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BOUNDS FOR HYPERSPHERES OF PRESCRIBED GAUSSIAN CURVATURE

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Abstract

Apriori C^0 bounds are established for solutions of A. D. Alexandrov's problem for prescribing the Gauss curvatures of hypersurfaces in Euclidean space. For smooth hyperspheres, this corresponds to finding apriori oscillation bounds for a certain homothety invariant Monge-Ampère equation on the sphere. Conditions are given which are both necessary and sufficient for the boundedness of the hypersphere. An example is given which shows how far Alexandrov's original condition for the existence of solutions must be strengthened to estimate the bound.

Apriori C^0 estimate for A. D. Alexandrov's problem are established in this work. By a convex hypersurface M^n of dimension n we mean the boundary of a convex domain K containing a neighborhood of the origin of Euclidean space $\mathbf{E}^{n+1}$. Suppose M^n is given as the graph of the unit sphere $\mathbf{S}^n$ about the origin with the radius function $r: \mathbf{S}^n \to \mathbf{R}^+$. A coordinate map $R: \mathbf{S}^n \to M^n$ is given by $R(x) = r(x)x$. Let $\nu: M^n \to \mathbf{S}^n$ be the Gauss or normal mapping. In general $\nu(Y)$ is the set of outward unit perpendiculars of supporting hyperplanes to M^n at the point Y. For smooth M it is the normal vector. Let da be the standard measure on $\mathbf{S}^n$, and write $|F|$ for the da measure of a Borel set F in $\mathbf{S}^n$. Let $O_n = |\mathbf{S}^n|$ and let $d(x, y)$ be the distance function on $\mathbf{S}^n$. For a set $F \subset \mathbf{S}^n$ let $F_\alpha = \{x \in \mathbf{X}^n : d(x, F) \le \alpha\}$ be the α-neighborhood of F. For convex hypersurface M^n, $|\nu(R(F))|$ is a nonnegative, completely additive function on the Borel subsets of $\mathbf{S}^n$ ([1], [4]) such that $|\nu(R(\mathbf{S}^n))| = O_n$. $|\nu(R(F))|$ is invariant under dilations of M. Alexandrov's problem is to reconstruct the hypersurface from this measure (see e.g., [11]).

Theorem 1. A. *If μ is a nonnegative, completely additive function on Ω, the set of Borel subsets of $\mathbf{S}^n$, such that $\mu(\mathbf{S}^n) = O_n$ and, for some*

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$0 < \alpha < \pi/2$,

$$\mu(F) \le |F_\alpha| \quad \textit{for all } F \in \Omega, \tag{1}$$

where F_α *is the* α*-neighborhood of* F, *then any convex hypersurface* M^n *satisfying*

$$\mu(F) = |\nu(R(F))| \quad \textit{for all } F \in \Omega \tag{2}$$

has the ratio of radii bounded

$$\rho(M) := \frac{\sup\{r(x) : x \in \mathbf{S}^n\}}{\inf\{r(x) : x \in \mathbf{S}^n\}} \le c(\alpha, n) < \infty. \tag{3}$$

B. *Conversely, suppose* M^n *is a convex hypersurface about the origin in Euclidean space* $\mathbf{E}^{n+1}$. *If the radii of* M^n *are bounded in the sense that* $\rho(M^n) < \infty$, *then there exists* $0 < \alpha(\rho) < \pi/2$ *so that*

$$|\nu(R(F))| \le |F_\alpha| \quad \textit{for all } F \in \Omega. \tag{4}$$

If M^n were C^2, then

$$|\nu(R(F))| = \int_{R(F)} \kappa \, da,$$

where κ is the Gauss-Kronecker curvature of M^n. If μ is given by integrating a function,

$$\mu(F) = \int_F f \, da,$$

then, from the expression for κ, Theorem 1 gives apriori boundedness of weak solutions (satisfying (2)) of the Monge-Ampère equation (e.g., [9])

$$\det(r^2\delta_{ij} + 2r_i r_j - r r_{ij}) = f(x) r^{n-1} (r^2 + |Dr|^2)^{(n+1)/2}, \tag{5}$$

where r_i and r_{ij} are the coefficients of the first and second covariant derivatives of r in an orthonormal frame on $\mathbf{S}^n$ (see [16]). Solutions of (5) are also invariant under dilations $r \mapsto kr$.

A priori C^0 bounds for the similar reduced higher mean curvature equations ([3], [5], [6], [10], [12]) all require additional decay assumptions that preclude homothety invariance. Condition (1) is more akin through Steiner's formula (27) to isoperimetric conditions on the solubility of the nonparametric mean curvature equations (e.g. [8]). In a future application of this work we will generalize our study [15] of the homothety invariant reduced mean curvature equation.

The question of existence of surfaces satisfying (2) was settled by A. D. Alexandrov [2], who also showed uniqueness up to dilation for such surfaces (e.g. [4, p. 30]). The existence of regular solutions to (5) was shown for $n = 2$ by Pogorelov [13, Chapter VIII], and for general n by Oliker [9]. Alexandrov's necessary and sufficient conditions that there *exist* solutions to (2) is

$$(6) \qquad \mu(F) < O_n - |F^*| \quad (= |F_{\pi/2}|) \text{ for all convex } F \subset \mathbf{S}^n,$$

where F^* denotes the dual to F. (See §1. There we check that (1) implies (6).)

Alexandrov's condition does not give an apriori C^0 bound for convex surfaces. $\rho(M)$ can be made arbitrarily large by choosing, say, M to be a sufficiently eccentric ellipsoid, or by moving M so that the origin is sufficiently close to the boundary, whereas (6) holds for any convex body with zero in the interior. In fact, the condition for boundedness strengthens (6) in two ways. First, inequality (6) must be uniformly strict. Second, the inequality must hold for more than just the convex subsets. To see that this second restriction is necessary, we show in §3 that there are a $\beta > 0$ and a sequence of convex surfaces M_i so that for each i,

$$(7) \qquad |\nu(R_i(F))| \le O_n - |F^*| - \beta \quad \text{for all convex } F \subset \mathbf{S}^n,$$

but $\rho(M_i)$ tends to infinity.

In §1 we list some facts about convex hypersurfaces and prove Theorem 1B. In §2 we prove Theorem 1A. Making a finite version of Alexandrov's argument, our proof depends on two elementary geometric results (Lemmas 2 and 6) about sets in the sphere which are interesting in their own right.

1. Preliminaries and the necessity of the condition

In this section we collect some facts about convex sets. A reference is [4]. If $F \in \Omega$ is any set, let $\mathbf{Cone}(F) = \{tX : X \in F, t \ge 0\}$ be the cone generated by F. For any cone $C \subset E^{n+1}$, let $C^* = \{X \in \mathbf{E}^{n+1} : X \cdot Y \le 0$ for all $0 \ne Y \in C\}$ be the dual cone. $F^* = (\mathbf{Cone}F)^* \cap \mathbf{S}^n$ is the dual angle. We check the equality in (6).

Lemma 1. *Suppose $F \subset \mathbf{S}^n$ is a convex set. Then $|F_{\pi/2}| = O_n - |F^*|$.*

Proof. By definition, $\operatorname{int}(F^*) \subset \mathbf{S}^n - F_{\pi/2}$. Since F^* is convex [4, p. 25] the result follows from $|\partial F^*| = 0$ (e.g. [7, pp. 271, 280]).

Corollary. *If μ satisfies* (1) *for some $0 < \alpha < \pi/2$, then it satisfies* (6).

By homothetic invariance, we may assume that M satisfies $1 \le r \le \rho$, In essence we utilize the fact that the unit ball about the origin $U_1(0)$

is contained in K. Alexandrov's condition depends only merely on the origin being in K.

Proof of Theorem 1B. For a convex hypersurface M^n about the origin such that $\rho(M) < \infty$ we claim condition (4) holds for $\alpha = \cos^{-1}(\rho^{-1})$ $(< \pi/2)$. To see this it suffices to show that $\nu(R(F)) = \bigcup\{\nu(R(x)) : x \in F\} \subset F_\alpha$. Choose $x \in F$ and $z \in \nu(R(x))$. Since z is an outward normal of K, we have $z \cdot (R(X) - p) \geq 0$ for all $p \in U_1$. By choosing $z = p$ we find $r(x) z \cdot x \geq z \cdot z = 1$ so that $d(z, x) = \cos^{-1}(z \cdot x) \leq \cos^{-1}(\rho^{-1})$. q.e.d.

We define the support function to be used later. Let $K \subset \mathbf{R}^n$ be a convex body. Let $\phi \in \mathbf{S}^{n-1} \subset \mathbf{R}^n$ denote both a coordinate in the standard unit sphere and a unit vector of $\mathbf{R}^n$. The support function $p : \mathbf{S}^n \to \mathbf{R}^n$ is the distance of the supporting plane in the ϕ direction to the origin,

$$p(\phi) = \sup\{x \cdot \phi : x \in K\}. \tag{8}$$

In particular, $|p(\phi)| \leq \operatorname{diam}(K)$, where the diameter of a set is the maximal distance between pairs of points in the set.

2. Proof of Theorem 1A

In the first part of the proof we show that there is a direction in which K has linear growth. Let $L_{s,\theta} = \{X \in \mathbf{E}^{n+1} : X \cdot \theta = s\}$ be the hyperplane in $\mathbf{E}^{n+1}$ normal to θ, and a distance s from the origin. Let $H_{s,\theta} = \bigcup\{L_{\sigma,\theta} : \sigma \leq s\}$ be a halfspace which it bounds. Let $K_{s,\theta} = L_{s,\theta} \cap K$. We show that for some θ_0 the diameter satisfies

$$\operatorname{diam} K_{s,\theta_0} \leq c_2 + c_3 s, \tag{9}$$

where c_2 and c_3 depend only on α and n.

To see this let $\beta = \alpha/2 + \pi/4$ and let the open geodesic ball $B_\beta((-1, -, \cdots, 0)) \subset \mathbf{S}^n$. By (1) applied to $\mathbf{S}^n - B_\beta$ and using [4, p. 26], we obtain

$$|\nu(R(B_\beta))| = O_n - |\nu(R(\mathbf{S}^n - B_\beta))| \geq O_n - |B_{\beta-\alpha}|) = c_4 > 0,$$

where c_4 depends only on α and n. To show that B_β contains a useful concentration of curvature we apply

Lemma 2. *Let $F \subset \mathbf{S}^n$ be a Borel set with $|F| \geq \mu > 0$. Then there are $n+1$ points $x_i \in F$ in an open hemisphere of $\mathbf{S}^n$ whose convex hull T contains a geodesic ball $B_\varepsilon(\theta)$ with $\varepsilon = c_5(n)\mu^n > 0$.*

Proof. Let $B_{\pi/4}$ be a fixed ball in $\mathbf{S}^n$ and $\{y_j : j = 1, \cdots, N\}$ be a maximal set of points in $B_{\pi/4}$ such that $d(y_j, y_k) \geq 4\delta$ for all $j \neq k$ and for some sufficiently small δ to be chosen later. By maximality, $\bigcup_j B_{4\delta}(y_j)$ is a cover of $B_{\pi/4}$. Hence $N|B_{4\delta}| \geq |B_{\pi/4}|$ so there is a constant $c_6(n) > 0$ so that

$$N \geq c_6 \delta^{-n}. \tag{10}$$

Let $C = \bigcup_j B_\delta(y_j)$. Let G be the group of isometries of $\mathbf{S}^n$, and dG the invariant unit measure on G. By Brother's Poincaré formula [14, p. 277] there is a $c_7(n) > 0$ such that

$$\int |F \cap gC|\, dG = c_7 N |F|\, |B_\delta|.$$

Hence, for some motion, say $g = \mathrm{id}$, by (10)

$$|F \cap C| \geq c_8 \mu,$$

where c_8 depends only on n. Thus there are at least k balls $\{B_\delta(y_{j'})\}$ so that $|B_\delta(y_{j'}) \cap F| > 0$, where $k|B_\delta| \geq c_8\mu$. On the other hand not all $B_\delta(y_{j'})$ lie in a 3δ neighborhood of any great $\mathbf{S}^{n-1}$ in $B_{\pi/4}$ because otherwise

$$c_8\mu \leq k|B_\delta| \leq 6\delta O_{n-1},$$

which is a contradiction for the choice $\delta = .1c_8\mu/O_{n-1}$.

By construction there are points $x_{j'} \in B_\delta(y_{j'}) \cap F$ which are pairwise 2δ apart. Now we show the existence of the desired $n+1$ points by iterating m, the number of points chosen. For $m = 2$ take any two $x_{j'}$'s. We may put $\varepsilon_1 = \delta$. Suppose m points have been chosen which lie in some great $\mathbf{S}^{m-1}$ and whose convex hull contains a $B'_{m-1} = B_{\varepsilon_{m-1}}(\theta_{m-1}) \cap \mathbf{S}^{m-1}$ for some $\theta_{m-1} \in \mathbf{S}^{m-1}$. Not all $B_\delta(y_{j'})$'s lie in a 3δ neighborhood of this $\mathbf{S}^{m-1}$. Hence there is an x_m in such an outlying ball so that $d(\mathbf{S}^{m-1}, x_m) > \delta$. Let γ denote the geodesic segment from θ_{m-1} to x_m, and q any point of $\mathbf{S}^{m-1} - \{\theta_{m-1}\}$. Let θ_m be the midpoint of γ and $\sin\varepsilon_m = .5\sin\delta\sin\varepsilon_{m-1}$. Then the ball $B'_m = B_{\varepsilon_m}(\theta_m) \cap \mathbf{S}^m$ is contained in the convex hull of B'_{m-1} and x_m, where $\mathbf{S}^m$ is the great sphere containing $\mathbf{S}^{m-1}$ and x_m. Let q' be the point on the geodesic extending the ray $\theta_{m-1}q$ closest to x_m. Let a be the angle from γ to the ray. By the law of sines restricted to the great $\mathbf{S}^2$ containing the ray and the segment,

$$\sin a \geq \sin a \sin d(x_m, \theta_{m-1}) = \sin d(x_m, q') \geq \sin\delta.$$

We show that $B'_m \cap \mathbf{S}^{m-1}$ is empty. To see this, let $q \in \mathbf{S}^{m-1}$ be the closest point to θ_m. If $q = \theta_{m-1}$, then, by the definition of ε_m, $d(q, \theta_m) \geq \varepsilon_m$. Otherwise, as before,

$$\begin{aligned}\sin d(\theta_m, q) &= \sin a \sin d(\theta_{m-1}, \theta_m) \\ &\geq \tfrac{1}{2}\sin^2\delta \geq \tfrac{1}{2}\sin\delta \sin\varepsilon_{m-1},\end{aligned}$$

and apply the triangle inequality. To see that B'_m is in the convex hull of B'_{m-1} and x_m, assume to the contrary that there is a geodesic ζ through x_m which meets B'_m but not B'_{m-1}. Let $y \in \zeta \cap B'_m$ be the closest point to θ_m and let ϕ be the angle from ζ to γ. By the law of sines for the triangle $yx_m\theta_m$,

$$\tfrac{1}{2}\sin\phi \sin d(x_m, \theta_{m-1}) \leq \sin\phi \sin d(\theta_m, x_m) \leq \sin\varepsilon_m.$$

On the other hand, letting $q = \zeta \cap \mathbf{S}^{m-1} \notin B'_{m-1}$ and b be the angle $x_m q \theta_{m-1}$ and noting the symmetry of the a estimate in q and θ_{m-1} yields

$$\sin\delta \sin\varepsilon_{m-1} \leq \sin b \sin\varepsilon_{m-1} < \sin d(\theta_{m-1}, x_m)\sin\phi,$$

a contradiction. When $m = n$ the iteration stops with $\theta = \theta_n$ and $\varepsilon = \varepsilon_n$. q.e.d.

To resume the proof of Theorem 1A, apply Lemma 2 to the set $\nu(R(B_\beta)) \subset \mathbf{S}^n$ with measure at least c_4 to show that it contains $n+1$ points z_i in $\nu(R(B_\beta))$ whose convex hull contains a ball $B_\varepsilon(-\theta)$, where ε is positive and depends on c_4 and n. K is contained in the supporting half-spaces, i.e.,

$$K \subset \bigcap_{i=0}^{n} H_{r(x_i), z_i} \subset U_{c_2} + \mathbf{Cone}(T)^* \subset U_{c_2} + \mathbf{Cone}(B_\varepsilon(-\theta))^*,$$

where $+$ is Minkowski addition of sets, and $\sup\{r(x_i)\colon i = 1, \cdots, n-1\} \leq c_2 \tan\beta$, since all the supporting planes intersect $\mathbf{Cone}(B_\beta) \cap H_{1,(-1,0,\cdots,0)}$. Thus in the θ direction, K has linear growth bounded by c_2 and $c_3 = \cot(\varepsilon)$. In particular $\operatorname{diam}(K_1, \theta) \leq c_9 = c_2 + c_3$.

In the second part of the proof, after rotating the coordinate axes so that θ is in the x_1 direction, we show that K has finite extent in the x_1 direction. For simplicity, denote $L_s = L_{s,\theta}$, $K_s = L_s \cap K$ and $H_s = H_{s,\theta}$. Since $K - H_s$ is convex, $F_x = R^{-1}(K_s)$ is convex in $\mathbf{S}^n$. Applying (1) we find

$$|\nu(R(F_s))| \leq |(F_s)_\alpha|.$$

To go further, let $p(s, \phi)$ be the support function for K_s in L_s as in (8). If we denote the spherical coordinate $\phi \in \mathbf{S}^{n-1} \subset L_s$, then $\{X \in L_s : X \cdot \phi = p(s, \phi)\}$ is the supporting plane to K_s with outward normal vector ϕ. Consider the convex hull $W_{s,t}$ of K_s and K_t in $\mathbf{E}^{n+1}$ for $s < t$. We wish to compute the total curvature of the right half of $W_{s,t}$, namely, $Q = \partial W_{s,t} - H_{(s+t)/2}$. In order to do this, note that given ϕ, the supporting plane to ∂Q has a normal vector which defines the angle $a(\phi)$,

$$n = \frac{\phi - \theta\Delta}{\sqrt{1+\Delta^2}} = \phi \sin a + \theta \cos a, \tag{11}$$

where $\Delta = (p(s, \phi) - p(t, \phi))/(s-t)$. If ξ denotes the Gauss map for $\partial W_{s,t}$,

$$\begin{aligned} |\xi(Q)| &= |\{\exp_\theta(s\phi) : 0 \le s \le a(\phi),\ \phi \in \mathbf{S}^{n-1}\}| \\ &= \int_{\mathbf{S}^{n-1}} \int_0^{a(\phi)} \sin^{n-1} s \, ds \, d\phi, \end{aligned} \tag{12}$$

where we have used (s, ϕ) as polar coordinates for $\mathbf{S}^n$ about θ and $\exp_\theta$ as the exponential map from the tangent space to $\mathbf{S}^n$. We substitute $u = \cos s$ to find

$$|\xi(Q)| = \tfrac{1}{2} O_n - \int_{\mathbf{S}^{n-1}} f_n(\cos a) \, d\phi,$$

where

$$f_n(v) = \int_0^v (1-u^2)^{(n-2)/2} \, du.$$

By convexity and using hypothesis (1), for any $\alpha \le \beta < \pi/2$ we have

$$|\xi(Q)| \le |\nu(M - H_s)| \le |(R^{-1}(M - H_s))_\beta|, \tag{13}$$

which may be computed like $|\xi(Q)|$ by viewing $s = 0$ and "$p(0, \phi) = 0$". Hence

$$|(R^{-1}(M - H_s))_\beta| = \tfrac{1}{2} O_n - \int_{\mathbf{S}^{n-1}} f_n(\cos(a + \beta - \pi/2)) \, d\phi, \tag{14}$$

since F_s is convex on $\mathbf{S}^n$. Now substituting (11), (12), and (14) in (13) and letting $s \to t-$, we obtain

$$\int_{\mathbf{S}^{n-1}} f_n(g(p_s)) \, d\phi \le \int_{\mathbf{S}^{n-1}} f_n\left(\sin\beta \, g\left(\frac{p}{s}\right) - \cos\beta \sqrt{1 - g^2\left(\frac{p}{s}\right)}\right) d\phi, \tag{15}$$

where p_s denotes the left derivate of $p(s, \phi)$ and

$$g(u) = \frac{u}{\sqrt{1+u^2}}. \tag{16}$$

By (9), $u = p/s$ satisfies $|u| \le c_{10} = \max\{c_9, \tan\alpha\}$ whenever $s \ge 1$. We may simplify the right side by a pointwise inequality.

Lemma 3. *For $m > 0$ there is a $\gamma(m)$, $0 < \gamma < \pi/2$, so that if $|u| \le m$, then*

$$(\sin\gamma)g(u) - \cos\gamma\sqrt{1 - g^2(u)} \le g((1-\varepsilon_1)u - \varepsilon_2),$$

for g given by (16) *and for any constants $1 - \varepsilon_1 \ge \sin\gamma$ and $\varepsilon_2 \le \cos^2\gamma$.*

Proof. Define $0 < \gamma < \pi/2$ by $g(m) = \sin\gamma$. For $u \ge 0$, by concavity and $\mathbf{Lip}(g) < 1$, we have

$$\begin{aligned} h_\gamma(u) &= (\sin\gamma)g(u) - \cos\gamma\sqrt{1 - g^2(u)} \\ &\le g(u\sin\gamma) - \cos^2\gamma \le g((1-\varepsilon_1)u - \varepsilon_2). \end{aligned}$$

If $u \le 0$, then by writing $g(u) = -\sin z$, $0 \le z \le \gamma$, $h_\gamma = -\cos(\gamma - z)$ and the result follows from the trigonometric inequality

$$g^{-1}(h_\gamma) = -\cot(\gamma - z) \le -\tan z - \cot\gamma \le (1-\varepsilon_1)u - \varepsilon_2. \quad \text{q.e.d.}$$

By Lemma 3 there is a $\gamma(c_{10}(\alpha, n)) \ge \alpha$ so that (15) becomes

$$\int_{\mathbf{S}^{n-1}} f_n \circ g(p_s) \le \int_{\mathbf{S}^{n-1}} f_n \circ g\left((1-\varepsilon_1)\frac{p}{s} - \varepsilon_2\right) \tag{17}$$

for any $1 - \varepsilon_1 \ge \sin\gamma$ and $\varepsilon_2 \le \cos^2\gamma$.

We now show $K_s = \varnothing$ for some $s < s_1$, with s_1 large and depending only on n and α (to be described). Suppose this were not the case. Then M^n must contain a long midsection in which it is nearly conical. This is first shown in an L^1 sense.

Lemma 4. *Let $p(s, \phi)$ be the support function for the sections K_s of a convex body $0 \in K$ as in* (8), *with $p(1, \phi) \le c_9$. Let*

$$P(x) = \int_{\mathbf{S}^{n-1}} \frac{p(s, \phi)}{s}\, d\phi.$$

Then for all $\delta > 0$ there exists $s_1(\delta, c_9)$ so that if $P(s) > 0$ for $1 \le s \le s_1$, then for some s_0, $1 \le s_0 \le 2s_0 \le s_1$,

$$(1-\delta)P(s_0) \le P(2s_0). \tag{18}$$

Proof. First observe $p_s \le p/s$. Fixing $\phi \in \mathbf{S}^{n-1}$, consider the orthogonal projection K' of K onto the (θ, ϕ) plane. Since K' is still convex and $p(s, \phi)$ is the distance from $s\theta$ to $\partial K'$ in the ϕ direction, we see that rays from the origin pierce outwards any secant of $\partial K'$, $(s, p(s, \phi))$ to $(t, p(t, \phi))$, $s < t$. Hence in the notation of (11), $\Delta \le p/s$ so $P_s \le 0$.

Changing to $s = 2^t$ and $u = \log P$ we still have $u_t \leq 0$. If $P(t) > 0$ for $0 \leq t \leq 2T + 2$ then we have $P(s) \geq 1/2$ for $s \leq T + 1$, since $P(s)$ is independent of the choice of origin of L_s, and K contains the convex hull of $B_1(0)$ and a point in K_{2T+2}. But $P(s)$ is absolutely continuous, so

$$\log u(0)+\log 2 \geq u(0)-u(T+1) = -\int_0^{T+1} u_t(x)\,dx \geq -\sum_{i=0}^{[T]} \int_i^{i+1} u_t(x)\,dx,$$

where $[T]$ is the greatest integer function. Hence for some i',

$$u(i') - u(i'+1) \leq \frac{\log c_9 + \log O_{n-1} + \log 2}{T}.$$

The result follows if we set $s_1 = 4(2c_9 O_{n-1})^{2/\delta}$. q.e.d.

Next we show that a long body K is nearly conical at K_{s_0} in measure. In L_{s_0}, by convexity, the two convex sets satisfy $K_1 = \frac{1}{2}K_{2s_0} \subset K_2 = K_{s_0}$ where we mean the Minkowski multiplication of sets in $\mathbf{E}^{n+1}$. The support functions q_i for K_i satisfy

$$q_1(\phi) = \tfrac{1}{2}p(2s_0, \phi) \leq q_2(\phi) = p(s_0, \phi).$$

By (9), $|q_i| \leq s_0 c_9$. We need to deduce from (18) that the support functions are close in measure.

Lemma 5. *Let $K_1 \subset K_2 \subset B_m(0) \subset \mathbf{R}^n$ be two convex sets. For all $\varepsilon > 0$ there is a $\delta(\varepsilon, n) > 0$ so that if*

$$\int_{\mathbf{S}^{n-1}} q_1(\phi)\,d\phi \geq (1-\delta) \int_{\mathbf{S}^{n-1}} q_2(\phi)\,d\phi$$

for the support functions q_i of K_i as in (8), *then there is a set $F \subset \mathbf{S}^{n-1}$, $|F| \leq \varepsilon$, such that for all $\phi \in \mathbf{S}^{n-1} - F$,*

$$q_1(\phi) \geq (1-\varepsilon)q_2(\phi) - 2m\varepsilon.$$

Proof. Let $F = \{\phi \in \mathbf{S}^{n-1} : q_1(q) < (1-\varepsilon)q_2(\phi) - 2m\varepsilon\}$. Then

$$\delta O_{n-1} m \geq \delta \int q_2 \geq \int q_2 - q_1 \geq \varepsilon \int_F 2m + q_2 \geq |F|\varepsilon m.$$

The lemma follows by choosing $\delta = \varepsilon^2/O_{n-1}$. q.e.d.

Thus there is a set $F \subset \mathbf{S}^{n-1}$, $|F| \leq \varepsilon$ and such that

$$\tfrac{1}{2}p(2s_0, \phi) \geq (1-\varepsilon)p(s_0, \phi) - 2\varepsilon c_9 s_0 \tag{19}$$

for all $\phi \in \mathbf{S}^{n-1} - F$ and any ε with corresponding δ and s_1. Hence for $\phi \notin F$, by convexity and (19) we may estimate

$$p_s(s_0, \phi) \geq \frac{p(2s_0, \phi) - p(s_0, \phi)}{s_0} \geq (1 - 2\varepsilon)\frac{p(s_0, \phi)}{s_0} - 4\varepsilon c_9.$$

Finally, there is a positive lower bound

$$\frac{d}{du} f_n(g(u)) \geq c_{11}(\alpha, n) > 0$$

for $|u| \leq c_9 + 1$. Since $|p/s| \leq c_9$ and $|f_n \circ g| \leq 1$,

$$\begin{aligned}\int_{\mathbf{S}^{n-1}} f_n \circ g(p_s) &\geq \int_{\mathbf{S}^{n-1}-F} f_n \circ g\left((1-2\varepsilon)\frac{p}{s} - 4\varepsilon c_9\right) - 2|F| \\ &\geq \int_{\mathbf{S}^{n-1}-F} f_n \circ g\left((1-2\varepsilon)\frac{p}{s} - 4\varepsilon c_9 - \frac{\varepsilon^{1/2}}{c_{11}}\right) \\ &\quad - 2\varepsilon + (O_{n-1} - \varepsilon)\varepsilon^{1/2} \\ &\geq \int_{\mathbf{S}^{n-1}} f_n \circ g\left((1-2\varepsilon)\frac{p}{s} - 4\varepsilon c_9 - \frac{\varepsilon^{1/2}}{c_{11}}\right) \\ &\quad - 4\varepsilon + (O_{n-1} - \varepsilon)\varepsilon^{1/2},\end{aligned}$$

provided that $4\varepsilon c_9 + \varepsilon^{1/2}/c_{11} \leq 1$. Now choosing ε depending only on α and n so small that $1 - 2\varepsilon \geq \sin\gamma$, $4\varepsilon c_9 + \varepsilon^{1/2}/c_{11} \leq \cos^2\gamma$ and $4\varepsilon < (O_{n-1} - \varepsilon)\varepsilon^{1/2}$, we get a contradiction to (17).

3. A counterexample to boundedness

The estimate of the bound in Theorem 1 requires that Alexandrov's condition be strengthened in two ways, by making the inequality uniformly strict and by requiring that the condition hold for more than the convex sets. In this section we given an example which shows that making the inequality in Alexandrov's condition strict by itself does not provide an apriori C^0 bound. We begin with an estimate of the width of a convex set in $\mathbf{S}^2$ in terms of its area.

Lemma 6. *Suppose $F \subset \mathbf{S}^2$ is a convex set. Let D be the diameter of F. Then there exist v_1, v_2 in $\mathbf{S}^2$, $d(v_1, v_2) = \lambda$, so that the lune*

$$V_\lambda = \{x \in \mathbf{S}^2 : x \cdot v_1 \leq 0,\ x \cdot v_2 \geq 0\} \supset F \tag{20}$$

provided that

$$|F| \leq \lambda \tan \tfrac{1}{4} D \tag{21}$$

or

(22) $$16\pi|F| \le \lambda^2.$$

Proof. Choose x, y in the closure of F, which realize the diameter. Extend the geodesic γ from x to y to a semi-great circle $exyf$ centered at the midpoint of γ. Consider the smallest lune V_λ with vertices e, f containing F. Let $V_{\lambda'}$ be one of the components of $V_\lambda - \gamma$, and z a point of contact between $\partial V_{\lambda'}$ and F not on γ. $V_{\lambda'}$ is divided into three disjoint geodesic triangles $T = xyz$, $T_1 = exz$ and $T_2 = fyz$. Let the lengths $d' = d(e, x)$, $d_1 = d(e, z)$ and $d_2 = d(z, f)$. We have $2d' + D = d_1 + d_2 = \pi$. By the triangle inequality $d_i \ge d'$ since $\pi - d_i = d_{3-i} \le d' + D = \pi - d'$. To estimate the area of T in terms of λ' we appy the area formula for spherical triangles to find

(23) $$\sin\lambda' \cot\tfrac{1}{2}|T_i| = \cot\tfrac{1}{2}d' \cot\tfrac{1}{2}d_i + \cos\lambda'.$$

Supposing $d_1 \le d_2 \le \pi - d'$ we find using (23) that $|T_2| \le \lambda'$. Hence we may estimate

(24) $$|T| = |V_{\lambda'}| - |T_1| - |T_2| \ge \lambda' - 2\cot^{-1}(\cot\tfrac{1}{2}d' \cot\tfrac{1}{2}d_1 \csc\lambda' + \cot\lambda').$$

For $\varepsilon, \lambda > 0$, using the concavity of $\sin[(1+\varepsilon)\lambda/(2+\varepsilon)]$ and calculus one finds

$$\frac{1+\varepsilon+\cos\lambda}{\sin\lambda} > \cot\left(\frac{\lambda}{2+\varepsilon}\right).$$

Applying this with $1+\varepsilon = \cot\frac{1}{2}d' \cot\frac{1}{2}d_1$ to (24) and using $d' = (\pi - D)/2$ and $d_1 \le \pi/2$ yield (21) after summing both sublunes of V_λ. Since some ball $B_D \supset F$ we have

$$|F| \le |B_D| = 4\pi\sin^2\tfrac{1}{2}D.$$

Combining this with $\tan\frac{1}{4}D \ge \frac{1}{2}\sin\frac{1}{2}D$ in (21) gives (22).

Theorem 2. *There are a positive β and a sequence of convex two-dimensional surfaces M_i about the origin for which the bound on the ratio of radii* (3) *satisfies $\rho(M_i) \to \infty$ as $i \to \infty$ but for all i,*

(25) $$|\nu_i(R_i(F))| < O_2 - |F^*| - \beta \quad \textit{for all convex } F \subset \mathbf{S}^2.$$

Proof. We show the condition is satisfied by surfaces of rotation M_i given for each positive integer i by rotating about the x-axis:

$$\begin{aligned} x &= -1 && \text{if } 0 \le y \le i \text{ and } x = -1, \\ y &= i + \sqrt{1-x^2} && \text{if } -1 < x \le 0, \\ y &= \sqrt{(i+1)^2 - x^2} && \text{if } 0 < x \le i+1. \end{aligned}$$

For these surfaces, $\rho(M_i) = i+1$. In general, the same argument will show that if M is any convex hypersurface which has a planar neighborhood about some $Y \in M$, then $M_i = M - X_i$, where X_i is a sequence tending to Y in $\mathbf{E}^{n+1}$, also satisfies (7) but $\rho_i \to \infty$. The differentiability provides the convenience of an absolutely continuous Gauss map so $|\nu_i(R_i(\partial F))| = 0$. For convex $F \in \mathbf{S}^2$ we have by Lemma 1,

$$|F^*| = O_2 - |F_{\pi/2}|. \tag{26}$$

Next, by Steiner's formula for space forms [14, p. 322],

$$|F_{\pi/2}| = l(\partial F) + 2\pi, \tag{27}$$

where l is the length. The isoperimetric inequality on spheres [14, p. 324] is

$$l(\partial F)^2 \geq 4\pi|F| - |F|^2. \tag{28}$$

We shall establish for some $\beta > 0$, but independent of i, and for $J(u) = \sqrt{4\pi u - u^2}$,

$$\begin{aligned} I(F) = |\nu(R_i(F))| & \left(= \int_{R_i(F)} \kappa_i \, da_i\right) \\ & \leq 2\pi + J(|F|) - \beta \quad \text{for all convex } F \in \mathbf{S}^2, \end{aligned} \tag{29}$$

which by (26), (27) and (28) is sufficient for (25).

We decompose sets into the disjoint union $F = F^- \cup F^+$, where $F^- = F \cap \{(x, y, z) \colon x < 0\}$. For the rest of the proof, let $\mu = |F|$. If H is an open half-space bounded by a plane containing the x-axis, by rotational symmetry, the Gauss images $\nu(M \cap H) = \mathbf{S}^2 \cap H$. Consider various cases of convex F in $\mathbf{S}^2$ depending on the area.

If $\mu_1 = \pi(2 - \sqrt{3}) < \mu$ $(\leq 2\pi)$, there is some half-space H bounded by a plane through the x-axis so that one of $\{H \cap F^{\pm}, F^{\pm} - H\}$ is null. Hence $I(F) \leq 3\pi = 2\pi + J(\mu_1) < 2\pi + J(\mu)$.

In case $\mu \leq \mu_2 = \pi(2 - \sqrt{2})$, writing the convex set $F = F^+ \cup F^-$ we claim if (29) holds for F^-, then it also holds for F. Using (29) and $\int_{R_i(f^+)} \kappa_i \, da_i = \mu^+ = |F^+|$ we conclude

$$2\pi + J(\mu^+ + \mu^-) - \int_{R_i(F^+ \cup F^-)} \kappa_i \, da_i > J(\mu^+ + \mu^-) - \mu^+ - J(\mu^-). \tag{30}$$

Under the hypothesis $\mu \leq \mu_2$ the right side of (30) is nonnegative, proving the claim.

Now if $0 < \mu_3 \leq \mu \leq \mu_2$, then $I(F^-) \leq 2\pi < 2\pi + J(\mu_3) \leq 2\pi + J(\mu)$.

Finally suppose $\mu \leq \mu_4 = \pi/64$. By Lemma 6, F^- is contained in a lune $V_{\pi/2}$. The support of the curvature in $(\mathbf{S}^2)^-$ is contained in a band of width $\cot^{-1} i$ about the $x = 0$ great circle. If the axis of the lune coincides with the x-axis, then $I(F^-) \leq I(V^-_{\pi/2}) \leq \pi/2$. Otherwise, let P be a plane containing the x-axis and vertices of $V_{\pi/2}$. If $V_{\pi/2}$ is on one side of P, then $I(F^-) \leq I(V^-_{\pi/2}) \leq \pi$, because P cuts the band in half. If not, since the lune is on both sides of P, one of the faces of the lune makes an angle of at most $\pi/4$ with P, thus avoiding the band of support at distances from P greater than $\pi/4$ on this side of P. In particular, if $i \geq 2$, the band width is small enough so that $\pi/4$ of the band remains untouched by the lune. Hence $I(F^-) \leq I(V^-_{\pi/2}) \leq \frac{7}{4}\pi < 2\pi$. Since each of the cases overlaps if $\mu_3 < \mu_4$, we may extract a β for (29).

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