_k(r_i)^* g_k - g|_g < \varepsilon + C(H, i_0, n, \bar{r})\varepsilon$$

on $[r_i, r_{i+1}] \times S \subset \Omega_k$.

Sublemma 5.38. *For any $\delta < 0$ there exists $N > 0$, such that if $k > N$, then*

$$|d(\psi_k(r)^{-1}(p), \psi_k(r)^{-1}(q)) - d(p, q)| < \delta C(H, i_0, n, \bar{r})$$

for all $p, q, \in \Omega_k$.

Proof. Letting $d^k(p, q) = d(\psi_k(r)^{-1}(p), \psi_k(r)^{-1}(q))$ and noting that

$$|D\psi_k| + |D\psi_k^{-1}| \le C(H, i_0, n, \bar{r}),$$

we have

$$|d^k(p, q) - d^k(p', q')| \le C(H, i_0, n, \bar{r})[d(p, p') + d(q, q')].$$

This implies that for each $(p, q) \in \overline{\Omega}_k \times \overline{\Omega}_k$, there exists a small neighborhood $O(p, q) = \{(p', q'), d(p, p') + d(q, q') < \delta\}$ such that

$$|d^k(p', q') - d^k(p, q)| \le C\delta,$$
$$|d(p', q') - d(p, q)| \le \delta.$$

The family of open sets $\{O_{p,q}\}$ covers $\overline{\Omega}_k \times \overline{\Omega}_k$, and then there is a finite covering $\{O(p_i, q_i), i = 1, 2, \cdots, l\}$ of $\overline{\Omega}_k \times \overline{\Omega}_k$. Taking N large, such that for each (p_i, q_i), we have Sublemma 5.36,

$$|d^k(p_i, q_i) - d(p_i, q_i)| < \delta \quad \text{for } k \ge N.$$

Thus for any $(p, q) \in \overline{\Omega}_k \times \overline{\Omega}_k$, there is a (p_i, q_i), such that $(p, q) \in O(p_i, q_0)$ and

$$|d^k(p, q) - d^k(p_i, q_i)| < C\delta, \qquad |d(p, q) - d(p_i, q_i)| < \delta,$$

which implies that for $k \ge N$,

$$|d^k(p, q) - d(p, q)| < C\delta \le C(H, i_0, n, \bar{r})\delta.$$

We now return to the proof of Sublemma 5.37. Taking $\delta = \varepsilon^3$ and $N(r)$ large, for $k \ge N$, we have

$$|d(\psi_k(r)^{-1}(p), \psi_k(r)^{-1}(q)) - d(p, q)| \le C(H, i_0, n, \bar{r})\varepsilon^3.$$

We take $N = \max\{N(r_i)\}$; then for $k \ge N$,

$$|d(\psi_k(r_i)^{-1}(p)\psi_k(r_i)^{-1}(q)) - d(p, q)| \le C\varepsilon^3$$

for all (p, q).

We now take a $k \ge N$ such that

$$d_k(p, q) \le \inf \operatorname{Lim} d_k(p, q) + \varepsilon,$$

and Sublemma 5.38 is satisfied for all r_i. We now fix such a $k \geq N$, and let γ be a curve from p to q in $B_{i_0/4}(y_k)$ such that

$$L_k(\gamma) \leq d_k(p, q) + \varepsilon \leq \inf\operatorname{Lim} d_k(p, q) + 2\varepsilon,$$

where $L_k(\gamma)$ is the length of γ in (Ω_k, g_k). Let p_i be the intersection of γ with $\{r_i\} \times S \subset \Omega_k$ (if they have more than one intersection, take p_i be one of them). Then $d^k(p_i, p_{i+1}) \geq \varepsilon^2$ and

$$|d^k(p_i, p_{i+1}) - d(p_i, p_{i+1})| \leq C\varepsilon\, d_k(p_i, p_{i+1}),$$

that is,

$$d(p_i, p_{i+1}) \leq d^k(p_i, p_{i+1}) + C\varepsilon\, d_k(p_i, p_{i+1}). \tag{5.39}$$

By Sublemma 5.38, we have

$$|g_k - \psi_k(r_i)^{-1*} g|_g \leq C(H, i_0, n, \bar{r})\varepsilon,$$

which implies that

$$\begin{aligned} d_k(p_i, p_{i+1}) &\geq d(\psi_k(r_i)^{-1}(p_i), \psi_k(r_i)^{-1}(p_{i+1})) - C\varepsilon\, d_k(p_i, i_{i+1}) \\ &\geq d^k(p_i, p_{i+1}) - C\varepsilon\, d_k(p_i, p_{i+1}). \end{aligned} \tag{5.40}$$

Therefore,

$$\begin{aligned} \sum d(p_i, p_{i+1}) &\leq \sum d^k(p_i, p_{i+1}) + C\varepsilon \sum d^k(p_i, p_{i+1}) \\ &\leq \sum d^k(p_i, p_{i+1}) + C\varepsilon \sum d_k(p_i, p_{i+1}) \\ &\leq L_k(\gamma) + C\varepsilon \sum d_k(p_i, p_{i+1}) \\ &\leq \inf\operatorname{Lim}\, d_k(p, q) + 2\varepsilon + C\varepsilon(\inf\operatorname{Lim}\, d_k(p, q)), \end{aligned}$$

and hence

$$d(p, q) \leq \inf\operatorname{Lim}\, d_k(p, q) + 2\varepsilon + C\varepsilon(\inf\operatorname{Lim}\, d_k(p, q)).$$

Since $\varepsilon > 0$ is arbitrary, we have $d(p, q) \leq \inf\operatorname{Lim}\, d_k(p, q)$, which proves Sublemma 5.37.

Lemma 5.41. *For $p, q \in \Omega_k$,*

$$\operatorname{Sup}\lim\, d_k(p, q) \leq d(p, q),$$

and thus

$$\lim_{k\to\infty} d_k(p, q) = d(p, q).$$

Proof. The proof is similar to that of Sublemma 5.32. Let γ be the minimal curve from p to q in $\overline{M}$, such that

$$L(\gamma) = d(p, q).$$

Note that $g_k \to g$ almost everywhere. We can find a nearby parallel geodesic γ' of γ in Ω_k, such that for any preassigned $\varepsilon > 0$

$$L(\gamma') \leq L(\gamma) + \varepsilon, \quad d(p, \gamma'(0)) < \varepsilon, \quad d(q, \gamma'(1)) > \varepsilon,$$

and $g_k \to g$ almost everywhere on γ'. Thus $L_k(\gamma') \to L(\gamma')$ and for large k,

$$d_k(\gamma'(0), \gamma'(1)) \leq L_k(\gamma') \leq L(\gamma') + \varepsilon,$$

$$d_k(p, q) \leq L(\gamma') + 3\varepsilon \leq L(\gamma) + 4\varepsilon \leq d(p, q) + 4\varepsilon.$$

Since $\varepsilon > 0$ is preassigned, we have

$$\sup\lim d_k(p, q) \leq d(p, q).$$

Now we are ready to prove the main lemma of this section.

Lemma 5.42. *For any $\bar{y} \in \overline{M}$, $B_{i_0/4}(\bar{y})$ is isometric to $(B_{i_0/4}(0), g)$, where $B_{i_0/4}(0) = \{x \in R^n; |x| \leq i_0/4\}$ and $g = dr^2 + s(r)^2 d\theta^2$, i.e., $(B_{i_0/4}(0), g)$ is the geodesic ball of radius $i_0/4$ in the constant sectional curvature space form.*

We start with the following Lemma of [16]. We write it in the form that we need, so that the easy proof is also provided.

Sublemma 5.43. *Let $\{X_i, i = 1, 2, \ldots\}$ and X be compact metric spaces with*

$$\sup\{\operatorname{diam}(X_i), \operatorname{diam} X\} \leq D.$$

If for each $\varepsilon > 0$, there exists a 2ε-net of X which is the limit of a sequence of a 4ε-net $N_i \subset X_i$ in Lipschitz distance, i.e., there exists a 2ε-net $\{x_p\}_{p\in P} \subset X$ and 4ε-net $N_i = \{y_p^i\}_{p\in P} \subset X_i$ such that

$$\sup_{p, q\in P} \left| \ln \frac{d^{X_i}(y_p^i, y_q^i)}{d^X(x_p, x_q)} \right| \to 0 \quad as\ i \to \infty,$$

then $X_i \xrightarrow{H} X$.

Proof. Let $Z_i = X \cup X_i$, and define the metric d on Z_i such that d is the same with d^X on X and with d^{X_i} on X_i, and for $x \in X$, $y \in X_i$

$$d(x, y) = \inf_{p\in P}[d^X(x, x_p) + d^{X_i}(y_p^i, y)] + \varepsilon. \tag{5.44}$$

We need to show that d is a metric on Z_i for large i. Note that $\varepsilon > 0$ is a preassigned number.

(a) $d(x, y(\leq d(x, x') + d(x', y)$ or $d(x, y) \leq d(x, y') + d(y', y)$ for $x, x' \in X$ and $y, y' \in X_i$. This is clear from (5.44).

(b) $d(x, x') \le d(x, y) + d(y, x')$ or $d(y, y') \le d(y, x) + d(x, y')$ for $x, x' \in X$ and $y, y' \in X_i$.

Proof. We take i large, such that

$$\max\left\{\sup_{p,q\in P} \frac{d^X(x_p, x_q)}{d^{X_i}(y+p, y_q)}, \sup_{p,q\in P} \frac{d^{X_i}(y_P, y_q)}{d^X(x_p, x_q)}\right\} = 1 + \eta_i \le 1 + \left(\frac{1}{4D}\right)\varepsilon,$$

and write $d(x, x') \le d(x, x_p) + d(x_p, x_q) + d(x_q, x')$, so we have

$$\begin{aligned} d(x, x') &\le d(x, x_p) + \left(1 + \frac{1}{4D}\right) d(y_p^i, y_q^i) + d(x_q, x') \\ &\le d(x, x_p) + d(y_p^i, y) + (d(y, y_q^i) + d(x_q, x')) + \varepsilon. \end{aligned}$$

This implies

$$d(x, x') \le d(x, y) + d(y, x').$$

Similarly, we have

$$D(y, y') \le d(y, x) + d(x, y'),$$

so (Z_i, d) is a metric space. Clearly, we have $d_H^Z i(X, X_i) < 5\varepsilon$; that is, $X_i \xrightarrow{H} X$.

Proof of Lemma 5.42. Let $X = (B_{i_0/4}(0) - \overset{\circ}{B}_{2\bar r}(x), g)$ and $X_k = (B_{i_0/4}(y,) - \overset{\circ}{B}_{2\bar r}(y_k), g_k)$. We claim that $X_k \xrightarrow{H} X$.

By Lemma 5.42, we identify X_k with $(\Omega(\bar r), g_k)$, where $\Omega(\bar r) = B_{i_0/4}(0) - \overset{\circ}{B}_{2\bar r}(0) = \{x \in R^n, 2\bar r \le |x| \le i_0/4\}$. Let $\{x_p\}_{p\in P}$ be a maximal subset of X, such that $d(x_p, x_q) \ge \varepsilon$. Then $\{x_p\}$ is a 2ε-net of X. We know that $\{y_p^k = x_p\}_{p\in P}$ is a 4ε-net of X_k for large k by Lemma 5.41. We also have

$$\sup_{p,q}\left|\ln \frac{d^X(x_p, x_q)}{d^{X_k}(y_p, y_q)}\right| \to 0 \quad \text{as } k \to \infty.$$

By Sublemma 5.43, we obtain $d_H(X, X_k) \le 5\varepsilon$, which implies that $X_k \xrightarrow{H} X$. On the other hand, from §3 it follows that

$$X_k \xrightarrow{H} (B_{i_0/4}(\bar y) - B - 2r(\bar y)).$$

Therefore, $B_{i_0/4}(\bar y) - \overset{\circ}{B}_{2\bar r}(\bar y) = (\Omega(\bar r), g)$. Since $\bar r$ can be any positive number, we thus have

$$B_{i_0/4}(\bar y) - \{\bar y\} = (B_{i_0/4}(0) - \{0\}, g).$$

Now clearly we can extend the metric on $B_{i_0/4}(\bar{y}) - \{\bar{y}\}$ to $B_{i_0/4}(\bar{y})$, and we finally obtain

$$D_{i_0/4}(\bar{y}) = (B_{i_0/4}(0), g),$$

which proves Lemma 5.42.

We are now ready to return to the proof of Theorem 5.4. We have the limit $\overline{M}$; for each $\bar{y} \in \overline{M}$, $B_{i_0/4}(\bar{y})$ is isometric to the geodesic ball of the space form. This implies that $\overline{M}$ is a Riemannian manifold of constant sectional curvature Δ, since any distance preserving map of any two Riemannian manifolds is a smooth isometry [23]. This proves Theorem 5.4.

Corollary 5.45. *Let* (M_k, g_k) *be as in Theorem* 5.4 *and* $\Delta = 1$. *Then*

$$\sup\lim \operatorname{diam}(M_k) < \pi.$$

Proof. For any subsequence of (M_k, g_k),we can find a subsequence which converges to a Riemannian manifold $\overline{M}$ of constant sectional curvature 1, and this implies that $\operatorname{diam}(\overline{M}) \leq \pi$, and therefore that the diameters of such a subsequence are uniformly bounded. This clearly proves Corollary 5.45.

Remark 5.46. From Corollary 5.45 we may assume that the diameters of (M_k, g_k) are uniformly bounded.

We are finally ready to prove Theorems 5.1 and 5.2.

Proof of Theorem 5.1 *and* 5.2. If the theorems are false, then we have a sequence of manifolds (M_k, g_k), which satisfy the conditions of Theorem 5.4, and for each k, M_k is not homotopic to a manifold of constant sectional curvature Δ. From Corollary 5.45, in either case, we assume that

$$\operatorname{Sup} \operatorname{diam}(M_k, g_k) \leq d < \infty.$$

By Theorem 5.4, we know that $(M_k, g_k) \xrightarrow{H} (\overline{M}, g)$ (passing to a subsequence if necessary), where $(\overline{M}, g)$ is a Riemannian manifold with constant sectional curvature Δ. From the proofs above, we have $\operatorname{inj}(\overline{M}) \geq i_0/4$. The rest of the proof is very similar to [18]; since $\operatorname{inj}(M_k) \geq i_0$ and $M_k \xrightarrow{M} \overline{M}$, the proof is much easier.

First we imbed all M_k to a compact metric space X, with metric d, as in §3 such that the distance function d_k of M_k is the same as the induced distance from X. Then M_k converges to a subset G of X in Haussdorff distance (passing to a subsequence if necessary), and G is isometric to $\overline{M}$; we can identify $\overline{M}$ with G.

Given $\varepsilon > 0$ small, we take a minimal ε-net $\{p_i\}$ of M, $i = 1, 2, \cdots$, N, i.e.,

(i) The open balls $B(p_i, \varepsilon)$, $i = 1, 2, \cdots, N$, cover $\overline{M}$.

(ii) The open balls $B(p_i, \varepsilon/2)$, $i = 1, 2, \cdots, N$, are disjoint.

Since $\overline{M}$ has constant sectional curvature $\Delta = 1, 0$, or -1, the following lemma is well known ([18], [16]).

Lemma 5.47. *There is a constant N_1 depending only on n and d, so that any minimal ε-net $p_1, \cdots, p_N$ in $\overline{M}$ has the property: For any $y \in \overline{M}$, the ball $B(x, \varepsilon)$ intersects at most N_1 of the balls $B(p_1, \varepsilon), \cdots$, $B(p_N, \varepsilon)$.*

We now take a sequence $\{p_i^k\}$ for each $i = 1, 2, \cdots, N$, so that $p_i^k \in M_k \subset X$, and $d(p_i^k, p_i) \to 0$ as $k \to \infty$. Since $\bigcup B(p_i, \varepsilon) = \overline{M}$, clearly there is a $0 < \delta < \varepsilon$, such that $\bigcup B(p_i, \varepsilon - \delta) = \overline{M}$. We take k large, so that

$$d_H^X(M_k, \overline{M}) \ll \delta/2, \qquad \text{i.e.}$$

$$\overline{M} \subset U_{\delta/2}(M_k) \subset X, \qquad M_k \subset U_{\delta/2}(\overline{M}) \subset X, \tag{5.48}$$

$$d(p_i^k, p_i) < \delta/4, \qquad i = 1, 2, \cdots, N.$$

Then it is easy to see that

$$\bigcup B(p_i^k, \varepsilon) = M_k.$$

As above, we may assume that $d(p_i, p_j) > \varepsilon/2$, and no pair of $\{p_i\}$ satisfies $d(p_i, p_j) = \varepsilon$ by shrinking the balls a little bit if necessary. Then by taking k even larger, we have

$$B(p_i^k, \varepsilon/2) \cap B(p_j^k, \varepsilon/2) = \varnothing, \qquad i \neq j,\ i, j = 1, 2, \cdots, N.$$

Now note that $d(p_i, p_j) < \varepsilon$ if and only if $B(p_i \varepsilon) \cap B(p_j, \varepsilon) = \varnothing$. By taking k large enough, we have

$$d(p_i^k, p_j^k) < \varepsilon \quad \text{if and only if} \quad d(p_i, p_j) < \varepsilon,$$

$$d(p_j^k, p_j^k) > \varepsilon \quad \text{if and only if} \quad d(p_i, p_j) > \varepsilon.$$

This implies that the minimal ε-net $\{p_i^k,\ i = 1, 2, \cdots, N\}$ of M_k has the same intersection pattern as $\{p_i\}$ of M, for large k, i.e.,

$$B(p_k^i, \varepsilon) \cap B(P`p_i^k, \varepsilon) \neq \varnothing \quad \text{if and only if} \quad B(p_i, \varepsilon) \cap B(p_j, \varepsilon) \neq \varnothing.$$

As in [18], we define the notion of center of mass.

Let $q \in M = M_k$ or $\overline{M}$, and consider a continuous map $f\colon T \to M$, where T is a topological space and $d(f(t), q) < i_0/4$ for all $t \in T$. The map $Cf(t, s) = \gamma_t(s)$, where $\gamma_t(s)$ is the unique minimal geodesic from $f(t)$ to q, induces a continuous map

(5.49) $$f: T \times I \to M, \ (t, s) \to C(t, s),$$

which we interpret as an extension of f to the CT, mapping the cone point to q.

We construct a center of mass for an order set of points $(p_0, p_1, \cdots, p_m)$ with weights $(\lambda_0, \lambda_1, \cdots, \lambda_m)$, $0 \le \lambda_0 \le 1$, $\Sigma\lambda_i = 1$.

Lemma 5.50. *Let $\varepsilon m < i_0/4$. If $d(p_i, p_j) < \varepsilon$, $i, j = 0, 1, \cdots, m$. Then a center of mass, $c = C_{(p_0, p_1, \cdots, p_m)}(\lambda_0, \lambda_1, \cdots, \lambda_m)$ with the following properties is defined*

(i) *C depends continuously on $(\lambda_0, \lambda_1, \cdots, \lambda_m)$,*

(ii) $$C_{(p_0, \cdots, p_0, \cdots, p_m)}(\lambda_0, \cdots, \lambda_i = 0, \cdots, \lambda_m) = C_{(p_0, \cdots, \hat{p}_i, \cdots, p_m)}(\lambda, \cdots, \hat{\lambda}_i, \cdots, \lambda_m),$$

(iii) $d(p_i, c) \le m\varepsilon$, $i = 0, 1, \cdots, m$.

Proof. The proof is by induction on m. For fixed $(p_0, \cdots, p_m)$, we will view C as a map from the standard m-simplies $\Delta^m \subset R^{m+1}$ into M. For $m = 1$, $c(\lambda_0, \lambda_1) = \gamma(1 - \lambda_0)$, where γ is the unique minimal $\Delta^m \subset R^{m+1}$ into M. For $m = 1$, $C(\lambda_0, \lambda_1) = \gamma(1 - \lambda_0)$, where γ is the unique minimal geodesic from p_0 to p_1. The induction step is completed by identifying Δ^m with the cone $C\Delta^{m-1}$ and appealing to (5.49).

We now first take $\varepsilon > 0$ small such that $\varepsilon N \le i_0/100$. Then we take the minimal ε-net of $\overline{M}$ as above so that

$$d(p_i, p_j) > \varepsilon/2 \quad \text{and} \quad d(p_i, p_j) \neq \varepsilon \quad \text{for all } p_i \neq p_j,$$

and take a minimal ε-net $\{p_i^k\}$ of M_k as above for k sufficiently large, so that $\{B(p_i^k, \varepsilon)\}$, $i = 1, \cdots, N$, has the same intersection pattern as $\{B(p_i, \varepsilon)\}$.

Let (λ_i^k) be a partition of unity subordinate to the covering $\{B(p_i^k, \varepsilon)\}$ of M_k. For $x \in M_k$, let $i_0 < i_1 < \cdots < i_s$ be the indices i for which $\lambda_i^k(x) \neq 0$. Note that $s + 1 \le N_1$ by Lemma 5.47, and use Lemma 5.50 to define $F: M_k \to \overline{M}$ by

$$F(x) = C_{(p_{i_0}, \cdots, p_{i_s})}(\lambda_{i_0}^k(x), \cdots, \lambda_{i_s}^k(x)).$$

Then $F: M_k \to \overline{M}$ is continuous.

Similarly, we can define $G: \overline{M} \to M_k$ by choosing a partition of unity (μ_i) for the cover $\{B(p_i, \varepsilon)\}$ of $\overline{M}$. By symmetry, it is sufficient to show that $G \cdot F$ is homotopic to the identity map Id_{M_k} of M_k.

Let us consider again $x \in M_k$ and suppose $\lambda_i(x) \neq 0$. From Lemma 5.50, we have $d(p_i, F(x)) \leq N_1\varepsilon$, and hence $d(p_i, p_j) < \varepsilon + N_1\varepsilon$ where $\mu_j(F(x)) \neq 0$. Similarly, $d(p_j^k, G(F(x))) \leq N_1\varepsilon$ and by (5.48), $d(p_i^k, p_j^k) \leq d(p_i^k, p_i) + d(p_i, p_j) + d(p_j, p_j^k) < 2\varepsilon + N_1\varepsilon$. This together with the triangle inequality yields

$$\begin{aligned} d(x, G(F(x))) &\leq d(x, p_i^k) + d(p_i^k, p_j^k) + d(p_j^i, G(F(x))) \\ &\leq \varepsilon + 2\varepsilon + N_1\varepsilon + N_1\varepsilon \leq 4N + 1\varepsilon < i_0/4. \end{aligned}$$

Then we can connect x with $G(F(x))$ by the unique minimal geodesic in M_k to construct a homotopy of $G \cdot F$ and Id_{M_k}, which shows that $G \cdot F$ is homotopic to Id_{M_k}. This contradiction proves the homotopy part of the theorems. For the case $\Delta = 1$, we can apply the above results to the universal cover $\tilde{M}$ of M, which satisfies the hypothesis of Theorem 5.1. Then clearly $\tilde{M}$ with the pull-back metric also satisfies the assumption of Theorem 5.1, and hence is homotopic to a Riemannian manifold of constant positive sectional curvature. This implies that $\tilde{M}$ is homotopic to S^n; by the solution of Poincaré conjectures ([29], [9]), $\tilde{M}$ is homeomorphic to S^n. Hence Theorems 5.1 and 5.2 are proved.

6. Miscellaneous results

In this section we study the problems with the curvature bound $\int_M |R_m|^{(n+2)/2}\, dg \leq C < \infty$ for an n-manifold M, but with a little more work, every result in this section can be proved for the general curvature bound $\int_M |R_m|^{p/2}\, dg \leq C$ for $p > n$. Using the exact same proof of Theorem 2.11, we can prove the following.

Theorem 6.1. *Let (M, g) be a compact Riemannian n-manifold which satisfies*

(i)

$$\int_m |R_m|^{(n+2)/2}\, dg \leq K_1 < \infty,$$

(ii) *for any small $0 < L \leq 1$ and $f \in C^\infty(M)$, we have the weak Sobolev inequality*

$$\begin{aligned} &\frac{1}{L^{2/(n+1)}} \left(\int_M |f|^{2(n+1)/(n-1)}\, dg \right)^{(n-1)/(n+1)} \\ &\qquad \leq K_2 \left[\int_M |\Delta f|^2\, dg + \frac{1}{L^2} \int_M f^2\, dg \right]. \end{aligned}$$

Then the evolution equation

$$\frac{\partial}{\partial t}h = -2\,\mathrm{Ric}(h)$$

on M *for* $h(0) = g$ *has a solution on* $[0, T]$, *where* $T = T(K_1, K_2, n) > 0$ *depends only on* K_1, K_2 *and* n, *and we have for* $\tau \in [0, T]$,

$$\max|\mathrm{Rm}(h)|(\tau) \le C\frac{(K_1, K_2, n)}{\tau^{1-1/(n_1)}}.$$

As an easy consequence of Theorem 6.1, we have the following.

Theorem 6.2. *There is a constant* $\mu = \mu(K_1, H, d, V, n) > 0$ *which depends only on* K_1, H, d, V *and* n, *such that if* (M, G) *is a Riemannian manifold of* $\dim M = n$, *and*

(i) $\mathrm{Ric}(g) \ge -Hg$,
(ii) $\int |\mathrm{Rm}(g)|^{(n_1)/2}\, dg \le K_1$,
(iii) $\mathrm{diam}(g) \le d$, $\mathrm{Vol}(g) \ge V > 0$,
(iv)(a) $\int_M |\mathrm{Ric}(g) = (R/n)g|\, dg \le \mu$, *or*
(iv)(b) $\int_M |R_{ijkl} - R(g_{ik}g_{jl} - g_{il}g_{jk})/[n(n-1)]|\, dg \le \mu$,

then M *admits an Einstein metric or a constant sectional curvature metric.*

Proof. We consider the manifold $M \times S^1$ with a product metric g. Then $\mathrm{Ric}(g) \ge -H$ and $\mathrm{diam}(M \times S^1) \le \max\{d, 1\}$, $\mathrm{Vol}(M \times S^1) \ge V$. The following Sobolev inequality follows from a combination of the results in [6] and [23]:

$$\left(\int_{M\times S^1} |f|^{2(n_1)/(n-1)}\right)^{(n-1)/n+1} \le C(H, d, V)\left(\int_{M\times X^1} |\Delta f|^2 + \int_{M\times S^1} f^2\right)$$

for each $f \in C^\infty(M \times S^1)$.

As in the proof of Theorem 2.3, we take $f \in C^\infty(M)$, and for any $0 < L \le 1$ and a cut-off function $\phi\colon S = [0, 1]/\{0, 1\} \to [0, 1]$, $\phi(0) = 0$, $\phi(t) = 0$ for $t \ge L$, $\phi(t) = 1$ for $L/4 \le t \le 3L/4$, and ϕ is linear on $[0, L/4]$ and $[3L/4, L]$. As in the proof of Theorem 2.3, we have

$$\frac{1}{L^{2/(n+1)}}\left(\int_M |f|^{2(n+1)/(n-1)}\, dg\right)^{(n-1)/(n+1)} \le C(H, d, V)\left(\int_M |\nabla f|^2\, dg + \frac{1}{L^2}\int_M f^2\, dg\right).$$

We now consider the evolution equation

$$\frac{\partial}{\partial t}h_{ij} = -2\,\mathrm{Ric}(h)_{ij}$$

on M, with $h(0) = g$. Using Theorem 6.1, we see that there exists $T = T(H, K_1, d, V, n) > 0$, such that h exists for $t \in [0, T]$ and

$$\max_M |\mathrm{Rm}(h)|(\tau) \leq \frac{C(H, K_1, d, V, n)}{\tau^{1-1/(n+1)}}. \tag{6.3}$$

As before, for case (iv)(a), we have

$$\frac{\partial}{\partial t}\left|\overset{\circ}{\mathrm{Ric}}\right| \leq \Delta\left|\overset{\circ}{\mathrm{Ric}}\right| + C|\mathrm{Rm}|\left|\overset{\circ}{\mathrm{Ric}}\right|,$$

where $\overset{\circ}{\mathrm{Ric}} = \mathrm{Ric} - (R/n)h$. This implies

$$\frac{\partial}{\partial t}\int\left|\overset{\circ}{\mathrm{Ric}}(h)\right| dh \leq C(H, K_1, d, V, n)\,\mu$$

and for $h(T)$,

$$\max_M |\mathrm{Rm}(h(T))| \leq C(H, K_1, d, V, n).$$

(6.3) yields

$$0 < C(H, K_1, d, V, n)g \leq h(t) \leq C(H, K_1, d, V, n)g,$$

and hence

$$\mathrm{diam}(h(T)) \leq Cd, \qquad \mathrm{Vol}(h(T)) \geq CV > 0.$$

As in §5, a smoothing theorem of Bemelmanns, Oo and Ruh [1] also gives

$$|\nabla \mathrm{Rm}(h(T))| \leq C(H, K_1, d, V, n).$$

If Theorem 6.2 is false, there exists a sequence of Riemannian manifolds (M_k, g_k) which satisfy (i)–(iv) for $\mu = 1/k$, and M_k does not admit any Einstein metric.

By the Gromov Convergence Theorem, $(M_k, h_k(T))$ (passing to a subsequence if necessary) converges to $(\overline{M}, h)$ in C^2-topology, and we have

$$\mathrm{Ric}(h) = (R/n)h,$$

which shows that h is an Einstein metric on M. This contradiction proves Theorem 6.2, since M is diffeomorphic to $\overline{M}$ for k large.

Similarly, we can prove case (b).

Remark 6.4. We can replace R by $\Delta = 1$, 0 or -1 in (iv) of Theorem 6.2. Then M admits an Einstein metric or a constant sectional curvature metric with $R = n(n-1)\Delta$. The proof is similar to that of Sublemma 5.21.

Remark 6.5. This section serves as a contrast with the case $\int_M |\mathrm{Rm}|^{n/2}\, dg \leq K$.

References

[1] J. Bemelmanns, M. Min-Oo & E. Ruh, *Smoothing Riemannian metrics*, Math. Z. **188** (1984) 69–74.

[2] M. Berger, *Les variétés riemanniennes* (1/4)*-pincees*, Ann. Scuola. Norm. Sup. Pisa Sci. Fis. Mat. (3) **14** (1960) 160–170.

[3] R. Bishop & R. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

[4] J. Cheeger & D. G. Ebin, *Comparison theorems in Riemannian geometry*, Math. Library, Vol. 9, North-Holland, Amsterdam, 1975.

[5] J. Cheeger, M. Gromov, & M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geometry **17** (1982) 15–54.

[6] S. Y. Cheng, P. Li, & S. T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math. **103** (1981) 1021–1063.

[7] C. B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) **13** (1980) 419–435.

[8] H. Federer, *Geometric measure theory*, Springer, Berlin, 1969.

[9] M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geometry **17** (1982) 357–454.

[10] L. Z. Gao, *Einstein metrics*, J. Differential Geometry **31** (1990).

[11] R. Greene & H. Wu, *Lipschitz convergence of Riemannian manifolds*, Pacific J. Math., to appear.

[12] D. Gromoll, *Differenzierbare Strukturen and Metriken positiver Krummung auf Spharen*, Math. Ann. **164** (1966) 353–371.

[13] D. Gromoll, W. Klingenberg & W. Meyer, *Riemannsahe Geometrie im Grossen*, 2nd ed., Lecture Notes in Math., no. 55, Springer, Berlin, 1975.

[14] M. Gromov, *Manifolds of negative curvature*, J. Differential Geometry **13** (1978) 223–230.

[15] ——, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci., 1986.

[16] M. Gromov, J. Lafontaine & P. Pansu, *Structure metrique pour les varietes Riemanniennes*, Cedic/Fernand, Nathan, 1981.

[17] M. Gromov & W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987) 1–12.

[18] K. Grove & P. Peterson, *Bounding homotopy types by geometry*, Ann. of Math. **128** (1988) 195–206.

[19] R. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) 255–306.

[20] G. Huisken, *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geometry **21** (1985) 47–62.

[21] M. A. Kervaire & J. W. Milnor, *Groups of homotopy sphere.* I, Ann. of Math. (2) **77** (1963) 504–537.

[22] K. Klingenberg, *Uber Riemannsche Mannigfaltigkeiten mit positiver Krummung*, Comment. Math. Helv. **35** (1961) 47–54.

[23] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Wiley, New York, 1963.

[24] P. Li, *On Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) **13** (1980) 451–469.

[25] S. Peters, *Convergence of Riemannian manifolds*, Compositio Math. **62** (1987) 3–16.

[26] H. E. Rauch, *A contribution to differential geometry in the large*, Ann. of Math. (2) **54** (1951) 38–55.

[27] E. A Ruh, *Curvature and differential structures on spheres*, Comment. Math. Helv. **46** (1971) 127–136.
[28] ——, *Riemannian manifolds with bounded curvature ratios*, J. Differential Geometry **17** (1982) 643–653.
[29] S. Smale, *Generalized Poincaré conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961) 391–406.

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