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REDUCTION AND THE TRACE FORMULA

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1. This is the second paper of a series in which we explore the relationships between the spectrum of an elliptic operator with symmetries and the procedure of reduction in classical mechanics. The results were announced in [8], while the particular case when the symmetry group is the circle was considered in [9]. The present paper is essentially self-contained, and, in fact, some points in [9] are clarified.

Let M be a closed, smooth n-dimensional manifold, and G a compact connected Lie group with Lie algebra $\mathfrak{g}$. We assume given a representation of G in the Hilbert space of square-integrable half densities of M, $L^2(M)$, by Fourier integral operators. Associated to this representation, there is a Hamiltonian action of G on $X = T^*M - \{0\}$ with moment map $\Phi\colon X \to \mathfrak{g}^*$ which we assume is positive homogeneous of degree one. We also consider a first-order, elliptic, self-adjoint, classical pseudodifferential operator, P, with positive symbol, and which commutes with the G-representation. We are interested in the relationship between the way the spectrum of P is distributed along the various isotypical subspaces of $L^2(M)$ and the reduction (in the sense of classsical mechanics) of the bicharacteristic flow of P.

More precisely, let $\mathscr{O} \subset \mathfrak{g}^*$ be an integral coadjoint orbit of G. By the Borel–Weil theorem, associated to $\mathscr{O}$ there is an irreducible representation of G, which we denote by κ. For every positive integer m, let κ_m be the representation corresponding to the integral coadjoint orbit $m\mathscr{O}$, and let $\mathfrak{H}_m \subset L^2(M)$ denote the isotypical subspace associated to κ_m. Since P commutes with the representation of G, it maps each space $\mathfrak{H}_m$ into itself, and hence it maps

$$(1.1) \qquad \mathfrak{H} = \bigoplus_{m=1}^{\infty} \mathfrak{H}_m$$

into itself. The representation of G in (1.1) is called a *ladder*. For every

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m, let

$$\mathfrak{H}_m = \bigoplus_{k=1}^{\infty} \mathfrak{H}_{m,k} \tag{1.2}$$

be the decomposition of $\mathfrak{H}_m$ into eigenspaces of $P \mid \mathfrak{H}_m$. Then each of the subspaces $\mathfrak{H}_{m,k}$ is G-invariant, and hence it is isomorphic as a G-module to a direct sum of copies of κ_m:

$$\mathfrak{H}_{m,k} = \bigoplus_{l=1}^{\mu(m,k)} \kappa_m. \tag{1.3}$$

We will denote by $\{\lambda_{m.j}\,;\, j = 1, 2, \ldots\}$ the eigenvalues of the restriction of P to $\mathfrak{H}_m$, arranged in increasing order, and where the eigenvalue corresponding to the eigenspace $\mathfrak{H}_{m,k}$ appears precisely $\mu(m, k)$ times. (Thus we are getting rid of superfluous multiplicities arising from the fact that the eigenspaces of P are G-modules.) As we will see, the joint distribution of the eigenvalues $\{(\lambda_{m,j}, m)\,;\, m, j = 1, 2, \ldots\}$ is very closely related to the Marsden-Weinstein reduction of the Hamilton flow of the symbol P with respect to $\mathcal{O}$.

In order to state our theorem, we recall the construction of the Marsden-Weinstein reduced system. The group G acts in a Hamiltonian fashion on the product $\mathcal{O}^- \times X$, with moment map

$$\begin{aligned} \Phi_0 \colon \mathcal{O}^- \times X &\to \mathfrak{g}^* \\ (f, x) &\mapsto \Phi(x) - f \end{aligned} \tag{1.4}$$

We shall make the assumption

$$\begin{gathered} \text{zero is a regular value of } \Phi_o \\ \text{and } G \text{ acts freely on } \Phi_o^{-1}(0). \end{gathered} \tag{H.1}$$

As a consequence, the quotient space

$$X_{\mathcal{O}} = \Phi_o^{-1}(0)/G \tag{1.5}$$

is a smooth manifold; it carries a natural symplectic structure. Since the principal symbol of P, p, is invariant under the action of G on X, it induces a smooth function on $X_{\mathcal{O}}$ which we will continue to denote by p. By definition, the Hamilton flow of p on $X_{\mathcal{O}}$ is the reduction of the Hamilton flow of p on X with respect to the coadjoint orbit $\mathcal{O}$. Notice that the symplectic manifold (1.5) is not conic, so the reduced flow of p will, in general, be very different on different energy surfaces. Let us fix a value of the energy, $E > 0$, and pick a smooth test function $\varphi \in C^{\infty}(\mathbf{R})$

with compactly supported Fourier transform. Our theorem will relate the singularities of the periodic distribution

$$\Upsilon = \sum_{m,j=1}^{\infty} \varphi(\lambda_{m,j} - mE)e^{im\theta} \tag{1.6}$$

and the periodic trajectories of the reduced flow of p on the energy surface $\{p = E\}$. Notice that, since φ is a Schwartz function, the main contribution to (1.6) comes from the eigenvalues $\lambda_{m,j}$ which are approximately equal to mE. Microlocally, these correspond to closed trajectories on the reduced space with energy E.

Our first result describes the location of the singularities of Υ, in terms of the geometry of reduction. To state it, let $f \in \mathscr{O}$ and let G_f be its isotropy subgroup. It is well known that an alternative way of constructing the reduced space $X_{\mathscr{O}}$ is by taking the quotient

$$X_{\mathscr{O}} \cong \Phi^{-1}(f)/G_f.$$

Indeed, one has the commutative diagram

$$\begin{tikzcd}[column sep=small] \{f\} \times \Phi^{-1}(f) \arrow[rr, hook] \arrow[dr, "\pi_f"'] & & \Phi_o^{-1}(0) \arrow[dl, "\pi"] \\ & X_{\mathscr{O}} & \end{tikzcd} \tag{1.7}$$

where π_f (resp. π) is the projection induced by the G_f (resp. G) action. The fact that $\mathscr{O}$ is an integral coadjoint orbit means that there is a Lie group morphism

$$\chi_f \colon G_f \to S^1 \tag{1.8}$$

such that its differential at the identity is the infinitesimal character of $\mathfrak{g}_f$ defined by $f \colon \eta \mapsto 2\pi i\langle f, \eta\rangle$. We let Z

$$\begin{tikzcd} S^1 \arrow[r] & Z \arrow[d] \\ & X_{\mathscr{O}} \end{tikzcd} \tag{1.9}$$

be the circle bundle associated to the G_f-bundle π_f via the character (1.8). It is not hard to see that the canonical one-form of $X = T^*M - \{0\}$ induces a connection on (1.9), and that different choices of $f \in \mathscr{O}$ define naturally isomorphic circle bundles with connection (see (4.16) and the discussion that follows it). Given a closed curve, $\gamma \colon [0, T] \to X_{\mathscr{O}}$, we will denote by $h(\gamma) \in S^1$ its holonomy with respect to (1.9). We can now state our first theorem.

1.1. Theorem. *Let $J \subset \mathbf{R}$ be the support of $\hat{\varphi}$, the Fourier transform of φ, and*

$$\mathfrak{P}_{E,\varphi} = \{(x, T) \in X_{\mathscr{O}} \times J | p(x) = E, \exp(T\xi_p)(x) = x\}, \tag{1.10}$$

where ξ_p is the Hamilton field of p in $X_{\mathscr{O}}$. Then, under the assumptions (H.1) *and* (H.2), *the wave-front set of Υ is contained in*

$$\begin{aligned}\Sigma = \{(\omega, r) \in S^1 \times \mathbf{R}^+ | \exists (x, T) \in \mathfrak{P}_{E,\varphi}, \\ \omega = h(\exp(t\xi_p)(x); t \in [0, T])\}.\end{aligned} \tag{1.11}$$

The statement of the hypothesis (H.2) can be found in §4; it says that the given representation of G is by Fourier integral operators. Now under clean intersection hypotheses (see (H.3), §4), one can say more; namely, Υ is then a Fourier integral distribution and we compute its symbol. For simplicity of exposition we will only state this result in a particular case. Let $\mathfrak{s}$ be the subprincipal symbol of P. The restriction of $\mathfrak{s}$ to $\Phi^{-1}(f)$ can be averaged over the fibers of π_f to yield a function, $\mathfrak{s}^{\mathrm{av}}$, on $X_{\mathscr{O}}$. We will use C to denote a power of 2π which only depends on the dimensions of the spaces involved, and which may not be the same at each occurrence.

1.2. Theorem. *Assume the dynamical conditions* (H.1) *and* (H.3). *Then, for every φ with compactly supported Fourier transform,* (1.11) *is an embedded Lagrangian submanifold of T^*S^1. Assume furthermore* (H.2). *Then Υ is a Lagrangian distribution associated with* (1.11), *and hence there is an asymptotic expansion as $m \to \infty$,*

$$\sum_{j=1}^{\infty} \varphi(\lambda_{m,j} - mE) \sim \sum_{k=0}^{\infty} m^{d-k} a_k(\hat{\varphi}, m), \tag{1.12}$$

where each a_k is bounded in m (and rapidly oscillatory), $d = \frac{1}{2}\dim(X_{\mathscr{O}}) - 1$, and

$$a_0 = C\hat{\varphi}(0)\,\mathrm{Vol}, \tag{1.13}$$

where Vol *is the Liouville measure of $\{x \in X_{\mathscr{O}} | p(x) = E\}$. If, moreover, $0 \notin \operatorname{supp}\hat{\varphi}$, and $\mathfrak{P}_{E,\varphi}$ consists of the closed trajectories $\gamma_1, \cdots, \gamma_q$ with periods $T_1, \cdots, T_q$, then* (1.12) *holds, with $d = 0$ and*

$$a_0(\hat{\varphi}, m) = C \sum_{j=0}^{q} \omega_j^m \hat{\varphi}(T_j) \frac{T_j^{\#}}{2\pi} \frac{e^{i\pi\sigma_j/4}}{|\det(I - P_j)|^{1/2}} e^{i\int_{\gamma_j} \mathfrak{s}^{\mathrm{av}}}, \tag{1.14}$$

where $T_j^{\#}$ is the primitive period of γ_j, $\omega_j \in S^1$ is the holonomy of γ_j with respect to (1.9), *P_j is its (linearized) Poincaré map, and σ_j is an integer associated to γ_j (Maslov factor).*

The clean intersection conditions ensure that the trajectories $\gamma_1, \cdots, \gamma_q$ are nondegenerate. The coefficients a_k are distributions supported in the period spectrum of the reduced flow on the energy surface $\{p = E\}$. This theorem can be easily extended to the case when $\mathfrak{P}_{E,\varphi}$ contains families of closed trajectories. This comes about in the following way. The dynamical assumptions (H.1) and (H.3) imply that

$$\mathfrak{P}_E = \{(x, T) \in X_{\mathcal{O}} \times \mathbf{R} | p(x) = E \text{ and } \exp(T\xi_{\mathfrak{P}})(x) = x\} \tag{1.15}$$

is a disjoint union of smooth submanifolds of $X_{\mathcal{O}} \times \mathbf{R}$, and it carries a natural density, δ. Although the periods T need not be locally constant, one can easily show that the holonomy map $\mathfrak{P}_E \to S^1$ is. A connected component of $\mathfrak{P}_E$, Y, containing periods in the support of $\hat{\varphi}$, contributes to the asymptotic expansion (1.12); the leading contribution is

$$C\omega^m m^{(\dim Y - 1)/2} \int_Y \hat{\varphi}(T)\delta, \tag{1.16}$$

where ω is the (common) holonomy of a periodic trajectory in $\mathfrak{P}_E$.

One can also study the case when E is not a regular value of $p\colon X_{\mathcal{O}} \to \mathbf{R}$; this means that the reduced flow has equilibrium positions with energy E. All these matters were carefully discussed in [9] for the case of the circle group. As we will see, the study of Υ for a general compact group G reduces in some sense to the circle group case (indeed this will be the method of proof of our theorems). Thus we will not bother to describe all the details of these generalizations, but simply refer the reader to [9].

A few words about the organization of the paper. We will spend the next two sections explaining why Υ is a natural object to consider, and why (1.12) is the natural generalization of the trace formula of [3] to the setting of reduction. We do this by placing the construction of Υ in the general context of 'symplectic analogies' which are backed-up by the machinery of microlocal analysis. In connection with this material, it is a pleasure to thank Steve Zelditch for forcing us to motivate our constructions as clearly as possible. Then in §4 we sketch the proof of the above theorems. As mentioned, the main idea is to restrict P to the ladder (1.1) and reduce the problem to the case of the circle. This is possible because the ladder construction has very good microlocal properties, which we explore in §5. As an application of these results we study, in §6, the particular case when M is a principal G-bundle over a base manifold Y. Given a connection and a Riemannian metric on Y, one obtains an invariant metric on M, and the reduced system describes the movement of a particle in the background of the Yang-Mills field of the connection. (The operator P being the square root of the Laplacian on M.)

2. We begin our attempt to explain Theorems 1.1 and 1.2 by considering some symplectic analogies of standard concepts and constructions with Hilbert spaces and group representations. These analogies reflect theorems in microlocal analysis, as well as ideas from geometric quantization. We will be, however, somewhat vague, as the full depth of the connections between analysis and symplectic geometry cannot be easily captured with concrete statements.

We begin by defining the *symplectic category*, following Alan Weinstein [13]. In fact we will need a somewhat beefed-up version of the symplectic category of [13]:

2.1. Definition. A weighted manifold is a pair (M, μ), consisting of a smooth manifold M and a smooth half-density μ.

For example every symplectic manifold has a canonical weighting with μ equal to the square root of Liouville measure. Also, up to a positive scalar factor, every unimodular Lie group has a canonical weighting which is bi-invariant.

Now let X be a symplectic manifold with symplectic form ω. We will denote by X^- the symplectic manifold which, as an abstract manifold, is identical with X but has as its symplectic form $(-\omega)$.

2.2. Definition. The category $\mathfrak{C}$ is the category whose objects are symplectic manifolds and whose morphisms are weighted canonical relations. Thus, in the category $\mathfrak{C}$, a morphism, Γ, from object X to Y, is a weighted Lagrangian submanifold of $X^- \times Y$.

To make $\mathfrak{C}$ into a category, we have to show how to compose a pair of morphisms $\Gamma\colon X \to Y$ and $B\colon Y \to Z$. To see how this is done, consider the fiber product, F, in the diagram

$$\begin{array}{ccc} F & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

i.e., by definition $F = \{(x, y, z) | (x, y) \in \Gamma, (y, z) \in B\}$. Let $\pi\colon F \to X \times Z$ be the projection $(x, y, z) \mapsto (x, z)$. Note that the image of π is the set-theoretic composite relation $B \cdot \Gamma$.

2.3. Definition. The diagram (2.1) is said to be clean iff F is a submanifold of $\Gamma \times B$ and if, in addition, for every $f = (x, y, z) \in F$, the derived diagram

$$\begin{array}{ccc} T_f F & \longrightarrow & T_\gamma \Gamma \\ \downarrow & & \downarrow \\ T_b B & \longrightarrow & T_y Y \end{array} \tag{2.2}$$

with $\gamma = (x, y)$ and $b = (y, z)$ is a fiber product.

For the following see [3, §5].

2.4. Theorem. *If* (2.1) *is clean and all arrows are proper, then* $B \cdot \Gamma$ *is an immersed Lagrangian submanifold of* $X^- \times Z$, *and* $\pi\colon F \to B \cdot \Gamma$ *is a fiber mapping with compact fibers.*

Next we will show that there is a natural way to weight $B \cdot \Gamma$. For this one needs the following elementary symplectic linear algebra lemma, for the proof of which we refer to [3, Lemma 5.2].

Lemma. *Let* U, V, *and* W *be symplectic vector spaces, and let* T_γ *and* T_β *be Lagrangian subspaces of* $U^- \times V$ *and* $V^- \times W$, *respectively. Let* T_f *be the fiber product*:

$$\begin{array}{ccc} T_f & \longrightarrow & T_\gamma \\ \downarrow & & \downarrow \\ T_\beta & \longrightarrow & V \end{array}$$

Let $\pi\colon T_f \to U \times W$ *be the mapping* $(u, v, w) \mapsto (u, w)$, *and* $\operatorname{Ker}\pi$ *and* $\operatorname{Im}\pi$ *be its kernel and its image, respectively. Then there is a canonical identification of spaces of half-densities*:

$$|T_\gamma|^{1/2} \otimes |T_\beta|^{1/2} \cong |\operatorname{Ker}\pi| \otimes |\operatorname{Im}\pi|^{1.2}.$$

Applying this lemma to the diagram (2.2), one sees that for each $f \in F$, $T_f \to F$ is equipped with an object of the form $\mu_f \otimes \nu_f$, where μ_f is a density on the tangent space to the fiber of the map $\pi\colon F \to B \cdot \Gamma$ and ν_f is a half-density on the tangent space to $B \cdot \Gamma$ at $\pi(f)$. Integrating μ over each fiber one obtains a half-density ν on $B \cdot \Gamma$.

2.5. Definition. The morphism $B \cdot \Gamma\colon X \to Z$ defined by the weighted canonical relation $(B \cdot \Gamma, \nu)$ is the composition of the morphisms $\Gamma\colon X \to Y$ and $B\colon Y \to Z$.

Concerning this definition, we must insert at this point a couple of cautionary remarks:

1. By Theorem 2.4, $B \cdot \Gamma$ is an immersed canonical relation. To ensure that it be imbedded in $X^- \times Z$ one has to impose some further hypotheses (of a global nature) on the diagram (2.1).

2. Strictly speaking, $\mathfrak{C}$ is not a category. As we have just seen, clean intersection hypotheses are needed to make morphisms compose. Thus every time we want to compose two morphisms we must check that the composition is legitimate.

2.6. Let us denote by $\mathfrak{Q}$ the category whose objects are Hilbert spaces and whose morphisms are operators from one Hilbert space to another. It has been very useful to think that there is an equivalence of categories between $\mathfrak{C}$ and $\mathfrak{Q}$. Although of course this is not true, for large classes of symplectic manifolds one knows how to associate a Hilbert space to a symplectic manifold. For example, if $X = T^*M - \{0\}$ with its natural symplectic structure, the corresponding Hilbert space is the space, $L^2(M)$, of square-integrable half-densities on M. If $\mathcal{O} \subset \mathfrak{g}^*$ is an integral coadjoint orbit of a compact Lie group G, the corresponding Hilbert space is the (finite-dimensional) representation space constructed by the Borel-Weil theorem. The theory of Fourier integral operators tells us how to construct operators associated to certain morphisms of $\mathfrak{C}$. These $\mathfrak{C}$-$\mathfrak{Q}$ correspondences are a fruitful source of theorems. In what follows we will pretend that there is a well-defined correspondence between $\mathfrak{C}$ and $\mathfrak{Q}$, and list some analogies between these categories.

2.7. We have already observed that every symplectic manifold X has a symplectic dual X^-. Similarly, every Hilbert space H has associated with it the dual Hilbert space H^*.

2.8. To the operation of forming the tensor product of two Hilbert spaces corresponds the construction of the product symplectic manifold: $(X, \omega_X) \times (Y, \omega_Y)$ is the manifold $X \times Y$ equipped with the symplectic form $\pi_X^*\omega_X + \pi_Y^*\omega_Y$, where π_X (resp. π_Y) is the projection of $X \times Y$ onto X (resp. Y).

2.9. To every bounded operator $A: H \to K$ corresponds the adjoint operator $A^*: K \to H$. Similarly, to every weighted canonical relation $\Gamma: X \to Y$ corresponds the transposed canonical relation $\Gamma^t: Y \to X$ defined by $(y, x) \in \Gamma^t \Leftrightarrow (x, y) \in \Gamma$. If $\tau: \Gamma^t \to \Gamma$ is the map $\tau(y, x) = (x, y)$ and ν is the weighting of Γ, then the weighting of Γ^t is by the complex conjugate of $\tau^*\nu$.

2.10. Given Hilbert spaces H and K, the Hilbert space tensor product

$$K^* \otimes H \tag{2.3}$$

can be thought of as the set of all operators of Hilbert-Schmidt type mapping K to H. By 2.7 and 2.8, the $\mathfrak{C}$ analogue of (2.3) is the product symplectic manifold $Y^- \times X$.

2.11. The category $\mathfrak{Q}$ is a *pointed* category, the point object being the one-dimensional Hilbert space $\mathbf{C}$. Thus, given a Hilbert space H, one can define a 'categorical' element or point of H to be a morphism $\alpha: \mathbf{C} \to H$. Setting $v = \alpha(1)$, we get a one-to-one correspondence between 'categorical' elements of H and elements of H (vectors) in the usual sense. If, however, X is a symplectic manifold and we play the same game in the category $\mathfrak{C}$, 'categorical elements' or points of X are *not* the same as points of X in the usual sense. In fact, if we designate the point object in $\mathfrak{C}$ to be the unique zero-dimensional symplectic manifold, $\{\cdot\}$, then 'categorical' points of X are morphisms of $\{\cdot\}$ into X, i.e., they are just weighted Lagrangian submanifolds of X. Notice that this clarifies 2.10; the categorical points of $Y^- \times X$ are indeed the morphisms from Y to X, as defined in 2.2.

2.12. In particular, the diagonal Δ is a Lagrangian submanifold of $X^- \times X$, and the identification $X \cong \Delta$ gives it a canonical weighting. If we think of Δ as a morphism from X to itself, then it is the obvious $\mathfrak{C}$ analogue of the identity mapping of a Hilbert space onto itself. However, we can also think of the diagonal as a morphism

$$\gamma: \{\cdot\} \to X^- \times X$$

and its transpose, $\tau = \gamma^t$, as a morphism

$$\gamma: X^- \times X \to \{\cdot\}. \tag{2.4}$$

We will see below that it is useful to think of (2.4) as the $\mathfrak{C}$ analogue of the operation of taking the trace of an operator (cf. 2.10 with $H = K$).

2.13. let G be a Lie group. In the category $\mathfrak{Q}$, an 'action of G' is a linear representation of G on some fixed Hilbert space. In the category $\mathfrak{C}$, the appropriate analogue is a *Hamiltonian* action of G on some fixed symplectic manifold.

2.14. Pursuing the previous analogy further, the correct analogue of an irreducible representation should be a transitive action. We recall, however, that if G acts transitively and in a Hamiltonian fashion on X, then, by a theorem of Kostant, either X is a coadjoint orbit or it is a covering space of a coadjoint orbit. So irreducible G-spaces in the category $\mathfrak{Q}$ correspond to coadjoint orbits of G in the category $\mathfrak{C}$.

2.15. We turn now to some symplectic analogies which are particularly pertinent for us. Given a representation of G on a Hilbert space H, one can consider the subspace, H_G, of G-invariant elements of H. Moreover, one has an inclusion map

$$\imath: H_G \hookrightarrow H. \tag{2.5}$$

What corresponds to (2.5) in the category $\mathfrak{C}$? That is, given a symplectic manifold X and a Hamiltonian action of G on X, what is the classical analogue of H_G? We claim that a plausible candidate is the Marsden-Weinstein reduction X_G (at the zero coadjoint orbit) if we want to be consistent with 2.11 above. We argue as follows. For simplicity, let us assume that the action of G on X is free. This means that the moment map $\Phi: X \to \mathfrak{g}^*$ is a submersion and that

$$Z = \{x \in X | \Phi(x) = 0\} \tag{2.6}$$

is a closed, coisotropic submanifold of X. Recall that X_G is the base of the fibration $\pi: Z \to X_G$ of Z by the orbits of G. It is easy to see that the orbits of G are the leaves of the null-foliation of Z, so there exists a symplectic form, ω_G, on X_G, such that if ω is the symplectic form on X and $\iota: Z \hookrightarrow X$ is the inclusion, then $\pi^*\omega_G = \iota^*\omega$. In other words, the map

$$(\pi, \iota): Z \to X_G^- \times X$$

imbeds Z inside $X_G^- \times X$ as a Lagrangian submanifold. We will now show that if we fix a bi-invariant half-density on G, we obtain thereby, in a canonical way, a G-invariant half-density on Z: Let NZ be the normal bundle of Z in X. By (2.6), NZ is a trivial bundle with fibers naturally isomorphic with $\mathfrak{g}^*$. Therefore, in particular, we have an exact sequence of vector bundles

$$0 \to TZ \to (TX)_Z \to Z \times \mathfrak{g}^* \to 0, \tag{2.7}$$

where $(TX)_Z$ is the tangent bundle of X restricted to Z. However, the symplectic form in X gives us a trivialization of the half-density bundle of X, $|TX|^{1/2} \cong \mathbf{R}$; so from (2.7) we get a trivialization of the half-density bundle of Z:

$$|TZ|^{1/2} \cong |\mathfrak{g}|^{1/2}. \tag{2.8}$$

Finally, if μ is a G-bi-invariant half-density on G, then μ_e is an Ad-G-invariant half-density on the tangent space, $\mathfrak{g}$, to G at the identity element e; so by (2.8) it determines a half-density ν on Z. Now let

$$\Gamma_r: X_G \to X \tag{2.9}$$

be the morphism in $\mathfrak{G}$ associated with the weighted canonical relation (Z, ν). We will henceforth refer to Γ_r as the *reduction morphism.* We will briefly explain why, on categorical grounds, (2.9) is the correct $\mathfrak{C}$ analogue of (2.5). We have already seen that points of X, in the categorical sense, are weighted Lagrangian submanifolds of X. Thus, from the categorical point of view, a fixed point of G is a G-invariant morphism $\gamma\colon \{\cdot\} \to X$ (or, in other words, a pair (Λ, ν) consisting of a G-invariant Lagrangian submanifold, $\Lambda \subset X$, and a G-invariant half-density, μ, on it). For the following see [6].

Theorem. *If G is semisimple, every G-invariant morphism $\gamma\colon \{\cdot\} \to X$ is of the form $\Gamma_r \cdot \gamma_1$, where γ_1 is a morphism of $\{\cdot\}$ into X_G.*

Thus X_G bears the same categorical relation to X as H_G does to H.

A concrete realization of the correspondence between X_G and H_G is provided by the following theorem. Suppose that X is the punctured cotangent bundle of a compact manifold M, $H = L^2(M)$, and that the representation of G is by Fourier integral operators (more precisely, assume (H.2), §4). Assume furthermore that G itself is compact and that the action of G on Z is free.

Theorem. *Under the previous assumptions, the leaf relation of the null-foliation of Z is an embedded Lagrangian submanifold of $X^- \times X$, and the orthogonal projection $\pi\colon L^2(M) \to L^2(M)_G$ is a Fourier integral operator associated with it.*

In [5], compressions of pseudodifferential operators on M to $L^2(M)_G$ are considered, and the point is made that the symbols of such compressions should be interpreted as functions on the reduced space X_G. Thus X_G is the correct phase space corresponding to $L^2(M)_G$.

Important Remark. Back to the discussion of symplectic analogies, by 2.10, if X and Y are G-spaces in $\mathfrak{C}$, then $(Y^- \times X)_G$ is the classical analogue of the space of intertwining operators from K to H. This will be particularly relevant for our purposes.

2.16. Given a unitary representation T of G on the Hilbert space H, there is a $(G \times G)$-invariant operator

$$T\colon C_0^\infty(G) \to B(H) \tag{2.10}$$

defined by

$$T_f = \int_G f(g)T(g)\,d\mu_g,$$

where μ is Haar measure. A plausable candidate for the $\mathfrak{C}$ analogue of this operator is the *moment Lagrangian*:

$$\Gamma_G\colon T^*G \to X^- \times X. \tag{2.11}$$

We briefly review how (2.11) is defined.[1] By means of the left action of

[1]For the case of induced representations, there is a very simple and direct connection between (2.10) and (2.11). See [13].

G on itself, we get a trivialization of the cotangent bundle of G: $T^*G \cong G \times \mathfrak{g}^*$, and modulo this identification (2.11) is just the set

$$\Gamma_G = \{(x, g \cdot x, g, \Phi(x)) | x \in X, g \in G\}, \tag{2.12}$$

where Φ is, as before, the moment mapping. Weinstein has proved that this is a Lagrangian submanifold of $(T^*G)^- \times X^- \times X$. Moreover, Γ_G is diffeomorphic to $X \times G$ by the diffeomorphism $(x, g) \mapsto (x, g \cdot x, g, \Phi(x))$. Therefore, since X and G are equipped with half-densities, so is Γ_G. This makes Γ_G a morphism in $\mathfrak{C}$. We will refer to it from now on as the *moment morphism.*

2.17. Continuing with the previous discussion, let us find the $\mathfrak{C}$ analogue of the character of the representation T. Recall that the character of T is defined when the image of (2.10) is contained in the space of trace-class operators on H. If that is the case, then the character of T is the composition of (2.10) with the trace morphism Tr: $\mathrm{Tr}(H) \to \mathbf{C}$. By (2.4), the corresponding $\mathfrak{C}$ analogue is then the composition of the moment morphism and the diagonal viewed as a morphism:

$$\tau: X^- \times X \to \{\cdot\}.$$

One easily sees that the composition

$$\chi = \tau \cdot \Gamma_G: T^*G \to \{\cdot\}$$

is (as a manifold) equal to

$$\chi = \{(g, \alpha) \in G \times \mathfrak{g}^* | \exists x \in X \ g \cdot x = x, \ \alpha = \Phi(x)\}. \tag{2.13}$$

We will call (2.13) the *character morphism.* It is easy to see that it is $\mathrm{Ad}\, G$-invariant.

2.18. We will now discuss the $\mathfrak{C}$ analogue of the notion of 'ladder representation' of a compact Lie group G. Let $\mathscr{O} \subset \mathfrak{g}^*$ be an integral coadjoint orbit, and κ the corresponding irreducible representation. For every positive integer m, let κ_m be the irreducible representation associated to the coadjoint orbit $m\mathscr{O}$, and let K_m be the finite-dimensional Hilbert space of κ_m. On the Hilbert space direct sum

$$K = K_1 \oplus K_2 \oplus \cdots \tag{2.14}$$

we can let G act by letting it act as κ_m in the mth summand. This is the *ladder representation* of G associated with $\mathscr{O}$. There turns out to be a natural $\mathfrak{C}$ analogue of this notion. Namely, let $\omega_{\mathscr{O}}$ be the symplectic form of $\mathscr{O}$. Then (see [10]) there exists a principal circle bundle with connection $\pi: Q \to \mathscr{O}$ such that the connection form α satisfies $d\alpha = \pi^*\omega_{\mathscr{O}}$. Moreover, the action of G on $\mathscr{O}$ lifts to an action of G on Q by bundle morphisms that preserve α. Now consider the following subset of T^*Q:

$$W_{\mathscr{O}} = \{(q, r\alpha_q) | q \in Q, \ r \in \mathbf{R}^+\}. \tag{2.15}$$

This set is a symplectic submanifold of T^*Q, and G acts on it in a Hamiltonian fashion; the corresponding moment map is

$$\begin{aligned} W_{\mathscr{O}} &\to \mathfrak{g}^* \\ (q, r\alpha_q) &\mapsto r\pi(q). \end{aligned} \tag{2.16}$$

The Borel-Weil theorem, together with the microlocal theory of Toeplitz operators, imply that we should regard (2.15) as the correct $\mathfrak{C}$ analogue of (2.14).

Notice that the circle group acts on (2.15) by the lift of the bundle action on Q, and that this action commutes with the action of G. This circle action is also Hamiltonian, with moment map a: $(q, r\alpha_q) \mapsto r$. Again by the Borel-Weil theorem, this corresponds to the representation of S^1 on (2.14) which, when restricted to the mth summand, is multiplication by $e^{im\theta}$. Notice that this representation commutes with the representation of G as well. We will take over this discussion in the next section.

3. We now look at the construction of Υ, from the point of view of the symplectic analogies which we discussed in the previous section. One of our goals is to put Theorems 1.1 and 1.2 in the context of the multiplicity results of [6, 7], as well as to relate them to the microlocal trace formula of [3]. For simplicity, we will temporarily ignore half-densities.

A. We begin by considering a compact Lie group G, and a unitary representation in a Hilbert space H. We assume that this representation is *elliptic*, in the sense that each irreducible representation of G appears in H with finite multiplicity, and the multiplicities do not grow too fast so that the character of the representation is well-defined distribution on G. the corresponding object in $\mathfrak{C}$ is a Hamiltonian G-space X, which is elliptic in the sense that zero is not in the image of the moment map Φ: $X \to \mathfrak{g}^*$. Recall the discussion of 2.18; we will continue to use the same notation. Thus $\mathscr{O} \subset \mathfrak{g}^*$ is an integral coadjoint orbit, and we assume that Φ is transversal to it. It follows that G acts in a locally free fashion on $W_{\mathscr{O}}^- \times X$, and hence $(W_{\mathscr{O}}^- \times X)_G$ is a V-manifold. By the last remark of 2.15,

$$\operatorname{Hom}_G(K, H) \quad \text{corresponds to} \quad (W_{\mathscr{O}}^- \times X)_G. \tag{3.1}$$

Let us denote by $\mu(\kappa_m, H)$ the multiplicity with which the irreducible representation κ_m appears in H. The following are standard facts from representation theory:

$$\mathrm{Hom}_G(K, H) = \bigoplus_{m=1}^{\infty} \mathrm{Hom}_G(K_m, H), \tag{3.2}$$

and

$$\text{for all } m, \text{ the dimension of the space } \mathrm{Hom}_G(K_m, H) \text{ is equal to } \mu(\kappa_m, H). \tag{3.3}$$

Now let $\mathfrak{H} \subset H$ be the subspace spanned by the vectors transforming according to some representation in the ladder K. Thus, $\mathfrak{H}$ is the image of the evaluation map

$$\mathrm{ev}\colon K \otimes \mathrm{Hom}_G(K, H) \to H. \tag{3.4}$$

It is easy to see that (3.4) is injective. The direct sum decomposition of K (2.14) has an analogue for $\mathfrak{H}$:

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots, \tag{3.5}$$

where, for every m, $\mathfrak{H}_m$ is the image of $K_m \otimes \mathrm{Hom}_G(K_m, H)$ under (3.4). Thus, as G-space, $\mathfrak{H}_m$ is the direct sum of $\mu(\kappa_m, H)$ copies of κ_m.

Notice that the circle group acts on everything in sight. As we remarked at the end of 2.18, it acts on $W_{\mathscr{O}}$. This action is Hamiltonian, with moment map

$$\begin{aligned} a\colon W_{\mathscr{O}} &\to \mathbf{R} \\ (q, r\alpha_q) &\mapsto r. \end{aligned} \tag{3.6}$$

We then let S^1 act on $W_{\mathscr{O}}^-$ by letting it act trivially on X. Since this action commutes with the G-action, we get an induced circle action on $(W_{\mathscr{O}}^- \times X)_G$. There is a corresponding representation of the circle group in $\mathrm{Hom}_G(K, H)$, which is induced by the representation of S^1 on K which we have already discussed. Thus we can refine (3.1) to:

$$\mathrm{Hom}_G(K, H) \text{ and } (W_{\mathscr{O}}^- \times X)_G \text{ are corresponding elliptic } S^1 \text{ spaces.} \tag{3.7}$$

In some sense, the main question that we are interested in is: What is the character of these S^1 spaces? It is easy to write down an answer in the category $\mathfrak{Q}$.

3.1. Lemma. *The character of the representation of the circle group in* $\mathrm{Hom}_G(K, H)$ *is*

$$\chi(\theta) = \sum_{m=1}^{\infty} \mu(\kappa_m, H) e^{im\theta}. \tag{3.8}$$

This follows immediately from (3.3). It is not so immediately clear what the character morphism of the circular space $(W_{\mathcal{O}}^{-} \times X)_G$ is. To explain the answer, recall that if $f \in \mathcal{O}$ and G_f is the isotropy subgroup of f, there is a unique character $\chi_f \colon G_f \to S^1$ whose differential at the identity is the infinitesimal character satisfying $\forall \xi \in \mathfrak{g}_f$, $d\chi_f(\xi) = 2\pi i \langle f, \eta \rangle$. Using the functorial construction of the character described in 2.17, one easily computes:

3.2. Lemma. *The Lagrangian manifold underlying the character morphism of the circular action on* $(W_{\mathcal{O}}^{-} \times X)_G$ *is the set* Σ *of all* $(\omega, r) \in S^1 \times \mathbf{R}^+$ *satisfying the following condition*:

$$\exists f \in \mathcal{O},\ x \in \Phi^{-1}(rf) \text{ and } g \in G_f \text{ such that } g \cdot x = x \text{ and } \omega = \chi_f(g).$$

Proof. Let $\Psi \colon W^{-} \times X \to \mathfrak{g}^*$ be the moment map for the G-action on $W^{-} \times X$. This is the map

$$\Psi \colon ((q, r\alpha_q)x) \mapsto \Phi(x) - r\pi(q),$$

where $\pi \colon Q \to \mathcal{O}$ is the natural projection. Thus $\Psi^{-1}(0)$ is the coisotropic submanifold

$$L = \{(q, r\alpha_q, x) | \Phi(x) = r\pi(q)\}. \tag{3.9}$$

Recall that $(W_{\mathcal{O}}^{-} \times X)_G$ is the quotient of (3.9) by G. Extend the moment map for the circular action, (3.6), to L in the obvious way. By (2.12), it is clear that

$$\Sigma = \{(\omega, r) \in S^1 \times \mathbf{R} | \exists l \in L,\ g \in G \text{ such that } \omega \cdot l = g \cdot l \text{ and } r = a(l)\}.$$

Now the actions of G and S^1 are given by

$$\omega \cdot p = (\omega \cdot q, r\alpha_{\omega \cdot q}, x),$$

and

$$g \cdot p = (g \cdot q, r\alpha_{g \cdot q}, g \cdot x).$$

Thus the condition on p is equivalent to the following conditions on x and q:

$$\Phi(x) = r\pi(q), \quad g \cdot x = x, \quad \text{and} \quad g \cdot q = \omega \cdot q.$$

Since the moment map is equivariant, $g \in G_f$, where $f = \pi(q)$. Now the action of an element of G_f on the fiber of π above f is by the character χ_f (that is how the quantizing bundle is defined). Thus the third condition above is equivalent to $\omega = \chi_f(g)$.

As we have mentioned, $(W_{\mathcal{O}}^{-} \times X)_G$ is in general a V-manifold. To ensure that it is an ordinary manifold, we must assume that the action of G on L is free. In this case, no nontrivial g's can appear in (3.10).

3.3. Corollary. *If G acts freely on L, then $\Sigma = \{1\} \times \mathbf{R}^+ \subset T^*S^1$.*

Having described the formal relation between (3.8) and Σ, let us place ourselves in a more concrete situation; namely, assume $H = L^2(M)$ with M compact, $X = T^*M - \{0\}$, and (H.2), §4. If G acts freely on L, one knows [7] that the orthogonal projection operator Π: $L^2(Q \times M) \to L^2(Q \times M)_G$ is a Fourier integral operator associated with the leaf relation of the null-foliation of L (a special case of the second theorem of 2.15). Denote by ρ the representation of the circle in $L^2(Q \times M)$ which is given by the circle action on $Q \times M$ (trivial on M). Then the representation of the circle of which (3.8) is the character is the compressed representation $\Pi\rho\Pi$. It is clear that one can compute microlocally the character of this representation. Under favorable (clean intersection) conditions, (3.8) will be a Fourier integral distribution on the circle associated with the conormal space to $1 \in S^1$. It follows that its Fourier coefficients, i.e., the multiplicities $\mu(\kappa_m, H)$, have an asymptotic expansion in decreasing powers of m, as m goes to infinity. Since the multiplicities are integers, necessarily they are equal to a *polynomial* in m, for m sufficiently large. The polynomial is basically equal to the Riemann-Roch number of the reduction of $(W_{\mathcal{O}}^{-} \times X)_G$ with respect to the S^1 action at the energy level $a = m$ (see [6, 7]).

B. Let us now see to what extent we can replace G by a noncompact Lie group, say by the real line. Thus let X and H be associated elliptic $\mathbf{R}$ spaces. Of course, one cannot hope that the previous arguments will apply directly without substantial modifications. Let us see where trouble arises. The coadjoint orbit $\mathcal{O}$ gets replaced by a point $E \in \mathbf{R}$, and so its quantizing bundle Q is simply a copy of the circle. The Hilbert space K of (2.14) is simply the Hardy space of the circle, where the real line is represented by the rule $t \cdot e^{im\theta} = e^{im(\theta + tE)}$. The symplectic manifold $W_{\mathcal{O}}$ is just the positive part of the cotangent bundle of the circle, that is $W_{\mathcal{O}} = \{(e^{i\theta}, r\,d\theta) | r > 0\}$. Let iP: $H \to H$ be the infinitesimal generator of the representation of $\mathbf{R}$ in H so that P is self-adjoint. Assume also that it has pure point spectrum and each eigenvalue has finite multiplicity. If we were to follow the previous construction verbatim, then $\mathfrak{H}_m$ would be the eigenspace of P corresponding to the eigenvalue mE. Now with probability one mE is *not* an eigenvalue of P, i.e., with probability one $\mu(m, H) = 0$, and the 'ladder space' $\mathfrak{H} \subset H$ is likely to be empty, or, at any rate, very thin. Hence in general there will not be practically *any* nontrivial intertwining operators from K to H. On the $\mathfrak{C}$ side, this is reflected on the fact that the null-foliation of the coisotropic manifold corresponding to (3.9), namely

$$L = \{(e^{i\theta}, r\,d\theta; x) \in T^*S^1 \times X | p(x) = rE\}, \tag{3.10}$$

is almost never fibrating. To be more concrete, assume that $X = T^*M - 0$, where M is a compact manifold and P is a first-order, self-adjoint pseudodifferential operator on M. Then L is the characteristic variety of the operator

$$R = i\partial_\theta \otimes I + I \otimes P$$

which is an operator of real principal type (one can safely ignore the fact that R is not a classical pseudodifferential operator due to 'corners'). Now the Schwartz kernels of operators that intertwine $-i\partial_\theta$ and P are those distributions U that satisfy the equation $P(U) = 0$. It is know [5], that if the null-foliation of L is fibrating, then the orthogonal projection operator onto the null-space of a suitable bounded perturbation of R is a Fourier integral operator; its underlying canonical relation is the leaf relation of L. This is the case if the bicharacteristic flow of P is periodic of period $2\pi/E$; this happens very rarely.

Now in general the leaves of the null-foliation of L are the trajectories of the Hamilton flow of the defining function of L, that is, the trajectories of the flow

$$f_t(e^{i\theta}, r\,d\theta; x) = (e^{i(\theta+tE)}, r\,d\theta; \phi_t(x)),$$

where ϕ_t is the given flow on X, and hence the null-foliation of L will be rarely fibrating. The way around this difficulty is by considering just some portions of the null-foliation at a time, as we now explain; this corresponds to viewing the multiplicities as *distributions*. Let $\psi \in C_0^\infty(\mathbf{R})$ be a test function, φ its Fourier transform, and consider the operator

$$\varphi(R) = \int_{\mathbf{R}} \psi(t) \exp(itR)\, dt. \tag{3.11}$$

If $X = T^*M - 0$, etc., one can easily show that this is a Fourier integral operator associated with the canonical relation

$$\{(\lambda, \lambda') \in L | \exists t \in \operatorname{supp}(\psi) \text{ such that } \lambda' = f_t(\lambda)\}. \tag{3.12}$$

The operator (3.11) commutes with ∂_θ, and is 'transversely of trace class', i.e., for each integer m, $\varphi(R)$ restricted to the mth eigenspace of $-i\partial_\theta$ is of trace class, and its trace is equal to

$$\mu(m, H, \varphi) = \sum_{j=1}^{\infty} \varphi(\lambda_j - mE). \tag{3.13}$$

As a distribution in φ this is the multiplicity of m in H. Then the analogue of the character (3.8) is

$$\sum_{m=1}^{\infty} \mu(m, H, \varphi)e^{im\theta} = \sum_{m, j=1}^{\infty} \varphi(\lambda_j - mE)e^{im\theta}. \tag{3.14}$$

Let us try to understand the construction of (3.14) better. Let $\{v_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of H satisfying $P(v_j) = \lambda_j v_j$ for all j. Then define the operator $\pi_{m,\varphi}: H \to H$ by

$$\pi_{m,\varphi}(w) = \sum_{j=1}^{\infty} \varphi(\lambda_j - mE)\langle w, v_j\rangle v_j. \tag{3.15}$$

This operator generalizes the orthogonal projection into $\mathfrak{H}_m$. Thus, the generalization of the representation of S^1 in $\mathfrak{H}$ (which is to say in $\mathrm{Hom}_G(K, H)$ since $K_m = \mathbf{C}$) is the family of operators

$$\rho_\varphi(e^{i\theta}) = \sum_{m=1}^{\infty} e^{im\theta}\pi_{m,\varphi}. \tag{3.16}$$

(In case X is the punctured cotangent bundle of a compact manifold M, P a first-order pseudodifferential operator on M, and $H = L^2(M)$, one can easily prove that for each φ this is a Fourier integral operator on $S^1 \times M$.) Although ρ_φ is certainly not a representation, in some sense it behaves like one: $\rho_\varphi(1)$ is the approximate projection operator (3.15), while

$$\rho_\varphi(e^{i\theta}) \cdot \rho_\varphi(e^{i\gamma}) = \rho_{\varphi^2}(e^{i(\theta+\gamma)}).$$

At any rate, we should view this as a generalization of the S^1 space $\mathrm{Hom}_G(K, H)$ when $G = \mathbf{R}$. This point of view can be formalized by using Alain Connes' construction of a C^* algebra of a foliation, where this 'representation' of the circle should correspond to an action of S^1 on the C^* algebra of the null-foliation of L. (Recall that the C^* algebra of the null-foliation models the singular space of leaves and compare with (A).) Although we cannot go into any detail here, we mention that the elements of the C^* algebra of the null-foliation of L are very closely related to the symbols of the Fourier integral operators $\varphi(R)$, and, more generally, of FIOs associated with pieces of the leaf relation of L such as (3.12). We hope to explore the connection with the Connes theory in the future.

It still makes sense to ask whether (3.14) is in some sense associated with the character Lagrangian Σ (see Lemma 3.1). It turns out that, at least in the case when P is an elliptic operator on a compact manifold M and $X = T^*M - \{0\}$, (3.14) is a distribution with wave-front set contained in

$$\Sigma\{(e^{itE}, r) | r > 0 \text{ and } \exists x \in X \text{ such that } \phi_t(x) = x\},$$

and, as before, under clean intersection conditions it is a Fourier integral distribution. Then its Fourier coefficients have an asymptotic expansion

$$\sum_{j=1}^{\infty} \varphi(\lambda_j - mE) \sim \sum_{k=1}^{\infty} m^{d-k} c_k(\hat{\varphi}\,; m) \tag{3.17}$$

as m goes to infinity. (Here the coefficients c_k are, as functions of m, finite sums of rapidly oscillatory exponentials.) This theorem was proved in [3], where it is also shown that the coefficients c_k are distributions supported in the period spectrum of the action of $\mathbf{R}$ on X.

C. The case that concerns us in this paper is a mixture of the cases G compact and $G = \mathbf{R}$. We want to consider a group of the form $G \times \mathbf{R}$ and two objects, H and X, where this group is represented in an elliptic fashion. Pick an integral coadjoint orbit $\mathscr{O} \subset \mathfrak{g}^*$, a constant $E \in \mathbf{R}$, and consider $\mathscr{O} \times \{E\} \subset \mathfrak{g}^* \times \mathbf{R}$. We can define the subspace $\mathfrak{H} \subset H$ corresponding to the ladder of G defined by $\mathscr{O}$, and then form approximate projection operators analogous to (3.14). Specifically, for every m let $\{v_{m,k}\}$ be an orthonormal basis of $\mathfrak{H}_m$ consisting of eigenvectors of P; say $P(v_{m,k}) = \lambda'_{m,k} v_{m,k}$. Then form the approximate projection operator

$$\pi_{m,\varphi} = \sum_{k=1}^{\infty} \varphi(\lambda'_{m,k} - mE) v_{m.k} \otimes v^*_{m,k}. \tag{3.18}$$

It should be clear that the 'character' of the circle group corresponding to this construction (once we have eliminated redundant multiplicities) is precisely the expression Υ of (1.6). Let us work out what the character Lagrangian, Σ, is. The coadjoint orbit $\mathscr{O} \times \{E\}$ is quantized by the quantizing circle bundle of $\mathscr{O}$, $Q \to \mathscr{O}$. Then

$$W_{\mathscr{O}\times\{E\}} = \{(q, r\alpha_q) | q \in Q \text{ and } r \in \mathbf{R}^+\},$$

where $G \times \mathbf{R}$ acts in a Hamiltonian action with moment map $(q, r\alpha_q) \mapsto (r\pi(q), rE)$. The set L is given by

$$L = \{(q, r\alpha_q\,; x) | (\Phi(x), p(x)) = (r\pi(q), rE)\}.$$

The following is proved easily from these facts:

3.4. Proposition. *Σ is the set of all $(\omega, r) \in S^1 \times \mathbf{R}$ such that there exist $f \in \mathscr{O}$, $x \in (\Phi, p)^{-1}(rf, rE)$, and $(g, t) \in G_f \times \mathbf{R}$ such that*

$$x = t \cdot g \cdot x \quad \textit{and} \quad \omega = \chi_f(g) e^{itE}.$$

Let us assume that the action of G on L *is* free, and let $Y = L/G$. The action of $\mathbf{R}$ on X induces a flow on Y, and it is closed trajectories of this reduced flow that give rise to points in Σ. Proposition 3.4 describes the singularities of Υ on the circle for every φ. In our setting, these singularities will be interpreted as the holonomies of the closed trajectories of the flow on Y with respect to a natural connection. We now see that the asymptotic expansion (1.12) is completely analogous to the main trace formula of [3] (notice that the distributions a_k on the right-hand side of (1.12) are supported in the period spectrum of the reduced flow).

4. We will now sketch the proof of the theorems stated in §1. We are given a representation of G on $L^2(M)$; this can be thought of as an operator from $L^2(M)$ to $L^2(G \times M)$. As such, it has a Schwartz kernel which is a distributional half-dentistry on $G \times M \times M$. We will denote this distribution by $\mathscr{K}_0 \in C^{-\infty}(G \times M \times M)$. We are also given a Hamiltonian action of G on $X = T^*M - \{0\}$, with moment map $\Phi\colon X \to \mathfrak{g}^*$. As we saw in §2, the object in $\mathfrak{C}$ that corresponds to the representation is the moment Lagrangian

$$(4.1) \qquad \Gamma_0 = \{(g, f; x; g \cdot x) | f = \Phi(x)\} \subset T^*G \times X \times X.$$

We can now state precisely what we mean by this statement: we will henceforth assume that

(H.2)
Φ is positive homogeneous of degree one, and
$\mathscr{K}_0$ is in the Hörmander space $I^{-n_1/4}(G \times M \times M; \Gamma_0)$,

where n_1 is the dimension of G. Notice that the first part of (H.2) implies that Γ_0 is a homogeneous Lagrangian submanifold, so the second part makes sense. The assumption on the order is equivalent to assuming that every pseudodifferential operator in the derived representation of $\mathfrak{g}$ has order one. We will assume that all these operators have vanishing subprincipal symbol; it follows that the half-density part of $\mathscr{K}_0$ is the half-density attached to a moment Lagrangian as explained in 2.16.

Now let P be a first-order, self-adjoint pseudodifferential operator on M commuting with the G representation. By taking the product of the representation of G with the representation of $\mathbf{R}$ given by $t \mapsto \exp(itP)$, we obtain a representation of $\mathbf{R} \times G$ on $L^2(M)$. Denoting the Hamilton flow on X of p, the principal symbol of P, by $\{\phi_t\}$, this representation corresponds to the moment Lagrangian of the product action of $\mathbf{R} \times G$ on X, namely

$$(4.2) \qquad \Gamma = \{(t, \tau; g, f; x; \phi_t(g \cdot x)) | \tau = p(x) \quad \text{and} \quad f = \Phi(x)\},$$

which is a Lagrangian submanifold of $T^*\mathbf{R} \times T^*G \times X \times X \cong (\mathbf{R} \times \mathbf{R}) \times (G \times \mathfrak{g}^*) \times X \times X$.

4.1. Lemma. *let $\mathscr{K} \in C^{-\infty}(\mathbf{R} \times G \times M \times M)$ denote the Schwartz kernel of the product representation of $\mathbf{R} \times G$. Then the hypothesis* (H.2) *implies that $\mathscr{K}$ is a Fourier integral distribution associated with the conic Lagrangian submanifold Γ. In fact,*

$$\mathscr{K} \in I^{-(n_1+1)/4}(\mathbf{R} \times G \times M \times M; \Gamma).$$

Proof. We will briefly sketch why $\mathscr{K}$ is Lagrangian, leaving the details to the reader. The unitary representation of $\mathbf{R}$ that we are considering has a Schwartz kernel $\mathscr{K}_1 \in C^{-\infty}(\mathbf{R} \times M \times M)$ which is a Fourier integral distribution associated to the canonical relation

$$\mathscr{C}_1 = \{(t, \tau; x; \phi_t(X)) | \tau = p(x)\}.$$

In fact, $\mathscr{K}_1 \in I^{-1/4}(\mathbf{R} \times M \times M; C_1)$ (see [3, §1]). Considering the transpose of $\mathscr{K}_1$, we obtain the kernel of an operator from half-densities on $\mathbf{R} \times M$ to half-densities on M. Composing this with the operator from M to $G \times M$ that is defined by $\mathscr{K}_0$, we obtain a Fourier integral operator from $\mathbf{R} \times M$ to $G \times M$. Its Schwartz kernel, $\widehat{\mathscr{K}}$, is a Lagrangian distribution on $G \times M \times \mathbf{R} \times M$. It is easy to see that under one of the obvious diffeomorphisms $G \times M \times \mathbf{R} \times M \cong \mathbf{R} \times G \times M \times M$, the distribution $\widehat{\mathscr{K}}$ goes over to $\mathscr{K}$.

The half-density symbol of $\mathscr{K}$ is easy to describe. There is a natural parametrization of Γ by the product $\mathbf{R} \times G \times X$. Using this parametrization, the half-density symbol of $\mathscr{K}$ at the point (t, g, x) is

$$|dt|^{1/2} \otimes |dt|^{1/2} \otimes |dx|^{1/2} \otimes |dx|^{1/2} e^{i\int_0^t \mathfrak{s}(\phi_s(g\cdot x))\, ds}, \tag{4.3}$$

where $|dx|$ is Liouville measure, and the last $|dx|^{1/2}$ in (4.3) takes place at the half-density line at $\phi_t(g \cdot x)$. (Recall that $\mathfrak{s}$ is the subprincipal symbol of P.)

We will show that Υ is the result of applying to $\mathscr{K}$ a Fourier integral operator, $\mathscr{T}$, from $C^\infty(\mathbf{R} \times G \times M \times M)$ to $C^\infty(S^1)$. The clean intersection conditions alluded to in the statement of Theorem 1.2 will ensure that $\Upsilon = \mathscr{T}(\mathscr{K})$ is a Fourier integral distribution on the circle.

The simplest way to describe $\mathscr{T}$ is as the composition of three operators: $\mathscr{T} = \mathscr{T}_3 \circ \mathscr{T}_2 \circ \mathscr{T}_1$. The operator $\mathscr{T}_1$ maps half-densities on $\mathbf{R} \times G \times M \times M$ to half-densities on $\mathbf{R} \times G$; when applied to $\mathscr{K}$ it produces the character of the product representation of $\mathbf{R} \times G$. Its Schwartz kernel is simply the delta function along

$$\operatorname{diag}(\mathbf{R} \times G \times M) \subset \mathbf{R} \times G \times M \times M \times \mathbf{R} \times G,$$

where diag(N) denotes the diagonal imbedding of N, in the obvious sense. The role of the operator $\mathcal{T}_2$ is to localize the character of the product representation along the ladder of G defined by the coadjoint orbit $\mathcal{O}$, and it only depends on the choice of this orbit. This is the step alluded to in §1 as 'reduction to the S^1 case'; the operator $\mathcal{T}_2$ maps half-densities on $\mathbf{R} \times G$ to half-densities on $\mathbf{R} \times S^1$, and in fact reduces the problem to the circular case studied in [9]. Finally, the operator $\mathcal{T}_3$, from half-densities on the cylinder $\mathbf{R} \times S^1$ to half-densities on the circle, localizes along the 'fuzzy ladder' $\lambda_{m,j} - mE$ (with the notation of §1; see also §3). $\mathcal{T}_3$ is a Fourier integral operator depending only on the parameter E and test function φ that appear in Υ, and is identical to the operator that we called G in [9]. We will now describe the operator $\mathcal{T}_2$.

The definition of $\mathcal{T}_2$ is independent of the representation of $\mathbf{R} \times G$ on $L^2(M)$; in fact it only depends on the choice of coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. $\mathcal{T}_2$ is of the form

$$\mathcal{T}_2 = 1_{\mathbf{R}} \otimes \mathcal{L}, \tag{4.4}$$

where $\mathcal{L}: C^\infty(G) \to C^\infty(S^1)$ is a Fourier integral operator which we will define shortly. Thus $\mathcal{T}_2$ is not an FIO in the strict sense of the word: however, its Schwartz kernel is a Lagrangian distribution microlocally away from 'vertical' and 'horizontal' covectors. Since the Lagrangians of $\mathcal{T}_1$ and $\mathcal{T}_3$ do not contain any covectors of this type, we can (and will) treat $\mathcal{T}_2$ as if it were an ordinary FIO. To define $\mathcal{L}$, for every positive integer m let κ_m be the irreducible representation of G corresponding to $m\mathcal{O}$, and let $\rho_m \in C^\infty(G)$ be the corresponding character.

Definition. $\mathcal{L}: C^\infty(G) \to C^\infty(S^1)$ is the operator whose Schwartz kernel is

$$\mathcal{L}(e^{i\theta}, g) = \sum_{m=1}^{\infty} e^{im\theta} \rho_m(g). \tag{4.5}$$

In §5 we will study the microlocal properties of (4.5), thereby substantiating the symplectic analogies of 2.18 and part A of §3. They are summarized by the following.

4.2. Proposition. *The distribution* (4.5) *is a Lagrangian distribution associated with the 'ladder Lagrangian',* $\Lambda_{\mathcal{O}} \subset T^*(S^1 \times G) \cong S^1 \times \mathbf{R} \times G \times \mathfrak{g}^*$, *equal to*

$$\Lambda_{\mathcal{O}} = \{(w, r; g, rf) | f \in \mathcal{O}, g \in G_f, \omega = \chi_f(g)\}. \tag{4.6}$$

More precisely, $\mathcal{L}$ *is in the Hörmander space* $I^{(1-n_1)/4}(S^1 \times G; \Lambda_{\mathcal{O}})$. *Recall that* n_1 *is the dimension of* G, $G_f \subset G$ *is the stabilizer of* f, *and* $\chi_f: G_f \to S^1$ *is the global character with differential* $d\chi_f(\xi) = 2\pi\langle f, \chi\rangle$.

The half-density attached to (4.6) is described in 5.3. It is constructed, in a natural way, from Haar measure and Liouville measure on $\mathscr{O}$. Thus we can treat $\mathscr{T}_2$ as an FIO of order $(1-n_1)/4$, associated to the product of $\Lambda_{\mathscr{O}}$ and the conormal to the diagonal in $\mathbf{R}\times\mathbf{R}$.

We now define the operator $\mathscr{T}_3$.

Definition. Given $E\in\mathbf{R}$ and a Schwartz function φ on the real line, the operator $\mathscr{T}_3\colon C^\infty(\mathbf{R}\times S^1)\to C^\infty(S^1)$ is defined by the property that

$$\forall(\lambda, m)\in\mathbf{R}\times\mathbf{Z}\quad \mathscr{T}_3(e^{i(\lambda t+m\theta)})(s)=\varphi(\lambda-Em)e^{im\theta}. \tag{4.7}$$

It is very easy to see that $\mathscr{T}_3$ is a Fourier integral operator associated to the canonical relation

$$\{(t, rE; \omega, r; \omega e^{itE}, r)\}\subset T^*(\mathbf{R}\times S^1)\times T^*(S^1) \tag{4.8}$$

and of order $-1/4$. In fact, the Schwartz kernel of $\mathscr{T}_3$ is the oscillatory integral

$$\mathscr{T}_3(\theta', t, \theta)=(2\pi)^{-2}\hat{\varphi}(t)\int_{\mathbf{R}}e^{i\mu(\theta'-\theta-tE)}\,d\mu. \tag{4.9}$$

Putting all this together, one easily proves

4.3. Lemma. *The operator $\mathscr{T}$ is a Fourier integral operator of order $-n_1/4$ associated to the canonical relation*

$$\mathscr{A}=\{(t, Er; g, f; x; x; \omega, r)|f\in\mathscr{O}, g\in G_f, \omega=\chi_f(g)e^{itE}\} \tag{4.10}$$

in $T^*(\mathbf{R}\times G)\times X\times X\times T^*(S^1)$.

Will will omit the proof, which is an elementary exercise in FIO theory. All the compositions are transversal, and hence the order of $\mathscr{T}$ is simply the sum of the orders of $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3$. Let us now check that the operator $\mathscr{T}$ produces Υ from $\mathscr{K}$.

4.4. Lemma. *Indeed one has $\Upsilon=\mathscr{T}(\mathscr{K})$.*

Proof. For every irreducible representation κ of G, let ρ_κ denote its character. Then

$$\mathscr{T}_1(\mathscr{K})=\sum_{\kappa\in G, \lambda\in\sigma(P)}\mu(\kappa, \lambda)\rho_\kappa(g)e^{i\lambda t}, \tag{4.11}$$

where $\sigma(P)$ is the spectrum of P *without* multiplicities, an $\mathrm{d}\,\mu(\kappa, \lambda)$ is the multiplicity with which κ appears in the λ-eigenspace of P. By (4.4), (4.5), and orthogonality of characters,

$$\mathscr{T}_2\circ\mathscr{T}_1(\mathscr{K})=\sum_{m>0, \lambda\in\sigma(P|\mathfrak{H})}e^{i\lambda t}e^{im\theta}\mu(\kappa, \lambda). \tag{4.12}$$

It is now clear that the lemma follows property (4.7) of $\mathscr{T}_3$.

Proof of Theorem 1.1. From the calculus of wave-front sets, and Lemmas 4.1, 4.3, and 4.4, it follows that the wave-front set of Υ is contained in the set Σ of Proposition 3.4. It remains to prove that this set agrees with the set (1.11). By homogeneity, it suffices to check equality for the value $r = 1$ of the fiber variable. Notice that, by (4.10), no points where $t \notin \operatorname{supp} \hat{\varphi}$ contribute to the singular support of Υ. All that remains to be done is to prove that, for a closed trajectory of the reduced flow of p with energy E, arising from the equation

$$x = \phi_T(g \cdot x), \tag{4.13}$$

as in Proposition 3.4, its holonomy with respect to the natural connection on (1.9) is equal to $\chi_f(g)\exp(itE)$. More explicitly, let $f \in \mathcal{O}$, $g \in G_f$, $T \in \mathbf{R}$, and $x \in \Phi^{-1}(f)$ satisfy (4.13). Denote the Hamilton flow of p in the reduced space $X_{\mathcal{O}}$ by $\{\psi_t\}$, and let

$$\pi_f \colon \Phi^{-1}(f) \to X_{\mathcal{O}} \tag{4.14}$$

be the natural G_f-bundle. (See diagram (1.7).) Since π_f intertwines the flow of p on $\Phi^{-1}(f)$ and ψ, (4.14) implies that the trajectory of $\pi_f(x)$ in $X_{\mathcal{O}}$ is periodic of period T. A converse statement is obviously true: every point in (1.10) arises in this fashion. Let γ denote the trajectory $\psi_t(\pi_f(x))$, with t between zero and T. We must compute the holonomy of γ with respect to the natural connection on the circle bundle Z (cf. (1.9)) associated to (4.14) via the character χ_f of G_f. It will be useful to keep in mind the following diagram, which defines Z:

$$\begin{array}{ccc} \Phi^{-1}(f) \times S^1 & \longrightarrow & \Phi^{-1}(f) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \pi_f} \\ Z & \longrightarrow & X_{\mathcal{O}} \end{array} \tag{4.15}$$

The vertical arrows are principal G_f-bundles, while the horizontal ones are circle bundles. The connection on Z is defined as follows: with α^0 the canonical one-form on X, and $\iota\colon \Phi^{-1}(f) \hookrightarrow X$ the inclusion, let $\alpha^1 = \iota^*(\alpha^0)$. The one-form $\alpha = \alpha^1 + d\theta$ on $\Phi^{-1}(f) \times S^1$ is G_f-invariant and horizontal, and hence induces a one-form on Z, which is the connection form of the canonical connection. Let Ξ denote the restriction to $\Phi^{-1}(f)$ of the Hamilton vector field of p. By homogeneity, the contraction of α^1 with Ξ at x is equal to E. Hence the vector field $\Xi - E\partial_\theta$ on $\Phi^{-1}(f) \times S^1$ descends to the horizontal lift to Z of the Hamilton field of p on $X_{\mathcal{O}}$. Flow the vector field $\Xi - E\partial_\theta$ for time T starting at $(x, 1)$. The final point of this trajectory is $(g \cdot x, e^{-iTE})$. When we project this trajectory down to Z via ν, we obtain the horizontal lift of γ, starting at $\nu(x, 1)$ and ending at

$$\nu(g \cdot x, e^{-iTE}) = \nu(x, \chi_f(g^{-1})e^{iTE}) = \nu(x, 1) \cdot \chi_f(g^{-1})e^{-iTE}.$$

Now we must find composition conditions that will ensure that Υ is a Lagrangian distribution. For the sake of simplicity, we will only state *sufficient* conditions. These will not cover the case when there are equilibrium positions of the reduced flow on the energy surface $\{p = E\}$. However, an analysis of that case can be done (we did so in [9] for the circle case); we hope to return to this problem in the future. Here we will assume:

(H.3) E is a regular value of p, and the reduced flow of the energy surface $\{p = E\}$ is clean.

Proof of Theorem 1.2. To compose $\mathscr{T}$ and $\mathscr{K}$, one must look at the diagram

$$\begin{array}{ccc} \mathfrak{F} & \to & \mathscr{A} \\ \downarrow & & \downarrow \\ \Gamma & \hookrightarrow & T^*(\mathbf{R} \times G) \times X \times X \end{array} \tag{4.16}$$

where $\mathscr{A}$ is (4.10), Γ the moment Lagrangian (4.2), and $\mathfrak{F}$ the fibre product. It is very easy to see that $\mathfrak{F}$ is diffeomorphic to a product, $\mathfrak{F} \cong F \times \mathbf{R}^+$, with

$$F = \{(f, x; g; t) \in \Phi_0^{-1}(0) \times G \times \mathbf{R} | g \in G_f \text{ and } \phi_t(g \cdot x) = x\}. \tag{4.17}$$

(Recall that $\Phi_0 \colon \mathscr{O}^- \times X \to \mathfrak{g}^*$ is the moment map of the product action.) An argument identical to the one in Lemma 4.5 in [9] proves that (H.1) and (H.3) are sufficient to guarantee that (4.16) is clean. Notice that G acts on F, by the product of the action on $\Phi_0^{-1}(0)$, the adjoint action of G on itself, and the trivial action on $\mathbf{R}$. (H.1) implies that this action is free, and the orbit space is naturally diffeomorphic with the period set $\mathfrak{P}_E$ of (1.15). At this point the proof is a trivial generalization of the one presented in §4 of [9], so we omit the details. Notice that if $Y \subset \mathfrak{P}_E$ is a connected component, its preimage in F has dimension $d = \dim(Y) + n_1$. A simple counting argument shows that the excess of (4.16) at a given point is the dimension of the connected component of F containing that point. Thus, at $Y \subset \mathfrak{P}_E$, the excess is $d = \dim(Y) + n_1$. Hence, at the corresponding singularity, Υ is of order

$$-(n_1 + 1)/4 - n_1/4 + d/2 = \dim(Y)/2 - 1/4,$$

which justifies the degrees of the asymptotic expansions in Theorem 1.2, as well as (1.16).

5. We will now prove Proposition 4.2. Let $\mathscr{O} \subset \mathfrak{g}^*$ be an integral coadjoint orbit of a connected compact Lie group G, and $\pi: Q \to \mathscr{O}$ the quantizing circle bundle. Denote the connection form of Q by α, and let

$$\Sigma = \{(q, r\alpha_q) | q \in Q, r \in \mathbf{R}^+\}, \tag{5.1}$$

which is a symplectic submanifold of T^*Q. The bundle π is acted on by G, and the action on Q lifts to an action on Σ, which is Hamiltonian with moment map

$$\begin{aligned} \Sigma &\to \mathfrak{g}^* \\ (q, r\alpha_q) &\mapsto r\pi(q). \end{aligned} \tag{5.2}$$

It will be useful to understand the action of G on Q more concretely. Let $q \in Q$, $f = \pi(q)$, and denote the isotropy subgroup of f (under the coadjoint action) by G_f. The Lie algebra of G_f is equal to

$$\mathfrak{g}_f^* = \{\xi \in \mathfrak{g} | \forall \eta \in \mathfrak{g}, \ \langle [\xi, \eta], f\rangle = 0\},$$

and, because $\mathscr{O}$ is integral, there is a character $\chi_f: G_f \to S^1$ such that $\forall \xi \in \mathfrak{g}_f^* \, d\chi_f(\xi) = 2\pi i \langle \xi, f\rangle$. Then one has

$$\forall g \in G_f, q \in \pi^{-1}(f), \quad g \cdot q = q \cdot \chi_f(g), \tag{5.3}$$

where the circle action on the right is the bundle action on Q. It follows that if $H_f \subset G_f$ denotes the kernel of χ_f, the isotropy subgroup of $q \in Q$ is precisely $H_{\pi(f)}$.

The manifold (Q, α) is a contact manifold; as such it has a natural volume form. Thus we can identify functions and half-densities in a canonical way; $L^2(Q)$ will denote the space of square-integrable functions on Q with respect to the natural measure. Q can be identified with the circle bundle of a holomorphic hermitian line bundle over $\mathscr{O}$. As such, it is the boundary of the unit disc bundle and hence a strictly pseudoconvex domain. Let $\mathscr{H} \subset L^2(Q)$ be the Hardy space, and $S: L^2(Q) \to L^2(Q)$, $\mathrm{Im}(S) = \mathscr{H}$, the Szegö projector. the action of G induces a representation T on $L^2(Q)$ by the standard formula $T_g(u)(q) = u(g^{-1} \cdot q)$. This representation preserves the Hardy space $\mathscr{H}$. Our first goal is to compute the character of the restriction of T to $\mathscr{H}$, that is, the distribution on G formally defined by

$$\chi(g) = \mathrm{trace}(T_g \circ S). \tag{5.4}$$

Notice that, with the notation of §4, $\chi = \sum_{m=1}^{\infty} \chi_m$. We can understand (5.4) microlocally thanks to the theory of Toeplitz operators, as developed in [1].

To compute (5.4), we proceed in what is by now a standard way. Consider T as a single operator from $C^\infty(Q)$ to $C^\infty(G \times Q)$. As such, its Schwartz kernel is the delta function along the graph of the action $G \times Q \to Q$. Consider now the composition

$$T_{\mathscr{H}}: L^2(Q) \xrightarrow{S} L^2(Q) \xrightarrow{T} L^2(G \times Q).$$

In [1] it was shown how to associate, to every isotropic submanifold, $\Theta \subset T^*Y - \{0\}$ of the punctured cotangent bundle of a manifold Y, and to every real number s, a space of distributions on Y, $J^s(Y, \Theta)$. If Θ is in fact Lagrangian, then $J^s(Y, \Theta)$ is identical with the Hörmander space $I^{s-\dim(Y)/4}(Y, \Theta)$ of [4, Chapter 25]. (In [1], what we are calling $J^s(Y, \Theta)$ was denoted $I^s(Y, \Theta)$; we are changing the notation because of the slight discrepancy in the definition of order in the Lagrangian case.) Operators whose Schwartz kernels are in one of these spaces are called Hermite operators. One of the theorems of Boutet de Monvel and Sjöstrand in [2] is that the Szegö projector S is a Hermite operator. In fact, by Theorem 11.1 in [1], the Schwartz kernel of S in the space $J^{1/2}(Q \times Q; \Sigma \times^\Delta \Sigma)$, where

$$\Sigma \times^\Delta \Sigma = \{(q, r\alpha_q; q, -r\alpha_q) | q \in Q, r \in \mathbf{R}^+\} \subset T^*Q \times T^*Q. \tag{5.5}$$

On the other hand, as said earlier, G acts on Σ in a Hamiltonian fashion, and hence there is an associated moment morphism

(5.6)

$$\mathfrak{M} = \{(g, rf; q, r\alpha_q; g \cdot q, -r\alpha_{g\cdot q}) | g \in G, q \in Q, r = \pi(q), r \in \mathbf{R}^+\}$$

which carries a natural half-density, as discussed in 2.16. The composition of S and T is transversal; hence one can apply Theorem 9.5 in [1] to obtain the following.

5.1. Lemma. *The operator $T_{\mathscr{H}}$ is a Hermite operator associated with the isotropic manifold $\mathfrak{M} \subset T^*G \times T^*Q \times T^*Q$. More precisely, the Schwartz kernel of $T_{\mathscr{H}}$ is in the space $J^{1/2}(G \times Q \times Q; \mathfrak{M})$. Moreover, its half-density symbol is the half-density associated with $\mathfrak{M}$ as a moment morphism.*

Let $\Delta: C^\infty(G \times Q \times Q) \to C^\infty(G)$ be the operator 'integration along the diagonal in $Q \times Q$'. Obviously the Schwartz kernel of Δ is the delta function along the product of the diagonals of G and Q, and hence it is a very simple Lagrangian distribution. It is also obvious that χ is the result of applying Δ to the Schwartz kernel of $T_{\mathscr{H}}$. An application of Theorem 7.5 of [1] yields:

5.2. Proposition. *χ is a distribution in the Hörmander space $I^{(2-n_1)/4}(G;\mathscr{C})$, where*

$$\mathscr{C}=\{(g\,,rf)|f\in\mathscr{O}\,,\ g\in H_f \text{ and } r\in\mathbf{R}^+\}.$$

5.3. We will now describe the half-density symbol of χ. Pick an element $f_0\in\mathscr{O}$, and let $G_0=G_{f_0}$, $H_0=H_{f_0}$. Denote the adjoint action of G_0 on H_0 by β (this makes sense since H_0 is a normal subgroup of G_0). Let G_0 act on $G\times H_0$ by acting on the right on G and by β on H_0. It is very easy to see that the quotient $(G\times H_0)/H_0$ is naturally identified with $\mathscr{C}_0=\{(f\,,g)|f\in\mathscr{O}$ and $g\in H_f\}$. In fact, one has a commutative diagram

$$\begin{array}{ccc} G\times H_0 & \longrightarrow & \mathscr{C}_0 \\ \downarrow & & \downarrow{\scriptstyle\nu} \\ G & \longrightarrow & \mathscr{O} \end{array} \tag{5.7}$$

where the vertical arrows are principal H_0-bundles, while the horizontal ones are G_0-bundles. Using ν, there is an obvious way of putting a measure on C_0 starting with Haar measure on H_0 and Liouville measure on $\mathscr{O}$. If μ denotes the measure, then the symbol of χ is the half-density $|\mu|^{1/2}\otimes|dr|^{1/2}$ on $\mathscr{C}=\mathscr{C}_0\times\mathbf{R}^+$.

Proof of Proposition 4.2. To obtain Proposition 4.2, we apply the previous results to a product $S^1\times G$, and to an integral coadjoint orbit $\{1\}\times G$. The quantizing circle bundle is Q itself, where $S^1\times G$ acts on it by the product of the natural G-action and the bundle action. The corresponding character is easily seen to be equal to $\mathscr{L}(\omega^{-1}\,,g)$ (cf. (4.5)). (The reason for the power (-1) is that the sections of the line bundle associated to Q via the character $\omega\mapsto\omega^m$ are functions on Q transforming according to the character $\omega\mapsto\omega^{-m}$.) Given $f\in\mathscr{O}$, the isotropy subgroup of $(1\,,f)$ is $S^1\times G_f$, while the corresponding character $\chi_{(1\,,f)}\colon S^1\times G_f\to S^1$ is given by $\chi_{(1\,,f)}(\omega\,,g)=\omega\chi_f(g)$. Hence the kernel of $\chi_{(1\,,f)}$ equals

$$H_{(1\,,f)}^{S_1\times G}=\{(\chi_f(g)^{-1}\,,g)\in S^1\times G_f\}.$$

Proposition 4.2 follows from this and Proposition 5.2.

6. Henceforth we assume that the representation of G on $L^2(M)$ arises from a free action of G on M. Letting $Y = M/G$,

$$\begin{array}{ccc} G & \longrightarrow & M \\ & & \downarrow \\ & & Y \end{array} \tag{6.1}$$

is a principal G-bundle. Choose a Riemannian metric on Y, a bi-invariant metric on G, and a connection on (6.1). Then there is a unique G-invariant metric on M which makes (6.1) into a Riemannian submersion, and the fibers isometric to G. One calls this metric the Kaluza-Klein metric of the connection. We will also choose once for all a Higgs field, Ψ, which is, by definition, a section of the vector bundle associated to (6.1) via the adjoint representation of G.

Consider a particle with configuration space Y, kinetic energy given by the Riemannian metric on Y, and which reacts to the action of the Yang-Mills-Higgs field according to the principles of minimal coupling. Such a particle has internal degrees of freedom, which, quantum mechanically, are described by the vectors in an irreducible representation of G. The corresponding classical phase space of internal degrees of freedom is an integral coadjoint orbit $\mathscr{O} \subset \mathfrak{g}^*$. There is a symplectic manifold which serves as a phase space for the particle, incorporating both the internal degrees of freedom and the external ones (Y variables). In the description of Weinstein [12], this manifold is $X_{\mathscr{O}}$, the Marsden-Weinstein reduction of T^*M with respect to $\mathscr{O}$. On the quantum-mechanical side, the dynamics of the particle is described by a suitable version of the Schrödinger equation on sections of the vector bundles associated with (6.1) ('particle fields'). We wish to show how the theorems from §1 apply to this situation. We will begin by looking at the differential geometry involved.

The connection on (6.1) induces a fiber map $T^*M \to T^*Y$, which in turn induces a map $X_{\mathscr{O}} \to T^*Y$. Using this, the kinetic energy Hamiltonian can be pulled back to $X_{\mathscr{O}}$. Weinstein showed that the resulting Hamilton flow is identical with the Wong equations of motion [14], which describe the semiclassical limit of a particle minimally coupled to a Yang-Mills field. As we will now see, the Higgs field also defines, in a natural way, a function on $X_{\mathscr{O}}$, which should be added to the kinetic term to obtain the Hamiltonian of a particle under the Yang-Mills-Higgs field. By definition, Ψ is an equivariant, smooth map $\Psi\colon M \to \mathfrak{g}$. Thus the function

$$\begin{aligned} \Psi_0\colon \mathscr{O} \times M &\to \mathbf{R} \\ (f, x) &\mapsto \langle \Psi(x), f \rangle \end{aligned}$$

is well defined and G-invariant. Thus it defines a function on $X_{\mathscr{O}}$ which we will continue to denote by Ψ_0.

A useful alternative description of $X_{\mathscr{O}}$, due to Sternberg [11], is the following. As a manifold, $X_{\mathscr{O}}$ is the fiber product in the following diagram:

$$\begin{array}{ccc} X_{\mathscr{O}} & \xrightarrow{\pi_1} & T^*Y \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow \\ \mathscr{O}\times_G M & \longrightarrow & Y \end{array} \tag{6.2}$$

The symplectic structure of $X_{\mathscr{O}}$ is equal to $\pi_1^*(\omega_Y)+\pi_2^*(\Omega)$, where ω_Y is the canonical symplectic structure on T^*Y, and Ω is a two-form on $\mathscr{O}\times_G M$ which we now describe. This picture will be very useful in understanding how the results from §1 apply to the present situation.

Let $Q\to\mathscr{O}$ be the quantizing circle bundle of $\mathscr{O}$, with canonical connection form $\alpha\in\Lambda^1(\mathscr{O})$. Pulling this back to $\mathscr{O}\times M$, one obtains a G-invariant circle bundle with connection, which we will continue to denote by $\nu\colon Q\to\mathscr{O}\times M$. Let θ be the connection one-form on M. Then define a real-valued one-form β on $\mathscr{O}\times M$, by the formula

$$\forall (u,v)\in T_{(f,x)}\mathscr{O}\times M \quad \beta(u,v)=\langle\theta_x(v),f\rangle \tag{6.3}$$

(recall that θ is $\mathfrak{g}$-valued). We can now translate the connection on Q by this one-form, β. The resulting circle bundle with connection $(Q,\alpha+\nu^*\beta)$ is G-invariant and defines a circle bundle with connection on $\mathscr{O}\times_G M$. We will call this the *canonical circle bundle with connection* $\mathscr{O}\times_G M$. By definition, its curvature form is Ω. These definitions readily imply the

6.1. Lemma. *Let γ be a closed curve in $X_{\mathscr{O}}$, and let $\gamma_j=\pi_j\circ\gamma$, $j=1,2$. Then the holonomy of γ with respect to the connection on the circle bundle* (1.9) *is equal to*

$$h(\gamma)=h_{M(\gamma_2)}e^{i\int_{\gamma_1}\eta},$$

where η is the canonical one-form in T^*Y and h_M is the holonomy with respect to the canonical circle bundle with connection on $\mathscr{O}\times_G M$.

We argued in [9] that the logarithm of $h(\gamma)$ should be viewed as i times the *action* of the trajectory γ. Then Lemma 6.1 says that the action of the particle is the sum of two terms: $(-i)\log h_M(\gamma_2)$ and $\int_{\gamma_2}\eta$. The first is the contribution to the action of the vector potential, i.e., the connection on (6.1), while the second represents the kinetic energy.

Let us now turn our attention to the various operators associated with our data. For every positive integer m, let $\mathscr{V}_m\to Y$ be the Hermitian vector bundle with connection associated to the irreducible representation κ_m of G corresponding to $m\mathscr{O}$. Sections of V_m can be naturally identified with equivariant maps from M to κ_m. The Laplacian on M, with respect to the Kaluza-Klein metric, can be written as a sum

$$\Delta_{KK} = \Delta_h + \Delta_f,$$

where Δ_f is the Laplacian on the fibers. Since (6.1) is a Riemannian submersion with totally geodesic fibers, Δ_h and Δ_f commute. Under the identification of sections of V_m with equivariant functions of M to κ_m, the Laplacian on sections of V_m goes over to Δ_h. More precisely, Δ_h preserves the image of the evaluation map

$$\text{(6.4)} \qquad \mathrm{ev}\colon \kappa_m \otimes C^\infty_G(M, \kappa_m) \to C^\infty(M)$$

and intertwines with the operator which is the identity on κ_m and the Laplacian on $\mathrm{Hom}_G(M, \kappa_m)$. (The subindex G will stand for 'equivariant maps'.) The (closure of the) image of (6.4) is the isotypical subspace of κ_m. Hence the spectrum of the Laplacian on sections of V_m is identical with the spectrum of Δ_h on the image of (6.4), provided we throw away, in the latter, redundant multiplicities as we did in §1.

Let us now review how the Higgs field Ψ defines zeroth order operators on each of the associated bundles $\mathscr{V}_m$. As stated, Ψ is a smooth, equivariant map $\Psi\colon M \to \mathfrak{g}$. Let $\kappa\colon G \to V$ be a representation of G. The differential of κ at the identity is a representation of the Lie algebra, $d\kappa\colon \mathfrak{g} \to \mathrm{End}(V)$. Suppose now that $u\colon M \to V$ is an equivariant map (i.e., a section of the bundle associated to κ). Then the formula

$$\text{(6.5)} \qquad \Psi(u)(x) = d\kappa(\Psi(x))[u(x)]$$

defines a linear operator $\Psi\colon C^\infty_G(M, V) \to C^\infty_G(M, V)$. Notice that this is indeed a zeroth order differential operator on sections of the associated vector bundle $\mathscr{V}$; essentially by (6.5), Ψ defines a bundle map $\mathscr{V} \to \mathscr{V}$. In terms of the identification between sections of associated bundles and functions on M provided by the evaluation map (6.4), the operators induced by Ψ can be realized *simultaneously* by a vector field on M. (Notice that, on M, the induced operator becomes *first order*.) Ψ defines a vector field Θ on M by the recipe $\forall x \in M \ \ \Theta_x = \widehat{\Phi}(x)$. By (6.5), the operator $i\Theta$ on M has the stated property.

Notation. For every m, S_m (resp. T_m) will denote the operator on sections of $\mathscr{V}_m$ corresponding to $\Delta_h - \Theta^2$ (resp. $\Delta_h + i\Theta$).

One can study the spectral properties of these operators by applying the results of §1 to the operators

$$P = \sqrt{\Delta_{KK} - \Phi^2}, \tag{6.6}$$

and

$$Q = \sqrt{\Delta_{KK} + i\Theta}, \tag{6.7}$$

respectively. Notice that, in each case, the operator inside the square root has positive principal symbol, and hence its spectrum is positive when restricted to the orthogonal complement of a finite-dimensional subspace of $L^2(M)$. By the square root we mean any square root which is positive in the complement of this finite-dimensional subspace. (The statements of our results are unchanged by a finite-rank perturbation of the operators.)

Example. Take $G = S^1$. Then Ψ is just an S^1-invariant smooth function on M. Thus it is of the form $\Psi = \pi^*(q)$, for some $q \in C^\infty(Y)$. Then $\Theta = q\partial_\theta$. If furthermore we take $M = Y \times S^1$, then $\Delta_{KK} - \Theta^2 = \Delta_Y - (q^2 + 1)\partial_\theta^2$. When restricted to $\mathscr{H}_m$, the eigenspace of ∂_θ with eigenvalue im, this operator coincides with the Schrödinger operator $\Delta_Y + m^2(q^2+1)$ on $L^2(Y)$. Its eigenvalues are m^2 times the eigenvalues of the standard Schrödinger operator, with potential $q^2 + 1$, and a value of Planck's constant equal to $\hbar = m^{-2}$ (see [9]).

The application of the theorems of §1 to (6.6) and (6.7) is immediate. For every m, let μ_m denote the eigenvalue of the Laplacian on the group G in the representation κ_m, and let c be the length squared of any of the elements $f \in \mathscr{O}$ with respect to the chosen bi-invariant metric. Finally, let $H \in C^\infty(X_{\mathscr{O}})$ be the pull-back to $X_{\mathscr{O}}$ of the square of the Riemannian norm function on T^*Y.

6.2. Theorem. *For every $m > 0$, let $\{\mu_{m,j} | j = 1, 2, \cdots\}$ be the eigenvalues of S_m (resp. T_m), with multiplicities, and define Υ by* (1.6), *where*

$$\lambda_{m,j} = \sqrt{\mu_{m,j} + \mu_m}.$$

Then the conclusions of Theorem 1.2 *hold, with the Hamiltonian $p \in C^\infty(X_{\mathscr{O}})$ equal to $p = \sqrt{H + \Psi_0^2 + c}$ for S_m, and $p = \sqrt{H + c}$ for T_m, assuming E is a regular value of p and its flow on the energy surface $\{p = E\}$ is clean. The subprincipal symbol factor in* (1.14) *equal one for S_m, while $\mathfrak{s}^{\mathrm{av}} = \Psi_0$ for T_m.*

The meaning of holonomy was discussed in Lemma 6.1.

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