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# $T$-EQUIVARIANT $K$-THEORY OF GENERALIZED FLAG VARIETIES

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## 0. Introduction

To any (not necessarily symmetrizable) generalized $l \times l$ Cartan matrix $A$, one associates a Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbf{C}$ and group $G = G(A)$. $G$ has a "standard unitary form" $K$. If $A$ is a classical Cartan matrix, then $G$ is a finite dimensional semi-simple simply-connected algebraic group over $\mathbf{C}$ and $K$ is a maximal compact subgroup of $G$. We refer to this as the finite case. In general, one has subalgebras of $\mathfrak{g}$: $\mathfrak{h} \subset \mathfrak{b} \subseteq \mathfrak{p}$, the Cartan subalgebra, the Borel subalgebra, and a parabolic subalgebra, respectively. One also has the corresponding subgroups: $H \subset B \subseteq P$, the complex maximal torus, the Borel subgroup, and a parabolic subgroup, respectively. We denote by $T$ the compact maximal torus $H \cap K$ of $K$. Let $W$ be the Weyl group associated to $(\mathfrak{g}, \mathfrak{h})$ and let $\{r_i\}_{1 \le i \le l}$ denote the set of simple reflections. The group $W$ operates on the compact maximal torus $T$ (as well as on $H$) and hence on the group algebra $R(T) := \mathbf{Z}[X(T)]$ of the character group $X(T)$ of $T$ and also on the quotient field $Q(T)$ of $R(T)$.

For any $W$-field $F$, we can form the smash product $F_W$ of the group algebra $\mathbf{Z}[W]$ with $F$. In [19] we took, for $F$, the field $Q = Q(\mathfrak{h}^*)$ of all the rational functions on $\mathfrak{h}$ and defined an appropriate subring $R \subset Q_W$, and showed that $R$ and its "appropriate" dual $\Lambda$, along with a certain $R$-module structure on $\Lambda$, replace the study of the cohomology algebra of $G/B$ together with the various operators defined on $H^*(G/B)$. Hence the problem of understanding $H^*(G/B)$, especially the cup product structure and other operators on $H^*(G/B)$, reduced to a purely combinatorial (and hopefully more tractable) problem of understanding the ring $R$ and its "dual" $\Lambda$, defined purely and explicitly in terms of the Coxeter group $W$ and its representation on $\mathfrak{h}^*$.

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Our aim in this paper is to prove similar results for the $T$-equivariant $K$-theory of $G/B$ as well as the $K$-theory of $G/B$, where $T$ acts on $G/B$ by the left multiplication. A parallel approach for other cohomology theories is not possible, as is shown by Bressler-Evens and Gutkin [4, 10, 13].

We replace $Q(\mathfrak{h}^*)$ by the $W$-field $Q(T)$ and analogously define a certain subring $Y$ of $Q(T)_W$, again purely and explicitly, in terms of the Coxeter group $W$ and its action on the torus $T$. We prove a structure theorem for $Y$ analogous to the corresponding structure theorem for $R$ [19, Theorem 4.6]. Our next main result is that the dual $\Psi$ of $Y$, which is also a $Y$-module, is "canonically" isomorphic with $K_T(G/B)$ and, moreover, under this isomorphism, the Weyl group action as well as certain operators $\{D_w\}_{w\in W}$ on $K_T(G/B)$, which are similar to the Demazure operators defined on $R(T)$, correspond to the action of certain well-defined elements in $Y$. The ring $\Psi$ "evaluated" at 1 does the same for $K(G/B)$. Similar results are true for any $G/P$ and in fact for any Schubert subvariety of $G/P$.

As a particular case, we obtain the above-mentioned results in the finite case. *We believe that the main results of this paper are new in the finite case as well.* As an application of our results in this case, we can easily deduce some of the important (though known) results.

Now let us describe the contents of the paper in more detail.

§1 is devoted to recalling some standard facts from Kac-Moody theory and setting up the notation to be followed throughout the paper.

In §2 we let $Q_W = Q(T)_W$ be the smash product of the $W$-field $Q = Q(T)$ with the group ring $\mathbf{Z}[W]$ (cf. §2.1). Then $Q_W$ is an associative ring with identity, which is an algebra over the $W$-invariants $Q^W$ (but not over $Q$). The ring $Q_W$ admits an involuntary anti-automorphism $t$ (cf. $(\mathrm{I}_2)$). For any simple reflection $r_i \in W$, we define a certain element $y_i = y_{r_i} \in Q_W$ (cf. $(\mathrm{I}_4)$). These elements satisfy the braid relations (cf. Proposition 2.4), and as a consequence we have a well-defined element $y_w \in Q_W$ for any $w \in W$.

The ring $Q_W$ has a natural representation in $Q$ (cf. $(\mathrm{I}_3)$). We define our basic subring $Y \subset Q_W$ as the stabilizer of the subring $R(T)$ of $Q$. It is easy to see that $y_w \in Y$, and moreover $Y$ is stable under the left (as well as the right) multiplication with $R(T)$. But conversely, we prove the crucial structure theorem for $Y$ (Theorem 2.9); which asserts that $Y$ is a free $R(T)$-module under left (as well as right) multiplication, with a basis $\{y_w\}_{w\in W}$ (and this is our first main theorem). This theorem is analogous to our structure theorem for the ring $R$ [19, Theorem 4.6] and its proof also is similar. But let us point out that the structure theorem for $Y$ is proved here even 'over $\mathbf{Z}$' in contrast to [19], where the corresponding theorem for $R$ was proved only 'over $\mathbf{C}$ (or $\mathbf{Q}$)'. In fact, in the appendix of this paper, we show that this is false 'over $\mathbf{Z}$' already in the finite case. We analyze this question in somewhat more detail in the appendix. We introduce a coproduct structure $\Delta$ in $Q_W$ (in §2.14) which is used to study the product in $K_T(G/B)$.

We dualize the above objects and define $\Omega = \Omega(T) := \mathrm{Hom}_Q(Q_W, Q)$, where $Q_W$ is considered as a $Q$-module under the right multiplication. The coproduct $\Delta$ in $Q_W$ makes $\Omega$ into an associative and commutative algebra over $Q$. Since $Q_W$ has a $Q$-basis $\{\delta_w\}_{w\in W}$, $\Omega$ can also be thought of as the space of *all* the functions $W \to Q$. Under this identification, the algebra structure on $\Omega$ is nothing but the pointwise addition, scalar multiplication, and pointwise multiplication of functions. Using the involution $t$ of $Q_W$, $\Omega$ gets equipped with a natural left $Q_W$-module structure defined in $(\mathrm{I}_{17})$. Now 'dualizing' $Y$, we get an $R(T)$-subalgebra $\Psi := \{\psi \in \Omega: \psi(Y^t) \subset R(T)\}$ of $\Omega$, which will play an important role in the paper. It is easy to see that the action of $Y \subset Q_W$ on $\Omega$ keeps $\Psi$ stable, in particular, the elements $\delta_w$ and $y_w$ act on $\Psi$. The $R(T)$-algebra $\Psi$ has a 'basis' $\{\psi^w\}_{w\in W}$ dual to the basis $\{y_w\}$ of $Y$. (Actually $\Psi$ is the direct product $\prod_{w\in W} R(T)\psi^w$ (cf. Proposition 2.20).) We introduce the $W\times W$ matrix $E = (e^{v,w})_{v,w\in W}$, where $e^{v,w} := \psi^v(\delta_w)$. We collect various properties of the matrix $E$ in Proposition 2.22. In particular it is 'upper triangular'. We show (cf. Proposition 2.22(e)) that the '$l(v)$th degree component' of $e^{v,w}$ is precisely equal to $(-1)^{l(v)}d_{v,w}$, where $d_{v,w}$ is as in [19, §4.21]. So the $E$-matrix determines the $D$-matrix of [19]. The action of $y_{r_i}$ on $\Psi$ is explicitly given by Proposition 2.22(d), and moreover the action of $\delta_w$ as well as the product in $\Psi$ is explicitly written down (in the $\{\psi^w\}$-'basis') in terms of the $E$-matrix (cf. Proposition 2.25).

Finally we show (cf. Proposition 2.30) that the ring $\Psi$ has a 'natural' filtered ring structure, such that the associated graded ring $\mathrm{Gr}(\Psi)$ (rather $\mathbf{C}\otimes_{\mathbf{Z}} \mathrm{Gr}(\Psi)$) is canonically isomorphic with the ring $\Lambda$ introduced in [19]. (We recall that the ring $\Lambda$ is the 'cohomological analogue' of the ring $\Psi$.) In particular, by the results of §3, we get that $\mathbf{C}\otimes_{\mathbf{Z}} K_T(G/B)$ has a filtration such that the associated graded ring is canonically isomorphic with the equivariant cohomology (over $\mathbf{C}$) $H_T^*(G/B)$.

§3 is devoted to the study of $T$-equivariant $K$-theory of $G/P$, where $G$ is any Kac-Moody group with any parabolic subgroup $P$ and $T$ acts on $G/P$ by the left multiplication. *In particular, the results apply to the based loop group* $\Omega_e(G_0)$ *of a compact simply-connected Lie group* $G_0$.

Motivated by the Demazure operators on $R(T)$, we define certain operators $\{D_w\}_{w\in W}$ on $K_T(G/B)$ (and $K(G/B)$). It may be mentioned that Kazhdan and Lusztig have recently defined similar but more general operators in the finite case (acting on equivariant $K$-theory of Springer fibres) and used them to prove the Deligne-Langlands conjecture [18]. The Weyl group $W$, being isomorphic with $N_K(T)/T$, acts on $K/T \approx G/B$ (cf. §3.11). Moreover the $W$-action commutes with the action of $T$ on $G/B$, and hence we get an action of $W$ on $K_T(G/B)$ (and $K(G/B)$).

Our second main theorem of the paper (Theorem 3.13) is that there is a 'canonical' $R(T)$-algebra isomorphism $\gamma\colon K_T(G/B) \to \Psi$, such that the action of the Weyl group element $w$ (resp. the operator $D_w$) on $K_T(G/B)$ corresponds, under $\gamma$, to the action of the element $\delta_w$ (resp. $y_w$) on $\Psi$. About the proof; we only mention that it crucially uses the localization theorem of Atiyah-Segal, and a certain consequence of the equivariant Thom isomorphism (which can be viewed as a generalization of Bott-periodicity). We also prove (Theorem 3.28) that $\gamma$ induces an isomorphism $\gamma_1\colon K(G/B) \to \mathbf{Z} \otimes_{R(T)} \Psi$, where $\mathbf{Z}$ is considered as an $R(T)$-module under the standard augmentation map. Similar results are also obtained for $K_T(G/P)$ (and $K(G/P)$) and, in fact, even more generally for any left $B$-stable closed subspace $V_\Theta$ of $G/P$ (cf. Corollary 3.20 and Theorems 3.23 and 3.29). By transporting the 'basis' $\{\psi^w\}$ of $\Psi$ via $\gamma^{-1}$, we get a 'basis' $\{\tau^w\}$ of $K_T(G/B)$. In particular, the Weyl group action, the product, and the action of the operators $D_w$ on $K_T(G/B)$ can be explicitly written down in the $\{\tau^w\}$ 'basis' in terms of the $E$-matrix. We give a characterization of this 'basis' in Proposition 3.39. As a consequence we show that, in the finite case, the basis $\{\varepsilon(\tau^w)\}$ of $K(G/B)$ (where $\varepsilon$ is the canonical map $K_T(G/B) \to K(G/B)$) is essentially the basis given by Demazure in [7].

§4 is devoted to specializing the earlier results to the finite case. We show that some of the important (though known) results can be easily deduced from our Theorem 3.13 (which identities $K_T(G/B)$ with $\Psi$). In particular, for any compact simply-connected Lie group $G_0$ with maximal torus $T$, we deduce that: (a) $K_T(G_0/T)$ is canonically isomorphic with $R(T) \otimes_{R(G_0)} R(T)$ (cf. Theorem 4.4), and (b) the Atiyah-Hirzebruch homomorphism $R(T) \to K(G_0/T)$ is surjective (cf. Theorem 4.6). The fact that $K^*(G_0)$ is torsion free can also be easily deduced from our Theorem 3.13.

*The main results of this paper are announced in* [20].

The second named author has proved that, for any $v \le w \in W$, the ring of functions of the tangent cone $T_v(X_w)$ at $v$ for $X_w$, which is canonically a $T$-module, has character (defined appropriately) $*b_{w^{-1}, v^{-1}}$ (cf. $(\mathrm{I}_5)$), where $*$ is the involution of $Q(T)$ induced by the map $e^\lambda \mapsto e^{-\lambda}$ for any $e^\lambda \in X(T)$, and $X_w$ is the Schubert variety $\overline{BwB/B} \subset G/B$.

This result is used to connect the singularity of the Schubert varieties with the $B$-matrix (cf. §2.7), which in turn 'controls' the $T$-equivariant $K$-theory of the flag variety $G/B$.

As another consequence, one obtains that $b_{w,v} \neq 0$ if and only if $w \ge v$. The details will appear elsewhere.

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## 1. Preliminaries and notation

(1.1) **Kac-Moody algebra (definitions and basic properties) [16, 25].** Let $A = (a_{ij})_{1\le i,j\le l}$ be any *generalized Cartan matrix* (i.e., $a_{ii} = 2$, $-a_{ij} \in \mathbf{Z}_+$ for all $i \neq j$, where $\mathbf{Z}_+$ is the set of nonnegative integers, and $a_{ij} = 0$ if and only if $a_{ji} = 0$). Choose a triple $(\mathfrak{h}, \pi, \pi^\vee)$, unique up to isomorphism, where $\mathfrak{h}$ is a vector space over $\mathbf{C}$ of dimension $(2l - \operatorname{rank} A)$, $\pi = \{\alpha_i\}_{1\le i\le l} \subset \mathfrak{h}^*$, and $\pi^\vee = \{h_i\}_{1\le i\le l} \subset \mathfrak{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra over $\mathbf{C}$, generated by $\mathfrak{h}$ and the symbols $e_i$ and $f_i$ $(1 \le i \le l)$ with the defining relations $[\mathfrak{h}, \mathfrak{h}] = 0$, $[h, e_i] = \alpha_i(h)e_i$, $[h, f_i] = -\alpha_i(h)f_i$ for $h \in \mathfrak{h}$ and all $1 \le i \le l$, $[e_i, f_j] = \delta_{ij}h_j$ for all $1 \le i, j \le l$, and

$$(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0 = (\operatorname{ad} f_i)^{1-a_{ij}}(f_j) \quad \text{for all } 1 \le i \neq j \le l.$$

In the above, we can replace $\mathbf{C}$ by any field $k$ of characteristic 0 and obtain a Kac-Moody Lie algebra $\mathfrak{g}_k$ over the field $k$. If $k$ is a subfield of $\mathbf{C}$, then of course $\mathfrak{g}_k \otimes_k \mathbf{C} \cong \mathfrak{g}$.

$\mathfrak{h}$ is canonically embedded in $\mathfrak{g}$ and is called the *Cartan subalgebra* of $\mathfrak{g}$.

One has the *root space decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, where, for any $\lambda \in \mathfrak{h}^*$, $\mathfrak{g}_\lambda := \{x \in \mathfrak{g}: [h, x] = \lambda(h)x$, for all $h \in \mathfrak{h}\}$, and $\Delta_+ := \{\alpha \in \sum_{i=1}^l \mathbf{Z}_+\alpha_i : \alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0\}$. Define $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_- := -\Delta_+$. The subset $\Delta_+$ (resp. $\Delta_-$) of $\mathfrak{h}^*$ is called the set of *positive* (resp. *negative*) *roots*. The roots $\{\alpha_i\}_{1 \leq i \leq l}$ are called the *simple roots* and the elements $h_i$ $(1 \leq i \leq l)$ are called the *simple coroots*.

We fix a subset $S$ (including $S = \emptyset$) of $\{1, \cdots, l\}$. Put $\Delta_+^S = \Delta_+ \cap \{\sum_{i \in S} \mathbf{Z}\alpha_i\}$, and define the following Lie subalgebras of $\mathfrak{g}$:

$$\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \qquad \mathfrak{u} = \mathfrak{u}_S = \sum_{\alpha \in \Delta_+ \setminus \Delta_+^S} \mathfrak{g}_\alpha,$$

$$\mathfrak{r} = \mathfrak{r}_S = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+^S} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{p} = \mathfrak{p}_S = \mathfrak{r} \oplus \mathfrak{u}.$$

Since $[\mathfrak{r}_S, \mathfrak{u}_S] \subset \mathfrak{u}_S$, $\mathfrak{r}_S$ acts on $\mathfrak{u}_S$.

Associated to $(\mathfrak{g}, \mathfrak{h})$ there is the *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}^*)$, generated by the 'simple' reflections $\{r_i\}_{1 \leq i \leq l}$, where $r_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$ for any $\lambda \in \mathfrak{h}^*$. As is known, $(W, \{r_i\}_{1 \leq i \leq l})$ is a *Coxeter group*, and hence we can talk of the *Bruhat ordering* $\leq$ and *length* of elements of $W$. We denote the length of $w$ by $l(w)$. The Weyl group $W$ preserves $\Delta$. The set of *real roots* $\Delta^{\mathrm{re}}$ is defined to be $W \cdot \pi$, and the set of *imaginary roots* $\Delta^{\mathrm{im}}$ is, by definition, $\Delta \setminus \Delta^{\mathrm{re}}$. For $\alpha \in \Delta^{\mathrm{re}}$, $\dim \mathfrak{g}_\alpha = 1$. We set $\Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap \Delta_+$; similarly $\Delta_-^{\mathrm{re}} := \Delta^{\mathrm{re}} \cap \Delta_-$. By dualizing, we get a representation of $W$ in $\mathfrak{h}$. Explicitly $r_i(h) = h - \alpha_i(h)h_i$ for $h \in \mathfrak{h}$ and $1 \leq i \leq l$.

For any $S \subset \{1, \cdots, l\}$, let $W_S$ be the subgroup of $W$ generated by $\{r_i\}_{i \in S}$ and define a subset $W_S^1$, of the Weyl group $W$, by $W_S^1 = \{w \in W: \Delta_+ \cap w\Delta_- \subset \Delta_+ \setminus \Delta_+^S\}$. Then $W_S^1$ can be characterized as the set of elements of minimal length in the cosets $W_S w$ $(w \in W)$ (each coset contains a unique element of minimal length).

There is a ($\mathbf{C}$-linear) involution $\omega$ of $\mathfrak{g}$ defined (uniquely) by $\omega(f_i) = -e_i$ for all $1 \leq i \leq l$, and $\omega(h) = -h$ for all $h \in \mathfrak{h}$. It is easy to see that $\omega$ leaves $\mathfrak{g}_R$ stable (where $\mathbf{R} \subset \mathbf{C}$ is the subfield of real numbers). Let $\omega_0$ be the conjugate-linear involution of $\mathfrak{g}$, which coincides with $\omega$ on $\mathfrak{g}_R$.

(1.2) **Integral form of the Cartan subalgebra.** We fix, once and for all, an integral lattice $\mathfrak{h}_{\mathbf{Z}} \subset \mathfrak{h}$ (i.e. $\mathfrak{h}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C} = \mathfrak{h}$) satisfying:

($\mathrm{P}_1$) $h_i \in \mathfrak{h}_{\mathbf{Z}}$ for all $1 \leq i \leq l$,

($P_2$) $\mathfrak{h}_{\mathbf{Z}}/\sum_{i=1}^{l} \mathbf{Z}h_i$ is torsion free, and
($P_3$) $\mathfrak{h}_{\mathbf{Z}}^* := \mathrm{Hom}_{\mathbf{Z}}(\mathfrak{h}_{\mathbf{Z}}, \mathbf{Z})$ $(\subset \mathfrak{h}^*)$ contains $\{\alpha_i\}$.

(The choice of $\mathfrak{h}_{\mathbf{Z}}$, as above, is possible.) Clearly $\mathfrak{h}_{\mathbf{Z}}^*$ is $W$-stable. It is called the *weight lattice* and its elements *integral weights.*

We make a choice of the *fundamental weights* $\rho_i \in \mathfrak{h}_{\mathbf{Z}}^*$ $(1 \leq i \leq l)$ satisfying $\rho_i(h_j) = \delta_{i,j}$, for all $1 \leq i, j \leq l$. This is possible because of ($P_2$). We further set $\rho = \sum_{i=1}^{l} \rho_i$. Of course in the case when $A$ is nondegenerate (i.e., $\operatorname{rank} A = l$) $\mathfrak{h}_{\mathbf{Z}} = \sum_{i=1}^{l} \mathbf{Z}h_i$ and the $\rho_i$'s are uniquely determined.

(1.3) **Kac-Moody group and its parabolic subgroups.** The construction, given below, is due to Kac-Peterson [17]. It should be mentioned that there are other constructions of the group(s) associated to any Kac-Moody Lie algebra $\mathfrak{g}$, due to Moody-Teo, Marcuson, Tits, Slodowy, etc. Even though these groups may differ from each other, the corresponding 'generalized flag varieties $G/P$' are 'essentially' the same. Since, in this paper, we will mainly be interested in the flag varieties $G/P$, we could have used either of these constructions.

A $\mathfrak{g}$-module $(V, \pi)$ $(\pi: \mathfrak{g} \to \operatorname{End} V)$ is called *integrable* if $\pi(x)$ is locally nilpotent whenever $x \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta^{\mathrm{re}}$ and, as an $\mathfrak{h}$-module, $V$ decomposes as the (direct) sum $\sum_{\chi \in \mathfrak{h}^*} V_\chi$ of its weight spaces, with the additional requirement that any $\chi$ such that $V_\chi \neq 0$ belongs to $\mathfrak{h}_{\mathbf{Z}}^*$. Observe that for any integrable $\mathfrak{g}$-module $(V, \pi)$, the $\mathfrak{h}$-module structure on $V$ integrates to give a representation of the multiplicative group $H := \mathfrak{h}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}^*$ on $V$, which we again denote by $\pi$. Let $G^*$ be the free product of the additive groups $\{\mathfrak{g}_\alpha\}_{\alpha \in \Delta^{\mathrm{re}}}$ and the group $H$, with canonical inclusions $i_\alpha: \mathfrak{g}_\alpha \to G^*$ and $i: H \to G^*$. For any integrable $\mathfrak{g}$-module $(V, \pi)$, define a homomorphism $\pi^*: G^* \to \operatorname{Aut}_{\mathbf{C}} V$ by $\pi^*(i_\alpha(x)) = \exp(\pi(x))$ for $x \in \mathfrak{g}_\alpha$ and $\pi^*(i(t)) = \pi(t)$ for $t \in H$. Let $N^*$ be the intersection of all $\operatorname{Ker} \pi^*$, where $\pi$ ranges over all the integrable representations of $\mathfrak{g}$. Put $G = G^*/N^*$. Let $q$ be the canonical homomorphism $G^* \to G$. It can be seen that the canonical map $H \to G$ is injective. For $x \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$, put $\exp(x) = q(i_\alpha x)$, so that $U_\alpha := \exp \mathfrak{g}_\alpha$ is an additive one-parameter subgroup of $G$. Denote by $U$ (resp. $U^-$) the subgroup of $G$ generated by the $U_\alpha$'s with $\alpha \in \Delta_+^{\mathrm{re}}$ (resp. $\alpha \in \Delta_-^{\mathrm{re}}$). We put a topology on $G$ as given in [17, 4(G)]. Then $G$ becomes a (Hausdorff) topological group, which may also be viewed as an (possibly infinite dimensional) affine algebraic group in the sense of Šafarevič with Lie algebra $\mathfrak{g}$ [17]. (Actually Kac-Peterson constructed a slightly different group which corresponds to the commutator subalgebra $\mathfrak{g}^1$.) We call $G$ the *Kac-Moody group* (*associated to the Kac-Moody Lie algebra* $\mathfrak{g}$).

The conjugate-linear involution $\omega_0$ of $\mathfrak{g}$, on 'integration', gives rise to an involution $\tilde{\omega}_0$ of $G$. Let $K$ denote the fixed point set of this involution. Then $K$ is called the *standard unitary form of* $G$.

For each $1 \le i \le l$, there exists a unique homomorphism $\beta_i\colon \mathrm{SL}_2(\mathbf{C}) \to G$, satisfying

$$\beta_i \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = \exp(ze_i) \quad \text{and} \quad \beta_i \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \exp(zf_i)$$

(for all $z \in \mathbf{C}$), where $e_i$ and $f_i$ are as in §1.1. Define

$$H_i = \beta_i \left\{ \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} : z \in \mathbf{C}^* \right\}, \qquad G_i = \beta_i(\mathrm{SL}_2(\mathbf{C})),$$

$N_i$ = Normalizer of $H_i$ in $G_i$, and $N$ the normalizer of $H$ in $G$. We call $H$ the *complex maximal torus* of $G$. Of course its Lie algebra is $\mathfrak{h}$. There is a group isomorphism $\tau\colon W \xrightarrow{\sim} N/H$, such that $\tau(r_i)$ is the coset $n_iH$, where $n_i$ is the (unique) nontrivial element of $N_i$ (mod $H_i$). *We will, sometimes, identify* $W$ *with* $N/H$ *under* $\tau$ *and hence* $w \in W$ *can also be thought of as an element of* $N$ (mod $H$).

Put $B = HU$ and $P = P_S = BW_SB$. Then $B$ is called the *standard Borel subgroup* and $P_S$ the *standard parabolic subgroup* of $G$ (associated to the subset $S$). (Since $H$ normalizes $U$, $B$ is a subgroup and $P_S$ is a subgroup because $(B, N)$ is a Tits system in $G$.) Define $T = B \cap K$; then $T$ is compact connected and is contained in $H$. Moreover the complexified Lie algebra of $T$ = Lie $H = \mathfrak{h}$. We call $T$ as the (standard) *compact maximal torus* of $K$ (or $G$).

The canonical inclusion $K/K_S \hookrightarrow G/P_S$, where $K_S$ is (by definition) $K \cap P_S$ and $K$ is given the subspace topology, is a (surjective) homeomorphism [17, Theorem 4(d)].

(1.4) **Bruhat decomposition.** Fix any subset $S \subset \{1, \cdots, l\}$. Then $G$ can be written as a disjoint union

$$G = \bigcup_{w \in W_S^1} (Uw^{-1}P_S), \quad \text{so that } G/P_S = \bigcup_{w \in W_S^1} (Uw^{-1}P_S/P_S).$$

Further $G/P_S$ is a CW complex with cells $\{Uw^{-1}P_S/P_S\}_{w \in W_S^1}$, and moreover $\dim_{\mathbf{R}}(Uw^{-1}P_S/P_S) = 2l(w)$.

## 2. Definition of the basic ring $Y$ and its structure

Throughout this section (and the next) $G$ denotes any (not necessarily symmetrizable) Kac-Moody group over $\mathbf{C}$, with the standard unitary form

$K$, the standard Borel subgroup $B$, the complex maximal torus $H \subset B$, and the compact maximal torus $T = H \cap K$. Let $W$ be the Weyl group associated to $(G, H)$ and let $\{r_i\}_{1\le i\le l}$ denote the set of simple reflections in $W$ (cf. §1). Let $R(T) := \mathbf{Z}[X(T)]$ be the group algebra $/\mathbf{Z}$ of the character group $X(T)$ of $T$ (i.e. $R(T)$ is the representation ring of the torus $T$) and $Q = Q(T)$ be its quotient field. Of course $\mathbf{C} \otimes_{\mathbf{Z}} R(T)$ can also be viewed as the ring of regular functions $\mathbf{C}[H]$ on the complex affine variety $H$. For any integral weight $\lambda$ (cf. §1.2), the notation $e^{\lambda}$ means the corresponding character of $T$ (or $H$).

*The treatment in this section is parallel to the one in* [19, §4].

(2.1) **Definition of the ring $Q_W$.** The Weyl group $W$ operates on the torus $T$ and hence on $R(T)$ and its quotient field $Q = Q(T)$ (by field automorphisms). Let $Q_W = Q(T)_W$ be the *smash product* of the $W$-field $Q$ with the group algebra $\mathbf{Z}[W]$, i.e., $Q_W := \mathbf{Z}[W] \otimes_{\mathbf{Z}} Q$, and the multiplication[1] is given by:

$(\mathrm{I}_1)$

$$(\delta_{w_1}q_1).(\delta_{w_2}q_2) = \delta_{w_1w_2}(w_2^{-1}q_1)q_2 \quad \text{for } q_1, q_2 \in Q \text{ and } w_1, w_2 \in W,$$

where we write (here and henceforth) $\delta_w q$ for $\delta_w \otimes q$. This makes $Q_W$ into an associative ring with identity $\delta_e$. Since $Q = \delta_e Q$ is not central in $Q_W$, $Q_W$ is not an algebra over $Q$, but clearly $Q_W$ is an algebra over the $W$-invariants $Q^W$ in $Q$.

The ring $Q_W$ admits an involuntary anti-automorphism $t$, defined by

$$(\mathrm{I}_2)\qquad (\delta_w q)^t = \delta_{w^{-1}}(wq) \quad \text{for } w \in W \text{ and } q \in Q.$$

Clearly $Q$ has a natural left $Q_W$-module structure, given explicitly by

$$(\mathrm{I}_3)\qquad (\delta_w q)\cdot q' = w(qq') \quad \text{for } w \in W \text{ and } q, q' \in Q.$$

For any simple reflection $r_i$, $1 \le i \le l$, define a certain element

$$(\mathrm{I}_4)\qquad y_i = y_{r_i} := (\delta_e + \delta_{r_i})\frac{1}{(1 - e^{-\alpha_i})} = \frac{1}{1 - e^{-\alpha_i}}(\delta_e - e^{-\alpha_i}\delta_{r_i}) \in Q_W,$$

where $\alpha_i$ is the (positive) simple root associated with the simple reflection $r_i$.

(2.2) **Remark.** The notation $Q$ and $Q_W$ in this paper, and also the subsequent notation $\Omega$ (§2.17), should not be confused with the corresponding notation in [19, §4], where they have somewhat different meaning.

[1] We will often drop the dot for multiplication.

We record the following simple lemma.

(2.3) **Lemma.** (a) $y_i^2 = y_i$ *for any* $1 \le i \le l$.

(b) $y_i q = (r_i q) y_i + ((q - r_i q)/(1 - e^{-\alpha_i}))\delta_e$ *for any* $q \in Q$.

(c) $\delta_{r_i} y_j = e^{\alpha_i} y_j + (1 - e^{\alpha_i}) y_i y_j$ *for any* $1 \le i, j \le l$.

(d)

$$y_j \delta_{r_i} = \begin{cases} (1 - e^{r_j \alpha_i}) y_j y_i + e^{r_j \alpha_i} y_j + \left(\frac{e^{\alpha_i} - e^{r_j \alpha_i}}{1 - e^{-\alpha_j}}\right)(\delta_e - y_i) & \text{if } i \neq j, \\ (1 + e^{\alpha_i})\delta_e - e^{\alpha_i} y_i & \text{if } i = j. \end{cases}$$

One has the following very useful proposition.

(2.4) **Proposition.** *Let* $w \in W$ *and let* $w = r_{i_1} \cdots r_{i_m}$ *be a reduced decomposition. Then the element* $y_{i_1} \cdots y_{i_m} \in Q_W$ *does not depend upon the particular choice of the reduced decomposition of* $w$.

We define $y_w = y_{i_1} \cdots y_{i_m} \in Q_W$. We further denote $\overline{y}_w = y^t_{w^{-1}}$.

*Proof.* By a result of Matsumoto [6, Proposition 5, p. 16], it suffices to check the braid relations:

For any two simple reflections $r_i, r_j$ $(i \neq j)$ such that $r_i r_j$ is of finite order $m_{ij}$, we need to check that

$$\frac{y_i y_j y_i y_j \cdots}{m_{ij} \text{ factors}} = \frac{y_j y_i y_j y_i \cdots}{m_{ij} \text{ factors}}.$$

Now, as is well known [16, Proposition 3.13], the only possibilities for $m_{ij}$ are 2, 3, 4, 6, and $\infty$. The proof of the proposition can now be completed by an explicit case by case checking (cf. [7], [10], [13] or [18, §3]). □

As an immediate consequence of the above proposition, together with Lemma 2.3, we have the following.

(2.5) **Corollary.** (a) $y_v y_w = y_{vw}$ *if* $l(vw) = l(v) + l(w)$.

(b) $y_v y_{r_i} = y_v$ *if* $l(vr_i) < l(v)$.

(c) $\sum_{w \in W} R(T) y_w = \sum_{w \in W} y_w R(T)$, *and it is a subring of* $Q_W$.

(2.6) **Proposition.** *For any* $v \in W$, *write*

$$(\mathrm{I}_5) \qquad y_{v^{-1}} = \sum_w b_{v,w} \delta_{w^{-1}} \textit{ for some (unique)} b_{v,w} \in Q.$$

*Then*

(a) $b_{v,w} = 0$, *unless* $w \le v$.

(b) $b_{v,v} = \prod_{\nu \in \Delta_+ \cap v^{-1}\Delta_-} (1 - e^{\nu})^{-1}$.

*In particular,* $b_{v,v} \neq 0$.

*Proof.* (a) is an easy consequence of [8, Theorem 1.1] and (b) follows from [26, §2].

(2.7) **Corollary.** *Define the* $W \times W$*-matrix* $B = (b_{v,w})_{v,w \in W}$, *where* $b_{v,w}$ *is as in* $(\mathrm{I}_5)$.

*By the above proposition,* $B$ *is a lower triangular matrix* (*with respect to the usual Bruhat partial ordering* $\leq$ *in* $W$) *with nonzero diagonal entries, and hence* $\{y_v\}_{v \in W}$ *is a left* (*as well as right*) $Q$*-basis for* $Q_W$.

The notation $B$ as above is not likely to cause any confusion with the same notation used for Borel subgroups.

(2.8) **Definition.** Recall from $(\mathrm{I}_3)$ that $Q$ is naturally a left $Q_W$-module. Now we define our very basic subring $Y \subset Q_W$ by

$$Y = \{y \in Q_W : y \cdot R(T) \subset R(T)\}.$$

It is easy to see that $y_i$, for any $1 \leq i \leq l$ (and hence any $y_w$), belongs to $Y$, and of course $Y$ is stable under the left (as well as the right) multiplication by the elements of $R(T)$. Conversely, we have the following crucial structure theorem analogous to [19, Theorem 4.6]. The proof given below also is similar; but we give the details for completeness.

(2.9) **Theorem.** *With the notation as above, the ring*

$$Y = \sum_w R(T) y_w = \sum_w y_w R(T).$$

*In particular the elements* $\{y_w\}_{w \in W}$ *form a* $R(T)$*-basis of* $Y$ *under the left* (*as well as the right*) *multiplication.*

(2.10) **Remark.** See the appendix.

Recall that the affine ring $\mathbf{C}[H]$ of the complex torus $H$ is a unique factorization domain. Also recall that $\mathbf{C}[H]$ can be identified with $\mathbf{C} \otimes_{\mathbf{Z}} R(T)$.

As a preparation for the proof of Theorem 2.9, we prove the following lemmas.

(2.11) **Lemma.** *Let* $f \in \mathbf{C}[H]$ *be irreducible and let* $\{f_w\}_{l(w) \leq k}$ *be certain elements in* $\mathbf{C}[H]$, *such that any nonzero* $f_w$ *is coprime to* $f$, $f_w \neq 0$ *for some* $w$ *of length* $k$, *and* $(\sum_{l(w) \leq k} f_w y_w) \cdot \mathbf{C}[H] \subset f\mathbf{C}[H]$.

*Then* $Z(f) \subset I_{v_0 r_i v_0^{-1}}$ *for some* $v_0 \in W$ *and some simple reflection* $r_i$, *where* $Z(f)$ *is the zero set* $\subset H$ *of* $f$ *and, for any* $v \in W$, $I_v := \{t \in H : vtv^{-1} = t\}$.

*In particular,* $f$ *divides* $(1 - e^{-v_0 \alpha_i})$. (Observe that in general $1 - e^{-v_0 \alpha_i}$ is not an irreducible element of $\mathbf{C}[H]$.)

*Proof*. Write $y = \sum_{l(w)\le k} f_w y_w = \sum_{l(w)\le k} q_w \delta_w$ for some $q_w \in Q$. By Proposition 2.6,

$$(\mathrm{I}_6) \qquad q_w = f_w b_{w^{-1},w^{-1}} \quad \text{if } l(w) = k.$$

Define $V = \bigcup_{v\neq e} I_v$. We claim that $Z(f) \subset V$. For, if not, choose any $t_0 \in Z(f) \setminus V$. Fix any $w_0$ of length $k$ and choose $f_0 \in R(T)$ such that $(w_0 f_0)(t_0) = 1$ and $(w f_0)(t_0) = 0$ for all those (finitely many) $w \neq w_0$ satisfying $q_w \neq 0$. (This is possible since the point $t_0$ has no $W$-isotropy.) Evaluating $y \cdot f_0$ at $t_0$, we get $q_{w_0}(t_0) = 0$ (observe that for any real root $\beta$ and any $t_0$ not in $V$, $(1 - e^\beta)(t_0) \neq 0$ and hence any $q_w$ does not have a pole at $t_0$), i.e., by $(\mathrm{I}_6)$, $f_{w_0}(t_0) = 0$. Hence $f$ divides $f_{w_0}$. A contradiction to the assumption of the lemma! So we obtain that $Z(f) \subset V$, and since $f$ is irreducible, we actually have $Z(f) \subset I_v$ for some $v \neq e \in W$. In particular, $Z(f)$ being a hypersurface, $I_v$ is of codim. 1 in $H$, i.e., the element $v$ fixes pointwise a hyperplane (the Lie algebra: Lie $I_v$ of $I_v$) in Lie $H$. Hence, by [19, Lemma 4.8], $v = v_0 r_i v_0^{-1}$ for some $v_0 \in W$ and some simple reflection $r_i$, and of course Lie $I_v = \mathrm{Ker}(v_0\alpha_i)$.

Now we prove that $I_{v_0 r_i v_0^{-1}} \subset Z(1 - e^{-v_0\alpha_i})$. Take $t \in I_{v_0 r_i v_0^{-1}}$ and write $t = \exp h$ for $h \in$ Lie $H$. Since $t \in I_{v_0 r_i v_0^{-1}}$, we get $r_i v_0^{-1} t v_0 r_i = v_0^{-1} t v_0$, i.e., $\exp(r_i v_0^{-1} h) = \exp(v_0^{-1} h)$. Hence $\exp(-\alpha_i(v_0^{-1}h)h_i) = 1$, where $h_i$ is the $i$th simple coroot. Taking $e^{\rho_i}$ (where $\rho_i$ is an $i$th fundamental weight; cf. §1.2) of both the sides, we get $e^{-\alpha_i(v_0^{-1}h)} = 1$. This proves the lemma.

(2.12) **Lemma.** *Let* $\{f_w\}_{l(w)\le k}$ *and* $f$ *be certain elements in* $\mathbf{C}[H]$ *such that* $(\sum_{l(w)\le k} f_w y_w) \cdot \mathbf{C}[H] \subset f\mathbf{C}[H]$. *Assume further that* $f$ *is irreducible and* $Z(f) \subset I_{r_i}$ *for some simple reflection* $r_i$. *Then* $f$ *divides all the* $f_w$*'s.*

*Proof*. Denote $y = \frac{1}{f}\sum f_w y_w$ and write $y = y^+ + y^-$, where $y^+$ (resp. $y^-$) $= \frac{1}{2}(y + \delta_{r_i} y)$ (resp. $\frac{1}{2}(y - \delta_{r_i} y)$). Now $y^+$ also satisfies $y^+ \cdot \mathbf{C}[H] \subset \mathbf{C}[H]$, and $y^+$ is again of the form $\frac{1}{f}\sum f'_w y_w$ for some $f'_w \in \mathbf{C}[H]$ (use Lemma 2.3 and the fact that $Z(f)$ is $r_i$-fixed and hence $f/r_i f \in \mathbf{C}[H]$). (A similar statement is true for $y^-$.) So we can assume that either $\delta_{r_i} y = y$ or $-y$.

Fix $w_0$ of length $k$ such that $f_{w_0} \neq 0$, and write:

$$(\mathrm{I}_7) \qquad fy = \sum_{l(w)\le k} f_w y_w = y_0 + q_{w_0}\delta_{w_0} + q_{r_i w_0}\delta_{r_i w_0},$$

where

$$(\mathrm{I}_8) \qquad q_{w_0} = f_{w_0} b_{w_0^{-1}, w_0^{-1}} \quad \text{(by Proposition 2.6)},$$

$$y_0 = \sum_{w \notin \{w_0, r_i w_0\}} q_w \delta_w \quad \text{for some } q_w \in Q, \text{ and}$$

$q_{r_i w_0}$ is some element in $Q$.

Fix any $t_0 \in Z(f) \subset I_{r_i}$ with the property that the set $\{v \in W : v t_0 v^{-1} = t_0\}$ coincides with $\{e, r_i\}$ and $(1 - e^{-\nu})(t_0) \neq 0$ for any positive real root $\nu \neq \alpha_i$. Such a choice is possible:

If possible, assume that $Z(f) \subset I_{v_0}$ for some $v_0 \neq r_i$ and $e$. Then for some $h_0 \in \operatorname{Lie} H$, $\exp(\operatorname{Ker}\alpha_i + h_0) \subset I_{v_0}$. This implies that

$$\exp(v_0(h + h_0) - (h + h_0)) = 1 \text{ for all } h \in \operatorname{Ker}\alpha_i,$$

which is possible only if $v_0 h = h$ for all $h \in \operatorname{Ker}\alpha_i$. A contradiction! Similarly, if possible, assume that $Z(f) \subset Z(1 - e^{-\nu})$ for some positive real root $\nu \neq \alpha_i$. Then $\exp(\operatorname{Ker}\alpha_i + h_0) \subset Z(1 - e^{-\nu})$, i.e., $e^{-\nu(h+h_0)} = 1$ for all $h \in \operatorname{Ker}\alpha_i$, which is possible only if $\nu(h) = 0$ for all $h \in \operatorname{Ker}\alpha_i$. Again a contradiction (since $\nu \neq \alpha_i$)!

Now choose $f_0 \in \mathbf{C}[H]$, such that $f_0(w_0^{-1} t_0 w_0) = 1$ and $f_0(w^{-1} t_0 w) = 0$ (in fact a zero of sufficiently high multiplicity) for all those $w \neq w_0$ and $r_i w_0$, satisfying $q_w \neq 0$.

In the case when $\delta_{r_i} y = y$ (resp. $\delta_{r_i} y = -y$), we have

$$\frac{f}{r_i f}(r_i q_{w_0}) = q_{r_i w_0} \quad \left(\text{resp. } \frac{f}{r_i f}(r_i q_{w_0}) = -q_{r_i w_0}\right).$$

In particular, in either case, $r_i w_0 < w_0$. Denote by $a = f / r_i f$ or $-f / r_i f$ according as we are in the first or the second case, respectively. Of course $a \in \mathbf{C}[H]$. We have, by $(\mathrm{I}_7)$ and $(\mathrm{I}_8)$, in either case:

$$(\mathrm{I}_9) \quad (1 - e^{-\alpha_i}) f y = (1 - e^{-\alpha_i}) y_0 + (1 - e^{-\alpha_i}) q_{w_0} \delta_{w_0} + (1 - e^{-\alpha_i}) a (r_i q_{w_0}) \delta_{r_i w_0}.$$

Take a reduced expression $w_0 = r_i r_{i_2} \cdots r_{i_m}$ (starting with $r_i$). Then, by Proposition 2.6 and $(\mathrm{I}_8)$–$(\mathrm{I}_9)$, we get

$$(\mathrm{I}_{10}) \quad (1 - e^{-\alpha_i}) f y = (1 - e^{-\alpha_i}) y_0 - f_{w_0}(r_i b) e^{-\alpha_i} \delta_{w_0} + a (r_i f_{w_0}) b \delta_{r_i w_0},$$

where $b = \prod_{\nu \in \{\alpha_{i_2}, r_{i_2}\alpha_{i_3}, \cdots, r_{i_2} \cdots r_{i_{m-1}} \alpha_{i_m}\}} (1 - e^{\nu})^{-1}$.

Evaluating $((1 - e^{-\alpha_i}) f y) \cdot f_0$ at $t_0$, we get from $(\mathrm{I}_{10})$:

$$0 = -b(t_0) f_{w_0}(t_0) + b(t_0) a(t_0) f_{w_0}(t_0) \quad (\text{since } e^{-\alpha_i}(t_0) = 1).$$

But, by the choice of $t_0$, $b(t_0) \neq 0$. Hence

$$(\mathrm{I}_{11}) \qquad f_{w_0}(t_0) = a(t_0) f_{w_0}(t_0).$$

From $(\mathrm{I}_{10})$, we have

$$fy = y_0 - f_{w_0} \frac{e^{-\alpha_i}}{1 - e^{-\alpha_i}} (r_i b) \delta_{w_0} + \frac{1}{1 - e^{-\alpha_i}} ba(r_i f_{w_0}) \delta_{r_i w_0}.$$

Applying it to the function $(1 - e^{-w_0^{-1}\alpha_i}) f_0$, we get

$$\begin{aligned} fy \cdot ((1 - e^{-w_0^{-1}\alpha_i}) f_0) &= y_0 \cdot ((1 - e^{-w_0^{-1}\alpha_i}) f_0) - f_{w_0} e^{-\alpha_i} (r_i b)(w_0 f_0) \\ &\quad + ba(r_i f_{w_0}) \left( \frac{1 - e^{\alpha_i}}{1 - e^{-\alpha_i}} \right) (r_i w_0 f_0). \end{aligned}$$

Evaluating at $t_0$, we get

$$0 = f_{w_0}(t_0) b(t_0) + b(t_0) a(t_0) f_{w_0}(t_0).$$

But since $b(t_0) \neq 0$, we get

$$(\mathrm{I}_{12}) \qquad f_{w_0}(t_0) + a(t_0) f_{w_0}(t_0) = 0.$$

Adding $(\mathrm{I}_{11})$–$(\mathrm{I}_{12})$, we get $f_{w_0}(t_0) = 0$. This proves the lemma.

(2.13) *Proof of Theorem* 2.9. Let $y \in Y$. By Corollary 2.7, we can write $y = \frac{1}{f} \sum f_w y_w$, where $f, f_w \in R(T) \hookrightarrow \mathbf{C}[H]$. We can further assume, without loss of generality, that $f \in \mathbf{C}[H]$ is irreducible. By Lemma 2.11, $Z(f) \subset I_{v_0 r_i v_0^{-1}} = v_0 I_{r_i} v_0^{-1}$ for some $v_0 \in W$ and some simple reflection $r_i$. Since $\delta_{v_0} Y = Y$ and, by Lemma 2.3, $\delta_{v_0}(\sum_w R(T) y_w) = \sum_w R(T) y_w$, we can assume that $Z(f) \subset I_{r_i}$. But then Lemma 2.12 proves that $y \in \sum_w \mathbf{C}[H] y_w$.

We next observe that

$$(*) \qquad Q_W \cap \left( \sum_{w \in W} \mathbf{C}[H] y_w \right) = \sum_{w \in W} \mathbf{Q}[H] y_w,$$

where $\mathbf{Q}$ is the field of rational numbers, and $\mathbf{Q}[H] := \mathbf{Q} \otimes_{\mathbf{Z}} R(T)$.

The inclusion $\sum_w \mathbf{Q}[H] y_w \subset Q_W \cap (\sum_w \mathbf{C}[H] y_w)$ is obviously true. To prove the reverse inclusion, take $y' = \sum_{l(w) \le k} g_w y_w$ in $Q_W$, where $\{g_w\}_{l(w) \le k} \subset \mathbf{C}[H]$. Then it suffices to show that $g_{w_0} \in \mathbf{Q}[H]$ for any $w_0$ with $l(w_0) = k$: Write $y' = \sum_{l(w) \le k} q_w \delta_w$, where $q_w \in Q(T)$ (since $y' \in Q_W$). Then, by Proposition 2.6, $q_{w_0} = g_{w_0} b_{w_0^{-1}, w_0^{-1}}$. But since $b_{w_0^{-1}, w_0^{-1}} \in Q(T)$, we obtain that $g_{w_0} \in \mathbf{C}[H] \cap Q(T)$. Further (as is easy to see, e.g., by taking a basis of the $\mathbf{Q}$-vector space $\mathbf{C}$) $\mathbf{C}[H] \cap Q(T) = \mathbf{Q}[H]$. This proves the assertion $(*)$. In particular we obtain that $Y \subset \sum_{w\in W} \mathbf{Q}[H]y_w$.

So finally it suffices to show that if there is a prime integer $p$ and elements $f_w \in R(T)$ such that

$$y \cdot R(T) \subset pR(T), \quad \text{where } y = \sum_{l(w)\le k} f_w y_w ,$$

then $\frac{1}{p} f_w$ itself is in $R(T)$ for all $w$.

Fix any field $F$ of characteristic $p$. Write

$$(\mathrm{I}_{13}) \qquad y = \frac{1}{f} \sum_w a_w \delta_w ,$$

where $a_w \in R(T)$ and $f$ is of the form $\prod_\beta (1 - e^\beta)$ for $\beta$ running over some finite set of (not necessarily distinct) real roots. (This is possible, as is easy to see.) Moreover, by Proposition 2.6,

$$(\mathrm{I}_{14}) \qquad \frac{1}{f} a_{w_0} = f_{w_0} \prod_{\nu\in\Delta_+\cap w_0\Delta_-} (1 - e^\nu)^{-1} \quad \text{for any } w_0 \text{ with } l(w_0) = k.$$

Of course $(fy) \cdot R(T) \subset pR(T)$. But, by $(\mathrm{I}_{13})$, $fy = \sum_{l(w)\le k} a_w \delta_w$ and hence $(\sum_{l(w)\le k} a_w(p)\delta_w(p)) \cdot F[H] = 0$, where $F[H] := F \otimes_{\mathbf{Z}} R(T)$, $a_w(p)$ denotes the reduction $\bmod p$ of the element $a_w \in R(T)$, and $\delta_w(p)$ denotes the reduction $\bmod p$ of the operator $\delta_w \colon R(T) \to R(T)$. But the canonical representation $W \to \mathrm{Aut}(F[H])$, given by $w \mapsto \delta_w(p)$, is clearly injective and hence by [2, Corollary on p. 35], $a_w(p) = 0$ for all $w$. (Even though this corollary is stated for fields, the same proof gives its validity for integral domains, i.e., when, in the notation of loc. cit., $E$ and $E'$ are integral domains.) In particular, by $(\mathrm{I}_{14})$, $f_{w_0}(p) = 0$ since $F[H]$ is a domain and, for any real root $\beta$, $(1-e^\beta)$ is a nonzero element of $F[H]$. This proves the theorem completely.

(2.14) **Coproduct structure in $Q_W$.** Let $\Delta\colon Q_W \to Q_W \otimes_Q Q_W$ (where the tensor product over $Q$ is taken with respect to the $Q$-module structure given by the right multiplication by $Q$ on both the copies of $Q_W$) be the diagonal map defined by

$$(\mathrm{I}_{15}) \qquad \Delta(\delta_w q) = \delta_w \otimes \delta_w q = \delta_w q \otimes \delta_w \quad \text{for } w \in W \text{ and } q \in Q.$$

Clearly $\Delta$ is $Q$-linear and it is easy to see that the coproduct $\Delta$ is associative and commutative with a counit defined by $\varepsilon(\delta_w q) = q$.

We introduce an associative product structure, denoted by $\odot$, in $Q_W \otimes_Q Q_W$, making $\Delta$ into a ring homomorphism:

$$(\delta_v q_v \otimes \delta_w q_w) \odot (\delta_{v'} q_{v'} \otimes \delta_{w'} q_{w'}) = \delta_{v'(w'^{-1})vw'} q_{v'}(w'^{-1} q_v) \otimes \delta_{ww'}(w'^{-1} q_w) q_{w'}.$$

Observe that the product $\odot$ introduces a left (resp. right) $Q_W$-module structure on $Q_W \otimes_Q Q_W$ by the left (resp. right) multiplication under the ring homomorphism $\Delta$. The right $Q_W$-module structure takes a particularly simple form:

$$(y \otimes z).(\delta_w q) = y\delta_w q \otimes z\delta_w \quad \text{for } y, z \in Q_W, \ w \in W, \text{ and } q \in Q.$$

Recall the definition of $\overline{y}_w$ from Proposition 2.4. The following proposition describes the $\Delta$-map in terms of the $\{\overline{y}_w\}$ basis.

(2.15) **Proposition.** *For any* $w \in W$,

$$\Delta(\overline{y}_w) = \sum_{u, v \leq w} \overline{y}_u \otimes \overline{y}_v a^w_{u,v}$$

*for some (unique)* $a^w_{u,v} \in R(T)$. *Moreover* $a^w_{u,v}$, *considered as an element of* $\mathbf{C}[H]$, *has a zero of multiplicity*[2] $\geq l(u) + l(v) - l(w)$ *at* $1$.

*Proof.* We prove the proposition by induction on $l(w)$. By the definition,

$$y^t_{r_i} = (\delta_e - \delta_{r_i} e^{-\alpha_i})\frac{1}{1 - e^{-\alpha_i}}.$$

So

$$(\mathrm{I}_{16}) \qquad \begin{aligned} \Delta(y^t_{r_i}) &= \delta_e(1 - e^{-\alpha_i})^{-1} \otimes \delta_e + \delta_{r_i} \otimes \delta_{r_i}(1 - e^{\alpha_i})^{-1} \\ &= \overline{y}_{r_i} \otimes \overline{y}_{r_i}(1 - e^{\alpha_i}) + \delta_e \otimes \overline{y}_{r_i} e^{\alpha_i} \\ &\quad + \overline{y}_{r_i} \otimes \delta_e e^{\alpha_i} - \delta_e \otimes \delta_e e^{\alpha_i}. \end{aligned}$$

Now write $w = w' r_i$, with $w' < w$. Then

$$\begin{aligned} \Delta(\overline{y}_w) &= \Delta(\overline{y}_{w'}) \odot \Delta(\overline{y}_{r_i}) \\ &= \left(\sum_{u', v' \leq w'} \overline{y}_{u'} \otimes \overline{y}_{v'} a^{w'}_{u', v'}\right) \\ &\quad \odot \left(\delta_e \otimes \delta_e \frac{1}{1 - e^{-\alpha_i}} - \delta_{r_i} \otimes \delta_{r_i} \frac{e^{-\alpha_i}}{1 - e^{-\alpha_i}}\right) \end{aligned}$$

(by the induction hypothesis)

[2]This, by definition, is the multiplicity at 1 of the divisor of $f$.

$$= \sum_{u',v'\le w'} \bar{y}_{u'} \otimes \bar{y}_{v'} \frac{a^{w'}_{u',v'}}{1-e^{-\alpha_i}} + \sum_{u',v'\le w'} \bar{y}_{u'}\delta_{r_i} \otimes \bar{y}_{v'}\delta_{r_i} \frac{(r_i a^{w'}_{u',v'})}{1-e^{\alpha_i}}$$

$$= - \sum_{u',v'\le w'} \bar{y}_{u'} \otimes \bar{y}_{v'} e^{\alpha_i}(y_{r_i} \cdot (r_i a^{w'}_{u',v'}))$$

$$+ \sum_{u',v'\le w'} \bar{y}_{u'}\bar{y}_{r_i} \otimes \bar{y}_{v'}\bar{y}_{r_i}(1-e^{\alpha_i})(r_i a^{w'}_{u',v'})$$

$$+ \sum_{u',v'\le w'} \bar{y}_{u'}\bar{y}_{r_i} \otimes \bar{y}_{v'} e^{\alpha_i}(r_i a^{w'}_{u',v'})$$

$$+ \sum_{u',v'\le w'} \bar{y}_{u'} \otimes \bar{y}_{v'}\bar{y}_{r_i}(r_i a^{w'}_{u',v'})e^{\alpha_i},$$

where $\cdot$ is as in $(\mathrm{I}_3)$.

Now the proposition follows by using Corollary 2.5 and the fact that, for any $f \in \mathbf{C}[H]$, $y_{r_i} \cdot f$ has zero (at 1) of multiplicity $\ge$ (multiplicity of zero at 1 for $f$) $-$ 1.

(2.16) **Remark.** We will determine the coefficients $a^w_{u,v}$ explicitly in Proposition 2.25.

(2.17) **Dualizing $Q_W$.** Regarding $Q_W$ as a $Q$-module under the right multiplication, define $\Omega = \Omega(T) := \mathrm{Hom}_Q(Q_W, Q)$. Then $\Omega$ is canonically a $Q$-module under $(q\psi)(y) = q.\psi(y)$ for $q \in Q$, $y \in Q_W$ and $\psi \in \Omega$. Further the coproduct structure $\Delta$ in $Q_W$, defined in §2.14, makes $\Omega$ into an associative and commutative algebra over $Q$ with identity (since $\Delta$ has the corresponding properties).

Since any $\psi \in \Omega$ is determined uniquely by its restriction to the basis $\{\delta_w\}$ (and conversely), we can (and often will) regard $\Omega$ as the space of *all* the functions $W \to Q$. It is easy to see that (under this correspondence) the addition, scalar multiplication (by elements of $Q$), and the multiplication in $\Omega$ correspond respectively to the pointwise addition, scalar multiplication, and pointwise multiplication of functions $W \to Q$. The (multiplicative) identity, denoted by 1, (under this correspondence) is the function which takes any $w \in W$ to 1.

We also introduce the structure of a left $Q_W$-module on $\Omega$ as follows:

$$(\mathrm{I}_{17}) \qquad (y \cdot \psi)y' = \psi(y^t.y') \quad \text{for } \psi \in \Omega \text{ and } y, y' \in Q_W.$$

(Observe that the action of $y$ is $Q$-linear.)

In particular $\Omega$ gets equipped with the *Weyl group action* (which is the action of $\delta_w \in Y \subset Q_W$) and also the *Hecke operators* (which is the action of $y_w \in Y \subset Q_W$). Let us describe the action of $Y_{r_i}$, for a simple reflection $r_i$, explicitly:

$$(\mathrm{I}_{18})\qquad \begin{aligned}(y_{r_i} \cdot \psi)(\delta_w) &= \psi\left[(\delta_e - \delta_{r_i} e^{-\alpha_i})\frac{1}{1-e^{-\alpha_i}}\delta_w\right]\\ &= \frac{\psi(\delta_w) - \psi(\delta_{r_i w})e^{-w^{-1}\alpha_i}}{1-e^{-w^{-1}\alpha_i}}.\end{aligned}$$

(2.18) **Remark.** Observe that $\Omega$ has two $Q$-module structures, one coming from the scalar multiplication by elements of $Q$ (viewing $\Omega$ as the space of functions $W \to Q$) and the other coming from the action of $Q\delta_e = \delta_e Q \subset Q_W$ defined in $(\mathrm{I}_{17})$. We caution that these two $Q$-module structures are in general different. *Whenever we speak of $\Omega$ as a $Q$-module, we will always mean the first $Q$-module structure. The other $Q$-structure is distinguished by denoting it with a solid dot.*

Now we are ready to define the dual of the ring $Y$, which will play an important role in the whole paper.

(2.19) **Definition.** Let $\Psi = \{\psi \in \Omega\colon \psi(Y^t) \subset R(T)\}$; recall that $Y$ is the ring defined in Definition 2.8.

(Notice the difference in the definition of $\Psi$ with the definition of the analogous ring $\Lambda$ in [19, §4.19], where we put, in addition, some finiteness condition.)

Define certain elements $\psi^w \in \Psi$ (for any $w \in W$) by

$$(\mathrm{I}_{19})\qquad \psi^w(\overline{y}_v) = \delta_{v,w} \quad \text{for } v, w \in W,$$

where $\overline{y}_v$ is as defined in Proposition 2.4.

By Corollary 2.7, $\psi = \sum_w q^w \psi^w$ is a well-defined element of $\Omega$ for arbitrary (infinitely many of them are allowed to be nonzero) choices of $q^w \in Q$. Of course if all the $q^w$'s belong to $R(T)$, then $\psi \in \Psi$.

We have the following proposition on the structure of $\Psi$.

(2.20) **Proposition.** (a) $\Psi$ *(as defined above) is an $R(T)$-subalgebra of $\Omega$.*

(b) $\Psi$ *is stable under the (left) action of $Y \subset Q_W$. In particular, for any $w \in W$, the elements $\delta_w$ and $y_w$ act on $\Psi$.*

(c) $\Psi$ *is the direct product $\prod_w R(T)\psi^w$, i.e., any element of $\Psi$ can be uniquely written as $\sum_w f^w \psi^w$, with $f^w \in R(T)$, where infinitely many of $f^w$'s are allowed to be nonzero.*

*Proof.* (a) follows from Proposition 2.15, (b) follows from the fact that $Y$ is a subring of $Q_W$, and (c) follows from the structure theorem (Theorem 2.9) for $Y$.

(2.21) **Definition of the matrix E.** Define the $W \times W$ matrix $E = (e^{v,w})$ by $e^{v,w} = \psi^v(\delta_w)$.

The relevance of the matrix $E$ to the study of $T$-equivariant $K$-theory of generalized flag varieties will be clear in the next section.

Recall the definition of the associative algebra $\mathscr{B} = \mathscr{B}_W$ over $Q$ from [19, §4.23]. We collect some of the basic properties of the 'basis' $\{\psi^w\}$ in the following:

(2.22) **Proposition.** *For any $v, w \in W$, we have:*

(a) $e^{v,w}$ *belongs to* $R(T)$. *Moreover they are uniquely determined by the following*:

$$\delta_{v^{-1}} = \sum_{u \in W} e^{u,v} y_{u^{-1}}.$$

(b) $e^{v.w} = 0$, *unless* $v \leq w$ *and*

$$e^{w,w} = \prod_{\nu \in w^{-1}\Delta_- \cap \Delta_+} (1 - e^\nu).$$

*In particular, the matrix $E$ is upper triangular (and hence $E \in \mathscr{B}_W$). Further, since $E$ has nonzero diagonal entries, $E$ is invertible as an element of $\mathscr{B}_W$.*

(c) $B^t = E^{-1}$, *where the matrix $B$ is as in Corollary* 2.7, *and $B^t$ denotes its transpose.* (Observe that, by Proposition 2.6(a), $B^t \in \mathscr{B}_W$.)

(d) *For any simple reflection $r_i$, we have*

$$y_{r_i} \cdot \psi^w = \begin{cases} \psi^w + \psi^{r_i w} & \text{if } r_i w < w, \\ 0 & \text{otherwise.} \end{cases}$$

(e) *The element $e^{v,w} \in R(T) \subset \mathbf{C}[H]$ has a zero of multiplicity $\geq l(v)$ at the point* 1. *Moreover the $l(v)$th homogeneous component*[3] *of $e^{v,w}$ is precisely equal to $(-1)^{l(v)} d_{v,w}$, where $d_{v,w}$ is as defined in* [19, §4.21].

(f) $\psi^e(\delta_w) = e^{\rho - w^{-1}\rho}$.

(g) $(e^{-\rho}\delta_{r_i} e^\rho) \cdot \psi^w = \psi^w$ *provided* $r_i w > w$.

(h) $\psi^v \psi^w = \sum_{v,w \leq u} a^u_{v,w} \psi^u$, *where $a^u_{v,w}$ is as defined in Proposition* 2.15.

(i) *For any $\psi_1, \psi_2 \in \Omega$,*

$$y_{r_i} \cdot (\psi_1 \psi_2) = \psi_1 (y_{r_i} \cdot \psi_2) + (y_{r_i} \cdot \psi_1 - \psi_1)(\delta_{r_i} \cdot \psi_2).$$

[3]For any $f = \sum_{\lambda \in \mathfrak{h}^*_{\mathbf{Z}}} n_\lambda e^\lambda \in \mathbf{C}[H]$ and any $d \in \mathbf{Z}_+$, by the $d$ th degree homogeneous component $(f)_d$ of $f$, we mean the element $\sum_\lambda n_\lambda \lambda^d / d!$ of $S(\mathfrak{h}^*)$. Recall that the smallest $d$, such that $(f)_d \neq 0$, is multiplicity $\mathrm{mult}_1(f)$ of the zero of $f$ at 1.

*Proof.* (a) follows from the definition of $e^{v,w}$.

(b) Assume that $v \nleq w$ and assume further, by induction, that for any $u < w$, we have $e^{v,u} := \psi^v(\delta_u) = 0$. By Proposition 2.6, we can write

$$(\mathrm{I}_{20})\quad \delta_{w^{-1}} = \left(\prod_{\nu \in w^{-1}\Delta_- \cap \Delta_+} (1 - e^{\nu})\right) y_{w^{-1}} + \sum_{u<w} q_u \delta_{u^{-1}} \quad \text{for some } q_u \in Q.$$

Taking $t$ and then taking $\psi^v$ (and $\psi^w$) of both the sides of $(\mathrm{I}_{20})$, we get (b).

(c) follows from (a) and $(\mathrm{I}_5)$.

(d) For any $v \in W$, $(y_{r_i} \cdot \psi^w)(\overline{y}_v) = \psi^w((y_{v^{-1}} y_{r_i})^t)$. Hence by Corollary 2.5

$$(\mathrm{I}_{21})\qquad (y_{r_i} \cdot \psi^w)(\overline{y}_v) = \begin{cases} \psi^w(\overline{y}_v) & \text{if } r_i v < v, \\ \psi^w(\overline{y}_{r_i v}) & \text{otherwise.} \end{cases}$$

In particular, $(y_{r_i} \cdot \psi^w)(\overline{y}_v) = 0$, unless $v = w$ or $r_i w$.

*Case* I. $r_i w < w$: In this case $(y_{r_i} \cdot \psi^w)(\overline{y}_w) = (y_{r_i} \cdot \psi^w)(\overline{y}_{r_i w}) = 1$.

*Case* II. $r_i w > w$: In this case $(y_{r_i} \cdot \psi^w)(\overline{y}_w) = (y_{r_i} \cdot \psi^w)(\overline{y}_{r_i w}) = 0$.

This proves (d).

(e) Assume, by induction, that $e^{v_1, w_1}$ satisfies the assertions in (e), provided either $l(v_1) < l(v)$ or $v_1 = v$ and $l(w_1) < l(w)$. (The induction starts by (b).) Write $w = r_i w_1$, such that $w_1 < w$. By $(\mathrm{I}_{18})$,

$$(\mathrm{I}_{22})\qquad (y_{r_i} \cdot \psi^v)(\delta_{w_1}) = \frac{e^{v,w_1} - e^{v,w} e^{-w_1^{-1}\alpha_i}}{1 - e^{-w_1^{-1}\alpha_i}}.$$

Now there are two cases to consider:

*Case* I. $r_i v > v$: In this case, by (d) and $(\mathrm{I}_{22})$,

$$(\mathrm{I}_{23})\qquad e^{v,w_1} = e^{v,w} e^{-w_1^{-1}\alpha_i}.$$

*Case* II. $r_i v < v$: In this case, again by (d) and $(\mathrm{I}_{22})$,

$$e^{v,w_1} + e^{r_i v, w_1} = \frac{e^{v,w_1} - e^{v,w} e^{-w_1^{-1}\alpha_i}}{1 - e^{-w_1^{-1}\alpha_i}},$$

i.e.,

$$(\mathrm{I}_{24})\qquad e^{v,w_1} - (1 - e^{-w_1^{-1}\alpha_i})(e^{v,w_1} + e^{r_i v, w_1}) = e^{v,w} e^{-w_1^{-1}\alpha_i}.$$

So in either case, by the induction hypothesis, the first part of assertion (e) follows. The second part follows similarly by using the analogous result for $d_{v,w}$'s as deduced from [19, Proposition 4.24(b) and $\mathrm{I}_{52}$].

(f) follows by induction on $l(w)$, using $(\mathrm{I}_{23})$.

(g) follows trivially from the (d) part, if we use the identity

$$\delta_e - \delta_{r_i} = e^{\rho}(1 - e^{-\alpha_i})y_{r_i}e^{-\rho}.$$

(h) is a consequence of Proposition 2.15.

(i) follows from direct calculation by using the right $Q_W$-module structure on $Q_W \otimes_Q Q_W$, as given in §2.14, and the identity

$$\Delta(\overline{y}_{r_i}) = 1 \otimes \overline{y}_{r_i} + \overline{y}_{r_i} \otimes \delta_{r_i} - 1 \otimes \delta_{r_i}.$$

(2.23) **Remark.** The elements $\{\psi^w\}_{w\in W}$ are uniquely determined if we assume that they satisfy (d) and (f) of the above proposition and, in addition, $\psi^w(\delta_e) = 0$ for all $w \neq e$.

The proof of this remark is similar to the proof of the (e) part of the above proposition.

(2.24) **Lemma.** *For any $u, v \in W$, write*

$$(\mathrm{I}_{25}) \qquad \delta_u \cdot \psi^v = \sum_w c^u_{v,w}\psi^w \quad \textit{for some (unique) } c^u_{v,w} \in R(T)$$

(which is possible by Proposition 2.20). *Then $c^u_{v,w} = 0$ unless $l(w) \geq l(v) - l(u)$, and moreover $c^u_{v,w}$, as an element of $\mathbf{C}[H]$, has a zero at $1$ of multiplicity $\geq l(v) - l(w)$.*

*Proof.* Choose a $w_0$ such that $w_0$ is of minimal length among those $w$ satisfying $c^u_{v,w} \neq 0$. Then

$$\psi^v(\delta_{u^{-1}}\delta_{w_0}) = (\delta_u \cdot \psi^v)(\delta_{w_0}) = c^u_{v,w_0}\psi^{w_0}(\delta_{w_0})$$

(by Proposition 2.22(b) and $(\mathrm{I}_{25})$), i.e., $\psi^v(\delta_{u^{-1}w_0}) = c^u_{v,w_0}\psi^{w_0}(\delta_{w_0})$. Thus, again by Proposition 2.22(b), $v \leq u^{-1}w_0$ and hence $l(w_0) \geq l(v) - l(u)$.

The assertion about multiplicity follows similarly (by induction on $l(w)$) using Proposition 2.22(e). □

Recall the definition of $a^w_{u,v}$ (resp. $c^w_{u,v}$) from Proposition 2.15 (resp. Lemma 2.24). Even though $a^w_{u,v}$ was defined only for $u, v \leq w$, we extend it for all $u, v, w \in W$ by putting $a^w_{u,v} = 0$ otherwise (i.e., if at least one of $u$ or $v$ violates the condition $u, v \leq w$). Now we will determine $\{a^w_{u,v}\}$ and $\{c^w_{u,v}\}$ explicitly in terms of the $E$-matrix.

(2.25) **Proposition.** *Fix $w \in W$.*

(a) *Define two $W \times W$ matrices $A_w$ and $E_w$ by $A_w(u, v) = a^v_{w,u}$ and $E_w(u, v) = \delta_{u,v}e^{w,v}$ for any $u, v \in W$.* (By Proposition 2.15, $A_w$

is upper triangular and hence $A_w \in \mathscr{B}_W$, and of course $E_w \in \mathscr{B}_W$.) *Then*

$$A_w = E.E_w.E^{-1}.$$

(b) *Similarly define two matrices* $C_w$ *and* $S_w \in \mathscr{B}_W$ *by* $C_w(u, v) = c^w_{u,v}$ *and* $S_w(u, v) = \delta_{wu,v}$. *Then*

$$C_w = E.S_w.E^{-1}.$$

(Observe that $C_w \in \mathscr{B}_W$, by Lemma 2.24.)

*Proof.*

$$\begin{aligned}(A_w.E)(u, v) &= \sum_{w'} a^{w'}_{w,u} e^{w',v} \\ &= (\psi^w \psi^u)(\delta_v) \quad \text{(by Proposition 2.22(h))} \\ &= e^{w,v} e^{u,v} = (E.E_w)(u, v),\end{aligned}$$

proving (a).

The proof of (b) is similar.

(2.26) **Definition.** Let $S \subset \{1, \cdots, l\}$ be any subset. Recall the definition of $W_S$ and $W^1_S$ from §1.1. We define $\Psi^S = \Psi^{W_S}$ to be the set of all the $W_S$-invariants in $\Psi$, i.e., $\Psi^S := \{\psi \in \Psi : \delta_{r_i} \cdot \psi = \psi$, for all the simple reflections $r_i$ with $i \in S\}$.

We have the following lemma describing the structure of $\Psi^S$.

(2.27) **Lemma.** $\Psi^S = \prod_{w \in W^1_S} R(T)(e^\rho \cdot \psi^w)$.

*Proof.* By Proposition 2.22(g), for any $w \in W^1_S$, $e^\rho \cdot \psi^w \in \Psi^S$. Hence $\prod_{w \in W^1_S} R(T)(e^\rho \cdot \psi^w) \subset \Psi^S$.

Conversely, take any $\psi \in \Psi^S$ and write

$$(\mathrm{I}_{26}) \qquad \psi = \sum_w f^w (e^\rho \cdot \psi^w) \quad \text{for some } f^w \in R(T).$$

By the definition of $\Psi^S$ and the identity used in the proof of Proposition 2.22(g), we get $(y_{r_i} e^{-\rho}) \cdot \psi = 0$ for any $i \in S$. Now, by Proposition 2.22(d) and $(\mathrm{I}_{26})$, we get

$$(y_{r_i} e^{-\rho}) \cdot \psi = \sum_{r_i w < w} f^w (\psi^w + \psi^{r_i w}).$$

Hence, by Proposition 2.20, $f^w = 0$ for all those $w$ such that $r_i w < w$ for some simple reflection $r_i$ with $i \in S$. This proves the lemma. □

Finally we show that the ring $\Psi$ admits a 'natural' filtration such that the associated graded ring is isomorphic with the ring $\Lambda$ defined in [19].

Recall the definition of $\Omega$ from [19, §4.17]. (We denote this $\Omega$ by $\Omega(\mathfrak{h})$ here to distinguish it from the $\Omega$ defined in §2.17.)

(2.28) **Definition.** Define a decreasing filtration $\{F_n = F_n(\Psi)\}_{n\geq 0}$ of the ring $\Psi$ by

$$F_n = \{\psi \in \Psi\colon \operatorname{mult}_1(\psi(\delta_w)) \geq n \text{ for all } w \in W\},$$

where, for any element $f \in \mathbf{C}[H]$, we denote by $\operatorname{mult}_1(f)$ the multiplicity of the zero of $f$ at $1$; in particular for $\psi \in \Psi$, since $\psi(\delta_w) \in R(T) \subset \mathbf{C}[H]$, $\operatorname{mult}_1(\psi(\delta_w))$ makes sense.

We clearly have

$$F_n.F_m \subset F_{n+m} \quad \text{for all } n, m \in \mathbf{Z}_+.$$

We define the associated graded ring $\operatorname{Gr}(\Psi) := \sum_{n\geq 0} F_n/F_{n+1}$. Further we define a map $\tilde{\mathfrak{e}}_n\colon F_n \to \Omega(\mathfrak{h})$ by

$$(\tilde{\mathfrak{e}}_n(\psi))(\delta_w) = (\psi(\delta_w))_n \quad \text{for } \psi \in F_n \text{ and } w \in W,$$

where $(\psi(\delta_w))_n$ is the $n$-th homogeneous component of $\psi(\delta_w)$ (cf. Proposition 2.22(e)). The map $\tilde{\mathfrak{e}}_n$ obviously factors through $F_n/F_{n+1}$ to give a map $\hat{\mathfrak{e}}_n\colon F_n/F_{n+1} \to \Omega(\mathfrak{h})$. These maps give rise to a ring homomorphism $\hat{\mathfrak{e}}\colon \operatorname{Gr}(\Psi) \to \Omega(\mathfrak{h})$ defined by $\hat{\mathfrak{e}}_{|(F_n/F_{n+1})} = \hat{\mathfrak{e}}_n$ for all $n \geq 0$.

(2.29) **Lemma.** *Image* $\hat{\mathfrak{e}} \subset \Lambda$, *where* $\Lambda$ *is the subring of* $\Omega(\mathfrak{h})$ *defined in* [19, §4.19].

We denote the map $\hat{\mathfrak{e}}$ considered as a map $\operatorname{Gr}(\Psi) \to \Lambda$ by $\mathfrak{e}$.

*Proof.* Let $\psi = \sum_w f^w \psi^w \in F_n$ (with $f^w \in R(T)$). Then we assert that

$$(*) \qquad \operatorname{mult}_1(f^w) \geq n - l(w) \quad \text{for any } w \in W.$$

For, otherwise, let $w_0$ be of minimal length violating $(*)$. By Proposition 2.22(b), $\psi(\delta_{w_0}) = f^{w_0}\psi^{w_0}(\delta_{w_0})$. But, by assumption, $\psi \in F_n$ and by Proposition 2.22(b), $\operatorname{mult}_1(\psi^{w_0}(\delta_{w_0})) = l(w_0)$. Hence $\operatorname{mult}_1(f^{w_0}) \geq n - l(w_0)$, contradicting the assumption! This proves $(*)$.

As a consequence of $(*)$ and Proposition 2.22(e), we obtain that $\tilde{\mathfrak{e}}_n(\psi) = \sum_{l(w)\leq n} (f^w)_{n-l(w)}(-1)^{l(w)}\xi^w$, where $\xi^w \in \Lambda$ is as defined in [19, Proposition 4.20]. In particular $\tilde{\mathfrak{e}}_n(\psi) \in \Lambda$. This proves the lemma. □

Since $\Lambda$ is a $\mathbf{C}$-vector space, by extension of scalars, we get a map

$$\mathfrak{e}_{\mathbf{C}}\colon \mathbf{C} \otimes_{\mathbf{Z}} \operatorname{Gr}(\Psi) \to \Lambda.$$

Now we have the following.

(2.30) **Proposition.** *The map* $\mathfrak{e}_{\mathbf{C}}\colon \mathbf{C} \otimes_{\mathbf{Z}} \operatorname{Gr}(\Psi) \to \Lambda$ *defined above is a ring isomorphism.*

*Proof*. Of course $\mathfrak{e}_{\mathbf{C}}$ is a ring homomorphism. We first prove the surjectivity of $\mathfrak{e}_{\mathbf{C}}$:

For any $w \in W$, by Proposition 2.22(e), $\psi^w \in F_{l(w)}$. Let $\overline{\psi}^w$ denote $\psi^w \bmod F_{l(w)+1}$. Then $\mathfrak{e}(\overline{\psi}^w) = (-1)^{l(w)}\xi^w$ (see the proof of the above lemma). Also for any $p \in S^n(\mathfrak{h}^*)$, there exists $f \in \mathbf{C}[H] \approx \mathbf{C} \otimes_{\mathbf{Z}} R(T)$ such that $\mathrm{mult}_1 f \geq n$ and $(f)_n = p$. In particular $p1 \in \mathrm{Image}(\mathfrak{e}_{\mathbf{C}})$. So the surjectivity of $\mathfrak{e}_{\mathbf{C}}$ follows from the structure of $\Lambda$ [19, Proposition 4.20].

The injectivity of $\mathfrak{e}_{\mathbf{C}}$ is easy to see.

## 3. Identification of $\Psi$ with the $T$-equivariant $K$-theory $K_T(G/B)$

*We continue to use the same notation and assumptions as in the first paragraph of* §2.

(3.1) **Definition.** Let $X$ be a compact (Hausdorff) topological space on which a compact group $G_0$ acts. For any $p \in \mathbf{Z}$, recall the definition of the $G_0$-equivariant $K$-group $K^p_{G_0}(X)$ from [29]. *In the sequel* $K_{G_0}(X)$ (*without a superscript*) *will always mean* $K^0_{G_0}(X)$. Let us just recall that $K_{G_0}(X)$ is the Grothendieck group associated to the semigroup, whose elements are the isomorphism classes of the $G_0$-equivariant complex vector bundles on the $G_0$-space $X$.

Now let $X$ be a Hausdorff (not necessarily compact) topological space on which the compact group $G_0$ acts. Assume further that $X$ has a filtration $\mathscr{X}\colon \varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots$, such that

(1) each $X_n$ is a compact subspace of $X$ which is $G_0$-stable, and

(2) topology of $X$ is the limit topology induced from the filtration $\mathscr{X}$.

Then we define, for any $p \in \mathbf{Z}$,

$$K^p_{G_0}(X) = \operatorname*{Inv\,lim}_{n\to\infty} K^p_{G_0}(X_n).$$

It is easy to see that $K^p_{G_0}(X)$ does not depend (up to a 'canonical' isomorphism) upon the particular choice of the filtration satisfying (1) and (2) as above (since any such filtration is cofinal in any other). Of course $K^*_{G_0}(X)$ is a graded algebra over $K_{G_0}$ (pt.), where pt. denotes a one point space.

In particular, for any Kac-Moody group $G$ and any standard parabolic subgroup $P = P_S$ (cf. §1.3), $K^*_T(G/P)$ makes sense, where $T$ is the standard compact maximal torus which acts on $G/P$ by the left multiplication.

Moreover $K_T^*(G/P)$ is an algebra over $K_T(\text{pt.}) \approx R(T)$. (The Bruhat decomposition, cf. §1.4, provides a desired filtration of $G/P$ as below.)

$$(\mathrm{I}_{27}) \qquad X_n(P) := \bigcup_{\substack{w \in W \\ l(w) \leq n}} (BwP/P).$$

*We often abbreviate* $X_n(B)$ *by* $X_n$ *itself*, where $B$ is the standard Borel subgroup of $G$.

(3.2) **Definition.** Fix a simple reflection $r_i$, and let $P_i := B \cup (Br_iB)$ be the corresponding (*standard*) *minimal parabolic subgroup*. The group $P_i$ has a natural two-dimensional representation $V_i$ ($V_i$ also denotes the underlying representation space) such that the 'unipotent radical' of $P_i$ (with Lie algebras $\sum_{\alpha \in \Delta_t \setminus \{\alpha_i\}} \mathfrak{g}_\alpha$) acts trivially on $V_i$, and the 'standard maximal reductive subgroup' of $P_i$ (of rank 1) (with Lie algebra $\mathfrak{h} \oplus \mathbf{C}e_i \oplus \mathbf{C}f_i$, cf. §1.1) acts by the highest weight $\rho_i$ (cf. §1.2).

(3.3) **Lemma.** *With the notation as above, the canonical* $\mathbf{P}^1$*-fibration* $\pi_i \colon G/B \to G/P_i$ *is* $G$*-equivariantly isomorphic with the projective bundle of the rank-two vector bundle on* $G/P_i$*, which is obtained from the principal* $P_i$*-bundle* $G \to G/P_i$ *by the representation* $V_i$ *defined above.*

*Proof.* We have the following commutative diagram:

$$\begin{tikzcd} G/B \arrow[rr, "\theta_i"] \arrow[dr, "\pi_i"'] & & \mathbf{P}(G \times_{P_i} V_i) \arrow[dl, "\tilde{\pi}_i"] \\ & G/P_i & \end{tikzcd}$$

where $\tilde{\pi}_i$ is the canonical projection, and $\theta_i$ is defined by $\theta_i(g \bmod B) = [g, v_i]$ (where $v_i$ is some fixed nonzero highest weight vector in $V_i$ and $[g, v_i]$ denotes the class of the element $(g, v_i)$ in $\mathbf{P}(G \times_{P_i} V_i)$).

It is easy to see that $\theta_i$ is a $G$-equivariant homeomorphism. □

Let us recall the following consequence of the equivariant Thom isomorphism (which can be viewed as a generalization of Bott-periodicity). (Even though a more general statement is true, the version given below is sufficient for our purposes.)

(3.4) **Proposition** [29, Proposition 3.9]. *Let* $p \colon E \to X$ *be a* $T$*-equivariant rank-two vector bundle on a compact space* $X$, *and let* $\mathbf{P}(E)$ *denote the corresponding projective bundle. Then* $K_T(\mathbf{P}(E))$ *is a free module over* $K_T(X)$ *with* (*free*) *generators* 1 *and the Hopf bundle* $H \in K_T(\mathbf{P}(E))$, *where, recall that, the Hopf bundle* $H$ *is the dual of the canonical line bundle on* $\mathbf{P}(E)$.

*In particular, the canonical map* $K_T(X) \to K_T(\mathbf{P}(E))$ *is injective.*

So we can identify $K_T(X)$ with its image in $K_T(\mathbf{P}(E))$.

As a consequence of the above proposition and Lemma 3.3, we get the following:

(3.5) **Corollary.** *For any* $n \in \mathbf{Z}_+$ *and* $1 \le i \le l$, $K_T(\pi_i^{-1}(X_n(P_i)))$ *is a free module over* $K_T(X_n(P_i))$ *with (free) generators* 1 *and the Hopf bundle* $H_i(n)$, *where* $X_n(P_i)$ *is defined in* $(\mathrm{I}_{27})$.

(3.6) **Definition.** For any $n \in \mathbf{Z}_+$ and $1 \le i \le l$, define an operator

$$D_{r_i}(n) \colon K_T(\pi_i^{-1}(X_n(P_i))) \to K_T(\pi_i^{-1}(X_n(P_i)))$$

by

$$D_{r_i}(n)(\sigma + H_i(n)\tau) = \sigma \quad \text{for } \sigma, \tau \in K_T(X_n(P_i)).$$

(3.7) **Lemma.** *For any* $n \in \mathbf{Z}_+$ *and any* $1 \le i \le l$, *the following diagram is commutative:*

$$\begin{array}{ccc} K_T(\pi_i^{-1}(X_{n+1}(P_i))) & \longrightarrow & K_T(\pi_i^{-1}(X_n(P_i))) \\ \uparrow{\scriptstyle D_{r_i}(n+1)} & & \uparrow{\scriptstyle D_{r_i}(n)} \\ K_T(\pi_i^{-1}(X_{n+1}(P_i))) & \longrightarrow & K_T(\pi_i^{-1}(X_n(P_i))) \end{array}$$

*where the horizontal maps are the canonical restriction maps.*

*Proof*. It suffices to show that $H_i(n+1)|_{\pi_i^{-1}(X_n(P_i))} = H_i(n)$. But this is clear from Lemma 3.3.

(3.8) **Definition.** For any simple reflection $r_i$, define an operator $D_{r_i} \colon K_T(G/B) \to K_T(G/B)$ as the inverse limit of the operators $D_{r_i}(n)$: $K_T(\pi_i^{-1}(X_n(P_i))) \to K_T(\pi_i^{-1}(X_n(P_i)))$ (cf. Lemma 3.7).

It can be easily seen that the operator $D_{r_i}$ does not depend upon the particular choice of the $i$th fundamental weight $\rho_i$, even though the isomorphism of Lemma 3.3 does depend on the choice of $\rho_i$ (as $V_i$ depends upon the choice of $\rho_i$).

Now, for $w \in W$, define $D_w \colon K_T(G/B) \to K_T(G/B)$ as the composite $D_w = D_{r_{i_1}} \circ \cdots \circ D_{r_{i_m}}$, where $w = r_{i_1} \cdots r_{i_m}$ is a reduced decomposition. We will see, during the proof of Theorem 3.13, that $D_w$ does not depend upon the choice of the reduced decomposition of $w$.

Of course, quite analogously, one can also define the operators (again denoted by) $D_w \colon K(G/B) \to K(G/B)$.

(3.9) **Remark.** Similar operators on $R(T)$ (see Definition 3.17(b)), introduced by Demazure [7, §5], provided motivation for our definition of the $D_{r_i}$'s.

Clearly $D_{r_i}$ satisfies the following:

(3.10) **Lemma.** $D_{r_i}^2 = D_{r_i}$.

(3.11) **Definition (Weyl group action on $K_T(G/B)$).** Recall that the Weyl group $W$ can be canonically identified with $N_K(T)/T$, where $N_K(T)$ denotes the normalizer of $T$ in the standard unitary form $K$ of $G$ (cf. §1.3). Now $W$ acts on $G/B \approx K/T$ by

$$(n \bmod T).(k \bmod T) = (kn^{-1}) \bmod T,$$
$$\text{for } n \bmod T \in W \approx N_K(T)/T \text{ and } k \in K.$$

Clearly the action of $W$ on $G/B$ commutes with the action of $T$ on $G/B$, and hence we obtain a left action of $W$ on $K_T(G/B)$ (and also on $K(G/B)$). (Since $K_T$ is a contravariant functor, action of the element $w \in W$ on $K_T(G/B)$ is induced from the action of $w^{-1}$ on $G/B$.)

(3.12) **Definition (the localization map).** For any $n \geq 0$, let $\hat{\gamma}_n: K_T(X_n) \to K_T(X_n^T)$ be the canonical restriction map; where $X_n^T$ is the set of all the $T$-fixed points in $X_n$, and $X_n = X_n(B)$ is as defined in $(\mathrm{I}_{27})$. Since the maps $\{\hat{\gamma}_n\}_{n\geq 0}$ are compatible, we get a map $\hat{\gamma}: K_T(G/B) \to K_T((G/B)^T)$.

Now the map $i: W \approx N_K(T)/T \to (G/B)^T$, given by $w \mapsto w^{-1}$ mod $B$, induces a homeomorphism; provided we put the discrete topology on $W$. Moreover, by [29, Proposition 2.2], $K_T(W)$ can be canonically identified (as an algebra over $R(T)$) with the $R(T)$-subalgebra of $\Omega$ (cf. §2.17) consisting of precisely those maps $W \to Q$ which have image $\subset R(T)$. Hence, on composition of $\hat{\gamma}$ with the induced map $i^*$, we get an $R(T)$-algebra homomorphism

$$\overline{\gamma}: K_T(G/B) \to \Omega.$$

Now we can state our second main theorem of this paper.

(3.13) **Theorem.** *Let $G$ be an arbitrary (not necessarily symmetrizable) Kac-Moody group with Borel subgroup $B$. Then the map $\overline{\gamma}: K_T(G/B) \to \Omega$, defined above, has its image precisely equal to $\Psi$ (see Definition* 2.19).

*Let $\gamma$ be the map $\overline{\gamma}$, considered as a map $K_T(G/B) \to \Psi$. Then the map $\gamma$ is an $R(T)$-algebra isomorphism. Further the action of the Weyl group element $w \in W$ (Definition* 3.11) *and the operator $D_w$ (Definition* 3.8) *correspond, under $\gamma$, to the action of $\delta_w$ and $y_w$ respectively* (cf. *Proposition* 2.20).

*Moreover $K_T^p(G/B) = 0$ for odd values of $p$.*

(3.14) **Remark.** A characterization of the $R(T)$-'basis' $\{\tau^w := \gamma^{-1}(\psi^w)\}_{w\in W}$ of $K_T(G/B)$ (cf. Proposition 2.20) will be given in Proposition 3.39.

As a preparation for the proof of the above theorem, we have the following.

(3.15) **Lemma.** *For any* $n \geq 0$, $K_T^p(X_n, X_{n-1}) = 0$ *for* $p$ *odd, and* $K_T^p(X_n, X_{n-1})$ *is a free* $R(T)$*-module for* $p$ *even.*

*In particular,* $K_T^p(X_n) = 0$ *for* $p$ *odd, and* $K_T^p(X_n)$ *is a free module over* $R(T)$ *for* $p$ *even.*

*Moreover,* $\mathrm{Rank}_{R(T)} K_T(X_n) = \#\{w \in W : l(w) \leq n\}$.

*Proof.* By [29, Proposition 2.9],

$$K_T^p(X_n, X_{n-1}) \approx K_T^p(X_n \setminus X_{n-1}) \approx \bigoplus_{l(w)=n} K_T^p(BwB/B).$$

Further the $T$-space $BwB/B$ is $T$-equivariantly homeomorphic with the $T$-module $\mathfrak{n}_w := \bigoplus_{\alpha \in \Delta_+ \cap w\Delta_-} \mathfrak{g}_\alpha$. (The homeomorphism is established by the exponential map.) Hence, by the Thom isomorphism [29, Proposition 3.2], $K_T^p(BwB/B) \approx K_T^p(\mathrm{pt.})$ as $R(T)$-modules. This gives the first part of the lemma.

The second part follows from the first by induction on $n$ and the long exact sequence associated to the pair $(X_n, X_{n-1})$ [29, §2]. □

(3.16) **Remark.** Let $P$ be any standard parabolic subgroup of $G$. Then the above lemma remains true (by the same proof; in view of the Bruhat decomposition for $G/P$) for $X_n$ replaced by $X_n(P)$ (cf. $(\mathrm{I}_{27})$). In this case

$$\mathrm{Rank}_{R(T)} K_T(X_n(P)) = \#\{w \in W_S^1 : l(w) \leq n\}.$$

We recall the following:

(3.17) **Definitions.** (a) *Atiyah-Hirzebruch homomorphism* . Let $\beta : R(T) \to K_T(G/B)$ be the additive map, which takes $e^\lambda \in X(T)$ to the $G$-equivariant (in particular a $T$-equivariant) line bundle on $G/B$ associated to the principal $B$-bundle $G \to G/B$ by the character $e^\lambda : B \to \mathbf{C}\backslash\{0\}$. (Although $e^\lambda$ is a character of $H$, it is extended to the whole of $B$ by defining it to be identically one on the commutator subgroup $[B, B]$.) Of course $\beta$ is a ring homomorphism, but it is *not* an $R(T)$-algebra homomorphism. One also has $\beta_1 : R(T) \to K(G/B)$, which is the composite of $\beta$ with the canonical homomorphism $K_T(G/B) \to K(G/B)$.

Further, we define a map $\overline{\beta} : R(T) \to \Psi \subset \Omega$ by $\overline{\beta}(f) = f \cdot 1$; where $1$ is the multiplicative identity of $\Psi$ and $\cdot$ is as defined in $(\mathrm{I}_{17})$. Let $\overline{\beta}_1 : R(T) \to \mathbf{Z} \otimes_{R(T)} \Psi$ be the composite of $\overline{\beta}$ with the canonical map $\Psi \to \mathbf{Z} \otimes_{R(T)} \Psi$, where $\mathbf{Z}$ is a $R(T)$-module under the standard augmentation map $er : R(T) \to \mathbf{Z}$ (which takes every $f \in R(T) \mapsto f(1)$).

It is easy to see that

$$(\mathrm{I}_{28}) \qquad \overline{\beta}(f)(y) = y^t \cdot f \quad \text{for any } y \in Q_W \text{ and } f \in R(T).$$

In particular,

$$(\mathrm{I}_{29}) \qquad \overline{\beta}(f)(\delta_w) = w^{-1} f \quad \text{for } w \in W.$$

By $(\mathrm{I}_{29})$, $\overline{\beta}$ is an injective ring homomorphism.

(b) *Demazure operators* [7]. For any simple reflection $r_i$, define

$$L_{r_i}(e^\lambda) := y_{r_i} \cdot e^\lambda = \frac{e^\lambda - e^{r_i\lambda - \alpha_i}}{1 - e^{-\alpha_i}} \quad \text{for } e^\lambda \in X(T),$$

and extend additively to $R(T)$. (It is easy to see that $L_{r_i}(e^\lambda) \in R(T)$.) Now set, for any $w \in W$, $L_w = L_{r_{i_1}} \circ \cdots \circ L_{r_{i_m}}$; where $w = r_{i_1} \cdots r_{i_m}$ is any reduced decomposition. Then, by Proposition 2.4, $L_w$ does not depend upon the particular choice of the reduced decomposition of $w$.

Now we have the following

(3.18) **Lemma.** *The following diagram is commutative*:

$$\begin{tikzcd} R(T) \arrow[rr, "\beta"] \arrow[dr, "\overline{\beta}"'] & & K_T(G/B) \arrow[dl, "\overline{\gamma}"] \\ & \Omega & \end{tikzcd}$$

*Further the maps $\beta$ and $\overline{\beta}$ commute with the Weyl group actions and moreover, for any $w \in W$ and $a \in R(T)$, $\overline{\beta}(L_w a) = y_w \cdot (\overline{\beta} a)$.*

(As a consequence of §3.19—Assertions I and III, we also have $\beta \circ L_w = D_w \circ \beta$.)

*Proof.* Fix any $v \in W$ and a representative $\tilde{v}$ for $v$ in $N_K(T)$. For any integral weight $\lambda$, let $\mathbf{C}_\lambda$ denote the one-dimensional representation of $B$ with the character $e^\lambda$. Then for any $t \in T$ and $x \in \mathbf{C}_\lambda$ (in the line bundle $\beta(e^\lambda)$),

$$t.(\tilde{v}, x) = (t\tilde{v}, x) = (\tilde{v}, (\tilde{v}^{-1} t \tilde{v}).x) = (\tilde{v}, e^{v\lambda}(t)x).$$

This gives that $(\overline{\gamma} \circ \beta(e^\lambda))(\delta_{v^{-1}}) = e^{v\lambda}$. In particular, by $(\mathrm{I}_{29})$, the commutativity of the above triangle follows.

The assertion that $\overline{\beta}$ commutes with $W$-actions follows from $(\mathrm{I}_{29})$, and the assertion $y_w \cdot (\overline{\beta} a) = \overline{\beta}(L_w a)$ follows from $(\mathrm{I}_{29})$ and $(\mathrm{I}_{18})$. Of course the map $\beta$ commutes with the $W$-actions.

With these preparations, we now come to the

(3.19) *Proof of Theorem* 3.13. The proof is slightly long and will be broken up into several subassertions:

**Assertion I.** *The map $\bar{\gamma}$ is injective*: It suffices to show that $\hat{\gamma}_n: K_T(X_n) \to K_T(X_n^T)$ is injective for all $n \in \mathbf{Z}_+$: By the localization theorem [29, Proposition 4.1], the localized map $\tilde{\gamma}_n: Q\otimes_{R(T)}K_T(X_n) \to Q\otimes_{R(T)}K_T(X_n^T)$ is an isomorphism, where (as in §2) $Q = Q(T)$ is the quotient field of $R(T)$. But, by Lemma 3.15, $K_T(X_n)$ is a free $R(T)$-module, and hence the canonical map $K_T(X_n) \to Q\otimes_{R(T)} K_T(X_n)$ is injective. Now the following commutative diagram proves the assertion:

$$\begin{array}{ccc} K_T(X_n) & \xrightarrow{\widehat{\gamma_n}} & K_T(X_n^T) \\ \big\downarrow & & \big\downarrow \\ Q\otimes_{R(T)} K_T(X_n) & \underset{\tilde{\gamma}_n}{\xrightarrow{\sim}} & Q\otimes_{R(T)} K_T(X_n^T) \end{array} \quad \square$$

Corresponding to any $1 \le i \le l$, there is a Hopf bundle $H_i \in K_T(G/B)$, which is the inverse limit (over $n$) of $H_i(n) \in K_T(\pi_i^{-1}(X_n(P_i)))$ (see the proof of Lemma 3.7). Also recall the definition of the map $\beta$ from Definition 3.17(a).

**Assertion II.** *The element $H_i \in K_T(G/B)$ is the same as $\beta(e^{-\rho_i})$, where $\rho_i$ is the $i$th fundamental weight* (*cf.* §1.2): Let us fix a nonzero highest weight vector $v_i \in V_i$, where $V_i$ is as defined in Definition 3.2. Consider the following commutative diagram:

$$\begin{array}{ccccc} G\times_B \mathbf{C}v_i & \xrightarrow{\tilde{\theta}_i} & \tilde{\pi}_i^*(G\times_{P_i} V_i) & \longrightarrow & G\times_{P_i} V_i \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G/B & \xrightarrow{\theta_i} & \mathbf{P}(G\times_{P_i} V_i) & \xrightarrow{\tilde{\pi}_i} & G/P_i \end{array}$$

where the maps $\theta_i$ and $\tilde{\pi}_i$ are as defined in the proof of Lemma 3.3, the vertical maps are the canonical projections, $\tilde{\pi}_i^*(G\times_{P_i} V_i)$ is the pull-back of the bundle $G\times_{P_i} V_i$ via the map $\tilde{\pi}_i$, and $\tilde{\theta}_i$ is induced from the canonical inclusion $G\times_B \mathbf{C}v_i \hookrightarrow G\times_B V_i$.

Now, from the definition of the Hopf bundle, it is easy to see that Image $\tilde{\theta}_i \subset H_i^*$ (where $H_i^*$ is the dual of the Hopf bundle). Further, since $\tilde{\theta}_i$ is an injective map, we have Image $\tilde{\theta}_i = H_i^*$, i.e., the bundle $G\times_B \mathbf{C}v_i$ represents the element $H_i^* \in K_T(G/B)$. But $\mathbf{C}v_i$ has character (as a $B$-module) $e^{\rho_i}$. This proves Assertion II.

**Assertion III.** *For any simple reflection $r_i$ and $\tau \in K_T(G/B)$, $\overline{\gamma}(D_{r_i}\tau) = y_{r_i} \cdot (\overline{\gamma}\tau)$*: We have the following commutative diagram:

$$\begin{array}{ccc} K_T(G/P_i) & \xrightarrow{\hat{\gamma}^{P_i}} & K_T((G/P_i)^T) \\ \downarrow & & \downarrow \\ K_T(G/B) & \xrightarrow{\hat{\gamma}} & K_T((G/B)^T). \end{array}$$

Let $\tau \in K_T(G/B)$ be in the image of $K_T(G/P_i)$. Then $D_{r_i}\tau = \tau$. Also, by the above diagram, $\hat{\gamma}(\tau)(w \bmod B) = \hat{\gamma}(\tau)(wr_i \bmod B)$, i.e., $\overline{\gamma}(\tau)(\delta_w) = \overline{\gamma}(\tau)(\delta_{r_iw})$ for any $w \in W$. Hence, by $(\mathrm{I}_{18})$, $y_{r_i} \cdot (\overline{\gamma}\tau) = \overline{\gamma}(\tau)$.

Further define $\Omega^{r_i} = \{\psi \in \Omega\colon \psi(\delta_w) = \psi(\delta_{r_iw})$ for all $w \in W\}$. Now $y_{r_i} \cdot (\psi\psi') = \psi(y_{r_i} \cdot \psi')$, for any $\psi \in \Omega^{r_i}$ and any $\psi' \in \Omega$ (by Proposition 2.22(i)). In particular, to establish Assertion III, it suffices to show that $y_{r_i} \cdot (\overline{\gamma}(H_i)) = 0$, where $H_i$ is the Hopf bundle as in Assertion II:

By Assertion II, Lemma 3.18, and the identity $(\mathrm{I}_{29})$, we get

$$(\mathrm{I}_{30}) \qquad \overline{\gamma}(H_i)(\delta_w) = e^{-w^{-1}\rho_i} \quad \text{for any } w \in W.$$

Hence, by $(\mathrm{I}_{18})$, $y_{r_i} \cdot (\overline{\gamma}(H_i)) = 0$.

**Remark.** Since the map $\overline{\gamma}$ is injective (by Assertion I), we get (by Proposition 2.4) that the operator $D_w$ (see Definition 3.8) is well defined, i.e., it does not depend upon the particular choice of a reduced decomposition of $w$.

**Assertion IV.** Image $\overline{\gamma} \subset \Psi$: Fix any $\tau \in K_T(G/B)$ and $w \in W$. By making successive use of Assertion III, we get that

$$y_w \cdot (\overline{\gamma}\tau) = \overline{\gamma}(D_w\tau).$$

In particular, $[y_w \cdot (\overline{\gamma}\tau)](\delta_e) = \overline{\gamma}(D_w\tau)(\delta_e)$. But, of course, $\overline{\gamma}(D_w\tau)(\delta_e) \in R(T)$ and hence $\overline{\gamma}(\tau) \in \Psi$ by the definition of $\Psi$ (cf. Definition 2.19) and the structure theorem (Theorem 2.9).

**Assertion V.** *Given any $w \in W$, there exists an element $\vartheta^w \in K_T(G/B)$ such that $\overline{\gamma}(\vartheta^w)(\delta_w) = \prod_{\nu \in \Delta_+ \cap w^{-1}\Delta_-}(1 - e^{\nu})$, and $\overline{\gamma}(\vartheta^w)(\delta_v) = 0$ if $l(v) \leq l(w)$ and $v \neq w$*: Let $l(w) = n$ and consider the exact sequence

$$0 \to K_T^0(X_n, X_n^w) \to K_T^0(X_n) \to K_T^0(X_n^w) \to 0,$$

where $X_n^w := \bigcup_{l(v) \leq n,\, v \neq w^{-1}} (BvB/B)$.

(The facts that $K_T^1(X_n, X_n^w)$ and $K_T^{-1}(X_n^w)$ are zero follow from the proof of Lemma 3.15.)

Now $K_T(X_n, X_n^w) \approx K_T(Bw^{-1}B/B) \approx K_T(\mathfrak{n}_{w^{-1}})$ (see the proof of Lemma 3.15). Recall from [29, §3] that there is the Thom isomorphism $\varphi_*: K_T(\text{pt.}) \xrightarrow{\sim} K_T(\mathfrak{n}_{w^{-1}})$. By the definition $\varphi_*(1) = \Lambda_E^{\cdot}$, where $E = \mathfrak{n}_{w^{-1}}$, $p: E \to \text{pt.}$ is the projection, $\varphi: \text{pt.} \to E$ is the zero section, and $\Lambda_E^{\cdot}$ is the Koszul complex on $E$ formed from $p^*(E)$ and the diagonal map $\delta: E \to p^*(E)$.

Since $K_T(\mathfrak{n}_{w^{-1}}) \approx K_T(X_n, X_n^w)$, we can think of $\varphi_*(1)$ as an element of $K_T(X_n, X_n^w)$ and hence, by restriction, we get an element $\overline{\varphi_*(1)} \in K_T(X_n)$. Lift $\overline{\varphi_*(1)}$ to an element $\vartheta^w$ of $K_T(G/B)$ (which is possible by Lemma 3.15). By the projection formula [29, §3], $\varphi^*\varphi_* f = f.\lambda_{-1}(E)$, for any $f \in K_T(\text{pt.})$, where

$$\lambda_{-1}(E) := \sum_k (-1)^k \Lambda^k(E) \in K_T(\text{pt.}).$$

Now it is easy to see that

$$\begin{aligned} \overline{\gamma}(\vartheta^w)(\delta_w) = \overline{\varphi_*(1)}_{|_{\{w^{-1} \bmod B\}}} &= \sum_k (-1)^k \operatorname{ch}_T(\Lambda^k(E)) \\ &= \prod_{\nu \in \Delta_+ \cap w^{-1}\Delta_-} (1 - e^{\nu}) \end{aligned}$$

(where ch denotes the character) and by the choice of $\vartheta^w$, $\overline{\gamma}(\vartheta^w)(\delta_v) = \overline{\varphi_*(1)}_{|_{\{v^{-1} \bmod B\}}} = 0$ if $l(v) \le l(w)$ and $v \ne w$.

**Assertion VI.** $\overline{\gamma}(K_T(G/B)) \supset \Psi$: Fix any $\psi \in \Psi$. We will construct, by induction on $n$, certain elements $\tau_n \in K_T(X_n)$ satisfying:

$C_1(n)$ $\qquad (\overline{\gamma}(\tau_n) - \psi)(\delta_w) = 0$ for all $l(w) \le n$, and

$C_2(n)$ $\qquad \tau_{n|_{X_{n-1}}} = \tau_{n-1}.$

Existence of $\tau_0$ satisfying $C_1(0)$ and $C_2(0)$ is trivial. Assume (by induction) the existence of $\tau_n$ (satisfying $C_1(n)$ and $C_2(n)$). Arbitrarily choose an element $\tilde{\tau}_n \in K_T(X_{n+1})$ such that $\tilde{\tau}_{n|X_n} = \tau_n$ (use Lemma 3.15). Now, for any $v \in W$ of length $n+1$, we have (from Assertion IV and Propositions 2.20 and 2.22(b)) $(\overline{\gamma}(\tilde{\tau}_n) - \psi)(\delta_v) = f^v e^{v,v}$ for some $f^v \in R(T)$, where $e^{v,v} = \prod_{\nu \in \Delta_+ \cap v^{-1}\Delta_-}(1 - e^{\nu})$. Now put $\tau_{n+1} = \tilde{\tau}_n - \sum_{l(v)=n+1} f^v(\vartheta^v_{|_{X_{n+1}}})$, where $\vartheta^v$ is as constructed in Assertion V. It is easy to see that $\tau_{n+1}$ satisfies $C_1(n+1)$ and $C_2(n+1)$.

By property $(C_2)$, the sequence $(\tau_n)_{n\geq 0}$ defines an element $\tau \in K_T(G/B)$. Further $\bar{\gamma}(\tau) = \psi$, since

$$\begin{aligned}(\bar{\gamma}(\tau) - \psi)(\delta_w) &= (\bar{\gamma}(\tau_n) - \psi)(\delta_w) \quad \text{for any } n \geq l(w) \\ &= 0 \quad \text{by } C_1(n).\end{aligned}$$

**Assertion VII.** $\bar{\gamma}$ *commutes with the Weyl group actions*: Observe that, for any $w \in W$, one has a commutative diagram:

$$\begin{array}{ccc} K_T(G/B) & \xrightarrow{\hat{\gamma}} & K_T((G/B)^T) \\ \big\downarrow{\scriptstyle w^*} & & \big\downarrow{\scriptstyle \tilde{w}^*} \\ K_T(G/B) & \xrightarrow{\hat{\gamma}} & K_T((G/B)^T) \end{array}$$

where $w^*$ (resp. $\tilde{w}^*$) denotes the map induced from the action of $w$ on $G/B$ (resp. the action of $w$ on $(G/B)^T$). This easily proves the assertion.

Now putting Assertions I–VII together, we get Theorem 3.13. □

As corollaries of Theorem 3.13, we deduce the following results.

(3.20) **Corollary.** *With the notation and assumptions as in Theorem 3.13, let $P = P_S$ be the standard parabolic subgroup of $G$ corresponding to any subset $S \subset \{1, \cdots, l\}$. Then there is a unique $R(T)$-algebra isomorphism $\gamma^P$ making the following diagram commutative*:

$$\begin{array}{ccc} K_T(G/P) & \xrightarrow{\gamma^P} & \Psi^S \\ \big\downarrow{\scriptstyle \pi_P^*} & & \big\downarrow \\ K_T(G/B) & \underset{\sim}{\xrightarrow{\gamma}} & \Psi \end{array}$$

*where $\Psi^S$ is as defined in Definition 2.26, and $\pi_P^*$ is induced from the canonical projection $\pi_P: G/B \to G/P$.*

*In particular, the map $\pi_P^*$ is injective with its image exactly equal to the $W_S$-invariants in $K_T(G/B)$. Taking $P = G$, we get that $[K_T(G/B)]^W \simeq R(T)$.*

*Further $K_T^p(G/P) = 0$ for odd $p$.*

*Proof*. The assertion, that $K_T^p(G/P) = 0$ for odd $p$, follows from Remark 3.16.

Since the map $\gamma$ commutes with the Weyl group actions, it suffices to show that the map $\pi_P^*$ is injective with its image exactly equal to the $W_S$-invariants in $K_T(G/B)$:

For any $w \in W_S$, we have the commutative triangle:

$$\begin{tikzcd} G/B \arrow[rr, "w."] \arrow[dr, "\pi_P"'] & & G/B \arrow[dl, "\pi_P"] \\ & G/P & \end{tikzcd}$$

where $w.$ denotes the action of $w$ on $G/B$ as in Definition 3.11. In particular, $\operatorname{Image} \pi_P^* \subset [K_T(G/B)]^{W_S}$.

We first prove the injectivity of $\pi_P^*$: We have the following commutative diagram, in which both the horizontal maps are injective (by §3.19—Assertion I):

$$\begin{tikzcd} K_T(G/P) \arrow[r, hook] \arrow[d, "\pi_P^*"] & K_T((G/P)^T) \arrow[d, "\tilde{\pi}_P^*"] \\ K_T(G/B) \arrow[r, hook] & K_T((G/B)^T) \end{tikzcd}$$

where $\tilde{\pi}_P^*$ is induced from the map $\tilde{\pi}_P : (G/B)^T \to (G/P)^T$. But the map $\tilde{\pi}_P$ is surjective; in fact under the isomorphism $i: W \to (G/B)^T$ (given in Definition 3.12) and a similar isomorphism $i_S: W_S\backslash W \to (G/P)^T$, the map $\tilde{\pi}_P$ is the canonical projection $W \to W_S\backslash W$. In particular, the map $\tilde{\pi}_P^*$ is injective and hence, by the above diagram, $\pi_P^*$ itself is injective.

Finally we prove the surjectivity of $\pi_P^*$ onto $[K_T(G/B)]^{W_S}$ or (what is the same as) the surjectivity of $\gamma \circ \pi_P^*$ onto $\Psi^S$. To achieve this, we first of all observe that in §3.19—Assertion V if we take $w \in W_S^1$ (cf. §1.1), then we can in fact choose $\vartheta^w \in \pi_P^*(K_T(G/P))$ (and satisfying the requirements in Assertion V). Now the desired surjectivity of $\gamma \circ \pi_P^*$ follows by an argument similar to the proof of Assertion VI.

(3.21) **Remark.** Recall that the structure of $\Psi^S$ is given in Lemma 2.27.

Actually one can improve upon the above corollary further.

(3.22) **Definition.** Fix a subset $S \subset \{1, \cdots, l\}$. Let $\Theta$ be a subset of $W$ with the following properties:

($P_1$) $\Theta$ is left $W_S$-stable, and

($P_2$) whenever $w \in \Theta$ and $w' \leq w$, then $w' \in \Theta$.

To any such $\Theta$, we can associate a (left) $B$-stable subspace $V_\Theta \subset G/P_S$ defined by

$$V_\Theta = \bigcup_{w \in \Theta} (Bw^{-1}P_S/P_S).$$

By Property $(\mathrm{P}_2)$, $V_\Theta$ is closed in $G/P_S$, and conversely any (left) $B$-stable closed subspace of $G/P_S$ is $V_\Theta$, for some appropriate choice of $\Theta$. In particular, the Schubert varieties $X_w^P := \overline{BwP/P} \subset G/P$ are such examples.

Let $\Omega_\Theta$ denote the $Q$-algebra of all the maps $\Theta \to Q$. There is of course the restriction map $r_\Theta\colon \Omega \to \Omega_\Theta$. Define $\Psi_\Theta^S = r_\Theta(\Psi^S)$.

Now we have the following corollary of Corollary 3.20.

(3.23) **Theorem.** *With the notation and assumptions as in Corollary 3.20, assume, in addition, that $\Theta$ is a subset of $W$ satisfying $(\mathrm{P}_1)$ and $(\mathrm{P}_2)$ as above. Then there is a unique $R(T)$-algebra isomorphism $\gamma_\Theta = \gamma_\Theta^P$, making the following diagram commutative:*

$$\begin{array}{ccc} K_T(G/P) & \xrightarrow[\sim]{\gamma^P} & \Psi^S \\ {\scriptstyle j_\Theta^*}\big\downarrow & & \big\downarrow{\scriptstyle r_\Theta} \\ K_T(V_\Theta) & \xrightarrow{\gamma_\Theta} & \Psi_\Theta^S \end{array}$$

*where $j_\Theta^*$ is induced from the inclusion $j_\Theta\colon V_\Theta \hookrightarrow G/P$.*

*Proof.* We first observe that the map $j_\Theta^*$ is surjective: Fix $\tau \in K_T(V_\Theta)$ and construct (by induction on $n$) elements $\tilde{\tau}_n \in K_T(X_n(P))$ satisfying

(1) $\tilde{\tau}_{n|_{X_n(P)\cap V_\Theta}} = \tau_{|_{X_n(P)\cap V_\Theta}}$, and

(2) $\tilde{\tau}_{n|_{X_{n-1}(P)}} = \tilde{\tau}_{n-1}$.

Having constructed $\tilde{\tau}_n \in K_T(X_n(P))$ as above, let

$$\tau'_{n+1} \in K_T(X_n(P) \cup (X_{n+1}(P) \cap V_\Theta))$$

be an (in fact unique) element such that $\tau'_{n+1|_{X_n(P)}} = \tilde{\tau}_n$ and $\tau'_{n+1|_{X_{n+1}(P)\cap V_\Theta}} = \tau_{|_{X_{n+1}(P)\cap V_\Theta}}$. The existence (and the uniqueness) of $\tau'_{n+1}$ is guaranteed from the following Mayer-Vietoris exact sequence:

$$\begin{aligned} 0 &\to K_T(X_n(P) \cup (X_{n+1}(P) \cap V_\Theta)) \\ &\to K_T(X_n(P)) \oplus K_T(X_{n+1}(P) \cap V_\Theta) \to K_T(X_n(P) \cap V_\Theta) \to 0. \end{aligned}$$

(The exactness of the Mayer-Vietoris sequence is known for any cohomology theory; see, e.g., [9, Chapter I]. Also use the fact that $K_T^p(X_n(P) \cap V_\Theta)$ as well as $K_T^p(X_n(P) \cup (X_{n+1}(P) \cap V_\Theta)) = 0$ for odd $p$; cf. the proof of Lemma 3.15.)

Now let $\tilde{\tau}_{n+1} \in K_T(X_{n+1}(P))$ be an arbitrary element such that

$$\tilde{\tau}_{n+1|_{X_n(P)\cup(X_{n+1}(P)\cap V_\Theta)}} = \tau'_{n+1}.$$

This completes the induction.

By property (1), the element $\tilde{\tau} \in K_T(G/P)$, determined by the compatible sequence $(\tilde{\tau}_n)_n$, of course satisfies $\tilde{\tau}_{|_{V_\Theta}} = \tau$, which proves the surjectivity of $j_\Theta^*$.

So it suffices to prove that $\gamma^P(\operatorname{Ker} j_\Theta^*) = \operatorname{Ker}(r_\Theta)$. Consider the following commutative diagram arising from the localization maps:

$$\begin{array}{ccc}
K_T(G/P) & \xrightarrow{\hat{\gamma}^P} & K_T((G/P)^T) \\
{\scriptstyle j_\Theta^*}\downarrow & & \downarrow{\scriptstyle \tilde{j}_\Theta^*} \\
K_T(V_\Theta) & \xrightarrow{\hat{\gamma}_\Theta} & K_T(V_\Theta^T)
\end{array}$$

By the localization theorem, the localization maps $\hat{\gamma}^P$ and $\hat{\gamma}_\Theta$ are both injective (see the proof of §3.19—Assertion I). Hence $\operatorname{Ker} j_\Theta^* = (\hat{\gamma}^P)^{-1}(\operatorname{Ker} \tilde{j}_\Theta^*)$. This readily gives that $\gamma^P(\operatorname{Ker} j_\Theta^*) = \operatorname{Ker}(r_\Theta)$.

The following lemma gives the structure of $\Psi_\Theta^S$.

(3.24) **Lemma.** $\Psi_\Theta^S \cong \prod_{w \in W_S^1 \cap \Theta} R(T)(r_\Theta(e^\rho \cdot \psi^w))$.

*Proof.* We first claim that for any $w \notin \Theta$, $r_\Theta(e^\rho \cdot \psi^w) = 0$: For $(e^\rho \cdot \psi^w)(\delta_v) = \psi^w(\delta_v).(v^{-1}\rho) = 0$, if $v \in \Theta$ (by Proposition 2.22(b) and property $(P_2)$ of $\Theta$). Further let

$$(\mathrm{I}_{31}) \qquad \sum_{w \in \Theta} f^w r_\Theta(e^\rho \cdot \psi^w) = 0 \quad \text{for some } f^w \in R(T).$$

(We allow infinitely many of $f^w$'s to be nonzero.) If possible, pick a $w_0 \in \Theta$ such that $f^{w_0} \neq 0$ and $w_0$ is of smallest length with this property. Now evaluating the identity $(\mathrm{I}_{31})$ at $\delta_{w_0}$ and applying Proposition 2.22(b), we get a contradiction!

So the lemma follows by using Lemma 2.27. □

Now we can prove the nonequivariant analogues of Theorem 3.13, Corollary 3.20, and Theorem 3.23 using the corresponding results in the equivariant case.

We first prove the following:

(3.25) **Proposition.** *The canonical map* $\tilde{\varepsilon}$: $\mathbf{Z} \otimes_{R(T)} K_T(G/B) \to K(G/B)$ *is an isomorphism, where* $\mathbf{Z}$ *is considered as an* $R(T)$*-module under the standard augmentation map* $R(T) \to \mathbf{Z}$ *(given by the evaluation at* 1).

(3.26) *Proof.* We break the proof into the following four assertions:

**Assertion I.** *The canonical map* $\tilde{\varepsilon}_n$: $\mathbf{Z} \otimes_{R(T)} K_T(X_n) \to K(X_n)$ *is an isomorphism for any* $n \geq 0$. We prove it by induction on $n$. We have, by

Lemma 3.15, a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbf{Z}\otimes_{R(T)} K_T(X_{n+1}, X_n) \arrow[r] \arrow[d] & \mathbf{Z}\otimes_{R(T)} K_T(X_{n+1}) \arrow[r] \arrow[d] & \mathbf{Z}\otimes_{R(T)} K_T(X_n) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K(X_{n+1}, X_n) \arrow[r] & K(X_{n+1}) \arrow[r] & K(X_n) \arrow[r] & 0 \end{tikzcd}$$

(The top horizontal sequence is exact because $K_T(X_n)$ is a free $R(T)$-module.) Further, by [29, Proposition 2.9],

$$K_T(X_{n+1}, X_n) \approx \sum_{l(w)=n+1} K_T(BwB/B),$$

and the same is true with $K_T$ replaced by $K$. Hence, by induction on $n$ and the five lemma, it suffices to show that the canonical map

$$(*) \qquad \mathbf{Z}\otimes_{R(T)} K_T(BwB/B) \to K(BwB/B)$$

is an isomorphism for any $w \in W$.

We have the following commutative diagram:

$$\begin{tikzcd} \mathbf{Z}\otimes_{R(T)} K_T(\mathrm{pt.}) \arrow[r, "\mathrm{Id}\otimes\varphi_*", "\sim"'] \arrow[d, "\wr"] & \mathbf{Z}\otimes_{R(T)} K_T(BwB/B) \arrow[d] \\ K(\mathrm{pt.}) \arrow[r, "\varphi_*", "\sim"'] & K(BwB/B) \end{tikzcd}$$

where the maps $\varphi_*$ are Thom isomorphisms (cf. §3.19—proof of Assertion V), and the left vertical map is an isomorphism since $K_T(\mathrm{pt.}) \approx R(T)$. This establishes the claim and hence Assertion I.

**Assertion II.** *The map $\tilde{\varepsilon}\colon \mathbf{Z}\otimes_{R(T)} K_T(G/B) \to K(G/B)$ is surjective.* Take any $\sigma = (\sigma_n) \in K(G/B)$, where $\sigma_n \in K(X_n)$. We assume, by induction on $n$, that we have constructed $\tau_n \in K_T(X_n)$ satisfying:

$$d_1(n) \qquad \tau_{n|X_{n-1}} = \tau_{n-1},$$

$$d_2(n) \qquad \varepsilon_n(\tau_n) = \sigma_n,$$

where $\varepsilon_n\colon K_T(X_n) \to K(X_n)$ is the canonical map.

One has the following commutative diagram (in which both the horizontal rows are exact):

$$\begin{tikzcd} 0 \arrow[r] & K_T(X_{n+1}, X_n) \arrow[r, "\eta_1"] \arrow[d, "\eta_3"] & K_T(X_{n+1}) \arrow[r, "\eta_2"] \arrow[d, "\varepsilon_{n+1}"] & K_T(X_n) \arrow[r] \arrow[d, "\varepsilon_n"] & 0 \\ 0 \arrow[r] & K(X_{n+1}, X_n) \arrow[r, "\eta_4"] & K(X_{n+1}) \arrow[r, "\eta_5"] & K(X_n) \arrow[r] & 0. \end{tikzcd}$$

Now choose any $\tilde{\tau}_{n+1} \in K_T(X_{n+1})$ such that $\eta_2(\tilde{\tau}_{n+1}) = \tau_n$. We can write $\varepsilon_{n+1}(\tilde{\tau}_{n+1}) - \sigma_{n+1} = \eta_4\eta_3(\hat{\tau}_{n+1})$ for some $\hat{\tau}_{n+1} \in K_T(X_{n+1}, X_n)$ (since $\eta_3$ is surjective; cf. proof of Assertion I). Put $\tau_{n+1} = \tilde{\tau}_{n+1} - \eta_1(\hat{\tau}_{n+1})$; then $\eta_2(\tau_{n+1}) = \eta_2(\tilde{\tau}_{n+1}) - \eta_2\eta_1(\hat{\tau}_{n+1}) = \tau_n$, and $\varepsilon_{n+1}(\tau_{n+1}) = \sigma_{n+1} + \eta_4\eta_3(\hat{\tau}_{n+1}) - \varepsilon_{n+1}\eta_1(\hat{\tau}_{n+1}) = \sigma_{n+1}$. So the induction is complete.

But then $(\tau_n)_n$ defines an element $\tau \in K_T(G/B)$ such that $\tilde{\varepsilon}(1 \otimes \tau) = \sigma$.

**Assertion III.** *Recall the definition of* $\tau^w$ *from Remark* 3.14. *Then for any* $n \geq 0$, $\{\tau^w{}_{|_{X_n}}\}_{l(w) \leq n}$ *is an* $R(T)$*-basis of* $K_T(X_n)$, *and* $\tau^w{}_{|_{X_n}} = 0$ *for any* $l(w) > n$: Take any $l(w) > n$. Since the localization map $K_T(X_n) \to K_T(X_n^T)$ is injective (cf. proof of Assertion I in §3.19), to prove that $\tau^w{}_{|_{X_n}} = 0$, it suffices to observe that $\gamma(\tau^w)(\delta_v) = \psi^w(\delta_v) = 0$ for any $l(v) \leq n$ (by Proposition 2.22(b)).

Since the restriction map $K_T(G/B) \to K_T(X_n)$ is surjective, $\{\tau^w{}_{|_{X_n}}\}_{l(w) \leq n}$ spans (over $R(T)$) $K_T(X_n)$ (by Theorem 3.13 and Proposition 2.20(c)). Further, by Lemma 3.15, $K_T(X_n)$ is a free $R(T)$-module of rank $= \#\{w \in W : l(w) \leq n\}$ and hence by (a subsequent) Lemma 4.5 the assertion follows.

**Assertion IV.** *The map* $\tilde{\varepsilon} : \mathbf{Z} \otimes_{R(T)} K_T(G/B) \to K(G/B)$ *is injective.* One has the canonical injective maps:

$$\delta : K_T(G/B) \hookrightarrow \prod_{n=0}^{\infty} K_T(X_n) \quad \text{and} \quad \delta_1 : K(G/B) \hookrightarrow \prod_{n=0}^{\infty} K(X_n).$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Z} \otimes_{R(T)} K_T(G/B) \arrow[rr, "\tilde{\varepsilon}"] \arrow[d, "\mathrm{Id} \otimes \delta"] & & K(G/B) \arrow[d, "\delta_1"] \\
\mathbf{Z} \otimes_{R(T)} (\prod_{n=0}^{\infty} K_T(X_n)) \arrow[rr, "\hat{\varepsilon} \circ \theta", "\sim"'] \arrow[dr, "\sim", "\theta"'] & & \prod_{n=0}^{\infty} K(X_n) \\
& \prod_{n=0}^{\infty} (\mathbf{Z} \otimes_{R(T)} K_T(X_n)) \arrow[ur, "\sim", "\hat{\varepsilon}"'] &
\end{tikzcd}$$

where the map $\theta$ is the canonical map, and $\hat{\varepsilon} = \prod_{n=0}^{\infty} \tilde{\varepsilon}_n$ (cf. Assertion I). By [5, p. 62, Exercise 9] the map $\theta$ is an isomorphism and, by Assertion I, the map $\hat{\varepsilon}$ is an isomorphism, and hence $\hat{\varepsilon} \circ \theta$ is an isomorphism.

So, to prove that $\tilde{\varepsilon}$ is injective, we need to show that $\mathrm{Id} \otimes \delta$ is injective: Let $1 \otimes \tau \in \mathrm{Ker}(\mathrm{Id} \otimes \delta)$ for some $\tau \in K_T(G/B)$, i.e., $1 \otimes \tau_n = 0$ as an element of $\mathbf{Z} \otimes_{R(T)} K_T(X_n)$ for all $n$, where $\tau_n$ is the restriction of $\tau$ to $X_n$. By Proposition 2.20(c) and Theorem 3.13, we can write $\tau = \sum_w f^w \tau^w$ for some (unique) $f^w \in R(T)$, where $\tau^w$ is as in Assertion III. By Assertion III, we obtain that $f^w \in R^+(T)$ for all $w \in W$, where $R^+(T)$ is the standard augmentation ideal of $R(T)$. Fix a finite set $\{f^j\}$

of generators of the $R(T)$-module $R^+(T)$, so that we can write $f^w = \sum_j f^j a^{j,w}$ for some $a^{j,w} \in R(T)$. Define the element $\tau^j = \sum_w a^{j,w}\tau^w \in K_T(G/B)$. Then $\tau = \sum f^j \tau^j$ and hence $1 \otimes \tau = \sum f^j \otimes \tau^j = 0$. This proves Assertion IV.

Now putting Assertions I–IV together, we get Proposition 3.25.

(3.27) **Remark.** An identical proof, as above, gives the following generalization of Proposition 3.25.

*The canonical map* $\mathbf{Z} \otimes_{R(T)} K_T(V_\Theta) \to K(V_\Theta)$ *is an isomorphism, where* $V_\Theta \subset G/P$ *is any* $B$*-stable closed subspace as in Definition* 3.22.

*In fact one can similarly prove that for any subtorus* $T' \subset T$, *the canonical map* $R(T') \otimes_{R(T)} K_T(V_\Theta) \to K_{T'}(V_\Theta)$ *is an isomorphism.*

As an immediate consequence of Theorem 3.13 and Proposition 3.25, we get the following nonequivariant analog of Theorem 3.13.

(3.28) **Theorem.** *With the notation and assumptions as in Theorem* 3.13, *there is a unique* $\mathbf{Z}$*-algebra isomorphism* $\gamma_1 : K(G/B) \to \mathbf{Z} \otimes_{R(T)} \Psi$ *making the following diagram commutative*:

$$\begin{array}{ccc} K_T(G/B) & \xrightarrow{\gamma} & \Psi \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow \\ K(G/B) & \xrightarrow{\gamma_1} & \mathbf{Z} \otimes_{R(T)} \Psi \end{array}$$

*where the vertical maps are the canonical maps.*

*Moreover the action of the Weyl group element* $w \in W$ (*Definition* 3.11) *and the operator* $D_w$ (*Definition* 3.8) *correspond, under* $\gamma_1$, *to the action of* $\mathrm{Id} \otimes \delta_w$ *and* $\mathrm{Id} \otimes y_w$ *on* $\mathbf{Z} \otimes_{R(T)} \Psi$ *respectively.* (Observe that the actions of $\delta_w$ and $y_w$ being $R(T)$-linear, $\mathrm{Id} \otimes \delta_w$ and $\mathrm{Id} \otimes y_w$ make sense.)

*Further* $K^p(G/B) = 0$ *for odd* $p$. $\square$

We also obtain the following nonequivariant analog of Theorem 3.23 as a consequence of Theorem 3.23 and Remark 3.27.

(3.29) **Theorem.** *With the notation and assumptions as in Theorem* 3.23, *there is a unique* $\mathbf{Z}$*-algebra isomorphism* $\gamma_{\Theta,1}$ *making the following diagram commutative*:

$$\begin{array}{ccc} K_T(V_\Theta) & \underset{\sim}{\xrightarrow{\gamma_\Theta}} & \Psi_\Theta^S \\ \downarrow & & \downarrow \\ K(V_\Theta) & \xrightarrow[\gamma_{\Theta,1}]{} & \mathbf{Z} \otimes_{R(T)} \Psi_\Theta^S \end{array}.$$

*If we take* $\Theta = W$, *we of course get the above theorem for* $G/P$.

(3.30) **Remark.** By virtue of Theorem 3.13 (resp. Theorem 3.28), study of the $R(T)$-algebra $K_T(G/B)$ (resp. $\mathbf{Z}$-algebra $K(G/B)$), together with the Weyl group action and the operators $D_w$, reduces to an algebraic (or combinatorial) problem of understanding the $R(T)$-algebra $\Psi$ along with the action of the ring $Y$ on $\Psi$ (which is defined purely and explicitly in terms of the Weyl group and its action on $R(T)$). In particular, the product (as well as the Weyl group action) in $K_T(G/B)$ in terms of the $\{\tau^w\}$-'basis' can explicitly be determined from the $E$-matrix by Proposition 2.25. Further, the action of the operators $D_w$ on $K_T(G/B)$ can be determined by Proposition 2.22(d). Of course the structure of $\Psi$ as an $R(T)$-module is given by Proposition 2.20.

Similarly, by Theorems 3.23 and 3.29, the study of $K_T(V_\Theta)$ (in particular $K_T(G/P)$) and $K(V_\Theta)$ reduces to the understanding of the $R(T)$-algebra $\Psi_\Theta^S$. Recall that the structure of $\Psi_\Theta^S$ (as an $R(T)$-module) is given by Lemma 3.24.

It may be mentioned that the proof of Theorem 3.13 (and consequently Theorems 3.23, 3.28, 3.29, and Corollary 3.20) did not require the structure theorem (Theorem 2.9), provided we replace the $R(T)$-algebra $\Psi$ by the algebra $(\Pi_{w\in W} R(T)\psi^w) \subset \Omega$. $\square$

The proofs given above can be adopted to the $T$-equivariant singular cohomology $H_T^*(.) = H_T^*(., \mathbf{Z})$ (with integer coefficients) to obtain the following results: Recall the definition of the ring $\Lambda$ and a basis $\{\xi^w\}_{w\in W}$ of $\Lambda$ from [19, §4]. Now let $\Lambda_{\mathbf{Z}} := \sum_w S_{\mathbf{Z}}\xi_w \subset \Lambda$, where $S_{\mathbf{Z}} = S(\mathfrak{h}_{\mathbf{Z}}^*)$ is the symmetric algebra of the weight lattice $\mathfrak{h}_{\mathbf{Z}}^*$ (cf. §1.2).

(3.31) **Theorem.** *Let $G$ be an arbitrary (not necessarily symmetrizable) Kac-Moody group with Borel subgroup $B$. Then:*

(a) *There is a 'natural' $S_{\mathbf{Z}} \approx H_T^*(\text{pt.})$-algebra isomorphism $\eta: H_T^*(G/B) \to \Lambda_{\mathbf{Z}}$, such that the action of the Weyl group element $w$ (resp. the analog of the BGG operators) on $H_T^*(G/B)$ corresponds under $\eta$ to the action of $\delta_w$ (resp. $x_w$) on $\Lambda_{\mathbf{Z}}$ defined in* [19, §4.17].

*More generally, there is a 'natural' $S_{\mathbf{Z}}$-algebra isomorphism $\eta_\Theta: H_T^*(V_\Theta) \to \Lambda_{\mathbf{Z},\Theta}^S$, where $V_\Theta$ is as defined in Definition* 3.22, *and $\Lambda_{\mathbf{Z},\Theta}^S$ is the image of $\Lambda_{\mathbf{Z}}^S$ (which is the set of $W_S$-invariants in $\Lambda_{\mathbf{Z}}$) under the map $r_\Theta$ defined in* [19, §5.14].

(b) *The canonical map $\mathbf{Z}\otimes_{S_{\mathbf{Z}}} H_T^*(G/B) \to H^*(G/B)$ is an isomorphism, where $\mathbf{Z}$ is a $S_{\mathbf{Z}}$-module under the canonical augmentation map $S_{\mathbf{Z}} \to \mathbf{Z}$ (given by the evaluation at* 0).

*More generally, the canonical map $\mathbf{Z}\otimes_{S_{\mathbf{Z}}} H_T^*(V_\Theta) \to H^*(V_\Theta)$ is an isomorphism.*

(3.32) **Remarks.** (a) The fact that $\eta$ is an isomorphism (as in Theorem 3.31) has recently been obtained by Arabia [1], but he takes the complex coefficients.

(b) Combining (a) and (b) of the above theorem, we can easily deduce [19, Theorem (5.12), Corollaries (5.13), and Theorem (5.16)], in fact over $\mathbf{Z}$ and for arbitrary Kac-Moody groups. (In [19] we had the symmetrizability restriction on $G$.) In particular, we obtain here a very different (and conceptually better!) proof of these results than given in [19]. □

Now we want to characterize the 'basis' $\{\tau^w\}$ of $K_T(G/B)$ given in Remark 3.14. Recall that we are denoting the (standard) complex maximal torus of $G$ by $H$.

(3.33) **Definition** [33]. For a (finite-dimensional) $H$-algebraic variety $X$ over $\mathbf{C}$ (i.e. $H$ acts on $X$ such that the action $H \times X \to X$ is algebraic), we denote by $K^0(H, X)$ (resp. $K_0(H, X)$) the Grothendieck group constructed from the semigroup whose elements are the isomorphism classes of $H$-equivariant locally free sheaves (resp. $H$-equivariant coherent sheaves) on $X$. (We have preferred to use the notation $K_0(H, X)$ instead of Thomason's $G_0(H, X)$.)

(3.34) **Bott-Samelson-Demazure-Hansen varieties.** Fix $v \in W$ and a reduced decomposition $v = r_{i_1} \cdots r_{i_m}$. Let $\mathfrak{v}$ denote the sequence $(r_{i_1}, \cdots, r_{i_m})$ of simple reflections, and (for any $1 \le j \le m$) $\mathfrak{v}[j] := (\gamma_{i_1}, \cdots, \gamma_{i_j})$. To the sequence $\mathfrak{v}$, there is associated a smooth projective $H$-variety $Z_{\mathfrak{v}}$ over $C$ of dimension $m$ (called the Bott-Samelson-Demazure-Hansen variety), and a continuous map $\theta_{\mathfrak{v}}: Z_{\mathfrak{v}} \to G/B$ (see, e.g., [21, §2.1] in the form convenient for our purposes). Further, denoting $\mathfrak{v}' = \mathfrak{v}[m-1]$, there is an $H$-equivariant $\mathbf{P}^1$-bundle $\pi_{\mathfrak{v}'}: Z_{\mathfrak{v}} \to Z_{\mathfrak{v}'}$, which is the pull-back of the $\mathbf{P}^1$-bundle $\pi_{i_m}: G/B \to G/P_{i_m}$ under the composite map $Z_{\mathfrak{v}'} \xrightarrow{\theta_{\mathfrak{v}'}} G/B \xrightarrow{\pi_{i_m}} G/P_{i_m}$. Moreover, the $\mathbf{P}^1$-bundle $\pi_{\mathfrak{v}'}$ is the projective bundle of a rank-2, $H$-equivariant algebraic vector bundle (i.e. $H$-equivariant locally free sheaf) on $Z_{\mathfrak{v}'}$.

In particular, making successive use of Proposition 3.4 for the $\mathbf{P}^1$-bundles:

$$Z_{\mathfrak{v}} \xrightarrow{\pi_{\mathfrak{v}[m-1]}} Z_{\mathfrak{v}[m-1]} \longrightarrow Z_{\mathfrak{v}[m-2]} \longrightarrow \cdots \longrightarrow Z_{\mathfrak{v}[1]} \longrightarrow \{\text{pt.}\},$$

and an analogous result for $K^0(H, \cdot)$ [33, Theorem 3.1], we easily obtain the following:

(3.35) **Proposition.** *With the notation as above, the canonical map $K^0(H, Z_{\mathfrak{v}}) \to K_T(Z_{\mathfrak{v}})$ is an isomorphism.*

For any $H$-equivariant locally free (more generally coherent) sheaf $\mathscr{S}$ on $Z_{\mathfrak{v}}$, the cohomology spaces $H^k(Z_{\mathfrak{v}}, \mathscr{S})$ are finite dimensional $H$-modules. Let $\operatorname{ch} H^k(Z_{\mathfrak{v}}, \mathscr{S}) \in R(T)$ define its character. As is standard, define

$$\chi(Z_{\mathfrak{v}}, \mathscr{S}) = \sum_k (-1)^k \operatorname{ch} H^k(Z_{\mathfrak{v}}, \mathscr{S}) \in R(T).$$

Clearly $\chi(Z_{\mathfrak{v}}, \cdot)$ extends to give a $R(T)$-linear map $K^0(H, Z_{\mathfrak{v}}) \to R(T)$.

Fix $v$ and $\mathfrak{v}$ as in §3.34, and take any $\tau \in K_T(G/B)$. Then, by the above proposition, the element $\theta_{\mathfrak{v}}^*(\tau)$ in $K_T(Z_{\mathfrak{v}})$ can also be thought of as a (unique) element in $K^0(H, Z_{\mathfrak{v}})$. In particular, $\chi(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*(\tau))$ makes sense. Also the operation which takes a vector bundle to its dual, gives a map $*: K_T(G/B) \to K_T(G/B)$. Similarly the ring $R(T)$ admits an involution (again denoted by) $*$; defined by $e^\lambda \mapsto e^{-\lambda}$ for any $e^\lambda \in X(T)$.

With this notation, we have the following.

(3.36) **Proposition.** *Fix any $v \in W$ and a reduced decomposition $v = r_{i_1} \cdots r_{i_m}$. Then, for any $\tau \in K_T(G/B)$,*

$$\chi(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*(*\tau)) = *([y_v \cdot \gamma(\tau)](\delta_e)),$$

*where $\mathfrak{v}$ is the sequence $(r_{i_1}, \cdots, r_{i_m})$, and the map $\gamma$ is as in Theorem* 3.13. *In particular, $\chi(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*(*\tau))$ does not depend upon the particular choice of reduced decomposition of $v$. Also*

$$\chi(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*(\beta(e^{-\lambda}))) = *(L_v(e^\lambda)),$$

*for any $e^\lambda \in X(T)$, where $\beta$ and $L_v$ are as defined in Definition* 3.17.

*Proof.* We first prove that

$$(*) \qquad \chi(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*(*\tau)) = \chi(Z_{\mathfrak{v}[m-1]}, \theta_{\mathfrak{v}[m-1]}^*(*(D_{r_{i_m}} \tau))).$$

Write

$$(\mathrm{I}_{32}) \qquad \tau = \pi_{i_m}^* \tau' + H_{i_m} \pi_{i_m}^* \tau'',$$

where $H_{i_m}$ is the Hopf bundle defined in §3.19—Assertion II, and $\tau'$, $\tau'' \in K_T(G/P_{i_m})$.

Let $\mathscr{S}$ be an $H$-equivariant locally free sheaf on $Z_{\mathfrak{v}'}$ ($\mathfrak{v}' := \mathfrak{v}[m-1]$). By the Leray spectral sequence for the $\mathbf{P}^1$-bundle $\pi_{\mathfrak{v}'}: Z_{\mathfrak{v}} \to Z_{\mathfrak{v}'}$ and the projection formula, we get

$$(\mathrm{I}_{33}) \qquad H^k(Z_{\mathfrak{v}}, \pi_{\mathfrak{v}'}^*(\mathscr{S})) \approx H^k(Z_{\mathfrak{v}'}, \mathscr{S}) \quad \text{for all } k.$$

Further the line bundle $\theta_{\mathfrak{v}}^*(*H_{i_m})$ on $Z_{\mathfrak{v}}$, which can canonically be given the structure of an algebraic line bundle, is of degree $-1$ along the fibres of $\pi_{\mathfrak{v}'}$ (by §3.19—Assertion II). Hence by a result of Grothendieck, the direct images $R^k\pi_{\mathfrak{v}'*}(\theta_{\mathfrak{v}}^*(*H_{i_m})) = 0$ for all $k \geq 0$. In particular, by the projection formula,

$$R^k\pi_{\mathfrak{v}'_*}(\theta_{\mathfrak{v}}^*(*H_{i_m}) \otimes \pi_{\mathfrak{v}'}^*(\mathscr{S})) \approx (R^k\pi_{\mathfrak{v}'_*}(\theta_{\mathfrak{v}}^*(*H_{i_m}))) \otimes \mathscr{S} = 0.$$

So, by the Leray spectral sequence,

$$(\mathrm{I}_{34}) \qquad H^k(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*(*H_{i_m}) \otimes \pi_{\mathfrak{v}'}^*(\mathscr{S})) = 0 \quad \text{for all } k \geq 0.$$

Now combining $(\mathrm{I}_{32})$–$(\mathrm{I}_{34})$ and using the definition of the operator $D_{r_{i_m}}$ (Definition 3.8), we obtain $(*)$.

Making successive use of $(*)$, together with Theorem 3.13, we get the first part of the proposition.

The assertion about $\chi(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*(\beta(e^{-\lambda})))$ follows from Lemma 3.18 and $(\mathrm{I}_{29})$.

(3.37) **Definition.** For any $v \in W$ and $\tau \in K_T(G/B)$, define the *'virtual' Euler-Poincaré characteristic* $\tilde{\chi}(X_v, \tau) := \chi(Z_{\mathfrak{v}}, \theta_{\mathfrak{v}}^*\tau) \in R(T)$, where $v = r_{i_1} \cdots r_{i_m}$ is a reduced decomposition, $\mathfrak{v}$ is the sequence $(r_{i_1}, \cdots, r_{i_m})$, and $X_v$ is the Schubert variety $\overline{BvB/B} \subset G/B$.

By the above proposition, $\tilde{\chi}(X_v, \tau)$ is well defined, i.e., it does not depend on the particular choice of reduced decomposition of $v$.

(3.38) **Remark.** As in [21, §1.8], we put the "stable variety structure" on $X_v$. Now take $\tau \in K_T(G/B)$. If $\tau_{|_{X_v}}$ is in the image of the canonical map $K^0(H, X_v) \to K_T(X_v)$ then, by [21, Theorem 2.16(3)] (or [22]), $\tilde{\chi}(X_v, \tau) = \chi(X_v, \tilde{\tau})$, where $\tilde{\tau}$ is any element in $K^0(H, X_v)$ such that $\tilde{\tau}$ goes to $\tau$ under the above map.

It is likely that, in the arbitrary Kac-Moody situation, $K^0(H, X_v) \to K_T(X_v)$ is always surjective (e.g. it is surjective in the finite case; as we will see in the next section, Theorem 4.4). In any case, any element in the image of the Atiyah-Hirzebruch homomorphism $\beta$ of course comes from $K^0(H, X_v)$.

As a corollary of Proposition 3.36, we have the following characterization of the 'basis' $\{\tau^w\}$ of $K_T(G/B)$ given in Remark 3.14:

(3.39) **Proposition.** *The 'basis' $\{\tau^w\}_{w\in W}$ of $K_T(G/B)$ satisfies the following*:

$$\tilde{\chi}(X_{v^{-1}}, *\tau^w) = \delta_{v,w}.$$

*Moreover, if* $\{\tilde{\tau}^w\}$ *is any indexed set of elements in* $K_T(G/B)$ *satisfying* $\tilde{\chi}(X_{v^{-1}}, *\tilde{\tau}^w) = \delta_{v,w}$, *then* $\tau^w = \tilde{\tau}^w$ *for all* $w \in W$.

*In particular, in the finite case, the basis* $\{a_w\}_{w \in W}$ *of the* $\mathbb{Z}$*-module* $K(G/B)$ *given by Demazure* [7, Proposition 7] *is related to our basis* $\{\tau^w\}$ *as follows*:

$$\varepsilon(*\tau^{w^{-1}}) = a_w \quad \textit{for any } w \in W,$$

*where* $\varepsilon\colon K_T(G/B) \to K(G/B)$ *is the canonical map.*

*Proof*. The assertion that $\tilde{\chi}(X_{v^{-1}}, *\tau^w) = \delta_{v,w}$ follows from Proposition 3.36 together with the definition of $\tau^w$ (i.e., $\gamma(\tau^w) = \psi^w$). Conversely, write

$$*\tilde{\tau}^w = \sum_v f^{v,w}(*\tau^v) \quad \text{for some } f^{v,w} \in R(T).$$

Then $\tilde{\chi}(X_{v^{-1}}, *\tilde{\tau}^w) = f^{v,w}$. But, by the assumption, $\tilde{\chi}(X_{v^{-1}}, *\tilde{\tau}^w) = \delta_{v,w}$ and hence $\tilde{\tau}^w = \tau^w$ for all $w$.

## 4. Consequences of the main results in the finite case

**(4.1).** *Unless otherwise stated we will assume, throughout this section, that we are in the finite case*, i.e., $G$ is a finite-dimensional, semisimple, connected, simply-connected, complex algebraic group, and we denote by $G_0$ (instead of $K$) any (fixed) maximal compact subgroup with a maximal torus $T$ and let $H$ be the complex torus $\subset G$ which is the complexification of $T$. We denote the longest element of $W$ by $w_0$.

The main aim of this section is to show that some of the important (though known) results in $K$-theory of $G/B$ (in the finite case) can be easily deduced from our Theorems 3.13 and 3.28.

(4.2) **Definitions.** Let $R(G_0)$ denote the representation ring of the compact group $G_0$. As in [15, p. 11], define a map

$$\varphi\colon R(T) \otimes_{R(G_0)} R(T) \to K_T(G_0/T), \quad \text{by } \varphi(f \otimes g) = f.\beta(g),$$

where $\beta$ is the Atiyah-Hirzebruch homomorphism defined in Definition 3.17(a). (Of course the notation $f.\beta(g)$ means the multiplication by $f \in R(T)$ in the $R(T)$-module $K_T(G_0/T)$.) It is easy to see that the map $\varphi$ is well defined, i.e., it factors through $R(T) \otimes_{R(G_0)} R(T)$.

We also define a map $\overline{\varphi}\colon R(T) \otimes_{R(G_0)} R(T) \to \Psi \subset \Omega$, by $\overline{\varphi}(f \otimes g) = f.\overline{\beta}(g)$, where the map $\overline{\beta}\colon R(T) \to \Psi$ is as defined in Definition 3.17(a).

Recall the definition of the Demazure operators $L_w$ on $R(T)$ from Definition 3.17(b). The action of $L_w$ (and also the Weyl group action) clearly commutes with the $R(G_0) \approx R(T)^W$-module structure on $R(T)$. In particular, we can define the operators $\mathrm{Id} \otimes L_w$ and $\mathrm{Id} \otimes \delta_w$ on $R(T) \otimes_{R(G_0)} R(T)$.

The following lemma follows fairly trivially from Lemma 3.18.

(4.3) **Lemma.** *The following diagram is commutative*:

$$\begin{tikzcd} R(T) \otimes_{R(G_0)} R(T) \arrow[rr, "\varphi"] \arrow[dr, "\overline{\varphi}"'] & & K_T(G_0/T) \arrow[dl, "\gamma"] \\ & \Psi & \end{tikzcd}$$

*where* $\gamma$ *is the map given in Theorem* 3.13.

*Moreover, for any* $w \in W$ *and* $x \in R(T) \otimes_{R(G_0)} R(T)$, $(\varphi \circ (\mathrm{Id} \otimes \delta_w))(x) = w \cdot \varphi(x)$ (*resp.* $(\overline{\varphi} \circ (\mathrm{Id} \otimes \delta_w))(x) = \delta_w \cdot (\overline{\varphi}(x)))$ *and* $\varphi \circ (\mathrm{Id} \otimes L_w) = D_w \circ \varphi$ (*resp.* $(\overline{\varphi} \circ (\mathrm{Id} \otimes L_w))(x) = y_w \cdot (\overline{\varphi}(x)))$, *where* $w \cdot \varphi(x)$ *denotes the action of* $w$ *on* $K_T(G_0/T)$.

Now we can prove the following, which was conjectured in [15, p. 11]. We thank V. Snaith from whom we subsequently learnt that it was already proved by John McLeod [23]. Recently Kazhdan-Lusztig [18] also have given a proof independently.

(4.4) **Theorem.** *With the assumptions as in* §4.1, *the map* $\varphi$, *defined in Definition* 4.2, *is an isomorphism.*

*Proof.* In view of Theorem 3.13, we need to prove that the map $\overline{\varphi}$ is an isomorphism. Now the image of $\overline{\varphi}$ is an $R(T)$-submodule of $\Psi$, which is stable under the action of $y_w$'s. So, to prove the surjectivity of $\overline{\varphi}$, by Proposition 2.22(d) it suffices to show that $\psi^{w_0}$ (where $w_0$ is the longest element of $W$) belongs to the Image of $\overline{\varphi}$:

Let $\{e_v\}_{v \in W}$ be the basis of $R(T)$ over $R(G_0) \approx R(T)^W$, given by Steinberg [32, Theorem 2.2]. Define the matrix $F = (f_{v,w})_{v,w \in W}$, where $f_{v,w} := we_v$. By [32, §2], the determinant of $F$, $\det F = ((-1)^{|\Delta_+|} e^{-\rho} \mathscr{D})^{|W|/2}$, where $\mathscr{D} := \prod_{\nu \in \Delta_+} (1 - e^\nu)$.

We want to find elements $(p_w)_{w \in W}$ in $R(T)$ such that

$$(\mathrm{I}_{35}) \qquad \overline{\varphi}\left(\sum_w p_w \otimes e_w\right) = \psi^{w_0}$$

which is equivalent, by Proposition 2.22(b), to solving the matrix equation in $\mathfrak{p}$ (over $R(T)$):

$$(\mathrm{I}_{36}) \qquad \mathfrak{p}.F = \mathfrak{q},$$

where $\mathfrak{p}$ is the row vector $(p_w)_{w\in W}$, and $\mathfrak{q}$ is the row vector with zeros everywhere except in the $w_0$th column, where it is equal to $\psi^{w_0}(\delta_{w_0}) = \mathscr{D}$. The equation $(\mathrm{I}_{36})$ has a unique solution for $\mathfrak{p}$ as a vector over the quotient field $Q(T)$ of $R(T)$ given by

$$(\mathrm{I}_{37}) \qquad \mathfrak{p} = \mathfrak{q}.\frac{\widetilde{F}}{\det F},$$

where $\widetilde{F} = (\tilde{f}_{v,w})$ is the matrix with $\tilde{f}_{v,w}$ equal to the (up to sign) determinant of the matrix $F^{v,w}$ obtained from $F$ by deleting the $v$th column and the $w$th row.

We next observe that $\det F^{v,w}$ is divisible by $\mathscr{D}^{(|W|/2)-1}$ (in $R(T)$) for any $v, w \in W$. To prove this, we use the Vandermonde determinant type argument:

Fix a positive root $\nu$, and let $r_\nu \in W$ be the reflection through the hyperplane given by the root $\nu$. Write $W\setminus\{r_\nu v, v\}$ as the disjoint union of the orbits under the left multiplication by $r_\nu$. Of course there are $(|W|/2)-1$ such orbits. Since for any $v'$, $w' \in W$, $r_\nu v' e_{w'} - v' e_{w'}$ is divisible (in $R(T)$) by $1-e^\nu$, we get (by subtracting the $r_\nu v'$ th column from the $v'$ th column) that $\det F^{v,w}$ is divisible by $(\prod_{\nu\in\Delta_+}(1-e^\nu))^{(|W|/2)-1} = \mathscr{D}^{(|W|/2)-1}$. (Observe that we have used the fact that $R(T)$ is a unique factorization domain and, for distinct $\nu$, $\nu' \in \Delta_+$, the elements $1-e^\nu$ and $1-e^{\nu'}$ are relatively prime in $R(T)$.)

Hence by $(\mathrm{I}_{37})$ the vector $\mathfrak{p}$ has its entries actually in $R(T)$, which proves the surjectivity of the map $\overline{\varphi}$.

Replacing $\mathfrak{q}$ by any other row vector over $R(T)$, one easily obtains (from $\mathrm{I}_{37}$) that the map $\overline{\varphi}$ is injective.

(To prove the injectivity of $\overline{\varphi}$, one can also use the following general lemma, which can easily be proved by using the determinants.)

(4.5) **Lemma.** *A surjective linear map of any two free modules of the same finite rank, over any commutative ring with identity, is an isomorphism.*

Of course as an immediate consequence of Theorem 4.4 together with Proposition 3.25 one obtains the following result, which was conjectured by Atiyah-Hirzebruch [3, §5.7] (who had checked its validity case-by-case for all the simple, simply-connected groups except for $E_6$, $E_7$, and $E_8$) and later proved independently by Seymour [30], Snaith [31], and Pittie [28]. (They all used Hodgkin's spectral sequence.)

(4.6) **Theorem.** *With the assumptions as in* §4.1, *the Atiyah-Hirzebruch homomorphism* $\beta_1\colon R(T) \to K(G_0/T)$, *defined in Definition* 3.17(a), *gives an isomorphism* $\mathbf{Z}\otimes_{R(G_0)} R(T) \to K(G_0/T)$.

*In particular,* $\beta_1$ *itself is surjective.*

One can also easily deduce the following result due to Hodgkin [14] from Theorem 4.4. We do not give the details here since they would appear elsewhere; where we intend to study $K^*(G_0)$ for the unitary form $G_0$ of a general Kac-Moody group.

(4.7) **Theorem.** *With the assumptions as in* §4.1, $K^*(G_0)$ *is a torsion-free* $\mathbf{Z}$*-module.*

We give below an alternative description of the operators $D_w$ (defined in Definition 3.8) in the finite case: For an $H$-variety $X$, recall the definition of $K^0(H, X)$ and $K_0(H, X)$ from Definition 3.33. In particular (in the finite case), $K^0(H, G/B)$ and $K_0(H, G/B)$ make sense; where $H$ is the complex torus (acting on $G/B$ by the left multiplication). Since $G/B$ is smooth, as a particular case of [33, Theorem 5.7], we have the following.

(4.8) **Proposition.** *The canonical map* $K^0(H, G/B) \to K_0(H, G/B)$ *is an isomorphism.*

For any $H$-stable closed subvariety $Y$ of $H$-variety $X$, let $\mathscr{O}_Y$ denote the structure sheaf of $Y$ extended to the whole of $X$ by defining it to be zero in $X \setminus Y$. Since $\mathscr{O}_Y$ is an $H$-equivariant coherent sheaf on $X$, it determines an element $[\mathscr{O}_Y] \in K_0(H, X)$. In particular, taking $X = G/B$ and $Y =$ Schubert variety $X_w$ $(= \overline{BwB/B})$ we get, for any $w \in W$, an element $[\mathscr{O}_w] = [\mathscr{O}_{X_w}] \in K_0(H, G/B)$.

Recall the filtration given in Definition 3.1:

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{\dim G/B} = G/B.$$

Since each $X_n$ is a $H$-stable closed subvariety of $G/B$, $X_n \setminus X_{n-1}$ is a disjoint union of affine cells $\{BwB/B\}_{l(w)=n}$, and moreover the action of $H$ on each $BwB/B$ can be linearized; by the $H$-equivariant analog (which is available because of the equivariant machinery developed in [33]) of a result due to Grothendieck [11, p. IV-31, Proposition 7] we get

(4.9) **Lemma.** *The elements* $\{[\mathscr{O}_w]\}_{w\in W}$ *form a* $R(T)$*-basis for the* $R(T) \approx R(H)$*-module* $K_0(H, G/B)$.

*In particular,* $K_0(H, G/B)$ *is a* $R(T)$*-free module of rank* $= |W|$.

One of course has a canonical map $\zeta\colon K^0(H, G/B) \to K_T(G/B)$, where $K_T(G/B)$ is the topological equivariant $K$-group as in §3.

(4.10) **Proposition.** *The map* $\zeta\colon K^0(H, G/B) \to K_T(G/B)$ *defined above is an isomorphism.*

*Proof.* Recall the definition of the map $\varphi : R(T) \otimes_{R(G_0)} R(T) \to K_T(G/B)$ from Definition 4.2. From the definition of $\varphi$, it is clear that Image $\varphi \subset$ Image $\zeta$. In particular, by Theorem 4.4, $\zeta$ is a surjective map. But

$K_T(G/B)$ (resp. $K^0(H, G/B)$) is a $R(T)$-free module of rank $|W|$ by Lemma 3.15 (resp. Proposition 4.8 and Lemma 4.9). Now, by Lemma 4.5, the proposition follows. □

As a consequence of Propositions 4.8 and 4.10, we can canonically identify $K_T(G/B)$, $K^0(H, G/B)$, and $K_0(H, G/B)$ with each other.

(4.11) **Proposition.** *Fix a simple reflection $r_i$ and let $P_i$ be the corresponding minimal parabolic* (cf. §3.2). *Then the operator $*D_{r_i}*$ (where $*$ is as in Proposition* 3.36, *and $D_{r_i}$ is as in Definition* 3.8) *is the composite*:

$$K_0(H, G/B) \xrightarrow{\pi_{i_!}} K_0(H, G/P_i) \approx K^0(H, G/P_i) \xrightarrow{\pi_i^*} K^0(H, G/B),$$

*where $\pi_i\colon G/B \to G/P_i$ is the canonical projection, $\pi_{i_!} := \sum_k (-1)^k [R^k \pi_{i_*}]$, and $\pi_i^*$ is the canonical pull-back.*

*Proof*. The idea of the proof is quite similar to the proof of Proposition 3.36: For $\tau \in K^0(H, G/B)$, write

$$(\mathrm{I}_{38}) \qquad *\tau = \pi_i^*(\tau') + H_i.\pi_i^*(\tau'') \quad \text{for } \tau', \tau'' \in K^0(H, G/P_i).$$

(Recall that $H_i$ is the Hopf bundle defined in §3.19—Assertion II.)

Hence

$$\tau = \pi_i^*(*\tau') + (*H_i).\pi_i^*(*\tau'').$$

By the projection formula, we obtain:

$$(\mathrm{I}_{39}) \qquad \pi_{i_!}(\tau) = *\tau' + (\pi_{i_!}(*H_i)).(*\tau'').$$

But

$$(\mathrm{I}_{40}) \qquad \pi_{i_!}(*H_i) = 0,$$

(see the proof of Proposition 3.36).

Combining $(\mathrm{I}_{38})$–$(\mathrm{I}_{40})$, we get the proposition. □

Recall the definition of the basis $([\mathscr{O}_w])_w$ of $K_T(G/B) \approx K_0(H, G/B)$ from Lemma 4.9.

(4.12) **Lemma.** *For any $w \in W$ and simple reflection $r_i$,*

$$D_{r_i}(*[\mathscr{O}_w]) = \begin{cases} *[\mathscr{O}_w] & \textit{if } wr_i < w, \\ *[\mathscr{O}_{wr_i}] & \textit{otherwise.} \end{cases}$$

*Proof*. Using the normality of $X_w$ and $\pi_i(X_w)$, it is easy to see that $\pi_{i_!}[\mathscr{O}_w] = [\mathscr{O}_{\pi_i(X_w)}]$ as elements of $K_0(H, G/P_i)$. Now the lemma follows from Proposition 4.11, if we observe the following simple fact:

Let $\pi\colon X \to Y$ be a surjective $H$-equivariant smooth morphism of smooth projective $H$-varieties, and let $Z \subset Y$ be a closed $H$-stable subvariety. Then $\pi^*[\mathscr{O}_Z] = [\mathscr{O}_{\pi^{-1}(Z)}]$, where $\pi^*\colon K^0(H, Y) \to K^0(H, X)$ is the canonical map, and $[\mathscr{O}_Z]$, which is an element of $K_0(H, Y)$, can also be thought of as an element of $K^0(H, Y)$ under the canonical isomorphism with $K_0(H, Y)$ (cf. Proposition 4.8). $\square$

Recall the definition of the $R(T)$-basis $\{\tau^w\}_{w\in W}$ of $K_T(G/B)$ from Remark 3.14. In particular, we have a **Z**-basis $\{\tau_1^w := \varepsilon(\tau^w)\}$ of $K(G/B)$, where $\varepsilon\colon K_T(G/B) \to K(G/B)$ is the canonical map. We also have another **Z**-basis $\{\sigma_1^w = \varepsilon(*[\mathscr{O}_w])\}_w$ of $K(G/B)$ (in the finite case) (cf. Lemma 4.9).

The following proposition describes how the basis $\{\sigma_1^w\}$ transforms with respect to the basis $\{\tau_1^w\}$ of $K(G/B)$.

(4.13) **Proposition.** *For any $v \in W$,*

$$\sigma_1^v = \sum_w m_{v,w}\tau_1^{w^{-1}w_0},$$

*where the matrix $M = (m_{v,w})_{v,w\in W}$ is defined as $m_{v,w} = 1$ if $v \geq w$, and $m_{v,w} = 0$ otherwise.*

*In particular, $\sigma_1^e = \tau_1^{w_0}$.*

Recall from [8, §3] that the transpose of the inverse matrix $M^{-1}$ is precisely the Möbius function associated to the pair $(W, \leq)$.

*Proof.* By Proposition 3.39,

$$(\mathrm{I}_{41})\qquad *[\mathscr{O}_v] = \sum_w *(\chi(X_{w_0w}, [\mathscr{O}_v]))\tau^{w^{-1}w_0}, \quad \text{as elements of } K_T(G/B).$$

But, by Proposition 3.36 and Remark 3.38,

$$(\mathrm{I}_{42})\qquad *(\chi(X_{w_0w}, [\mathscr{O}_v])) = (y_{w_0w}\cdot(\gamma(*[\mathscr{O}_v])))(\delta_e),$$

where $\gamma$ is the map defined in Theorem 3.13.

Further, by making successive use of Lemma 4.12 (see also the proof of Proposition 4.16), we get

$(\mathrm{I}_{43})$

$$D_{w_0w}(*[\mathscr{O}_v]) = \begin{cases} *[\mathscr{O}_{w_0}] & \text{if } v^{-1}w_0 \leq w^{-1}w_0, \text{ i.e., } w \leq v, \\ *[\mathscr{O}_{v'}] & \text{for some } v' = v'(v, w) < w_0, \text{ if } w > v. \end{cases}$$

Combining $(\mathrm{I}_{41})$–$(\mathrm{I}_{43})$, we obtain

$$\sigma_1^v = \sum_{w\leq v} ev(\gamma(*[\mathscr{O}_{w_0}])(\delta_e))\tau_1^{w^{-1}w_0} + \sum_{w\nleq v} ev(\gamma(*[\mathscr{O}_{v'(v,w)}])(\delta_e))\tau_1^{w^{-1}w_0},$$

where $ev\colon R(T) \to \mathbf{Z}$ is the augmentation map (cf. §3.17).

Now the proposition follows by the following simple lemma.

(4.14) **Lemma.** *For any $w \in W$, $ev(\gamma(*[\mathscr{O}_w])(\delta_e)) = 0$ unless $w = w_0$, in which case it is $1$.*

*Proof*. Let us resolve the $H$-equivariant coherent sheaf $\mathscr{O}_w$ on $G/B$ by $H$-equivariant locally free sheaves (which is possible since $G/B$ is smooth):

$$(\mathfrak{S}) \qquad 0 \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \to \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{O}_w \to 0.$$

It is easy to see that

$$ev(\gamma(*[\mathscr{O}_w])(\delta_e)) = ev(\gamma[\mathscr{O}_w](\delta_e)) = \sum(-1)^k \operatorname{rank}\mathscr{F}_k,$$

where $\operatorname{rank}\mathscr{F}_k$ denotes its rank as a vector bundle. If $w = w_0$, i.e., $\mathscr{O}_w = \mathscr{O}_{G/B}$, then $\mathscr{F}_k$ can be taken to be zero for all $k > 0$ and $\mathscr{F}_0 = \mathscr{O}_{G/B}$. Hence the assertion follows in this case. So assume that $w < w_0$, i.e., $X_w$ is properly contained in $G/B$. Now taking a point $\overline{g} \in (G/B) \setminus X_w$ and localizing the above sequence $(\mathfrak{S})$ at $\overline{g}$, we get the lemma.

(4.15) **Remark.** It will be interesting to see how the basis $\{*[\mathscr{O}_w]\}_{w\in W}$ of $K_T(G/B)$ itself transforms with respect to the basis $\{\tau^w\}$. □

Recall the definitions of the $W \times W$ matrices $B$ and $E$ from Corollary 2.7 and §2.21 respectively. Of course, by Proposition 2.22(c), one has $E = (B^t)^{-1}$. To conclude this section, we give another expression for the matrix $E$ in the finite case. Even though this expression again is in terms of the matrix $B$, but an interesting feature is that it does not require inverting $B$; instead it involves the Möbius function.

(4.16) **Proposition.** $E^t = \mathscr{D}B'.M^{-1}$, *where the matrix* $M$ *is defined in Proposition* 4.13, *the scalar* $\mathscr{D} := \prod_{\nu\in\Delta_+}(1-e^{\nu})$, *and* $B' = (b'_{v,w})_{v,w\in W}$ *is given by* $b'_{v,w} = v^{-1}(b_{w_0w^{-1},w_0v^{-1}})$.

*Proof*. Fix any $v, w \in W$. Then, by $(\mathrm{I}_5)$, one has

$$(\mathrm{I}_{44}) \qquad y_{v^{-1}}.y_{w^{-1}} = \sum_{v_1} b_{v,v_1}\delta_{v_1^{-1}}\left(\sum_{w_1} b_{w,w_1}\delta_{w_1^{-1}}\right).$$

Making successive use of Corollary 2.5, we get that $y_{v^{-1}}.y_{w-1} = y_{v^{-1}u}$ for some $u \in W$ satisfying $u \le w^{-1}$ and $l(v^{-1}u) = l(v^{-1}) + l(u)$.

Now for any sequence of simple reflections $\mathfrak{w} = (r_{i_1}, \cdots, r_{i_k})$, one of course has $y_{\mathfrak{w}} = y_{n(\mathfrak{w})}$ for some (unique) $n(\mathfrak{w}) \in W$, where $y_{\mathfrak{w}}$ is, by definition, $y_{r_{i_1}} \cdots y_{r_{i_k}}$. Further, by induction on $k - k'$, it is easy to see that if $\mathfrak{v} = (r_{i_{j_1}}, \cdots, r_{i_{j_{k'}}})$ is a subsequence of $\mathfrak{w}$, then $n(\mathfrak{v}) \le n(\mathfrak{w})$. These two observations together imply that $y_{v^{-1}}.y_{w^{-1}} = y_{w_0}$ if and only if $vw_0 \leq w^{-1}$. Hence, equating the coefficients of $\delta_{w_0}$ in both the sides of $(\mathrm{I}_{44})$, we get (by Proposition 2.6):

$$\sum_{v_1} b_{v,v_1} .(v_1^{-1} b_{w,w_0 v_1^{-1}}) = \begin{cases} 1/\mathscr{D} & \text{if } vw_0 \leq w^{-1}, \\ 0 & \text{otherwise}, \end{cases}$$

i.e., (replacing $w$ by $w_0 w^{-1}$),

$$\sum_{v_1} b_{v,v_1} .(v_1^{-1} b_{w_0 w^{-1}, w_0 v_1^{-1}}) = \begin{cases} 1/\mathscr{D} & \text{if } v \geq w, \\ 0 & \text{otherwise}. \end{cases}$$

Hence $\mathscr{D} B.B' = M$. Now the proposition follows from Proposition 2.22(c).

(4.17) **Remarks.** (a) As mentioned in Proposition 4.13, $(M^{-1})^t$ is precisely the Möbius function.

(b)[4] Recall the definition of the $W \times W$ matrix $C$ from [19, Corollary 4.5], and define a matrix $C' = (c'_{v,w})$ by $c'_{v,w} = v^{-1}(c_{w_0 w^{-1}, w_0 v^{-1}})$. By a proof exactly as above, one obtains that $C^{-1} = D^t = (\prod_{\nu \in \Delta_+} \nu) C'$, where $D$ is as defined in [19, §4.21].

(4.18) **Corollary** (of Proposition 4.16). *For any* $v, w \in W$, $\mathscr{D}.b_{v,w} \in R(T)$.

*Proof.* By Propositions 4.16 and 2.22(a), entries of the matrix $\mathscr{D} B'$ are in $R(T)$. Further, for any $w \in W$, $(w\mathscr{D})/\mathscr{D} \in R(T)$. This proves the corollary.

## 5. Appendix

*In this section, $G$ is an arbitrary Kac-Moody group.*

The aim of this appendix is to show that the structure theorem [19, Theorem 4.6] is false (in the sense made precise below) in general over $\mathbf{Z}$, unlike the corresponding structure theorem (Theorem 2.9 in this paper) for '$K$-theory'.

Let $\mathfrak{h}^*_{\mathbf{Z}} \subset \mathfrak{h}^*$ be the weight lattice (cf. §1.2) and, for any prime $p$, let $\mathfrak{h}^*_{\mathbf{Z}_p} := \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathfrak{h}^*_{\mathbf{Z}}$ (where $\mathbf{Z}_p$ is the prime field). Recall the definition of $Q_W$ from [19, §4.1] and certain elements $x_w \in Q_W$ from [19, Proposition 4.2], and let $R_{\mathbf{Z}}$ be the subring of $Q_W$ defined by

$$R_{\mathbf{Z}} = \{x \in Q_W : x \cdot S_{\mathbf{Z}} \subset S_{\mathbf{Z}}\},$$

where $S_{\mathbf{Z}} := S(\mathfrak{h}^*_{\mathbf{Z}})$, and $\cdot$ is defined by the same formula as $(\mathrm{I}_3)$.

[4] We thank A. Lascoux for a conversation which helped us to arrive at (b).

It is easy to see that (for any simple reflection $r_i$) $x_{r_i}$, and hence $x_w$ for any $w \in W$, belongs to $R_{\mathbf{Z}}$. Now define a $S_{\mathbf{Z}}$-submodule

$$\widehat{R}_{\mathbf{Z}} = \sum_{w \in W} S_{\mathbf{Z}} x_w \subset R_{\mathbf{Z}}.$$

From [19, Proposition 4.3] it follows that $\widehat{R}_{\mathbf{Z}}$ is in fact a subring of $R_{\mathbf{Z}}$, and by [19, Theorem 4.6] $R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}}$ is a torsion group.

*The question we are interested in is whether* $R_{\mathbf{Z}} = \widehat{R}_{\mathbf{Z}}$:

Let $(R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}})_p$ denote the $p$-torsion elements in $R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}}$, i.e.,

$$(R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}})_p := \{x \in R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}} : px = 0\}.$$

By analyzing the proof of Theorem 2.9 (as given in §2.13), together with [19, Theorem 4.6(a)], we obtain the following.

(5.1) **Lemma.** *Fix a prime* $p$. *Then* $(R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}})_p = 0$ *if both of the following two conditions are satisfied*:

(a) *none of the simple roots* $\alpha_i$ *are zero* mod $p$, *i.e., no* $\alpha_i$ *considered as an element of* $\mathfrak{h}^*_{\mathbf{Z}_p}$ *is* $0$, *and*

(b) *the canonical representation* $W \to \mathrm{Aut}(\mathfrak{h}^*_{\mathbf{Z}_p})$ *is injective.*

We also have the following very simple lemma, which does not use our [19, Theorem 4.6], instead uses [19, Lemma (6.2) and Remark 5.17(a)].

(5.2) **Lemma.** *Fix a prime* $p$. *If the characteristic homomorphism* $S(\mathfrak{h}^*_{\mathbf{Z}_p}) \to H^*(G/B, \mathbf{Z}_p)$ *is surjective, then again* $(R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}})_p = 0$.

*Also if* $S(\mathfrak{h}^*_{\mathbf{Z}}) \to H^*(G/B, \mathbf{Z})$ *is surjective, we have* $R_{\mathbf{Z}} = \widehat{R}_{\mathbf{Z}}$.

Finally we have the following (classical) result due to Minkowski.

(5.3) **Lemma** [24].[5] *For any odd prime* $p$ *and any* $n \geq 2$, *the kernel of the map* $\mathrm{SL}(n, \mathbf{Z}) \to \mathrm{SL}(n, \mathbf{Z}_p)$ *has no elements of finite order, where* $\mathrm{SL}(n, \mathbf{Z})$ *of course is the special linear group.*

Now combining Lemmas 5.1–5.3, we obtain the following.

(5.4) **Proposition.** *With the notation as above, we have the following*:

(a) *Let* $G$ *be of finite type. Then* $(R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}})_p = 0$ *for any odd prime* $p$.

(b) $\widehat{R}_{\mathbf{Z}} = R_{\mathbf{Z}}$ *for* $G$ *of type* $A_l$ $(l \geq 1)$, $C_l$ $(l \geq 2)$, $D_{2l+1}$ $(l \geq 1)$, *and* $E_6$.

(c) $(R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}})_p \neq 0$ *in the following cases*:

($c_1$) $p = 2$, *and* $G$ *of type* $B_l$ $(l \geq 3)$, $D_{2l}$ $(l \geq 2)$, $G_2$, $F_4$, $E_7$, *and* $E_8$.

[5]We thank A. Borel for providing this reference.

($c_2$) *$p$ any odd prime, and any Kac-Moody group $G$ which is not of finite type.*

($c_3$) *$p = 2$, and any Kac-Moody group $G$ which is not of finite type, provided no simple root is* $0 \bmod 2$.

*Proof*. (a) follows from Lemmas 5.1 and 5.3, and (b) follows from Lemma 5.2 for $G$ of type $A_l$, $C_l$. To prove the result for $D_{2l+1}$ and $E_6$, observe that no root is $0 \bmod 2$ and moreover $\varphi: W \to \mathrm{Aut}(\mathfrak{h}^*_{\mathbf{Z}_2})$ is injective for these. Injectivity of $\varphi$ for $D_{2l+1}$ follows from the explicit description of $W$ and its action on $\mathfrak{h}^*_{\mathbf{Z}}$; see, e.g., [6, Planche IV, p. 257]. Injectivity of $\varphi$ for $E_6$ follows from the fact that the subgroup of $W$ consisting of all the elements of even length is a simple group (cf. [6, Chapter VI, exercise §4-no. 2(d)]). Now use Lemma 5.1.

To prove (c), we first observe that for any prime $p$ (including $p = 2$) if the representation $\varphi: W \to \mathrm{Aut}(\mathfrak{h}^*_{\mathbf{Z}_p})$ is not injective but no simple root is $0 \bmod p$, then $(R_{\mathbf{Z}}/\widehat{R}_{\mathbf{Z}})_p \neq 0$:

Take $w \neq e \in \mathrm{Ker}\,\varphi$. Then clearly $\frac{1}{p}(\delta_w - \delta_e) \in R_{\mathbf{Z}}$. We claim that $\frac{1}{p}(\delta_w - \delta_e) \notin \widehat{R}_{\mathbf{Z}}$. For, otherwise, write

$$(*) \qquad \tfrac{1}{p}(\delta_w - \delta_e) = \sum_{v \in W} f_v x_v \quad \text{for some } f_v \in S_{\mathbf{Z}}.$$

By [19, Proposition 4.3(c)], $f_v = 0$ for all $v$ with $l(v) > l(w)$. Equating the coefficients of $\delta_w$ in both the sides of $(*)$, we get (by [19, Proposition 4.3(c)]) $\frac{1}{p} = f_w(\prod_{\nu \in w\Delta_- \cap \Delta_+} \nu)^{-1}$, i.e., $\prod_{\nu \in w\Delta_- \cap \Delta_+} \nu = p f_w$. So reducing $\bmod p$, we get $\prod \nu_p = 0$ ($\nu_p$ denotes $\nu$ reduced $\bmod p$), which contradicts the assumption that no simple (and hence no real) root is $0 \bmod p$. Further $\delta_w - \delta_e \in \widehat{R}_{\mathbf{Z}}$, since $\widehat{R}_{\mathbf{Z}}$ is a ring and $\delta_{r_i} \in \widehat{R}_{\mathbf{Z}}$ by [19, ($I_{24}$)].

Since any $G$, which is not of finite type, has an infinite Weyl group, ($c_2$) and ($c_3$) immediately follow. In the cases covered by ($c_1$), no root is $0 \bmod 2$, whereas $\varphi: W \to \mathrm{Aut}(\mathfrak{h}^*_{\mathbf{Z}_2})$ has indeed nontrivial kernel since the longest element of the Weyl group (in these cases) acts by $-1$ on $\mathfrak{h}^*_{\mathbf{Z}}$. So ($c_1$) also follows.

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