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CONVERGENCE OF RIEMANNIAN MANIFOLDS; RICCI AND $L^{n/2}$-CURVATURE PINCHING

L. ZHIYONG GAO

In this paper we consider classes of Riemannian manifolds (M, g) with bounds on the five fundamental geometric invariants: Ricci curvature Ric, $L^{n/2}$ -norm of curvature $\int |\mathrm{Rm}|^{n/2} dg$, injectivity radius $\mathrm{inj}(g)$, diameter $\mathrm{diam}(g)$ and volume $\mathrm{Vol}(G)$.

Let $G(H, i_0, K, n)$ denote the collection of all closed, connected n-dimensional Riemannian manifolds (M, g) satisfying $|\mathrm{Ric}| \leq H$, $\mathrm{inj}(g) \geq i_0 > 0$, and $\int_M |\mathrm{Rm}(g)|^{n/2} dg \leq K$.

One of the principal aims of this study is to understand the compactness property of $G(H, i_0, K, n)$. The well-known and fundamental result of Gromov states that the collection of all closed, connected n-dimensional Riemannian manifolds (M, g) with $|\mathrm{Rm}| \leq 1$, $\mathrm{diam} \leq D$ and $\mathrm{Vol} \geq V$ is precompact among "$C^{1,\alpha}$-Riemannian manifolds" [17], [12], [27].

Our first result says that we have a similar compactness result in $G(H < i_o, K, n)$, which roughly states that: Given a sequence $\{M_k\}$ of compact Riemannian n-manifolds with $|\mathrm{Ric}| \leq H$, $\mathrm{diam} \leq D$, $\int |\mathrm{Rm}|^{n/2} dg \leq K$, and $\mathrm{inj} \geq i_0 < 0$, then $\{M_k\}$ has a subsequence, away from finite number points, which converges to an n-manifold M with $C^{1,\alpha}$ metric g for an $\alpha \in (0, 1)$. The precise statement is the following $(n \geq 4)$.

Theorem 0.1. *Let $\{(M_k, g^k)\}$ be a sequence of Riemannian manifolds in $G(H, i_o, K, n)$, with $\mathrm{diam}(M_k) \leq D$. Then there exist a subsequence of $\{(M_k, g^k)\}$ (by renumbering, we still use $\{M_k, g^k\}$), and a sequence $\{r_l\}$, $r_l \to 0$ when $l \to \infty$, such that the following hold:*

(a) *There exists a C^∞-manifold M, such that M_k is diffeomorphic to M for each large k.*

(b) *There exist a C^∞-metric g on M, and finite number points $\{m_{f_1}, \cdots, m_h\}$ with $h \leq C(H, i_o, K, n)$, such that g is a $C^{1,\alpha}$-metric on $M - \{m_1, \cdots, m_k\}$, $0 < \infty < 1$.*

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(c) *For each l, we have open subsets $F_k(r_l) \subset M_k$, and an open subset $D(r_l) \subset M - \{m_1, \cdots, m_h\}$, such that there are diffeomorphisms*

$$f^k(r_l)\colon D(r_l) \to F_k(r_l)\ (k \le l),$$

and $f^k(r_l) * g^k$ converges to metric g on $D(r_l)$ in the $C^{1,\alpha}$ norm of M.

(d) $F_k(r_l) \subset F_k(r_{l+1})$.

(e) *There is an $\varepsilon(r_l) = \varepsilon(H, i_o, K, n, r_l)$, such that*

$$F_k(r_l) \cup \bigcup_{i=1}^{h} B^k(m_i^k, \varepsilon)) = M_k$$

and $\varepsilon(r_l) \to 0$ when $r_l \to 0$.

In four dimensions, we can use the Gauss-Bonnet formula for Euler number $\chi(M)$ to replace the L^2-norm of curvature by $\chi(M)$ and to get

Corollary 0.2. *Let $\{(M_k, g^i)\}$ be a sequence of Riemannian 4-manifolds with $|\mathrm{Ric}| \le H$, $\mathrm{diam} \le D$, $\mathrm{inj} \ge i_o > 0$, and $\chi(M_k) \le K$. Then there exists a subsequence of $\{M_k, g^k\}$ which satisfies* (a), (b), (c), *and* (d) *of Theorem* 0.1.

As an application of the above convergence theorems, we shall prove Ricci pinching theorems.

Theorem 0.3. *For $n \ge 4$, $i_o > 0$, and $K > 0$, there exists a small constant $\varepsilon = \varepsilon(i_o, K, n) > 0$ such that if (M, g) is a Riemannian n-manifold with $\mathrm{inj}(g) \ge i_o$,*

$$\int_M |\mathrm{Rm}(g)|^{n/2}\, dg \le K,$$

and ε-Ricci pinching

$$(1-\varepsilon)g \le \mathrm{Ric}(g) \le (1+\varepsilon)g,$$

then there exists an Einstein metric with $\mathrm{Ric} = 1$ on M.

Theorem 0.4. *Given $n \ge 4$, $i_o > 0$, $D > 0$, and $K > 0$, there exists a small constant $\varepsilon = \varepsilon(i_o, D, K, n) > 0$ such that if $M < g$ is a Riemannian n-manifold with $\mathrm{inj}(g) \ge i_o$, $\mathrm{diam}(g) \le D$, $\int_M |\mathrm{Rm}|^{n/2} dg \le K$, and*

$$(-\tau - \varepsilon)g \le \mathrm{Ric}(g) \le (-\tau + \varepsilon)g,$$

where $\tau = 1$ or 0, then there exists an Einstein metric with $\mathrm{Ric} = -\tau$ on M.

Remark. Similar results are proven with curvature bound $|\mathrm{Rm}| \le K$ and L^p norm of curvature in [26], [34].

Again in dimension 4, we have better results.

Corollary 0.5. *Given $i_o > 0$ and K, there exists a small constant $\varepsilon = \varepsilon(i_o, K) > 0$, such that if (M, g) is a Riemannian 4-manifold with* $\operatorname{inj}(g) \geq i_o$, $\chi(M) \leq K$, *and*

$$(1-\varepsilon)g \leq \operatorname{Ric}(g) \leq (1+\varepsilon)g,$$

then there exists an Einstein metric with $\operatorname{Ric} = 1$ *on* M.

Corollary 0.6. *Given $i_o > 0$, $D > 0$, and $K > 0$, there exists a small constant $\varepsilon = \varepsilon(i_o, D, K, n) > 0$, such that if (M, g) is a Riemannian 4-manifold with* $\operatorname{inj}(g) \geq i_o$, $\chi(M) \leq K$, $\operatorname{diam}(g) \leq D$, *and*

$$(-\tau-\varepsilon)g \leq \operatorname{Ric}(g) \leq (-\tau+\varepsilon g),$$

where $\tau = 1$ or 0, then there exists an Einstein metric with $\operatorname{Ric} = -\tau$ *on* M.

Our second application gives a generalization of Cheeger's finite theorem, which follows easily from Theorem 0.1.

Theorem 0.7 (*finite theorem*). *For any given numbers $D > 0$, $i_o > 0$, and $K > 0$ there are at most finitely many diffeomorphism classes of closed Riemannian manifolds M of a fixed dimension n such that*

$$|\operatorname{Ric}| \leq 1, \qquad \operatorname{diam}(M) \leq D,$$
$$\operatorname{inj}(M) \geq i_0, \qquad \int_M |\operatorname{Rm}(g)|^{n/2}\, dg \leq K.$$

With a slightly stronger condition on the $L^{n/2}$-norm of curvature, we can replace the lower bound of injectivity radius by the lower bound of volume as follows.

Theorem 0.8. *Let $\{M_k, g^k\}$ be a sequence of closed, connected Riemannian n-manifolds with* $|\operatorname{Ric}| \leq H$, $\operatorname{diam} \leq D$, *and* $\operatorname{Vol} \geq V > 0$. *Then there exists a small constant $\kappa = \kappa(H, D, V) > 0$, such that for any fixed $\rho > 0$, we have*

$$\int_{B^\kappa(x,\rho)} |\operatorname{Rm}(g^\kappa)|^{n/2}\, dg^\kappa \leq \kappa$$

for each κ and all $x \in M_\kappa$. Thus $\{M_\kappa, g^\kappa\}$ has a subsequence $\{M_l, g^l\}$ such that there exist a C^∞-manifold M, a $C^{1,\alpha}$-Riemannian metric g on M, $0 < \alpha < 1$, and diffeomorphisms

$$f_l = M \to M_l$$

with $f_l^* g^l$ converging to g on M in the $C^{1,\alpha}$-norm of a C^∞ manifold M.

As a corollary, we have

Theorem 0.9. *Assume that a sequence $\{M_\kappa, g^\delta\}$ of closed, connected Riemannian n-manifolds with $|\mathrm{Ric}| \le H$, $\mathrm{diam} \le D$, and $\mathrm{Vol} \ge V > 0$ is given, and that there exists a $p > n/2$ and a $K > 0$, such that*

$$\int_{M_\kappa} |\mathrm{Rm}(g^\kappa)|^p \, dg^\kappa \le K.$$

Then $\{M_k, g^k\}$ has a subsequence $\{M_l, g^l\}$ so that there exists a C^∞-manifold M, a $C^{1,\alpha}$-Riemannian metric g on M, $0 < \alpha < 1$, and diffeomorphisms $f_l \colon M \to M_l$ with $f_l^* g^l$ converging to g in the $C^{1,\alpha}$-norm of M.

These results can be used to improve the $L^{n/2}$ curvature pinching results in [10].

Theorem 0.10. *Given $n \ge 4$, $H > 0$, and $V > 0$, there exists a small constant $\kappa = \kappa(H, v, n) > 0$, such that for any closed Riemannian manifold (M, g) of dimension n with $|\mathrm{Ric}| \le H$, $\mathrm{Vol}(B(x, 1)) \ge V$ for all $x \in M$, and*

$$\int_{B(x,1)} |R_{ijkl} - (g_{ik}g_{jl} - g_{il}g_{jk})|^{n/2} \, dg \le \kappa$$

for all $x \in M$, there exists a constant sectional curvature metric with sectional curvature $\equiv 1$ on M.

Theorem 0.11. *Let $\Delta = 0$ or -1. Given $n \ge 4$, $H > 0$, $D > 0$, and $V > 0$, there exists a small constant $\kappa = \kappa(H, D, V, n) > 0$, such that for any Riemannian manifold (M, g) of dimension n with $|\mathrm{Ric}| \le H$, $\mathrm{Vol}(M) \ge V$, $\mathrm{diam}(M) \ge D$, and*

$$\int_M |R_{ijkl} - \Delta(g_{ik}g_{jl} - g_{il}g_{jk})|^{n/2} \, dg \le \kappa,$$

there exists a constant sectional curvature metric on M with sectional curvature $\equiv \Delta$.

We briefly describe here the method used in this paper (with some understandable unavoidable sacrifice of technical accuracy). The main theorems in the paper are Theorem 0.1 and Theorem 0.8, which are the generalizations of well-known Gromov convergence theorem [27], [12]. We want to replace the curvature pointwise bound by the pointwise Ricci curvature bound and the $L^{n/2}$ curvature bound. We use the same method of proving the Gromov convergence theorem, which covers the Riemannian manifolds by a small, controllable size *harmonic ball*, due to a result of Jost and Karcher [12], such a harmonic ball exists provided we have the pointwise curvature bound and the lower injectivity radius bound. Since we do not have the pointwise curvature bound, we have to prove a main technical result of the existence of the harmonic coordinate to replace the result of Jost and Karcher. We prove roughly that for any geodesic ball $B(x, 1)$ of radius 1 in a Riemannian manifold (M, g) with $|\text{Ricci curvature}| \leq H$, $\text{inj}(M) \geq 4$, and $\int_{B(x,1)} |\text{Rm}|^{n/2} dg \leq \varepsilon$, with ε sufficiently small, there exists a well-controlled harmonic coordinate on $B(x, 1)$ (for an exact statement see Lemma 1.1 and Theorem 3.0). We then can apply to smaller balls by rescaling.

The proof of the above result consists of two parts. The first part is mainly done in [10], which states that if we have $B^k(x, 1) \subset (M_k, g^k)$, with $|\text{Ric}| \leq H$, $\text{inj}(M_k, g^k) \geq 4$, and

$$\int_{B^k(x,1)} |\text{Rm}(g^k)|^{n/2} dg^k \to 0,$$

then by using geodesic coordinates to identify $B^k(x, 1)$ with the unit Euclidean ball $B(1)$, we can show $g^k|B(1)$ converges to the flat Euclidean metric $d\theta^2$ on $B(1)$ in $L^{n/2}$-norm.

The second part uses higher order estimates, roughly speaking. We can solve the Dirichlet problem of the Laplace equation on $B(1)$ for g^k with the boundary value equal to the value of the geodesic coordinate on the boundary of $B(1)$. We then have the solution F_k of $\Delta F_k = 0$ with the boundary value $F_k|_{\partial B(1)}$ equal to the boundary value of the geodesic coordinate. Since $g^k \to \theta^2$ in $L^{n/2}$-norm, we first show $F_k \to$ standard coordinate map of $B(1)$ in Euclidean space. For the higher order estimates, we use the Ricci identity and Bianchi identity (see 3.4 and 3.9 for details).

These, with the help of L^p estimates of elliptic theory, can show that if

$$\int_{B^k(x,1)} |\text{Rm}(g^k)|^{n/2} dg^k \to 0$$

and $|\text{Ric}(g^k)| \leq H$, we have the $C^{1,\alpha}$ estimates of the metric tensor in harmonic coordinates, and of course we first prove the existence of the harmonic coordinate (see §3). Since existence of the harmonic coordinate needs the $L^{n/2}$-norm of the curvature tensor to be small, we cannot have a nice coordinate on every ball of a covering, and the curvature tensor may concentrate near a finite number of points. In order to study the singularities of the limit metric, we blow up the metric at these points as in [11]. In the case of Theorem 0.8, if the curvature tensor is not concentrated near a finite number of points, then the $C^{1,\alpha}$ converges on the whole manifold.

1. Proof of the theorems

The Gromov convergence theorem is proved with optimal regularity properties by using harmonic coordinate [27], [12]. The convergence theorem in this paper is proved by using the same method. The main technical lemma is the existence of the harmonic coordinate.

Lemma 1.1. *Given a small η, $0 < \eta < 1$, there exists a small constant $\kappa = \kappa(H, i_o, n, \eta) > 0$, such that for any Riemannian manifold (M, g) of dimension n, $x_0 \in M$, $0 < \rho < i_o/4$, with $|\mathrm{Ric}| \le H$, $\mathrm{inj}(g) \ge i_o$, and*

$$\int_{B(x_0, \rho(1+2\eta))} |\mathrm{Rm}|^{n/2}\, dg \le \kappa .$$

Then, for any $0 < \delta < 1/3$ and $x \in B(x_o, (1+2\eta)\rho)$, so that $d(x_o, x) < 1 - \delta - 2\eta$, there exists a harmonic coordinate F with domain $F \supset B(x, \delta\rho)$ and image $F \supset B(\delta(1-\eta)\rho) = \{x \in \mathbb{R}^n |\ |x| \le \delta(1-\eta)\rho\}$, such that,

(a) $F^{-1}(B(\delta(1-\eta)\rho)) \supset B(x, \delta(1-2\eta)\rho)$.

Let $h^{ij} = \langle \nabla h^i, \nabla h^j \rangle$ be the metric tensor in such a harmonic coordinate, $F = (h^1, \cdots, h^n)$, $(n \ge 4)$. Then

(b) $|h^{ij} - \delta^{ij}|_{C_0} \le \eta^2/100n$,

(c) $|dh^{ij}|_{C^\alpha} \le C(H, i_o, n, \eta, \rho)$, $\qquad 0 < \alpha < 1$.

(d) $\|\partial^2 h^{ij}\|_{L^p} \le C(H, i_o, n, \eta, \rho)$, $\qquad p > n$.

Proof. This lemma follows directly by applying Theorem 3.0 to the metric $\overline{g} = (1/\rho^2)g$. From now on, the general constant C will depend on i_o, H, n and K, where we consider

$$G(H, i_o, K, n) = \left\{ (M, g) \mid |\mathrm{Ric}| \le H,\ \mathrm{inj}(g) \ge i_o,\ \mathrm{diam}\, M = n,\ \int |\mathrm{Rm}|^{n/2}\, dg \le K \right\}$$

and

$$G_D(H, i_o, K, n) = \{ (M, g) \mid (M, g) \in G(H, i_0, K, n),\ \mathrm{diam}(g) \le D \}.$$

We first prove a weak convergence result.

Proposition 1.2. *Let $\{M^k, g^k\}$ be a sequence in $G_D(H, i_o, K, n)$. Then there exist a subsequence of $\{M_k, g^k\}$ (by renumbering, we will use $\{M_k, g^k\}$), and a sequence $\{r_l\}$, $r_l \to 0$.*

(a) *For each* l, *we have open subsets* $F_k(r_l) \subset M_k$, *and an open* C^∞ *manifold* $D(r_l)$.

(b) *For each* l, *we have diffeomorphisms*

$$f^k(r_l)\colon D(r_l) \to F_K(r_l) \qquad (k \geq l)$$

such that $f^k(r_l)^* g^k$ *converges to a* $C^{1,\alpha}$ *metric* $g(r_l)$ *on* $D(r_l)$ *in* $C^{1,\alpha}$ *norm.*

(c) *There is* $\varepsilon(r_l)$, *such that*

$$F_k(r_l) \cup \bigcup_{i=1}^{h} B^k(m_i^k, \varepsilon(r_l)) = M_k$$

with $h \leq c(H, i_0, K, n, D)$ *and* $\varepsilon(r_l) \to 0$ *when* $r_l \to 0$.

Proof. Let $r < i_o/100$. Given a sequence $\{M_k, g^k\}$ in $G_D(H, i_o, K, n)$, let $Q(k)$ be the maximal number of disjoint geodesic balls of radius $r/4$. By the Gromov packing argument, we have

$$Q(k) \leq c(r).$$

By passing to a subsequence if necessary, we assume

$$Q(k) \equiv Q(r) \in \mathbb{Z}^+ \quad \text{for all } k.$$

Now fix k, and let $\{B^k(x_i, r/100)\}$, $i = 1, 2, \cdots, Q$, be a maximal family of disjoint geodesic balls of radius $r/4$. Then $\{B^k(x_i, r/2)\}$, $i = 1, \cdots, Q(r)$, is a covering of M_k. Let $h(k)$ be the maximal number such that any $h(k) + 1$ balls of $\{B^i(x_i, r, (1 + 2\eta)\}$ have empty intersection, where $\eta > 0$ is a small number to be determined later. By the Bishop-Gromov volume estimate, we have

$$h(k) \leq C,$$

so the $h(k)$ are uniformly bounded by c, which is independent of k and r. We would like to apply the estimates of Lemma 1.1 to each ball of $\{B^k(x_i, r(1 + 2\eta))\}$, but the hypothesis need not be met on all balls. We take $\eta = 10^{-10}$; for this η, we have an upper bound on the number of balls on which it fails. We have

$$\begin{aligned} h(k)K \geq h(k) \int_M |\operatorname{Rm}(g^k)\|^{n/2}\, dg^k &> \sum_{i=1}^{Q} \int_{B^k(x, r(1+2\eta))} |\operatorname{Rm}(g^k)|^{n/2}\, dg^k \\ &\geq N(k)\kappa, \end{aligned}$$

where $N(k)$ is the number of $B^k(x, r(1 + 2\eta))$ for which

$$\int_{B^k(x_i, r(1+2\eta))} |\operatorname{Rm}(g^k)|^{n/2}\, dg^k < \kappa,$$

and κ is given in Lemma 1.1. Thus

$$N(k) \le c(H, i_0, K, n, D).$$

So $N(k)$ is uniformly bounded by a constant c which is independent of r and k. We may assume $N(k) \equiv N \in \mathbb{Z}^+$, $h(k) = h \in \mathbb{Z}^+$ for all k, and $r < i_o/100$.

Remark 1.3. We will call a ball $B^k(x_i, r(1+2\eta))$ bad for M_k if

$$\int_{B^k(x_i, r(1+2\eta))} |\mathrm{Rm}(g^k)|^{n/2}\, dg^k > \kappa.$$

Otherwise, it is called good.

Let $Q' = Q - N$, and denote the good balls by

$$\{B^k(x_i, r(1+2\eta))\}, \qquad i = 1, \cdots, Q',$$

and bad balls by

$$\{B^k(x_i, r(1+2\eta)\}, \qquad i = Q'+1, \cdots, Q.$$

Now take $\frac{1}{3} > \delta > \frac{2}{7} > \frac{1}{4}$, and let $\bar{r} = \delta(1-\eta)r/10$. Using Lemma 1.1, we have a harmonic coordinate F_i^k with domain $\supset B^k(x_i, \delta^r)$ and $(F_i^k)^{-1}(B(10\bar{r})) \supset B^k(x_i, \delta(1-2\eta)r)$. We use $H_i^k(\rho)$ to denote the ball in harmonic coordinate $H_i^k(\rho) = (F_i^k)^{-1}(B(\rho))$, and let

$$U_k(r) = \bigcup_{i=1}^{Q'} B^k\left(x_i, \frac{r}{50}\right),$$

$$V_k(r) = \bigcup_{i=1}^{Q'} H_i^k(\bar{r}).$$

From (b) of Lemma 1.1, we have $V_k(r) \supset U_k(r)$, and if $H_i^k(\bar{r}) \cap H_j^k(\bar{r}) \neq \varnothing$, then $H_j^k(\bar{r}) \subset H_i^k(10\bar{r})$. Now the Hölder bounds of Lemma 1.1 are universal for the whole sequence. The transition functions can be considered as mappings $F_{ij}^k \colon B(\bar{r}) \to B(10\bar{r})$, and $|F_{ij}^k|_{C^{2+\alpha}} \le c(\bar{r})$ by (c) of 1.1. By Ascoli's theorem there exists a subsequence such that all pairs (i, j), for which a transition function exists, converge in the C^2-topology to limit functions $F_{ij}^\infty \colon B(\bar{r}) \to B(10\bar{r})$ of class $C^{2,\alpha}$. The metrics also converge—considered as functions on $B(\bar{r})$—to limit *metrics* g_i^∞ of class $C^{1,\alpha}$ on each coordinate ball, i.e., on each copy of $B(\bar{r})$. The distinct copies of $B(\bar{r})$ are now glued together via the transition functions F_{ij}^∞. Consider the restriction $F_{ij} = F_{ij}^\infty | (F_{ij}^\infty)^{-1}(B(\bar{r})) \cap B(\bar{r})$ and define

$x \sim y: \Leftrightarrow \exists F_{ij}$, such that $F_{ij}(x) = y$. Set $M(r) = \cup_{i=1}^{Q'} B(\bar{r})/\sim$. If F_i^∞ denotes the canonical projection, restricted to the i th copy of $B(\bar{r})$, $M(r)$ becomes a $C^{2,\alpha}$-manifold with the F_i^∞ as coordinates. $F_j^\infty (F_i^\infty)^{-1} = F_{ij}$ are the transition functions. By a classical result of Whitney's, $M(r)$ contains a smooth structure, which define $M(r)$ as a C^∞ manifold. We take any fix C^∞ Riemannian metric $\overline{g}(r)$ on $M(r)$. We define maps

$$f_i^k = (F_i^\infty)^{-1} \cdot (F_i^k).$$

Since $F_{ij}^i \xrightarrow{C^2} F_{ij}^\infty$, we have $|f_i^k - f_j^k|_{C^2} \to 0$ on $H_i^k(\bar{r}) \cap H_j^k(\bar{r})$. We can use the center of mass technic with respect to $\overline{g}(r)$ by using partition of unity with harmonic coordinate ball $H_i^k(\bar{r})$ to construct a C^∞ map $\phi^k(r)\colon F\{x \in V_k(r) | d(x, \partial V_k(r)) \geq \eta\bar{r}\} \to M(r)$, such that there exist an open subset $F_k(r) \supset \{x \in V_k(r) | d(x, \partial V_k(r)) \geq 2\eta\bar{r}\}$ and an open subset $D(r) \supset M(r)$ such that $\phi^k(r)\colon F_k(r) \to D(r)$ are diffeomorphisms, and

$$|\phi^k(r)|_{C^{2,\alpha}} \leq C(r)$$

and

$$\inf\{|d\phi^k(r)(v)|, |v| = 1\} \geq c > 0$$

in harmonic coordinates. We take $f^k(r) = (\phi(\eta))^{-1}$. We clearly have

$$|f^k(r)|_{C^{2,\alpha}} \leq c(r)$$

in harmonic coordinates, and

$$|f^k(r)^* g^k|_{C^{1,\alpha}} \leq c(r)$$

on $D(r)$. By taking subsequences once more, the $f^k(r)^* g^k$ converges to a limit metric $g(r)$ of class $C^{1,\alpha}$ on $D(r)$. Note

$$F_k(r) \cup \bigcup_{i=Q'+1}^{Q} B^k(x_i, r(1+2\eta)) = M_k,$$

which completes the proof of Proposition 1.2.

Now, for $r_l \to 0$, we have $F_k(r_l) \subset F_k(r_{l+1})$, and clearly there exist isometries

$$I_l = D(r_l) \to D(r_{l+1}).$$

Using these isometries and taking the direct limit of $D(r_l)$, we can define

$$M' = \coprod_{l+1}^{\infty} D(r_l).$$

M' is a C^∞ manifold with $C^{1,\alpha}$ metric g' such that $g'|D(r_l) = g(r_l)$. We then have the following.

Proposition 1.4. *For each l, there are diffeomorphisms $f^k(r_l)\colon F_k(r_l) \to D(r_l) \to\subset M'$ for $k \geq l$, such that $f^k(r_l)^* g^k \to g'|D(r_l)$ in $C^{1,\alpha}$ norm.*

Proof of Theorem 0.1. Applying the diameter estimate of a small geodesic sphere [10], [11, Theorem 4.23], we have for any $(M, g) \in G(H, i_0, K, n)$ and $x_0 \in M$,

$$\operatorname{diam}(S(x_0, r)) \leq C(H, i_0, n)r, \; r \leq \frac{i_0}{4},$$

which together with Proposition 1.4 clearly implies that the m_i are isolated point singularities of the $C^{1,\alpha}$ metric g'.

In order to finish the proof of Theorem 0.1, we first prove the following.

Lemma 1.5. *Let $M = M_k$ for large k. Then g' can be extended to a C° metric g on M.*

Proof. As in the proof above, we have

$$\int_{D(r_l)} |\operatorname{Rm}(\overline{g}^k)|^{n/2}\, d\overline{g}^k \leq K$$

for $\overline{g}^k = f^k(r_l)^* g^k$, once more taking subsequences. By the diagonalization process we may assume

$$\int_{D(r_l)} |\operatorname{Rm}(\overline{g}^k)|^{n/2} d\overline{g}^k \to \int_{D(r_l)} |\operatorname{Rm}|^{n/2}\, dg'$$

for a measurable function $|\operatorname{Rm}|$ on M', and

$$\int_{M'} |\operatorname{Rm}|^{n/2}\, dg' \leq K.$$

For any singular point $v = m_i$, first note the distance function d can be extended to M, define the neighborhood N of v in M as

$$N(v) = \{x \in M | \, d(x, v) \leq 2\varepsilon\}$$

for small $\varepsilon > 0$, and

$$A(\rho, 0) = \left\{x \in M' \middle| \, d(x, v) < 2\rho\right\}.$$

By considering the *tangent cone metric,* Lemma 1.5 clearly follows from the following.

Lemma 1.6. *$((1/\rho)A(\rho, 0), x_\rho)$ converges to $(U(2) - \{0\}, \vec{e})$ in $C^{1,\alpha}$ topology, where $U(2) = \{x \in \mathbb{R}^n | \; |x| < 2\}$, $\vec{e} \in \mathbb{R}^n$, $|\vec{e}| = 1$, and $x_\rho \in A(\rho, 0)$ with $d(x_\rho, v) = \rho$.*

Proof. For fixed N, let

$$A\left(\rho, \frac{\rho}{N}\right) = \left\{x \in M' \middle| \frac{\rho}{N} < d(x, v) < 2\rho\right\}.$$

We show

$$\left(\frac{1}{\rho}A(\left(\rho, \frac{\rho}{N}\right), x_\rho\right)$$

converges to a flat manifold D_N. Since (M_k, g^k) converges to M' away from the singularities, there exists a submanifold $(A^k_{\rho, N}, y_\rho)$ of (M_k, g^k) for large k_ρ such that

$$\left| \int_{A\rho, N} |\operatorname{Rm}(g^k)|^{n/2} dg^k - \int_{A(\rho, \rho/N)} |\operatorname{Rm}|^{n/2} dg \right| \leq \rho^2$$

and

$$\left\| \left(\frac{1}{\rho}A\left(\rho, \frac{\rho}{N}\right) x_\rho\right) - \left(\frac{1}{\rho}A^{k_\rho}_{\rho, N}, y_\rho\right) \right\|_{C^{1,\alpha}} < \rho.$$

The injectivity radius of $\frac{1}{\rho}A^{K_\rho}_{\rho, N}$ is bounded from below, and

$$\int_{A(\rho, \rho/N)} |\operatorname{Rm}|^{n/2} dg \to 0 \quad \text{when } \rho \to 0.$$

Thus $(\frac{1}{\rho}A^{k_\rho}_{\rho, N}, y_\rho)$ converges to a flat manifold D_N in $C^{i,\alpha}$ norm in the proof above and in the main theorem of [10]. This implies that $(\frac{1}{\rho}A(\rho, \rho/N), x_\rho)$ converges to (D_N, e_N) in $C^{1,\alpha}$ norm. As above, we can take the direct union of (D_N, e_N) to obtain $U(0), e)$ with an isolated point singularity. Since the injectivity radius of $(M_k, g^k) \geq i_0 > 0$, it is easily seen that $U(0)$ is a simply connected flat manifold, and $U(0) = U(2) - \{0\}$ [11]. This completes the proof of Lemma 1.6.

Secondly, for r_l small, we have

$$F_k(r_l) \cup \bigcup_{i=Q'+1}^{Q} B^k(x_i, r_l(1 + 2\eta)) = M_k,$$

where $B^k(x_i, r_l(1 + 2\eta))$ are diffeomorphism balls, and M_k is obtained from $D(r_l)$ by gluing a ball to each end of $D(r_l)$ for large l. There are only a finite number of ways to obtain different diffeomorphism classes by doing this [22]. By taking subsequences once more, we may assume the M_k are all diffeomorphic for large k. This clearly implies that M' is diffeomorphic to $M_k - \{m_1^k, \cdots, m_N^k\}$. Hence the proof of Theorem 0.1 is complete.

Proof of Theorems 0.3 *and* 0.4. Let $\tau = +1, 0$ or -1. Suppose Theorem 0.3 or 0.4 is false. Then we have a sequence of Riemannian manifolds $\{M_k, g^k\}$ with

$$(\tau - \varepsilon)g^k \leq \mathrm{Ric}(g^k) \leq (\tau + \varepsilon)g^k$$

and

$$\mathrm{inj}(g) \geq i_0, \quad \mathrm{diam}(g^k) \leq D, \quad \int_M |\mathrm{Rm}(g^k)|^{n/2} \, dg^k \leq K.$$

By taking subsequences, we may assume $\{M_k, g^k\}$ converges to a $C^{1,\alpha}$ Riemannian manifold (M', g') with a finite number of isolated point singularities on the C^∞ manifold M' such that $M' = M_k - \{m_1, \cdots, m_n\}$ for large k, and from the proof of Theorem 0.1, the weak curvature tensor of g' is well defined and we have

$$\int_{M'} |\mathrm{Rm}(g')|^{n/2} \, dg' \leq K.$$

First, for each point $x \in M'$, we have a harmonic coordinate near x with $C^{1,\alpha}$ metric tensor h^{ij}, and we have the weak equation

$$\Delta h^{ij} = 2\tau h^{ij} + \sum h^{kl} h^{pq} \Gamma^j_{lg}.$$

By the standard elliptic theory, h^{ij} is C^∞ and g' is an Einstein metric on M'. Then using the estimates of [11] for $n = 4$ and [31] for $n \geq 5$, we have

$$\sup_{M'} |\mathrm{Rm}(g')| \leq C.$$

We then can extend g' to a C^∞ Einstein metric on M_k for large k, this contradicts the fact that we suppose M_k has no Einstein metric.

Proof of Theorem 0.8. Let $\{M_k, g^k\}$ be a sequence of n-Riemannian manifolds with $|\mathrm{Ric}(g^k)| \leq H$, diam $\leq D$, and Vol $\geq V$, such that there exists a $\rho > 0$ with

$$\int_{B(x,\rho)} \|\mathrm{Rm}(g^k)\|^{n/2} \, dg^k \leq \frac{1}{k}$$

for all $x \in M_k$. If we take $\kappa(H, D, V) > 0$ sufficiently small, as in ([10, §VI or §II], or [34]), we can deform the metric g^k by the involution equation

$$\frac{\partial}{\partial t} h^k(t) = -2\,\mathrm{Ric}(h^k), \tag{1}$$

where $h^k(0) = g^k$, and $h^k(t)$ exists on $[0, T]$ for $T = T(H, D, V, \kappa, \rho) > 0$. We also have

(a) $$|\mathrm{Rm}(h^k(t))| \leq \tfrac{1}{t} C(H, D, V, \kappa, \rho),$$

(b) $$|\mathrm{Ric}(h^k(t))| \leq C(H, D, V, \kappa, \rho).$$

From (1) it follows that

(2) $$(1 - Ct)h^k(t) \leq g^k(t) \leq (1 + Ct)h^k(t).$$

By Bishop-Gromov volume comparison we obtain

$$\mathrm{Vol}(B^{g^k}(x, r)) \geq C(H, D, V)r^n,$$

which implies

$$\mathrm{Vol}(B^{h^k}(x, r)) \geq C(H, D, V, \kappa, \rho)r^n.$$

In order to show g^k to be convergent in $C^{1,\alpha}$ norm, it is sufficient to prove $L^2 g^k$ convergent in $C^{1,\alpha}$ norm for a large $L^2 > 0$. We take $L^2 = T/t$. Let $\overline{h}^k(t) = L^2 h^k(t)$, and $\overline{g}^k(t) = L^2 g^k(t)$. Then

(i) $$|\mathrm{Rm}(\overline{h}^k(t))| \leq C(H, D, V, \kappa, \rho),$$

(ii) $$\mathrm{Vol}(B^{\overline{h}^k}(x, r)) \geq C(H, D, V, \kappa, \rho)r^n,$$

(iii) $$(1 - Ct)\overline{h}^k(t) \leq \overline{g}^k(t) \leq (1 + Ct)\overline{h}^k(t).$$

By the local injectivity radius estimate of [5], and Rauch comparison estimate, there exists a $\rho_0 > 0$, such that on any geodesic $B^{\overline{h}^k}(x, \rho_0)$ of radius ρ_0, and any geodesic normal coordinates of $\overline{h}^k(t)$ on $B^{\overline{h}^k}(x, \rho_0)$, we have

$$|\overline{h}^k_{ij}(x) - \delta_{ij}| \leq Cr^2(x),$$

which together with (2) clearly implies that

$$(1 - Ct - Cr^2)\delta_{ij} \leq \overline{g}^k_{ij}(x) \leq (1 + Ct + Cr^2)\delta_{ij}.$$

Since the above inequality is uniform for all k and for small t and r, we can use it to replace Theorem 2.5. Since the arguments of §3 are still valid, we can obtain uniform harmonic coordinates at every point of M for all $\overline{g}^k$. This implies that $\{\overline{g}^k\}$ has a subsequence which is convergent in $C^{1,\alpha}$ norm. Thus the proof of Theorem 0.8 is finished.

Theorems 0.10 and 0.11 follow from 0.8 and [10].

2. The local metric estimates

Let $H \geq 0$ and $i_0 > 0$ be given numbers. We define the set of compact, connected Riemannian manifolds of dimension n as

$$G(H, i_o, \cdot, n) = \{(M, g) | |\mathrm{Ric}| \leq H, \mathrm{inj}(M) \geq i_0\}.$$

In this section we recall some results of [10] and will make some changes to suit our needs. Let $(M, g) \in G(H, i_0, \cdot, n)$, and for any $x_0 \in M$, the geodesic ball and geodesic sphere of M with center x_0 and radius ρ are denoted by $B^g(x_0, \rho)$ and $S^g(x_0, \rho)$, or simply by $B(x_0, \rho)$ and $S(x_0, \rho)$, respectively. For $\rho < i_0$, we consider the metric g in polar geodesic coordinates on $B(x_0, \rho)$. We have $g = dr^2 + \sum g_{ij}(\theta, r)\, d\theta^i\, d\theta^j$. Let $g_{x_0}(r)$ or simply $g(r)$ be the induced metric on the geodesic sphere $S(x_0, r)$. Then

$$g = dr^2 + g(r).$$

We use $\mathrm{Rm}(g)$ to denote the curvature tensor of g, and $\mathrm{Rm}(g(r)) = \mathrm{Rm}(r)$ to denote the curvature tensor of $g(r)$. We also agree to denote the second fundamental form of $S(x_0, r)$ by $A(X, Y) = \langle \nabla_X Y, \partial/\partial r\rangle$ for vector fields X, Y on $S(x_0, r)$, and the scalar curvature free curvature tensor of $g(r)$ by $\overset{\circ}{\mathrm{Rm}}(r) = \overset{\circ}{\mathrm{Rm}}(g(r))$,

$$\overset{\circ}{\mathrm{Rm}}_{ijkl}(r) = R_{ijkl}(r) - \frac{R(r)}{(n-1)(n-2)}(g_{ik}(r) g_{jl}(r) - g_{il}(r) g_{jk}(r)),$$

where $R(r)$ is the scalar curvature of $g(r)$, $n = \dim M$, and $g_{ij}(\theta, r)$.

Lemma 2.1. *For $\rho \leq i_0/4$, $0 < \bar\rho < \rho$, and $0 < \delta < i_0/2 - \rho$, we have*

$$\int_{B_\delta(x_0)} \left(\max_{\bar\rho \leq r \leq \rho} \int_{S(x, r)} \left| A_x(r) + \frac{1}{r} g_x(r) \right|^{n/2} dg_x(r) \right) dg(x) \leq C\left(H, \frac{1}{\delta}, n, \bar\rho, \rho\right) \int_{B(x_0, 2\delta+\rho)} |\mathrm{Rm}(g)|^{n/2}\, dg.$$

We sketch the proof of this lemma [10, 4.12(a)].

(1) For the minimal geodesic γ from x to y, let $\gamma(0) = x$, $\gamma(\rho) = y$, and $d(x, y) = \rho$. Then [10, 4.8(a), 4.9].

$$\max_{\bar\rho \leq r \leq \rho} \left| A_x(r) + \frac{1}{r} g_x(r) \right|^2 (\gamma(r)) \leq C(H, n, \bar\rho, \rho) \int_\gamma |\mathrm{Rm}(g)|^2.$$

(2) Now for each x, y, $d(x, y) = \rho + \delta \leq i_0/2$, we define the function

$$f(x, y) = \max_{\bar{\rho} \leq r \leq \rho} \left| A_x(r) + \frac{1}{r} g_x(r) \right|^{n/2} (\gamma(r))$$

for the minimal geodesic γ from x to y. Let

$$\Omega = \bigcup_{x \in B(x_0, \delta)} S(x, \rho + \delta) \subset M,$$

$$\Sigma = \bigcup_{x \in B(x_0, \delta)} (x, S(x, \rho + \delta)) \subset M \times M.$$

Then we have

$$\begin{aligned} \int_\Sigma \int f(x, y) &= \int_{B(x_0, \delta)} \left(\int_{S(x, \rho+\delta)} f(x, y)\, dg_x(y) \right) dg(x) \\ &= \int_\Omega \left(\int_{\Omega y} f(x, y)\, dg_y(x) \right) dg(y), \end{aligned}$$

where

$$\Omega_y = B(x_0, \delta) \cap S(y, \rho + \delta) \subset S(y, \rho + \delta),$$

$$\int_\Sigma \int f(x, y) \leq \int_\Omega \left(\int_{\Omega y} f(x, y)\, dg_y(x) \right) dg(y).$$

If γ is the geodesic from x to y and $\bar{\gamma}(t) = \gamma(t)$ for $t \leq \rho$, from (1) it follows that

$$\int_{\Omega y} f(x, y)\, dg_y(x) \leq C(H, \bar{\rho}, \rho) \int_{\Omega y} \left(\int_{\bar{\gamma}} |\operatorname{Rm}(g)|^{n/2} \right) dg_y.$$

Here $\int_{\bar{\gamma}} |\operatorname{Rm}(g)|^{n/2}$ is considered as a function of x and y with $d(x, y) = \rho + \delta$, and hence

$$\begin{aligned} &\int_{\Omega y} f(x, y)\, dg_y(x) \\ &\quad \leq C(H, \bar{\rho}, \rho) \int_\delta^{\rho+\delta} \left(\int_{\Omega y} |\operatorname{Rm}(g)|^{n/2})(\gamma(\rho + \delta - t))\, dg_y \right) dt. \end{aligned}$$

From [10, 1.11], we obtain $dg_y(\gamma(\rho + \delta - t)) \geq c(H, n, \rho/\delta)\, dg_y(x)$, which implies

$$\begin{aligned} \int_{\Omega y} f(x, y)\, dg_y(x) &\leq C\left(H, n, \bar{\rho}, \frac{\rho}{\delta}\right) \int_{C(y, \rho+\delta)} |\operatorname{Rm}(g)|^{n/2}\, dg \\ &\leq C\left(H, n, \bar{\rho}, \frac{\rho}{\delta}\right) \int_{B(x_0, \rho+2\delta)} |\operatorname{Rm}(g)|^{n/2}\, dg, \end{aligned}$$

where $C(y, \rho+\delta)$ is the geodesic cone over Ωy with vertex y. By noting $\Omega \subset B(x_0, \rho+\delta)$, we get

$$\int_\Sigma \int f(x, y) \le C\Big(H, n, \bar\rho, \frac{\rho}{\delta}\Big)\mathrm{Vol}(B(x_0, \rho+\delta)) \int_{B(x_0, \rho+2\delta)} |\mathrm{Rm}(g)|^{n/2}\, dg.$$

Lemma 2.2. *For* $0 \le \bar\rho < \rho \le i_0/4$, *we have*

$$\begin{aligned}\int_{\bar\rho}^{\rho} \left(\int_{S(x_0, r)} |\mathrm{Rm}(g(r))|_{g(r)}^{n/2}\, dg(r)\right) dr \\ \le C(H, n, \bar\rho, \rho) + C(H, n, \bar\rho, \rho) \int_{B(x_0, \rho)} |\mathrm{Rm}(g)|^{n/2}\, dg\end{aligned}$$

(*see* [10, 1.14).]

Lemma 2.3. *For* $0 \le \bar\rho < \rho \le i_0/4$, *we have*

$$\begin{aligned}&\int_{\bar\rho}^{\rho} \left(\int_{S(x_0, r)} \left|R_{ijkl}(g(r)) - \frac{1}{r^2}(g_{ik}(r)g_{jl}(r) - g_{il}(r)g_{jk}(r))\right|^{n/4} dg(r)\right) dr \\ &\quad\le C(H, n, \bar\rho, \rho)\left(\int_{B(x_0, \rho)} |\mathrm{Rm}(g)|^{n/2}\, dg\right)^{1/2} \\ &\qquad + C\left(\max_{\bar\rho \le r \le \rho} \int_{S(x_0, r)} \left|A(r) + \frac{1}{r} g(r)\right|^{n/2} dg(r)\right)^{1/2} \\ &\qquad + C(H, n, \bar\rho, \rho)\left(\max_{\bar\rho \le r \le \rho} \int_{S(x_0, r)} \left|A(r) + \frac{1}{r} g(r)\right|^{n/2} dg(r)\right).\end{aligned}$$

Proof. By the Gauss formula, we obtain

$$R_{ijkl}(g) = R_{ijkl}(g(r)) + (A_{ik}(r)A_{jl}(r) - A_{il}(r)A_{jk}(r)).$$

First, we have

$$\begin{aligned}&\max_{\bar\rho \le r \le \rho} \int_{S(x_0, r)} |A(r)|^{n/2}\, dg(r) \\ &\quad\le C(H, n, \bar\rho, \rho) + \max_{\bar\rho \le r \le \rho} \int_{S(x_0, r)} \left|A(r) + \frac{1}{r} g(r)\right|^{n/2} dg(r).\end{aligned}$$

Second, we derive

$$\begin{aligned}
&\int_{S(x_0,r)} \left| A_{ik}(r)A_{jl}(r) - \frac{1}{r^2} g_{ik}(r) g_{jl}(r) \right|^{n/4} dg(r) \\
&\le C \int_{S(x_0,r)} |A|^{n/4} \left| A + \frac{1}{r} g(r) \right|^{n/4} dg(r) + C \int_{S(x_0,r)} \left| A(r) + \frac{1}{r} \right|^{n/4} dg(r) \\
&\le C \left(\int_{S(x_0,r)} |A|^{n/2} dg(r) \right)^{1/2} \left(\int_{S(x_0,r)} \left| A + \frac{1}{r} g(r) \right|^{n/2} dg(r) \right)^{1/2} \\
&\quad + C \left(\int_{S(x_0,r)} \left| A(r) + \frac{1}{r} g(r) \right|^{n/2} dg(r) \right)^{1/2} \\
&\le \left(\int_{S(x_0,r)} \left| A(r) + \frac{1}{r} g(r) \right|^{n/2} dg(r) \right)^{1/2} \\
&\quad + C \int_{S(x_0,r)} \left| A(r) + \frac{1}{r} g(x) \right|^{n/2} dg(r)
\end{aligned}$$

which together with the Gauss formula clearly implies Lemma 2.3.

Lemma 2.4. *For* $0 < \bar{\rho} < \rho \le i_0/4$, *we have*

$$\begin{aligned}
&\int_{\bar{\rho}}^{\rho} \left(\int_{S(x_0,r)} \left| \overset{\circ}{\mathrm{Rm}}(r) \right|^{n/4} dg(r) \right) dr \\
&\le C(H, n, \bar{\rho}, \rho) \left(\int_{B(x_0,\rho)} |\mathrm{Rm}(g)|^{n/2} dg \right)^{1/2} \\
&\quad + C(H, n, \bar{\rho}, \rho) \left(\max_{\bar{\rho} \le r \le \rho} \int_{S(x_0,r)} \left| A(r) + \frac{1}{r} g(r) \right|^{n/2} dg(r) \right)^{1/2} \\
&\quad + C(H, n, \bar{\rho}, \rho) \left(\max_{\bar{\rho} \le r \le \rho} \int_{S(x_0,r)} \left| A(r) + \frac{1}{r} g(r) \right|^{n/2} dg(r) \right).
\end{aligned}$$

This follows easily from Lemma 2.3.

Theorem 2.5. *For* $0 \le \bar{\rho} < \rho \le i_0/4$ *and a sequence of* $\{M_k, g_k\} \subset G(H, i_0, \cdot, n)$, *let* $x_k \in M_k$, *and assume*

$$\eta k = \max_{\bar{\rho} \le r \le \rho} \int_{S(x_k,r)} \left| A_{x_k}(r) + \frac{1}{r} g_k(r) \right|^{n/2} dg_k(r) \to 0,$$

$$\mu_k = \int_{B(x_0,\rho)} |\mathrm{Rm}(g_k)|^{n/2} dg_k \to 0.$$

Then there exists a diffeomorphism $\phi_k : S(x_k, \rho) \to S(x_k, \rho)$ *for each* k, *such that on* $B(x_k \rho)$, $gk = dr^2 + gk(r)$, *and by identifying each* $S(x_k, r)$ *with the Euclidean unit sphere* $S(1)$ *with metric* $d\theta^2$, *we have*

$$\int_{S(1)} |\phi_k^* g_k(r) - r^2 d\theta^2|^{n/2} d\theta \to 0$$

uniformly for $\bar{\rho} \le r \le \rho$, *and* $|\phi_k^* g_k(\rho) - \rho^2 d\theta^2|_{C^0 \to 0}$.

In particular, since ϕ_k can trivially extend to $B(x_k, \rho) - B(x_k, \bar{\rho})$, for $\overline{g}_k = \phi_k^* g_k$ on $B(x_k, \rho) - B(x_k, \bar{\rho})$, by identifying $B(x_k, \rho) - B(x_k, \bar{\rho})$ with the Euclidean annulus $A(\bar{\rho}, \rho) = \{x \in \mathbb{R}^n | \bar{\rho} < |x| \le \rho\}$, we obtain

$$\int_{A(\bar{\rho}, \rho)} |\overline{g}_k - ds^2|^{n/2} ds \to 0,$$

where ds^2 is the Euclidean metric on $A(\bar{\rho}, \rho)$.

For the proof of this theorem, see [10, 5.18, 5.21, 5.25].

3. The harmonic coordinates

In this section, we shall prove the main technical result of this paper. Let $(M, g) \in G(H, 4, \cdot, n)$, and $x_0 \in M$. For simplicity write $B_\rho(x)$ as $B(x, \rho)$.

Theorem 3.0 (*Harmonic coordinate*). *For small* η, $0 < \eta < 1$, *there exists a small constant* $\kappa = \kappa(H, n, \eta) > 0$, *such that if*

$$\int_{B_{1+2\eta}(x_0)} |\,\mathrm{Rm}\,|^{n/2} dg \le \kappa,$$

then, for any $0 < \delta < \frac{1}{3}$ *and* $x \in B_{1+2\eta}(x_0)$ *so that* $d(x_0, x) < 1 - \delta - 2\eta$, *there exists a harmonic coordinate* F *with domain* $F \supset B_\delta(x)$ *and image* $F \supset B(\delta(1-\eta)) = \{x \in \mathbb{R}^n |\, |x| \le \delta(1-\eta)\}$, *such that*

(a) $$F^{-1}(B(\delta(1-\eta))) \supset B_{\delta(1-2\eta)}(x).$$

Let $h^{ij} = \langle \nabla n^i, \nabla h^j \rangle$ be the metric tensor in such a harmonic coordinate, $F = (h^1, \cdots, h^n)$. Then

(b) $$|h^{ij} - \delta^{ij}|_{C^0} \le \frac{\eta^2}{100n},$$

(c) $$|dh^{ij}|_{C^\alpha} \le C(H, n, \eta), \quad 0 < \alpha < 1, \quad \text{on } B(\delta(1-\eta)),$$

(d) $$\|h^{ij}\|_{L^{2,p}} \le C(H, n, \eta), \qquad p > n.$$

To prove Theorem 3.0 we start with the following.

Main Lemma. *For any* $0 < \eta < 1$, *there exists a small constant* $\kappa = \kappa(H, n, \eta) > 0$, *such that if*

$$\int_{B_{1+2\eta}(x_0)} |\operatorname{Rm}(g)|^{n/2}\, dg \le \kappa,$$

then there exist a point $x \in B_\eta(x_0)$ *and a diffeomorphism* F *from*

$$T_x(1+\tfrac{1}{2}\eta, \tfrac{3}{2}\eta) = \overline{B_{1+\frac{1}{2}\eta}(x) - B_{\frac{3}{2}\eta}(x)}$$

into

$$T(1+\tfrac{1}{2}\eta, \tfrac{3}{2}\eta) = \left\{x \in \mathbb{R}^n \,\middle|\, \tfrac{3}{2}\eta \le |x| \le 1+\tfrac{1}{2}\eta\right\},$$

such that

(a) F *is harmonic, i.e.,* $\Delta F = 0$.

(b) $F^{-1}(T(1+\frac{1}{4}\eta, \frac{1}{4}\eta+\eta)) \supset T_x(1-\eta, 2\eta)$, *and*

$$\textit{image } F \supset T(1+\tfrac{1}{4}\eta, \tfrac{5}{4}\eta).$$

Let $h^{ij} = \langle \nabla h^i, \nabla h^j\rangle$ *be the metric tensor in this harmonic coordinate, where* $F = (h^1, \cdots, h_n)$. *Then*:

(c) $|h^{ij} - \delta^{ij}|_{C^0} < \eta^2/100n$, *on* $T(1+\frac{1}{4}\eta, \frac{5}{4}\eta)$,

(d) $|dh^{ij}|_{C^\alpha} \le C(H, n, \eta)$, $0 < \alpha < 1$, *on* $T(1+\frac{1}{4}\eta, \frac{5}{4}\eta)$,

(e)

$$\left\| |F|^2 - r^2 \right\| \le \frac{\eta}{100n},$$

where $|F|^2 = \sum (h^i)^2$, *and* $r^2(y) = d(x, y)^2$ *is the square distance function of* (M, g) *from point* x.

(f)

$$\|ddh^{ij}\|_{L^p} \le C \quad \text{on } T\left(1+\tfrac{1}{4}\eta, \tfrac{5}{4}\eta\right), \ p > n.$$

We shall prove the main lemma by contradiction. It is easily seen that the main lemma follows from the following.

Lemma 3.1. *For any* $\eta, 0 < \eta < 1$, *small* $\delta > 0$, *and a sequence* $\{(M_k, g^k)\} \subset G(H, 4, \cdot, n)$, *there exists a point* $x_k \in M_k$ *for each* k, *such that*

$$\int_{B_{1+2\delta}(x_k)} |\operatorname{Rm}(g^k)|^{n/2}\, dg_k \le \frac{1}{k}.$$

Furthermore, for k *sufficiently large, there exists a point* $y_k \in B_\delta^k(x_k) \subset M_k$ *for each* k *and a diffeomorphism* $F_k = (h_k^1, \cdots, h_k^n)$ *from*

$$\Omega_k \supset T_{y_k}(1-\eta, 2\eta) = \overline{B_{1-\eta}(y_k) - B_{2\eta}(y_k)}$$

into $T(1, \eta) = \{x \in \mathbb{R}^n | \eta \leq |x| \leq 1\}$, *such that*

(a) F_k *is harmonic, i.e.,* $\Delta F_k = 0$ *on* $T_{yk}(1-\eta, 2\eta)$.

(b)

$$F_k^{-1}\left(1-\tfrac{3}{4}\eta, \tfrac{3}{4}\eta+\eta\right) \supset T_{yk}(1-\eta, 2\eta)$$

and image $F_k \supset T(1-\frac{3}{4}\eta, \frac{7}{4}\eta)$.

Let $H_k^{ij} = \langle \nabla h_k^i, \nabla h_k^j \rangle$ *be the metric tensor in this harmonic coordinate. Then*:

(c)

$$\left| h_k^{ij} - \delta^{ij} \right|_{C^0} < \eta^2/200n \quad on\ T\left(1-\tfrac{3}{4}\eta, \tfrac{3}{4}\eta+\eta\right),$$

(d)

$$\left| dh_k^{ij} \right|_{C^\alpha} \leq C(H, n, \eta, \delta), \quad 0 < \alpha < 1\ on\ T(\left(1-\tfrac{3}{4}\eta, \tfrac{3}{4}\eta+\eta\right),$$

(e)

$$\left| |F_k|^2 - r_k^2 \right| \leq \eta^2/200n,$$

where $r_k^2(y) = d(y_k, y)^2$ *is the square of the distance function of* (M_k, g^k) *from* y_k,

(f) $\|D^2 h^{ij}\|_{L^p} \leq C(H, n, \eta, \delta)$, $p > n$.

Remark. The main lemma follows from Lemma 3.1 by slightly changing parameters.

The proof of Lemma 3.1 is given through a series of lemmas.

First, using Lemma 2.1, there exists a point $y_k \in B_\delta^k(x_k)$ such that

$$\begin{aligned}\max_{\eta \leq r \leq 1} \int_{S_r^k(y_k)} \left| A_{y_k}^k(r) + \frac{1}{r} g_{y_k}^k(r) \right|_{g^k(r)}^{n/2} dg_{y_k}(r) \\ \leq C(H, \frac{1}{\delta}, \eta) \int_{B_{2\delta+1}(x_k)} |\operatorname{Rm}(g^k)|^{n/2} dg^k \leq C(H, \frac{1}{\delta}, \eta)\frac{1}{k}.\end{aligned}$$

Now we apply Theorem 2.5 in polar coordinates on $B_1^k(y_k)$. We have $g^k = dr^2 + \sum g_{ij}^k(\theta, r)\, d\theta^i\, d\theta^j$, and identify $B_1(y_k)$ with the unit Euclidean ball $B(1)$. By this geodesic polar coordinate, there exists a diffeomorphism $\phi_k \colon S(1) \to S(1) = \{x \in \mathbb{R}^n \mid |X| = 1\}$, such that

$$\int_{T(1, \eta)} |\phi_k^* g^k - g_0|^{n/2} dg_0 < 0\left(\frac{1}{k}\right)$$

for the Euclidean metric $g_0 = |dx|^2$ on $B(1)$. Here we extend ϕ_k trivially to $T(1, \eta)$, and will use $0(1/k)$ to denote a constant which converges to zero when $k \to \infty$.

Abusing notation slightly, we shall write g^k as $\phi_k^* g^k$, and drop all the k's in the remainder of this section. Moreover, we shall simply write g as g^k, Δ for the Laplacian operator of (M_k, g^k), etc.

Let $X = \{x^1, \cdots, x^n\}$ be the Euclidean coordinate on $B(1)$. For

$$g = \sum g_{ij}(x)\, dx^i\, dx^j,$$

we have

$$(1) \qquad \sum \int_{G(1,\eta)} |g_{ij}(x) - \delta_{ij}|^{n/2}\, dg_0 < 0\left(\frac{1}{k}\right).$$

Now, by solving the Dirichlet problem

$$\begin{cases} \Delta F = 0, \\ f|_{\partial T(1,\eta)} = X|_{\partial T(1,\eta)}, \end{cases}$$

we obtain $F\colon T(1, \eta) \to B(1)$.

The idea of the proof of the lemma is given through the following sequence of lemmas. We shall show, in fact, that F is the desired map.

Lemma 3.2.

$$\int_{T(1,\eta)} |\nabla F - \nabla x|_g^2\, dg < 0\left(\frac{1}{k}\right).$$

Proof. First we note

$$\begin{aligned} \Delta(h^l - x^l) &= \frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j}(h^l - x^l)\right) \\ &= -\frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^i}(\sqrt{g} g^{il} - \delta^{il}). \end{aligned}$$

Let $u = h^l - x^l$. Integrating by part gives

$$\begin{aligned} \int_{T(1,\eta)} |\nabla u|_g^2\, dg &= -\int_{T(1,\eta)} \Delta u \cdot u\, dg \\ &= \int^{T(1,\eta)} \frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{il} - \delta^{il}\right) u\, dg \\ &= -\int_{T(1,\eta)} \sum \left(\sqrt{g} g^{il} - \delta^{il})\right) \frac{\partial}{\partial x^i} u\, dg_0. \end{aligned}$$

By Hölder inequality, this implies

$$\int_{T(1,\eta)} |\nabla u|_g^2\, dg \le C \sum \int_{T(1,\eta)} \left|\sqrt{g} g^{il} - \delta^{il}\right|^2 dg_0.$$

Now, using [10, 1.11], we have for large k, $C^{-1} g_0 \le g \le c g_0$. This and (1) clearly imply the lemma.

Lemma 3.3. $|\nabla h^i|_g^2 \to |\nabla x^i|_{g_0}^2 = 1$ *almost everywhere on* $T(1, \eta)$.

Proof. This follows from (1) and Lemma 3.2.

Lemma 3.4. *For small* $\tau > 0$, *we have*

$$\left|\nabla h^i\right|^2 \le C(\eta, n, \tau) \quad \text{on } T(1-\tau, \tau+\eta).$$

Proof. Let us recall a well-known formula:

$$\Delta\left|\nabla h^i\right|^2 = 2\left|\nabla^2 h^i\right|^2 + 2\,\mathrm{Ric}(\nabla h^i, \nabla h^i).$$

Let $\phi = |\nabla h^i|$. Then we have

$$\Delta\phi \ge -2H\phi. \tag{2}$$

Noting the uniform Sobolev inequality on (M_k, g^k) and using the standard Moser iteration, we easily obtain the following estimate:

$$\max_{T(1-\tau, \tau+\eta)} \phi^2 \le C(\eta, n, \tau) \int_{T(1, \eta)} \phi^2 \, dg,$$

which together with Lemma 3.3 clearly implies Lemma 3.4.

Lemma 3.5. *For each* i, *we have*

$$|h^i - x^i|_{C^\circ(T(1-\tau, \tau+\eta))} < 0\left(\frac{1}{k}\right).$$

Proof. By Sobolev inequality and 3.2, we obtain

$$\begin{aligned}\left(\int_{T(1, \eta)} |h^i - x^i|^{2n/(n-2)} \, dg\right)^{(n-2)/n} &\le C \int_{T(1, \eta)} |\nabla h^i - \nabla x6i|^2 \, dg \\ &\le 0\left(\frac{1}{k}\right),\end{aligned}$$

which shows that h^i converges to x^i almost everywhere on $T(1, \eta)$. By 3.4, we have

$$|\nabla h^i| \le C(\eta, n, \tau) \quad \text{on } T(1-\tau, \tau+\eta),$$

which clearly implies Lemma 3.5.

Lemma 3.6.

$$\int_{T(1-2\tau, 2\tau+\eta)} |\nabla^2 h^i|^2 \, dg \le 0\left(\frac{1}{k}\right).$$

Proof. We shall use the formula

$$\Delta\phi^2 = 2|\nabla^2 h^i|^2 + 2\,\mathrm{Ric}(\nabla h^i, \nabla h^i)$$

for

$$\phi^2 = |\nabla h^i|^2.$$

Take a cut-off function $0 \le \mu(t) \le 1$, such that

$$\mu(t) = \begin{cases} 1, & \frac{7}{4}\tau + \eta \le t \le 1 - \frac{7}{4}\tau, \\ 0, & t \le \frac{5}{4}\tau + \eta \quad \text{or} \quad t \ge 1 - \frac{5}{4}\tau. \end{cases}$$

Then we have

$$|\mu'(t)| \le \frac{c}{\tau}\mu''(t)| \le \frac{c}{\tau^2}.$$

Let $f = \sum (h^i)^2$ and $\bar{\mu}(x) = \mu(f(x))$. Then

$$\int \Delta(\phi^2 - 1)\bar{\mu} = 2\int |\nabla^2 h^i|^2 \bar{\mu} + 2\int \mathrm{Ric}(\nabla h^i, \nabla h^i)\bar{\mu},$$

which gives

$$\int |\nabla^2 h^i|^2 \bar{\mu} \le \frac{1}{2}\int (\phi^2 - 1)|\Delta\bar{\mu}| + \int |\mathrm{Ric}|\phi^2 \cdot \bar{\mu}.$$

Now, note that for k large, by 3.5 we obtain

$$\operatorname{supp}\bar{\mu} \subset T(1 - \tau, \tau + \eta),$$
$$\bar{\mu} \equiv 1 \quad \text{on } T(1 - \tau, 2\tau + \eta),$$

which imply

$$\int_{T(1-2\tau, 2\tau+\eta)} |\nabla^2 h^i|^2 \, dg \le \frac{1}{2}\int (\phi^2 - 1)|\Delta\bar{\mu}| + \int_{T(1-\tau, \tau+\eta)} \phi^2 |\mathrm{Ric}| \, dg.$$

Since $\Delta f = 2\sum |\nabla h^i|^2 \le C(\eta, n, \tau)$, by 3.4, we clearly have

$$\begin{aligned} \int_{T(1-2\tau, 2\tau+\eta)} |\nabla^2 h^i|^2 \, dg &\le 0\left(\frac{1}{k}\right) + C\int |\mathrm{Ric}| \, dg \\ &\le 0\left(\frac{1}{k}\right) + C\left(\int |\mathrm{Ric}|^{n/2} \, dg\right)^{2/n} \le 0\left(\frac{1}{k}\right). \end{aligned}$$

Lemma 3.7. *From the Ricci formula, it follows that*

$$\begin{aligned} \Delta|\nabla^2 h^i|^2 = {} & 2|\nabla^3 h^i|^2 + 2\langle \mathrm{Rm} * \nabla^2 h^i, \nabla^2 h^i \\ & + 2\nabla\langle \mathrm{Rm} * \nabla h^i, \nabla^2 h^i\rangle - 2\langle \mathrm{Rm} * \nabla h^i, \nabla^3 h^i\rangle, \end{aligned}$$

where we have used $A * B$ to denote a bilinear form of A and B with constant coefficients.

Lemma 3.8.

$$\int_{T(1-3\tau, 3\tau+\eta)} |\nabla^2 h^i|^{2n/(n-2)} \, dg + \int_{T(1-3\tau, 3\tau+\eta)} |\nabla^3 h^i|^2 \, dg \le 0\left(\frac{1}{k}\right).$$

Proof. Take a cut-off function $\bar{\mu} = \mu(f)$ similar to the one in 3.6, such that

$$\operatorname{supp}\bar{\mu} \subset T(1 - 2\tau, 2\tau + \eta),$$
$$\bar{\mu} \equiv 1 \quad \text{on } T(1 - 3\tau, 3\tau + \eta).$$

Then we have

$$|\Delta\bar{\mu}| = |\mu'\Delta f + \mu''|\nabla f|^2| \leq \frac{c}{\tau^2}.$$

Using Lemma 3.7 and letting $\psi = |\nabla^2 h^i|^2$, we have

$$\int \psi \cdot \Delta\bar{\mu}^2 = 2\int |\nabla^3 h^i|^2\bar{\mu}^2 + 2\int \langle \mathrm{Rm} * \nabla^2 h^i\rangle \bar{\mu}^2 - 2\int \langle \mathrm{Rm} * \nabla h^i, \nabla^2 h^i\rangle \nabla\bar{\mu}^2 - 2\int \langle \mathrm{Rm} * \nabla h^i, \nabla^3 h^i\rangle \bar{\mu}^2.$$

By the Cauchy inequality $\varepsilon\alpha^2 + \frac{1}{\varepsilon}b^2 \geq 2ab$, we obtain

$$\int \psi\Delta\bar{\mu}^2 \geq 2\int |\nabla^3 h^i|^2\bar{\mu}^2 - C\int |\mathrm{Rm}|\psi\bar{\mu}^2 - C\int |\mathrm{Rm}|\,|\nabla h^i|\bar{\mu}^2 - \varepsilon\int |\nabla^3 h^i|^2\bar{\mu}^2 - C(\varepsilon)\int |\mathrm{Rm}|^2\bar{\mu}^2|\nabla h^i|.$$

Applying the Hölder inequality yields

(3)
$$(2-\varepsilon)\int \bar{\mu}^2|\nabla^3 h^i|^2 \leq \int \psi\Delta\bar{\mu}^2 + C\int |\mathrm{Rm}|^2\bar{\mu}^2 + C\left(\int |\mathrm{Rm}|^2\bar{\mu}^2\right)^{1/2}\left(\int \psi\bar{\mu}^2\right)^{1/2} + C\left(\int_{B(1)} |\mathrm{Rm}|^{n/2}\right)^{2/n}\left(\int (\bar{\mu}^2\psi)^{n/(n-2)}\right)^{(n-2)/n}.$$

Now, using the Sobolev inequality, we have

$$\begin{aligned}
\left(\int (\bar{\mu}^2\psi)^{n/(n-2)}\right)^{(n-2)/n} &= \left(\int \bar{\mu}|\nabla^2 h^i|)^{2n/(n-2)}\,dg\right)^{(n-2)/n} \\
&\leq C\int |\nabla(\bar{\mu}|\nabla^2 h^i|)|^2 + C\int \bar{\mu}^2|\nabla^2 h^i|^2 \qquad (4)\\
&\leq C\int |\nabla^3 h^i|^2\bar{\mu}^2 + C\int_{T(1-2\tau,\,2\tau+\eta)} |\nabla^2 h^i|^2.
\end{aligned}$$

(3) and (4) are combined to give

$$\begin{aligned}
(1-\varepsilon)\int |\nabla^3 h^i|^2\bar{\mu}^2 &+ \left(C - C\left(\frac{1}{k}\right)^{2/n}\right)\left(\int (\bar{\mu}^2\psi)^{n/(n-2)}\right)^{(n-2)/n} \\
&\leq C\int_{T(1-2\tau,\,2\tau+\eta)} \psi + C\left(\int |\mathrm{Rm}|^{n/2}\bar{\mu}^2\right)^{4/n} \\
&\quad + C\left(\int |\mathrm{Rm}|^{n/2}\bar{\mu}\right)^{4/n}\left(\int \psi\bar{\mu}\right)^{1/2}.
\end{aligned}$$

For k large, we clearly prove this lemma by 3.6.

Lemma 3.9.

$$\Delta\nabla^2 h^i = \nabla(\mathrm{Ric} * \nabla h^i) + Rm * \nabla^2 h^i.$$

Proof. For any x, choose the coordinate near x, such that $g_{ij}(x) = \delta_{ij}$. Apply the Ricci identity to obtain

$$\begin{aligned}
\Delta\nabla_k\nabla_l h^i &= \sum \nabla_j(\nabla_k\nabla_j\nabla_l h^i) + \sum \nabla_j(R_{jklm}g^{mp}\nabla_p h^i)\\
&= \sum \nabla_j(\nabla_k\nabla_l\nabla_j h^i) + \sum \nabla_j R_{jklm}g^{mp}\nabla_p h^i + \mathrm{Rm} * \nabla^2 h^i\\
&= \sum \nabla_k(\nabla_j\nabla_l\nabla_j h^i) + \sum \nabla_j R_{jklm}g^{mp}\nabla_p h^i + \mathrm{Rm} * \nabla^2 h^i\\
&= \sum \nabla_k\nabla_l\Delta h^i \sum \nabla_k(R_{jljm}g^{mp}\nabla_p h^i)\\
&\quad + \sum \nabla_j R_{jklm}g^{mp}\nabla_p h^i + \mathrm{Rm} * \nabla^2 h^i\\
&= \sum_k (R_{lm}g^{mp}\nabla_p h^i) + \sum \nabla_j R_{jklm}g^{mp}\nabla_p h^i + \mathrm{Rm} * \nabla^3 h^i.
\end{aligned} \tag{5}$$

Now using the second Bianchi identity yields

$$\begin{aligned}
\sum \nabla_j R_{jklm} = \sum \nabla_j R_{lmjk} &= -\sum \nabla_l R_{mjjk} - \sum \nabla_m R_{jljk},\\
&= +\nabla_l R_{mk} - \nabla_m R_{lk},
\end{aligned}$$

which together with (5) implies

$$\Delta\nabla^2 h^i = \nabla(\mathrm{Ric} * \nabla h^i) + \mathrm{Rm} * \nabla^2 h^i.$$

Lemma 3.10.

$$\int_{T(1-4\tau,\, 4\tau+\eta)} |\nabla^2 h^i|^q \, dg \le 0\left(\frac{1}{k}\right)$$

for any $q > m$.

Proof. Take $p > 0$ and a cut-off function μ, such that

$$\operatorname{supp}\mu \subset T(1-3\tau,\, 3\tau+\eta),$$

$$\mu \equiv 1 \quad \text{on } T(1-4\tau,\, 4\tau+\eta).$$

Then we have

$$|\nabla\mu| \le c/\tau.$$

Let $\psi = |\nabla^2 h^i|^2$. Using 3.9 and the Cauchy inequality $2ab \le a^2 + b^2$, we obtain

$$\begin{aligned}
\Delta\psi &= |\nabla^3 h^i|^2 + 2\langle \Delta\nabla^2 h^i , \nabla^2 h^i\rangle \\
&= 2|\nabla^3 h^i|^2 + 2\langle \nabla(\mathrm{Ric} * \nabla h^i), \nabla^2 h^i\rangle + 2\langle \mathrm{Rm} * \nabla^2 h^i , \nabla^2 h^i\rangle \\
&= 2|\nabla^3 h^i|^2 + 2\nabla\langle \mathrm{Ric} * \nabla h^i , \nabla^2 h^i\rangle - 2\langle \mathrm{Ric} * \nabla h^i , \nabla^3 h^i\rangle \\
&\quad + 2\langle \mathrm{Rm} * \nabla^2 h^i , \nabla^2 h^i\rangle \\
&\geq |\nabla^3 h^i|^2 + 2\nabla\langle \mathrm{Ric} * \nabla h^i , \nabla^2 h^i\rangle - C|\mathrm{Ric}|^2|\nabla h^i|^2 - C|\mathrm{Rm}|\psi.
\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}
&-\int \Delta\psi\psi^{2p-1}\mu^2 + \int |\nabla^3 h^i|^2\psi^{2p-1}\mu^2 \\
&\quad\leq 2\int \langle\langle \mathrm{Ric} * \nabla h^i , \nabla^2 h^i\rangle , \partial(\psi^{2p-1}\mu^2)\rangle \\
&\qquad + C\int |\mathrm{Ric}|^2\, |\nabla h^i|^2\psi^{2p-1}\mu^2 + C\int |\mathrm{Rm}|\psi^{2p}\mu^2 ,
\end{aligned}$$

$$\begin{aligned}
(2p-1)&\int |\nabla\psi|^2\psi^{2p_2}\mu^2 + \int |\nabla^3 h^i|^2\psi^{2p-1}\mu^2 \\
&\leq CH\int |\nabla^2 h^i||\nabla\psi|\psi^{2p-2}\mu^2 + CH\int |\nabla^2 h^i|\psi^{2p-1}|\nabla\mu^2| \\
&\quad + CH^2\int \psi^{2p-1}\mu^2 + C\int |\mathrm{Rm}|\psi^{2p}\mu^2 + C\int |\nabla\psi|\psi^{2p-1}|\nabla\mu^2|.
\end{aligned}$$

For $(2p-1)\geq 0$, use $2ab < \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ again to obtain

$$\begin{aligned}
\int |\nabla\psi|^2\psi^{2p-2}\mu^2 \leq C\varepsilon &\int |\nabla\psi|^2\psi^{2p-2}\mu^2 + \frac{C}{\varepsilon}\left(C\int \psi^{2p-1}\mu^2\right) \\
&+ C\int \mu|\nabla\mu|\psi^{2p-1/2} + C\int \psi^{2p-1}\mu^2 \\
&+ C\int |\mathrm{Rm}|\psi^{2p}\mu^2 \\
&+ C\varepsilon\int |\nabla\psi|^2\psi^{2p-2}\mu^2 + \frac{c}{\varepsilon}\int \psi^{2p}|\nabla\mu|^2.
\end{aligned}$$

which clearly implies that for ε small,

$$\begin{aligned}
\int |\nabla\psi|^2\psi^{2p-2}\mu^2 \leq C&\int \psi^{2p-1}\mu^2 \\
&+ C\int \psi^{2p}|\nabla\mu|^2 + C\int \mathrm{Rm}|\psi^{2p}\mu^2.
\end{aligned}$$

Now, using the Hölder and Sobolev inequalities, we get

$$\begin{aligned}\int|\nabla\psi|^2\psi^{2p-2}\mu^2 &\le C\int\psi^{2p-1}\mu^2 + C\int\psi^{2p}|\nabla\mu|^2\\ &\quad + C\left(\int|\mathrm{Rm}|^{n/2}\right)^{2/n}\left(\int(\mu\psi^p)^{2n/(n-2)}\right)^{(n-2)/n}\\ &\le C\int\psi^{2p-1}\mu^2 + C\int\psi^{2p}|\nabla\mu|^2\\ &\quad + C\left(\frac{1}{k}\right)^{2/n}\left\{C\int|\nabla(\mu\psi^p)|^2 + C\int\mu^2\psi^{2p}\right\}\\ &\le C\int\psi^{2p-1}\mu^2 + C\int\psi^{2p}(\mu^2+|\nabla\mu|^2)\\ &\quad + C\left(\frac{1}{k}\right)^{2/n}\int\mu^2|\nabla\psi^p|^2.\end{aligned}$$

For any fixed $q > 0$, we take k large, such that $q^2(1/k)^{2/n}$ is small. Then we have for $p \le q$

$$\int|\nabla\psi|^2\psi^{2p-2}\mu^2 \le C\int\psi^{2p-1}\mu^2 + C\int\psi^{2p}(\mu^2+|\nabla\mu|^2),$$

and the Sobolev inequality yields

$$\left(\int(\psi^p\mu)^{2n/(n-2)}\right)^{(n-2)/n} \le C\int\psi^{2p-2}\mu^2 + C\int\psi^{2p}(\mu^2+|\nabla\mu|^2),$$

which clearly can be iterated by using 3.8 to obtain

$$\left(\int(\mu\phi)^q\right) \le 0\left(\frac{1}{k}\right).$$

Lemma 3.11.

$$\left||\nabla h^i|^2 - 1\right| \le 0\left(\frac{1}{k}\right) \quad \textit{on } T(1-5\tau, 5\tau+\eta).$$

Proof. Let $f = |\nabla h^i|^2 - 1$. Then

$$\Delta f = 2|\nabla^2 h^i|^2 + 2\,\mathrm{Ric}(\nabla h^i, \nabla h^i) = u.$$

From 3.10 it follows that $\int_{T(1-4\tau, 4\tau+\eta)}|u|^q\,dg \le C(q)$ for $q > n$. Then we use Moser iteration with slight modifications. Define

$$f_+ = \max\{f, 0\}, \qquad f_- = \max\{-f, 0\},$$

and let $\bar{f} = f_+ + f_-$. For any cut-off function μ with $\operatorname{supp}\mu \subset T(1-4\tau, 4\tau+\eta)$, we have

$$-\int\Delta f\cdot f_\pm^{2p-1}\mu^2 = -\int\mu f_\pm^{2p-1}\mu^2.$$

Integrating by parts gives

$$\pm(2p-1)\int |\nabla f_\pm|^2 f_\pm^{2p-2}\mu^2 \pm \int \langle \nabla f, \nabla \mu^2\rangle f_\pm^{2p-1} = -\int u f_\pm^{2p-1}\mu^2,$$

$$(2p-1)\int |\nabla f_\pm|^2 f_\pm^{2p-2}\mu^2 \le \int |\nabla f_\pm|\,|\nabla\mu^2| f_\pm^{2p-1} + \int |u| f_\pm^{2p-1}\mu^2.$$

Let $\bar f = f_+ + f_-$. Then by adding the above inequalities we obtain

$$\begin{aligned}(2p-1)\int |\nabla \bar f|^2 \bar f^{2p-2}\mu^2 &\le \int |\nabla \bar f|\,|\nabla \mu^2| \bar f^{2p-1} + \int |u| \bar f^{2p-1}\mu^2\\ &\le \frac{1}{2}(2p-1)\int |\nabla \bar f|^2 \bar f^{2p-2}\mu^2\\ &\quad + \frac{2}{2p-1}\int |\nabla\mu|^2 \bar f^{2p} + \int |u| \bar f^{2p-1}\mu^2\end{aligned}$$

for $2p-1>0$, which together with the Hölder inequality yields

$$\int |\nabla \bar f^p|^2\mu^2 \le C\int |\nabla \mu|^2 \bar f^{2p} + \left(\int |u|^q\right)^{1/q}\left(\int (\mu^2 \bar f^{2p-1})^\kappa\right)^{1/\kappa},$$

where $1/q + 1/\kappa = 1$ and $\kappa < n/(n-2)$. Using the Sobolev inequality, we get

$$\left(\int (\mu \bar f^p)^{2n/(n-2)}\right)^{(n-2)/n} \le C\int (|\nabla\mu|^2 + \mu^2)\bar f^{2p} + C\left(\int (\mu^2 \bar f^{2p-1})^\kappa\right)^{1/\kappa}.$$

Since $\bar f \le |f|$ is bounded, we have

$$\begin{aligned}\left(\int (\mu \bar f^p)^{2n/(n-2)}\right)^{(n-2)/n} &\le C\int_{\operatorname{supp}\mu} \bar f^{2p-1} + C\left(\int_{\operatorname{supp}} \bar f^{(2p-1)\kappa}\right)^{1/\kappa}\\ &\le C\left(\int_{\operatorname{supp}\mu} \bar f^{(2p-1)\kappa}\right)^{1/\kappa}.\end{aligned}$$

Take $p_0 > 1$ and p_m, such that $(2p_m - 1)\kappa = 2p_{m-1}(n/n-2)$. Then we obtain

$$p_m = \frac{1}{2}\frac{\beta^m - 1}{\beta - 1} + \beta^m p_0, \qquad \beta = \frac{(n/(n-2))}{\kappa} > 1.$$

Iterating easily gives

$$\max_{T(1-5\tau,\,5\tau+\eta)} \bar f \le C\left(\int_{T(1-4\tau,\,4\tau+\eta)} \bar f^{2p_0(n/(n-2))}\right)^{q_0} \tag{6}$$

for some $q_0 = q_0(n, \kappa, p_0) > 0$.

From 3.3 and 3.4 it follows that $|f| \le C$, and $f \to 0$ almost everywhere on $T(1-4\tau, 4\tau+\eta)$ by the Lebesgue convergence theorem. Thus we obtain

$$\int_{T(1-4\tau, 4\tau+\eta)} |f|^p \, dg \le 0\left(\frac{1}{k}\right)$$

for any $p > 0$. By taking $p_0 = 2$ and $p = 4n/(n-2)$, (6) clearly implies 3.11.

Lemma 3.12.

$$|\langle \nabla h^i, \nabla h^j \rangle - \delta^{ij}| < 0\left(\frac{1}{k}\right)$$

on $T(1-5\tau, 5\tau+\eta)$.

Proof. Obviously we can apply the above estimates to $h = \lambda_i h^i + \lambda_j h^j$ for $\lambda_i^2 + \lambda_j^2 = 1$. Then $|\nabla h|^2 \to 1$, which together with the above estimates easily proves 3.12.

Now, for large k and $h^{ij} = \langle \nabla h^i, \nabla h^j \rangle$, we have

$$|h^{ij} - \delta^{ij}|_{C^0} < 0\left(\frac{1}{k}\right).$$

Therefore, for k sufficiently large, we obtain the following.

Lemma 3.13. $F = (h^i, \cdots, h^n)$ *defines a diffeomorphism from* $T(1-6\tau, 6\tau+\eta)$ *onto* $F(T(1-6\tau, 6\tau+\eta)) \subset B(1)$, *and*

$$F(T(1-6\tau, 6\tau+\eta)) \supset T(1-7\tau, 7\tau+\eta),$$
$$F^{-1}(T(1-7\tau, 7\tau+\eta)) \supset T(1-8\tau, 8\tau+\eta).$$

Lemma 3.14. *In the harmonic coordinate* $F = (h^1, \cdots, h^n)$, *we have*

$$|\partial h^{ij}|_{C^\alpha} \le C(H, K, \tau, \eta, \delta)$$

on $T(1-8\tau, 8\tau+\eta) \subset F(T(1-6\tau, 6\tau+\eta))$.

Proof. In such a coordinate system, we have

$$\Delta h^{ij} = 2h^{il}h^{jk}R_{kl} + 2\sum \Gamma^i_{pq}\Gamma^j_{kl}h^{pk}h^{ql},$$

which implies

$$|\Delta h^{ij}| \le C + C|dh^{kl}|^2.$$

Letting $G = (h^{ij}) \in \mathbb{R}^{n^2}$, we obtain

$$|\Delta G| \le C + C|dG|^2,$$

i.e.,

$$\left|\sum h^{kl}\partial_k\partial_l G\right| \le C + C|dG|^2,$$

where ∂ denotes the partial derivative with respect to the harmonic coordinate $F = (h^1, \cdots, h^n)$.

Now take a cut-off function μ, similar to the one in 3.6, with Supp $\mu \subset T(1-6\tau, 6\tau+\eta)$. Applying the L^p estimate of the Laplace operator Δ_0 of Euclidean space, we have

$$\|\partial^2(\mu^2 G)\|_{L^p} \leq C\|\Delta_0 G\|_{L^p},$$

which implies

$$\|\partial^2(\mu^2 G)\|_{L^p} \leq \|\Delta G\|_{L^p} + \left\|\sum (h^{kl} - \delta^{kl})\partial_k\partial_l(\mu^2 G)\right\|_{L^p}.$$

From 3.12, it follows that for large k,

$$\left|\langle \nabla h^i, \nabla h^j\rangle - \delta^{ij}\right| < 0\left(\frac{1}{k}\right),$$

which together with the above inequality yields

$$\|\partial^2(\mu^2 G)\|_{L^p} \leq C + C\|\mu^2|\partial G|^2\|_{L^p}.$$

By the Hölder inequality and 3.10, we have for $p > n$

$$\|\partial^2(\mu^2 G)\|_{L^p} \leq C + C \leq C.$$

Applying the Sobolev imbedding theorem we obtain

$$\|\partial(\mu G)\|_{C^\alpha} \leq C,$$

which implies 3.14.

Lemma 3.15. *If* $|\mathrm{Ric}(g)| \leq H$, *then*

$$\|\partial^2 h^{ij}\|_{L^p} \leq C$$

on $T(1-8\tau, 8\tau+\eta)$, *and* $p > n$.

Proof. From the proof in 3.14 it follows that

$$\|\partial^2(\mu G)\|_{L^p} \leq C|\mathrm{Ric}| + C\|dG\|^2_{L^{2p}},$$

which together with 3.10 clearly implies 3.15.

Proof of Lemma 3.1. We simply take $\tau = n/100$ in the above lemmas. This completes the proof of Lemma 3.1.

Lemma 3.16. *For any* $0 < \eta < 1$, *there exists a small constant* $\kappa = \kappa(H, n, \eta) > 0$ *such that if*

$$\int_{B_{1+2\eta}(X_0)} |\mathrm{Rm}(g)|^{n/2}\, dg \leq \kappa,$$

then there exists a diffeomorphism F *from* $\Omega(x_0) \supset T_{x_0}(1-\frac{1}{2}\eta, \frac{5}{2}\eta)$ *into* $T(1+\eta, \eta)$, *such that*

(a) F *is harmonic, i.e.,* $\Delta F = 0$,

(b) $F^{-1}(T(1+\frac{1}{4}\eta, \frac{5}{4}\eta)) \supset T_{x_0}(1-2\eta, 3\eta)$, *and*

$$\text{image } F \supset T\left(1+\tfrac{1}{4}\eta, \tfrac{5}{4}\eta\right).$$

Let $h^{ij} = \langle \nabla h^i, \nabla h^j \rangle$ *be the metric tensor in this harmonic coordinate, where* $F = (h^1, \cdots, h^n)$. *Then* (h^{ij}) *satisfies the conclusions of* (c), (d), *and* (e) *of the main lemma.*

Proof. This lemma follows from the main lemma by simply noting that in the main lemma we have

$$T_{x_0}\left(1-\tfrac{1}{2}\eta, \tfrac{5}{2}\eta\right) \supset T_x\left(1+\tfrac{1}{2}\eta, \tfrac{3}{2}\eta\right),$$
$$T_{x_0}(1-2\eta, 3\eta) \subset T_x(1-\eta, 2\eta).$$

Now we need to restate the above results in geodesic balls.

Lemma 3.17. *For small* η, $0 < \eta < 1$, *there exists a small constant* $\kappa = \kappa(H, n, \eta) > 0$, *such that if*

$$\int_{B_{1+2\eta}(X_0)} |\operatorname{Rm}(g)|^{n/2}\, dg \le \kappa,$$

then, for any $0 < \delta < \frac{1}{3}$ *and* $x \in B_{1+2\eta}(x_0)$, *so that* $d(d_0, x) < 1_{-\delta} - 2\eta$, *there exists a harmonic coordinate* F *with domain* $\supset B_\delta(x)$ *and image* $F \supset B(\delta(1-\eta)) = \{x \in \mathbb{R}^n \mid |x| \le \delta(1-\eta)\}$ *such that*

(a) $F^{-1}(B(\delta(1-\eta))) \supset B_{\delta(1-2\eta)}(x)$.

Let $H^{ij} = \langle \nabla h^i, \nabla h^j \rangle$ *be the metric tensor in such a harmonic coordinate, where* $F = (h^1, \cdots, h^n)$. *Then*

(b)
$$|h^{ij} - \delta^{ij}| \le \eta^2/100n.$$

(c)
$$|dh^{ij}|_{C^\alpha} \le C(H, \eta, n), \qquad 0 < \alpha < 1, \ \text{on } B(\delta(1-\eta)).$$

Proof. If $3\eta + \delta \le d(x_0, x) \le 1 - \delta - 2\eta$, then by Lemma 3.16 there exists a harmonic coordinate F on $T_{x_0}(1-2\eta, 3\eta)$. Since $B_\delta(x) \subset T_{x_0}(1-2\eta, 2\eta)$, we can use F and a translation in $\mathbb{R}^n$ to define a harmonic coordinate F on $B_\delta(x)$ with $F(x) = 0$. (b) and (c) follow from 3.16, and (a) follows from (b) by taking η small $< C(n)$.

Now, consider $3\eta + \delta > d(x_0, x)$. First, note that the constant $1+2\eta$ does not play an essential role in 3.16. Then we can replace $1+2\eta$ by $\frac{2}{3}-\eta$, and the constants change accordingly. For such x, we can find a point y on the extension of the geodesic from x to x_0, such that $d(y, x) = 3\eta+\delta$, and $d(y, \partial B(x)) > \frac{2}{3}-3\eta$. Since $B_{2/3-\eta}(y) \subset B_{1+2\eta}(x_0)$, we apply Lemma 3.16 to the ball $B_{2/3-\eta}(y)$. There exists a harmonic coordinate F with domain $F \supset T_y(\frac{2}{3} - 4\eta, 3\eta)$ and image $F \supset T(\frac{2}{3} - 3\eta, \frac{5}{4}\eta)$. Since $B_\delta(x) \subset T(\frac{2}{3} - 4\eta, 3\eta)$, we can use F and a linear translation of $\mathbb{R}^n$ to define a harmonic coordinate (still call it F) on $B_\delta(x)$ such that (b) and (c) are satisfied, and $F(x) = 0$. Thus (a) follows from (b) for η small $< c(n)$. This completes the proof of Lemma 3.17, and so also the proof of Theorem 3.0.

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