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ETA INVARIANTS AND HERMITIAN LOCALLY SYMMETRIC SPACES

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Introduction

Let L be an elliptic operator on a compact manifold M. According to the Atiyah-Singer index theorem, the index of L may be obtained by integrating a locally computable function $I(x)$—the index density—over M. By locally computable, we mean that, given a local coordinate system in a neighborhood of x, $I(x)$ may be expressed as a function of the symbol of L, the metric, and a finite number of derivatives of the symbol and the metric. When L is a classical geometric operator, $I(x)$ is an invariant curvature polynomial. We call $I(x)$ the Atiyah-Singer integrand.

Suppose now that D_+ denotes the signature operator with coefficients in a flat homogeneous vector bundle, $\mathscr{E}$, over a $\mathbf{Q}$-irreducible arithmetic variety $\Gamma\backslash G/K$. When the $\mathbf{Q}$-rank of G is greater than zero, $\Gamma\backslash G/K$ is noncompact and the Atiyah-Singer index theorem does not hold. In [20], we proved that the L^2-index of D_+ can be expressed as a sum of integrals of three types of data. These types are:

(i) data which are locally computable on $\Gamma\backslash G/K$—this is the Atiyah-Singer integrand constructed from local data as in the compact case (for example, see [3, 9]);

(ii) data which are locally computable on quotients $\hat{e}(P)$ (see (1.8)) of the maximal faces of the Borel-Serre boundary of $\Gamma\backslash G/K$—this is the product of the generalized zeta function (defined in [20, 5.2.15]) and invariant curvature polynomials;

(iii) the eta densities $\eta_P(0, x)$ defined in (1.10)—these terms are defined on the $\hat{e}(P)$ but are not locally computable there.

The object of this paper is to calculate $\eta_P(0, x)$. In particular, we wish to express it in terms of data of types (i) and (ii) on lower rank spaces, thus rendering it, in principle, computable. Our first result is the following.

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Theorem A. $\eta_P(0, x)$ *vanishes unless*

(i) G *is absolutely simple and*

(ii) $\hat{e}(P)$ *is an equal rank, maximal boundary component for some Satake compactification of* $\Gamma\backslash G/K$.

Moreover, $\eta_P(0, x)$ *is integrable over* $\hat{e}(P)$.

A precise statement of our main result requires too much notation to include here. Therefore, we will omit precise definitions, leaving them to the body of the paper.

We will define (in (1.9)) an elliptic operator $\widehat{D}$ which we call the boundary operator associated to D_+. In (1.11), we define a function $\hat{\eta}_P(\cdot, k, w)$ on elliptic operators which satisfies

$$\int_{\hat{e}(P)} \eta_P(0, x)\, dx = C(k) \int_0^\infty \hat{\eta}_P(\widehat{D}, k, w) w^{-3/2}\, dw,$$

for some constant $C(k)$ (defined in (5.8)). We obtain the following explicit recursive procedure for computing $\hat{\eta}_P(\widehat{D}, k, w)$.

Theorem B. *If* G *and* P *satisfy conditions (i) and (ii) of Theorem A, then*

$$\hat{\eta}_P(\widehat{D}, k, w) = \sum_\mu \mu^{1/2}(1 + \mu/w)^{1/2-k}$$
$$\times \left\{ \text{Index } D_\mu^+ + \sum_Q C(k - 1/2) \int_0^{w+\mu} \hat{\eta}_Q(\widehat{D}_\mu, k - 1/2, s) s^{-3/2}\, ds \right\}.$$

Here the first sum is finite and runs over the eigenvalues μ of a zero order operator Δ^n defined in §4. The second sum runs over the parabolic subgroups Q which parametrize the boundary components of $\hat{e}(P)$. The operators D_μ^+ and $\widehat{D}_\mu$ are defined in §5. These operators are of the same type (a signature operator and an associated boundary operator) as D_+ and $\widehat{D}$ but are defined on lower rank spaces. In particular, the index theorem of [20] and Theorem B above may be applied to them. Iterating the application of these formulae, we see that the index of D_+ may be computed explicitly in terms of curvature integrals over various boundary components and associated zeta functions. We emphasize that these latter terms can, in principle, be computed, whereas the eta density as it was initially defined did not, a priori, admit computation. (Recently, however, Moscovici and Stanton [15] have extended Millson's [13] computation of the Atiyah-Patodi-Singer eta invariants on compact locally symmetric spaces using (nonlocal) Selberg trace formula techniques.)

In §6, we compute the index for the real rank one cases and compare our results to those of [6]. These computations are independent of the rest of this paper.

I would like to thank N. Mok for encouraging me to focus on the Hermitian case, which he correctly predicted might be amenable to a local computation.

1. Notation and assumptions

We shall follow the convention of denoting the Lie algebra of a group by the corresponding lower case script letter. Let G be the identity component of the set of real points of a semisimple algebraic group defined over $\mathbf{Q}$, K a maximal compact subgroup, and Γ a neat arithmetic subgroup. We assume that the symmetric space $X \equiv G/K$ is Hermitian and $\mathbf{Q}$-irreducible. We endow X with the invariant metric determined by the Killing form of $\mathscr{g}$ and the Cartan involution, θ_K, corresponding to K. Denote by X_Γ the finite volume, locally symmetric space $\Gamma\backslash X$.

Let (ρ, E) be a finite-dimensional, complex irreducible representation of G with an admissible inner product (see [12]). Let $\mathscr{E}$ denote the corresponding flat vector bundle over X_Γ, and $L_2^{\cdot}(X_\Gamma, \mathscr{E})$ the space of square integrable $\mathscr{E}$-valued forms. Here the L^2-inner product is computed with respect to the admissible inner product on $\mathscr{E}$—not the flat one. Define the associated signature operator D_ρ by

$$D_\rho = d + d^*,$$

where d is the exterior derivative on $\mathscr{E}$-valued forms and d^* is its formal adjoint.

Let $\mathscr{g} = \mathscr{k} \oplus \mathscr{p}$ be the Cartan decomposition of $\mathscr{g}$. We may identify $L_2^{\cdot}(X_\Gamma, \mathscr{E})$ with $(L^2(\Gamma\backslash G) \otimes \Lambda^{\cdot}\mathscr{p}^* \otimes E)^K$—the elements of $L^2(\Gamma\backslash G) \otimes \Lambda^{\cdot}\mathscr{p}^* \otimes E$ invariant by the representation $R \otimes \sigma \otimes \rho$ of K. Here R denotes the right regular representation, and σ denotes the coadjoint representation on $\Lambda^{\cdot}\mathscr{p}^*$. Via this identification d and d^* have the form

$$d = \sum [R(X_i) \otimes \varepsilon(X_i) \otimes I + I \otimes \varepsilon(X_i) \otimes \rho(X_i)], \tag{1.1}$$

$$d^* = \sum [-R(X_i) \otimes \varepsilon^*(X_i) \otimes I + I \otimes \varepsilon^*(X_i) \otimes \rho(X_i)], \tag{1.2}$$

where $\{X_i\}_{i=1}^n$ is an orthonormal basis of $\mathscr{p}$, $\varepsilon(X_i)$ is exterior multiplication by X_i, and $\varepsilon^*(X_i)$ is its adjoint.

Let $*$ denote the Hodge star operator. We define the involution $\tau_{\mathscr{p}}$ of $\Lambda^{\cdot}\mathscr{p}^*$ by

$$\tau_{\mathscr{p}}\varphi = i^{k(k-1)+n/2} * \varphi \quad \text{for } \varphi \in \Lambda^k\mathscr{p}. \tag{1.3}$$

In [14], Moscovici defines a self-adjoint involution τ_E of E which satisfies

$$\begin{aligned} \tau_E\rho(X) &= -\rho(X)\tau_E \quad \text{for every } X \in \mathfrak{p}, \\ \tau_E\rho(Y) &= \rho(Y)\tau_E, \quad \text{for every } Y \in \mathfrak{k}. \end{aligned} \tag{1.4}$$

Set

$$\tau = I \otimes \tau_{\mathfrak{p}} \otimes \tau_E \tag{1.5}$$

to obtain an involution of $(L^2(\Gamma\backslash G) \otimes \Lambda^{\cdot}\mathfrak{p}^* \otimes E)^K$ which anticommutes with D_ρ. Let $\Omega_\pm$ denote the ± 1 eigenspaces of τ, and $D_\pm$ the restriction of D_ρ to $\Omega_\pm$. As τ satisfies

$$D_\rho\tau = -\tau D_\rho, \tag{1.6}$$

we have $D_+ : \Omega_+ \to \Omega_-$.

The L^2-index of D_+ is the signature of the L^2-cohomology of X_Γ with coefficients in $\mathscr{E}$ (see [14]). In order to recall the expression for this index, we first need to introduce more notation. If P is a maximal rational parabolic subgroup of G, let F_P denote the corresponding boundary component of the Baily-Borel-Satake compactification of X_Γ (see [5]), and let $n_{h,P}$ denote the dimension of F_P. Let L (respectively $L_{h,P}$) denote the Hirzebruch L-polynomial of X_Γ (respectively F_P), and let $\mathrm{Ch}\mathscr{E}^\pm$ (respectively $\mathrm{Ch}_{h,P}\mathscr{E}^\pm$) denote the Chern character of $\mathscr{E}^\pm$ (respectively the Chern character of the restriction of $\mathscr{E}^\pm$ to F_P). In [20], we show that the index is given by the formula

$$\begin{aligned} &L^2 - \mathrm{index}(D_+) \\ &\quad = 2^n \int_{X_\Gamma} [\mathrm{Ch}\,\mathscr{E}^+ - \mathrm{Ch}\,\mathscr{E}^-] L\,dx \\ &\qquad - \sum_P \int_{\hat{e}(P)} \{\eta_P(0, x) + 2^{n_{h,P}}[\mathrm{Ch}_{h,P}\mathscr{E}^+ - \mathrm{Ch}_{h,P}\mathscr{E}^-] L_{h,P} Z_P(x)\}\,dx, \end{aligned} \tag{1.7}$$

where the sum runs over Γ-conjugacy classes of rational maximal parabolic subgroups P of G. (There is a sign error in [20, Main Theorem]; δ_{D_+} should be $-\delta_{D_+}$.) We refer to [20, (5.2.13)] for the definition of $Z_P(x)$ (see also §6, where Z_P is calculated for certain examples). In order to define $\eta_P(0, x)$ and $\hat{e}(P)$, we must first recall some of the structure of the parabolic subgroups.

Let P be a maximal $\mathbf{Q}$-parabolic subgroup of G, with Langlands decomposition $P = NAM$. Here N denotes the unipotent radical of P,

AM is the θ_K-stable Levi subgroup, and A is the unique θ_K-stable lifting of the identity component of a maximal **Q**-split torus in the center of $N\backslash P$. Let $\mathfrak{p}_m \oplus \mathfrak{k}_m$ denote the Cartan decomposition of $\mathfrak{m}$. We set

$$\Gamma_P = \Gamma \cap P, \quad \Gamma_M = \Gamma \cap N\backslash\Gamma_P, \quad \hat{e}(P) = \Gamma_M\backslash M/K_M, \tag{1.8}$$

where $K_M = K \cap M$.

Let ρ_P denote the representation of M on $\Lambda^{\cdot}\mathfrak{n}^* \otimes E$ obtained by taking the tensor product of the coadjoint representation on $\Lambda^{\cdot}\mathfrak{n}^*$ with ρ. We denote by $\mathscr{E}^{\cdot}$ the corresponding flat vector bundle. Let δ_P denote one half the sum of the weights (counted with multiplicity) of $\mathfrak{n}$ with respect to $\mathfrak{a}$. We endow $\mathfrak{n}$ and N with the metric given by one half the metric determined by θ_K and the Killing form on G.

Let d_Z denote the exterior derivative on $\Gamma_M\backslash M/K_M$ with coefficients in $\mathscr{E}^{\cdot}$, and let d_n denote the coboundary operator of the Lie algebra $\mathfrak{n}$ with coefficients in E. We extend d_n to an operator $(-1)^p I \otimes I \otimes d_n$ (also denoted d_n) on $L_2^p(\hat{e}(P), \mathscr{E}^{\cdot})$, via the standard identification of $L_2^p(\hat{e}(P), \mathscr{E}^{\cdot})$ with $(L^2(\Gamma_M\backslash M) \otimes \Lambda^p \mathfrak{p}_m^* \otimes (\Lambda^{\cdot}\mathfrak{n}^* \otimes E))^{K_M}$. Define

$$\widehat{D} = d_Z + d_Z^* + d_n + d_n^*. \tag{1.9}$$

Setting $\Delta_Z = (d_Z + d_Z^*)^2$, $D_n = (d_n + d_n^*)$, and $\Delta^n = D_n^2$, we have (see [20, (1.4.7)])

$$\widehat{D}^2 = \Delta_Z + \Delta^n.$$

Let $\tilde{\tau}$ denote the composition of τ_E and Clifford multiplication (see (2.6)) by the volume form of NM/K_M. Let $\Gamma(\cdot)$ denote Euler's gamma function, and let k be any integer greater than the dimension of X. We can now define η_P (see [20, (4.9.9)]):

$$\begin{aligned} \eta_P(0, x) &= \frac{-\frac{1}{2}}{\Gamma(\frac{1}{2})} \int_0^\infty \operatorname{Regtr} \tilde{\tau}\widehat{D}e^{-t\widehat{D}^2}(x, x)t^{-\frac{1}{2}}\,dt \\ &= \frac{-\Gamma(k-\frac{1}{2})/2}{\Gamma(k-1)\Gamma(\frac{1}{2})} \int_0^\infty \operatorname{Regtr} \tilde{\tau}\widehat{D}(\widehat{D}^2 + w)^{\frac{1}{2}-k}(x, x)w^{k-2}\,dw \qquad (1.10) \\ &= \frac{-\Gamma(k-\frac{1}{2})/2}{\Gamma(k-1)\Gamma(\frac{1}{2})} \int_0^\infty \hat{\eta}_P(\widehat{D}, k, w, x)w^{-\frac{3}{2}}\,dw, \end{aligned}$$

where $\hat{\eta}_P(\widehat{D}, k, w, x) \equiv \operatorname{Regtr} \tilde{\tau}\widehat{D}(\widehat{D}^2 + w)^{1/2-k}(x, x)w^{k-1/2}$, and $e^{-t\widehat{D}^2}(x, y)$ (respectively $(\widehat{D}^2 + w)^{1/2-k}(x, y)$) denotes the Schwartz kernel of the operator $e^{-t\widehat{D}^2}$ (respectively $(\widehat{D}^2 + w)^{1/2-k}$). Regtr denotes the regularized trace (see [20, (4.9.8)]) obtained by subtracting from

$\operatorname{tr}\tilde{\tau}\widehat{D}e^{-t\widehat{D}^2}(x, x)$ those terms in its small t asymptotic expansion which are not integrable with respect to the measure $t^{-1/2}dt$. Similarly, we obtain

$$\operatorname{Regtr}\tilde{\tau}\widehat{D}(\widehat{D}^2 + w)^{\frac{1}{2}-k}(x, x)w^{k-\frac{1}{2}}$$

by subtracting off those terms in its large w asymptotic expansion which are not integrable with respect to $w^{-3/2}dw$. It will also be convenient to define

$$\hat{\eta}_P(\widehat{D}, k, w) = \int_{\hat{e}(P)} \hat{\eta}_P(\widehat{D}, k, w, x)\,dx. \tag{1.11}$$

2. Algebraic preliminaries

We will have frequent need to refer to Clifford algebras; therefore, we shall recall here the basic relations and introduce appropriate notation.

Let V be a vector space with inner product $(\cdot, \cdot)$. For $X \in V$, denote by $C(X)$ the automorphism of $\Lambda^{\cdot}V^*$ given by

$$C(X) = \varepsilon(X) - \varepsilon^*(X). \tag{2.1}$$

If $Y \in V$, we have the relation

$$C(X)C(Y) + C(Y)C(X) = -2(X, Y), \tag{2.2}$$

and if $\{X_1, \cdots, X_i\}$ is a collection of distinct orthonormal vectors in V, then (see [9, Theorem 1.8 or 20, (4.4.2)])

$$\operatorname{tr} C(X_1)\cdots C(X_i) = 0. \tag{2.3}$$

We define

$$\widehat{C}(X) = \varepsilon(X) + \varepsilon^*(X), \tag{2.4}$$

and observe that

$$\{C(X), \widehat{C}(Y)\} = 0 \quad \text{for all } X, Y \in V, \tag{2.5}$$

where $\{A, B\}$ denotes $AB + BA$.

Let $\{X_1, \cdots, X_{2m}\}$ be an ordered orthonormal basis for $\mathfrak{p}$. In terms of Clifford multiplication, the involution $\tau_{\mathfrak{p}}$ can be expressed as

$$\tau_{\mathfrak{p}} = i^m C(X_1)\cdots C(X_{2m}) \quad \text{and} \quad \tilde{\tau} = -C(\lambda)\tau_{\mathfrak{p}}\tau_E, \tag{2.6}$$

where λ is a unit vector generating $\mathfrak{a}$. The following relations are immediate consequences of (2.2) and (2.5):

$$\begin{aligned} &\text{(i)} \quad \tau_{\mathfrak{p}}C(X_i) = -C(X_i)\tau_{\mathfrak{p}}, \\ &\text{(ii)} \quad \tau_{\mathfrak{p}}\widehat{C}(X_i) = \widehat{C}(X_i)\tau_{\mathfrak{p}}, \\ &\text{(iii)} \quad \tilde{\tau}C(X_i) = C(X_i)\tilde{\tau} \quad \text{for } X_i \perp \lambda, \\ &\text{(iv)} \quad \tilde{\tau}\widehat{C}(X_i) = -\widehat{C}(X_i)\tilde{\tau}. \end{aligned} \tag{2.7}$$

These relations depend only on the parity of the dimensions of $\mathfrak{p}$ and NM/K_M. Hence, they hold whenever $\tau_{\mathfrak{p}}$ (respectively $\tilde{\tau}$) is replaced by Clifford multiplication by the volume form of an even- (respectively odd-) dimensional space, containing X_i. This fact will be used frequently in the remainder of this paper.

3. Structure of maximal parabolic subgroups

In order to simplify the expression for $\hat{\eta}_P$, we will first recall, in this section, a decomposition of maximal parabolic subgroups which is finer than the Langlands decomposition (see [2]). We will then use this decomposition to separate $\widehat{D}$ into terms which admit a simple analysis.

Let $P = NAM$ be a maximal rational parabolic subgroup. Denote by U the center of N, and set $\mathfrak{v} = \mathfrak{n} \cap \mathfrak{u}^{\perp}$. Let $d_u = \dim \mathfrak{u}$ and $d_v = \dim \mathfrak{v}$. Then d_v is even. Let G_l denote the group of automorphisms of a nondegenerate, self-adjoint homogeneous cone X_l in $\mathfrak{u}$. G_l may be identified with a subgroup of P containing A. Set $G_l' = G_l/A$. Finally, we may decompose P as the semidirect product (modulo a finite subgroup)

$$P = G_h \cdot G_l \times N,$$

where, modulo a compact factor, G_h is the group of automorphisms of a Hermitian symmetric space X_h (which covers F_P). The group G_h commutes with G_l and U. Similarly, we may write $\mathfrak{m} \oplus \mathfrak{a}$ as the orthogonal direct sum

$$\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}_h \oplus \mathfrak{g}_l, \quad \text{with } \mathfrak{a} \subset \mathfrak{g}_l.$$

Let $\mathfrak{p}_h \oplus \mathfrak{k}_h = \mathfrak{g}_h$ and $\mathfrak{p}_l \oplus \mathfrak{k}_l = \mathfrak{g}_l$ be the Cartan decompositions of $\mathfrak{g}_h$ and $\mathfrak{g}_l$ respectively. The nondegeneracy of X_l implies that the dimension of $\mathfrak{p}_l$ equals the dimension of $\mathfrak{u}$. Set $\mathfrak{p}_l' = \mathfrak{p}_l \cap \mathfrak{m}$. Let $\{Z_1, \cdots, Z_{d_h}\}$ and $\{X_1, \cdots, X_{d_l}\}$ be orthonormal bases of $\mathfrak{p}_h$ and $\mathfrak{p}_l'$ respectively. Let ρ_h and ρ_1 denote the restriction of ρ_P to $\mathfrak{g}_h$ and $\mathfrak{g}_l$.

With this notation we may decompose $\widehat{D}$ as

$$\widehat{D} = D_h + D_l + D_n,$$

where

$$(3.1) \qquad \begin{aligned} D_h &= \sum R(Z_i) \otimes C(Z_i) \otimes I + I \otimes \widehat{C}(Z_i) \otimes \rho_h(Z_i), \\ D_l &= \sum R(X_i) \otimes C(X_i) \otimes I + I \otimes \widehat{C}(X_i) \otimes \rho_l(X_i). \end{aligned}$$

Observe that D_h and D_l are formally signature operators with coefficients in $\mathscr{E}^{\cdot}$.

Henceforth, we will frequently omit the tensors and the R—denoting the right regular representation—in such expressions. Because $\mathfrak{g}_h$ and $\mathfrak{g}_l$ commute, D_h and D_l anticommute. Hence

$$\widehat{D}^2 = D_h^2 + D_l^2 + D_n^2.$$

Moreover, by Kuga's Lemma (see [12]), we have

$$(3.2) \qquad D_h^2 = -\Omega_h + \rho_h(\Omega_h) \quad \text{and} \quad D_l^2 = -\Omega_l + \rho_l(\Omega_l),$$

where Ω_h and Ω_l denote the Casimir operators of $\mathfrak{g}_h$ and $\mathfrak{g}_l'$. It will also be useful to define the operators

$$(3.3) \qquad \tau_h = C(Z_1)\cdots C(Z_{d_h}) \quad \text{and} \quad \tau_l = C(X_1)\cdots C(X_{d_l}).$$

Similarly, we define τ_n to be Clifford multiplication by the volume element of $\mathfrak{n}$. We shall adopt the convention that this acts on $L^2(e(\widehat{P})) \otimes \Lambda^{\cdot}\mathfrak{p}_m^* \otimes (\Lambda^{\cdot}\mathfrak{n}^* \otimes E)$ as $I \otimes I \otimes \tau_n$. In particular, it commutes with Clifford multiplication by elements of $\mathfrak{p}_m$.

The following lemma and its corollary will be used to obtain commutation results in §4 from which we will deduce vanishing theorems.

(3.4) **Lemma.** *The representations of M obtained by restricting σ to $\Lambda^{d_u}\mathfrak{u}^*$ and $\Lambda^{d_v}\mathfrak{v}^*$ are trivial.*

Proof. Let $T \subset P$ be a maximal torus. Let Φ_v, Φ_u, and Φ_m denote the weights of T on $\mathfrak{v}$, $\mathfrak{u}$, and $\mathfrak{m}$ respectively. The highest weights of σ on $\Lambda^{d_u}\mathfrak{u}^*$ and $\Lambda^{d_v}\mathfrak{v}^*$ are given by the sum of the roots in $-\Phi_v$ and $-\Phi_u$ respectively. If $\alpha \in \Phi_m$ and $\beta \in \Phi_v$ (respectively Φ_u), then $s_\alpha\beta \in \Phi_v$ (respectively Φ_u), where s_α denotes the reflection through α. Moreover, $(\alpha, \beta + s_\alpha\beta) = 0$. Hence, α has zero inner product with the sum of the roots in Φ_v or Φ_u, and, therefore, $T \cap M$ acts trivially on $\Lambda^{dv}\mathfrak{v}^*$ and on $\Lambda^{du}\mathfrak{u}^*$. The lemma now follows.

(3.5) **Corollary.** *Let $X \in \mathfrak{p}_m$. Then $\{\sigma(X), \tau_n\} = 0$.*

Proof. On forms of fixed degree, the Hodge star operator $*_n$ is a constant multiple of τ_n. Hence, as $\sigma(X)$ preserves degree, it suffices to prove that $\{\sigma(X), *_n\} = 0$. Let $g = \exp(tX)$, and let v_1 and $v_2 \in \mathfrak{n}$. Then

$$(\sigma(g^{-1})v_1)\hat{}\,\sigma(g^{-1})(*_n v_2) = \sigma(g^{-1})(v_1, v_2)d\,\mathrm{vol}_n = v_1\hat{}\, *_n v_2,$$

by the preceding lemma. We also have

$$(\sigma(g)v_1)\hat{}\, *_n v_2 = (\sigma(g)v_1, v_2)d\,\mathrm{vol}_n = (v_1, \sigma(g)v_2)d\,\mathrm{vol}_n = v_1\hat{}\, *_n \sigma(g)v_2.$$

Applying $\sigma(g^{-1})$ to the left-hand side of this equality, we obtain:

$$v_1\hat{}\,\sigma(g^{-1}) *_n v_2 = v_1\hat{}\, *_n \sigma(g)v_2.$$

Differentiating this expression with respect to t and evaluating at $t = 0$ gives the desired equality.

4. Vanishing lemmas

Let $\Delta = \widehat{D}^2$, $\Delta^h = D_h^2$, $\Delta^1 = D_1^2$, and $\Delta^n = D_n^2$. In order to understand $\hat{\eta}_p$, we write it as

(4.1) $\hat{\eta}_P(\widehat{D}, k, w, x) = \operatorname{Regtr} \tilde{\tau}(D_h + D_l + D_n)(\Delta + w)^{1/2-k}(x, x)w^{k-1/2}$,

and examine separately the contributions from D_h, D_l, and D_n. Our first goal is to show that the traces of the D_h and D_l terms vanish. We first recall the following elementary linear algebra lemma.

(4.2) **Lemma.** *Let A and B be two anticommuting endomorphisms of a finite-dimensional vector space. Then* $\operatorname{tr} AB = 0$.

(4.3) **Lemma.** $\operatorname{tr} \tilde{\tau} D_h(\Delta + w)^{1/2-k}(x, x) = 0$.

Proof. Write $D_h = D_h^1 + D_h^0$, where $D_h^1 = \sum C(Z_i)Z_i$ and $D_h^0 = \sum \widehat{C}(Z_i)\rho_h(Z_i)$. By the Clifford commutation rules (see (2.2) and (2.7)), $[\tau_h, \tau_l\tau_n\tau_E] = 0$ and $\{\tau_h, D_h^1\} = 0$. Here we are using the fact that $\mathfrak{p}_h$ is even dimensional. From (3.2) and (3.3), it is evident that τ_h and τ_l commute with Δ and, therefore, with $(\Delta + w)^{1/2-k}(x, x)$. Hence,

$$\{\tau_h, \tau_l\tau_n\tau_E D_h^1(\Delta + w)^{1/2-k}(x, x)\} = 0,$$

and by (4.2),

$$\operatorname{tr} \tau_h\tau_l\tau_n\tau_E D_h^1(\Delta + w)^{1/2-k}(x, x) = 0.$$

(4.4) **Remark.** Let M be a zero order operator which commutes with $(\Delta + w)^{1/2-k}$. Then

$$\int M(x)(\Delta + w)^{1/2-k}(x, y)f(y)\,dy = \int (\Delta + w)^{1/2-k}(x, y)M(y)f(y)\,dy$$

for every square integrable f. Hence,

$$M(x)(\Delta + w)^{1/2-k}(x, y) = (\Delta + w)^{1/2-k}(x, y)M(y),$$

and we may deduce the pointwise relationship $[M(x), (\Delta + w)^{1/2-k}(x, x)] = 0$ from the operator relationship $[M, (\Delta + w)^{1/2-k}] = 0$. Similarly, $\{M, (\Delta + w)^{1/2-k}\} = 0$ implies $\{M(x), (\Delta + w)^{1/2-k}(x, x)\} = 0$.

We now consider the D_h^0 term. According to our conventions, $[\tau_n\tau_E, \widehat{C}(Z)] = 0$ for all vectors Z. We also (see (1.4) and (3.5)) have $\{\tau_n\tau_E, \rho_h(Z)\} = 0$. This implies that

$$\{\tau_n\tau_E, D_h^0\} = 0 \quad \text{and} \quad \{\tau_n\tau_E, \tau_h\tau_l D_h^0(\Delta + w)^{1/2-k}(x, x)\} = 0.$$

We may, therefore, conclude that $\operatorname{tr}\tilde{\tau}D_h(\Delta+w)^{1/2-k}(x,x)=0$.

(4.5) **Lemma.** *If $\mathfrak{p}_l'$ is even dimensional, then*

$$\operatorname{tr}\tilde{\tau}D_l(\Delta+w)^{1/2-k}(x,x)=0.$$

Proof. Decompose D_l as $D_l=D_l^1+D_l^0$, where $D_l^1=\sum C(X_i)X_i$ and $D_l^0=\sum\widehat{C}(X_i)\rho_l(X_i)$. The D_l^1 term vanishes as in the preceding lemma, anticommuting τ_l through the trace in place of τ_h. The vanishing of the D_l^0 term follows exactly as in (4.3).

(4.6) **Lemma.** *If $\mathfrak{p}_l'$ is odd dimensional, then*

$$\operatorname{tr}\tilde{\tau}D_l(\Delta+w)^{1/2-k}=0.$$

Proof. The D_l^0 term vanishes exactly as in Lemmas (4.3) and (4.5). The parity of the dimension is not used in those arguments. To eliminate the D_l^1 term, we must decompose the trace further. Recall [11, (4.3.4)] that

$$[D_n,\rho_h(Z)]=[D_n,\rho_l(X)]=0 \tag{4.7}$$

for any $Z\in\mathfrak{p}_h$ and $X\in\mathfrak{p}_l'$. If $\mathfrak{p}_l'$ is odd dimensional, then $\mathfrak{n}$ is even dimensional. By (1.4) and the Clifford algebra commutation relations, $\{D_n,\tau_n\tau_E\}=0$ (when n is even dimensional), and $[D_n,\tau_h\tau_l D_l^1]=0$. We further note that τ, D_l^1, and $(\Delta+w)^{1/2-k}$ all commute with Δ^n. This allows us to diagonalize first with respect to Δ^n before taking the trace. Hence,

$$\operatorname{tr}\tilde{\tau}D_l^1(\Delta+w)^{1/2-k}=\sum_\mu\operatorname{tr}_\mu\tilde{\tau}D_l^1(\Delta^h+\Delta^l+\mu+w)^{1/2-k}, \tag{4.8}$$

where the sum runs over the eigenvalues of Δ^n, and tr_μ denotes the trace over the μth eigenspace.

(4.9) **Claim.** $\operatorname{tr}_\mu\tilde{\tau}D_l^1(\Delta^h+\Delta^l+\mu+w)^{1/2-k}=0$ for $\mu\neq0$.

To see this, express the trace as

$$\begin{aligned}&\tfrac{1}{\mu}\operatorname{tr}_\mu D_n^2\tilde{\tau}D_l^1(\Delta^h+\Delta^l+\mu+w)^{1/2-k}\\&\quad=-\tfrac{1}{\mu}\operatorname{tr}_\mu D_n\tilde{\tau}D_l^1(\Delta^h+\Delta^l+\mu+w)^{1/2-k}D_n\\&\quad=-\tfrac{1}{\mu}\operatorname{tr}_\mu D_n^2\tilde{\tau}D_l^1(\Delta^h+\Delta^l+\mu+w)^{1/2-k},\end{aligned}$$

and the claim follows.

We are thus left to calculate $\operatorname{tr}_0\tilde{\tau}D_l^1(\Delta^h+\Delta^l+w)^{1/2-k}$. The operator $\Pi\equiv D_h^0+D_l^0$ commutes with $(\Delta^h+\Delta^l+w)^{1/2-k}$, τ_h, and $\tau_lD_l^1$. It anticommutes with $\tau_n \tau_E$; hence, we may diagonalize with respect to Π^2 before taking the trace, yielding:

$$(4.10)\qquad \operatorname{tr}_0 \tilde{\tau} D_l^1 (\Delta^h + \Delta^l + w)^{1/2-k} = \sum_\nu \operatorname{tr}_{0,\nu} \tilde{\tau} D_l^1 (\Delta^h + \Delta^l + w)^{1/2-k},$$

where the sum runs over the eigenvalues of Π^2 and $\operatorname{tr}_{0,\nu}$ denotes the trace over the corresponding eigenspace. Using the relation $\{\Pi, \tau_n \tau_E\} = 0$, we may argue as before (commuting Π through the trace) to obtain

$$(4.11)\qquad \operatorname{tr}_0 \tilde{\tau} D_l^1 (\Delta^h + \Delta^l + w)^{1/2-k} = \operatorname{tr}_{0,0} \tilde{\tau} D_l^1 (\Delta^h + \Delta^l + w)^{1/2-k}.$$

Using $\{(\delta_p + \rho_P)(\lambda),\ \tilde{\tau} D_l^1 (\Delta^h + \Delta^l + w)^{1/2-k}\} = 0$ for $\lambda \in \mathfrak{a}$, we see that the trace vanishes in the complement of the zero $(\delta_P + \rho_P)(\lambda)$ weight space. In [18, Proposition (9.6)], however, we show that the $\Pi = 0$ eigenspace contains no elements with zero $(\delta_P + \rho_P)(\lambda)$ weight. Hence, the traces in (4.9) vanish, concluding the proof of the lemma.

Combining the preceding lemmas we have the following.

(4.12) Proposition. $\operatorname{tr} \tilde{\tau} \widehat{D} (\Delta + w)^{1/2-k} = \operatorname{tr} \tilde{\tau} D_n (\Delta + w)^{1/2-k}$.

5. The main theorem

Proposition (4.12) allows us to restrict our analysis to an operator which is formally similar to those studied in [20]. In this section, we seek to restrict the types of spaces and parabolic subgroups on which $\hat{\eta}_P$ can be nonzero, so that we may reduce our considerations to cases where the techniques of [20] apply. Then we compute to obtain Theorem (5.9).

(5.1) **Lemma.** *If G is not absolutely simple, then*

$$\operatorname{tr} \tilde{\tau} D_n (\Delta + w)^{1/2-k} = 0.$$

Proof. Let P be a maximal rational parabolic subgroup of G. When G is not absolutely simple, $G = R_{k/\mathbf{Q}} G'$ and $P = R_{k/\mathbf{Q}} P'$, where G' is an absolutely simple k-group, P' is a maximal parabolic subgroup of G', and k is a totally real number field. Here $R_{k/\mathbf{Q}}$ is the restriction of scalars functor (see [5, §3]). The center of $\mathfrak{p}_l'$ is, therefore, nontrivial. Hence, the proof of [20, (4.9.4)] implies that $\operatorname{tr} \tilde{\tau} D_n (\Delta + w)^{1/2-k} = 0$.

If G is not absolutely simple, we may conclude from this lemma that $\hat{\eta}_P = 0$. Henceforth, we assume that G is absolutely simple. At this stage, iterating the procedure which allowed us to calculate the signature defect in [20], we could, in principle, calculate the above trace in terms of local information on $\hat{e}(P)$ plus a defect term. Motivated by the Zucker conjecture to attempt to express the defect in terms local only on the Baily–Borel boundary components, $\hat{e}(P)/G_l'$, we first seek further vanishing results. As the expression now stands, we would expect to obtain terms proportional to the volume of $\Gamma_l \backslash G_l'/K_l$. These terms are eliminated by the following lemma and by Proposition (5.4).

(5.2) **Lemma.** *If* $\operatorname{rank} G_l' > \operatorname{rank} K_l'$ *or if* $G_l' = \mathrm{Sl}(2, \mathbf{R})$, *then*

$$\operatorname{tr} \tilde{\tau} D_n e^{-t\Delta}(x, x) = 0.$$

Proof. Clearly this trace vanishes if the dimension d_l of X_l is odd. Hence, we shall assume that $d_l = 2r$. Write

$$i^{-r} \operatorname{tr} \tilde{\tau} D_n e^{-t\Delta}(x, x) = \operatorname{tr}^+ \tau_h \tau_n \tau_E D_n e^{-t\Delta}(x, x) - \operatorname{tr}^- \tau_h \tau_n \tau_E D_n e^{-t\Delta}(x, x),$$

where $\operatorname{tr}^\pm$ denotes the trace over $L^\pm$, and $L^\pm$ is defined to be the ± 1 eigenspace of $i^r \tau_l$ acting on $\Lambda^\cdot(\mathfrak{p}_h^* \otimes \mathfrak{p}_l'^*) \otimes \Lambda^\cdot \mathfrak{n}^* \otimes E$. Let $s^\pm$ denote the representation of $K_h K_l$ on $L^\pm$ obtained by acting as the coadjoint representation on $\Lambda^\cdot(\mathfrak{p}_h^* \oplus \mathfrak{p}_l'^*) \otimes \Lambda^\cdot \mathfrak{n}^*$ and by ρ on E. According to [6, proof of (1.2.5)], when $\operatorname{rank} K_l < \operatorname{rank} G_l'$, there exists an isomorphism φ from L^+ to L^- which intertwines s^+ and s^-. When $G_l' = \mathrm{Sl}(2, \mathbf{R})$, $K_l = \mathrm{SO}(2, \mathbf{R})$ acts trivially on $\Lambda^2 \mathfrak{p}_l'^* \oplus \Lambda^0 \mathfrak{p}_l'^*$, and complex conjugation intertwines the coadjoint representation restricted to the ± 1 eigenspaces of $i\tau_l$ on $\Lambda^1 \mathfrak{p}_l'^* \otimes \mathbf{C}$. This intertwining operator extends to an isomorphism ψ which intertwines s^+ and s^- as above.

Let $e^{-t\Delta^\pm}$ denote the restriction of $e^{-t\Delta}$ to the subspace $(L^2(G_h G_l') \otimes L^\pm)^{K_h K_l}$ of $(L^2(G_h G_l') \otimes L^\pm)$ invariant under $R \otimes s^\pm(K_h K_l)$. Following the construction in [6, 2.1–2.6], we see that $e^{-t\Delta^\pm}$ has kernel

$$h_t^\pm(x, y) = \int_{K_h K_l} \int_{K_h K_l} p_t(xa, yb) s^\pm(a) e^{-t\Lambda^\pm} s^\pm(b)^{-1} \, da \, db, \tag{5.3}$$

where p_t is the heat kernel for the Laplacian on functions on $G_h G_l'$, and $\Lambda^\pm = -2s^\pm(\Omega_{K_h K_l}) + \rho(\Omega_{G_h G_l'}) + \Delta^n$. Using $\psi s^+ = s^- \psi$, and choosing ψ so that it commutes with $\rho(\Omega_{G_h G_l'})$ and D_n, we have $\psi^{-1} h_t^+(x, y) \psi = h_t^-(x, y)$. We construct kernels for the restrictions of $e^{-t\Delta}$ (on $\hat{e}(P)$) to $L^\pm$ by averaging $h_t^\pm(x, y)$ over Γ_M. Because $h_t^+(x, y)$ is conjugate to $h_t^-(x, y)$ by an operator which commutes with D_n, the difference of the two traces vanishes pointwise.

(5.4) **Proposition.** *If* G_l' *is neither* $\mathrm{SO}(2p, 1)$ *nor trivial, then*

$$\operatorname{tr} \tilde{\tau} D_n (\Delta + w)^{1/2-k}(x, x) = 0.$$

Proof. The only nontrivial, equal rank G_l' which may occur when G is absolutely simple are $G_l' = \mathrm{SO}(2p, 1)$ and $G_l' = \mathrm{Sl}(2, \mathbf{R})$ (see [19, pp. 115–118]). The trace vanishes in the latter case by the preceding lemma. When $\operatorname{rank} K_l$ is less than $\operatorname{rank} G_l'$, the result is also an immediate consequence of (5.2), via the functional calculus.

Thus, unless $G_l' = \mathrm{SO}(2p, 1)$, we may conclude that η_P vanishes unless F_P is a maximal boundary component (i.e., of maximal dimension). The case $G_l' = \mathrm{SO}(2p, 1)$ only arises when $G = \mathrm{SO}(2p+1, 2)$ and G_h is compact. It is easy to show in this case, however, that when $\mathscr{E}$ is trivial, η_P still vanishes because $D_n \equiv 0$. It is interesting to note that there exists a Satake compactification (satisfying the hypotheses of Borel's extension of the Zucker conjecture—see [21, Appendix]) for which the boundary component associated to such a P is a maximal boundary component.

Henceforth we assume that either G_l' is trivial or $G_l' = \mathrm{SO}(2p, 1)$ and G_h is compact.

(5.5) **Theorem.**

$$\operatorname{Regtr} \tilde{\tau} D_n (\Delta/w + 1)^{1/2-k}(x, x) = \operatorname{tr} \tilde{\tau} D_n (\Delta/w + 1)^{1/2-k}(x, x),$$

and this trace is integrable over $\hat{e}(P)$.

Proof. Decompose $\Lambda^{\cdot} n^* \otimes E$ into eigenspaces of Δ^n: $\Lambda^{\cdot} n^* \otimes E = \bigoplus_\mu E_\mu$, where $\Delta^n \varphi = \mu\varphi$ for all $\varphi \in E_\mu$. E_μ is a G_h-module. Let $\mathscr{E}_\mu$ denote the associated flat vector bundle. For $\mu \neq 0$, we may write E_μ as a direct sum $E_\mu = E_\mu^+ \oplus E_\mu^-$, where $E_\mu^\pm$ are the $\pm\mu^{1/2}$ eigenspaces of $\tau_n \tau_E D_n$. Let $\mathscr{E}_\mu = \mathscr{E}_\mu^+ \oplus \mathscr{E}_\mu^-$ be the corresponding decomposition of $\mathscr{E}_\mu$. Define the involution τ_μ on E_μ by letting $E_\mu^\pm$ be the ± 1 eigenspaces of τ_μ. Let Δ^μ denote the restriction of Δ^h (or Δ^1 when $G_l' = \mathrm{SO}(2p, 1)$ and G_h is trivial) to forms with coefficients in E_μ. With this notation we have

$$\begin{aligned} &w^{k-1/2} \operatorname{tr} \tilde{\tau} D_n (\Delta + w)^{1/2-k}(x, x) \\ &\qquad = \sum_\mu \mu^{1/2} \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + w + \mu)^{1/2-k}(x, x) w^{k-1/2}. \end{aligned} \tag{5.6}$$

Standard calculations (see [9 or 20]) imply that

$$\lim_{w \to \infty} \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + w + \mu)^{1/2-k}(x, x)(\mu + w)^{k-1/2} < \infty.$$

Hence,

$$\lim_{w\to\infty} \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + w + \mu)^{1/2-k}(x, x) w^{k-1/2} < \infty,$$

and the regularized trace is the usual trace. The integrability of this trace is a direct consequence of the arguments in [20]. (This may also be seen from the following paragraph, replacing $\hat{e}(P)$ by a sequence of compact subsets $\{B_j\}_{j=1}^{\infty}$ satisfying $B_j \subset B_{j+1}$ and $\bigcup_{j=1}^{\infty} B_j = \hat{e}(P)$).

The evaluation of $\int_{\hat{e}(P)} w^b \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + w)^{-b}(x, x)\, dx$ was studied extensively in [20]. In particular, note that $\hat{e}(P)$ is either a Hermitian locally symmetric space or (in the case $G = \mathrm{SO}(2p+1, 2)$) an equal rank, real rank one locally symmetric space. As in [20, 2.2], we have

$$\begin{aligned}\int_{\hat{e}(P)} & w^b \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + w)^{-b}(x, x)\, dx \\ &= \operatorname{Index} D_\mu^+ + \int_{\hat{e}(P)} \int_0^w \frac{d}{ds} s^b \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + s)^{-b}(x, x)\, ds\, dx.\end{aligned}$$

We may interpret the last integral as the integral of a divergence, exactly as in [20, (2.2.4)]. This integral may be computed by approximating $(\Delta^\mu + s)^{-k}$ by the sum of two operators H^1 and H^0 (see [20, §4]). The estimates of [20, §5] imply that, in this case, the H^1 terms vanish, and there is no zeta function contribution to the integral. (In [20], the H^1 terms do not vanish because the s integral runs from 0 to ∞ rather than 0 to w.) We are left to consider the terms arising from H^0. In order to describe these, we first introduce some notation.

Let $\mathscr{C}(P)$ denote the collection of Γ_M conjugacy classes of maximal parabolic subgroups of G_h. Let $Q \in \mathscr{C}(P)$ have Langlands decomposition $N_Q M_Q A_Q$. Let $D^1_{Q,\mu}$ denote the signature operator on $\hat{e}(Q)$ with coefficients in $\Lambda^\cdot \mathfrak{n}_Q^* \otimes \mathscr{E}_\mu$, and let $D_{n_Q,\mu} = d_{n_Q,\mu} + d^*_{n_Q,\mu}$, where $d_{n_Q,\mu}$ denotes the Lie algebra exterior derivative of n_Q with coefficients in $\mathscr{E}_\mu$. Set $\widehat{D}_\mu = D^1_{Q,\mu} + D_{n_Q,\mu}$. Then

$$\begin{aligned}\int_{\hat{e}(P)} & \int_0^w \frac{d}{ds} s^b \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + s)^{-b}(x, x)\, dw\, dx \\ (5.7) \quad &= \sum_{Q\in\mathscr{C}(P)} C(b) \int_{\hat{e}(Q)} \int_0^w \operatorname{Regtr} \tilde{\tau}_\mu \widehat{D}_\mu (\widehat{D}_\mu^2 + s)^{1/2-b} s^{b-2}\, dw\, dx \\ &= \sum_{Q\in\mathscr{C}(P)} C(b) \int_0^w \hat{\eta}_Q(\widehat{D}_\mu, b, s) s^{-3/2}\, ds,\end{aligned}$$

where

$$C(b) = \frac{-\Gamma(b - 1/2)/2}{\Gamma(b - 1)\Gamma(1/2)}. \tag{5.8}$$

Here $\tilde{\tau}_\mu$ denotes the composition of τ_μ and Clifford multiplication by the volume element of $\hat{e}(Q)$.

Thus we may conclude that

$$\begin{aligned}\int_{\hat{e}(P)} & w^b \operatorname{tr} \tau_h \tau_\mu (\Delta^\mu + w)^{-b}(x, x)\, dx \\ &= \text{Index } D_\mu^+ - \sum_{Q \in \mathscr{C}(P)} C(b) \int_0^w \hat{\eta}_Q(\widehat{D}_\mu, b, s) s^{-3/2}\, ds.\end{aligned}$$

Substituting this equality and (5.6) into the definition of $\hat{\eta}_P(\widehat{D}, k, w)$, we obtain the following theorem.

(5.9) **Theorem.** *If G is absolutely simple and either* (i) *$\hat{e}(P)$ is a maximal boundary component for the Baily-Borel compactification of X_Γ, or* (ii) *$G_l' = \mathrm{SO}(2p, 1)$ and G_h is compact, then*

$$\begin{aligned}\hat{\eta}_P(\widehat{D}, k, w) &= \sum_\mu \mu^{1/2} w^{k-1/2} (w + \mu)^{-k+1/2} \\ &\times \left\{ \mathit{Index}\ D_\mu^+ - \sum_{Q \in \mathscr{C}(P)} C(k - \tfrac{1}{2}) \int_0^{w+\mu} \hat{\eta}_Q(\widehat{D}_\mu, k - \tfrac{1}{2}, s) s^{-\frac{3}{2}}\, ds \right\}.\end{aligned}$$

Otherwise, $\hat{\eta}_P(\widehat{D}, k, w) = 0$.

This theorem yields a recursive procedure for computing the index of the signature operators considered in §1.

6. Real rank one computations

In this section, we compute the signature (with coefficients in $\mathscr{E}$) of the equal rank, real rank 1 locally symmetric spaces, and compare our results to those of [6]. We compute our zeta function term and find that (up to factors of two) it equals the contribution of the unipotent term in [6]. The eta term should, therefore, equal the weighted unipotent term in [6]. A discrepancy appears, however, due to an error (explained below) in the computation in [17] (used in [6]) of the weighted unipotent term of the trace formula.

In the real rank one cases, we have the following simplifications:

(i) For each parabolic subgroup P, $\hat{e}(P)$ is a point,

(ii) $\widehat{D} = D_n$, and thus,

(iii) $\eta_P(0) = (-1/2/\Gamma(1/2)) \int_0^\infty \operatorname{tr} \tilde{\tau} D_n e^{-tD_n^2} t^{-1/2}\, dt$.

Performing the t integration in (iii), we obtain

$$\eta_P(0) = -\tfrac{1}{2} \operatorname{tr} \tilde{\tau} D_n (D_n^2)^{-1/2} = -\tfrac{1}{2} \operatorname{signature}(\tilde{\tau} D_n).$$

We observe that if $\mathscr{E}$ is trivial, and the unipotent radical N of P is abelian, then D_n vanishes. Hence, for $\mathscr{E}$ trivial, $\eta_P(0) = 0$, unless $G = \mathrm{SU}(2n, 1)$ or $\mathrm{Sp}(n, 1)$. Simple root considerations (δ_P does not occur as a weight in $\Lambda^{\cdot} \mathfrak{n}^*$) imply $\eta_P(0) = 0$ for $\mathrm{Sp}(n, 1)$ (and $\mathscr{E}$ trivial). We are thus left to compute the case where $G = \mathrm{SU}(2n, 1)$. First we need some notation.

Recall that N is endowed with the invariant metric given by one half the metric determined by θ_K and the Killing form. This induces a metric on U, $\Gamma \cap N \backslash N$, and $\Gamma \cap U \backslash U$. Let $e_\kappa \in \mathfrak{u}$ be the vector defined in [19, p. 97, (2.9)] and computed in [19, p. 99], which corresponds to an idempotent in the Jordan algebra associated to $\mathfrak{u}$. Let ω denote the Kähler form of $\mathfrak{v}$ with respect to this metric (and the natural complex structure on $\mathfrak{v}$), and let L denote exterior multiplication by ω. Here we follow the convention that the Kähler form on $\mathbf{C}$ is, in the usual coordinates, $dx \wedge dy$. Let $\mathrm{Sp}(\omega)$ denote the subgroup of $\mathrm{Gl}(\mathfrak{v})$ which preserves ω. Let $\mathfrak{t}$ denote the Lie algebra of a maximal real split torus in $\mathrm{Sp}(\omega)$.

From [19, p. 102, Lemma 3.2], $d_n(e_\kappa^*) = \|e_\kappa\|^{-2} \omega$. Thus,

$$\|e_\kappa\|^2 D_n = L\varepsilon^*(e_\kappa) + L^* \varepsilon(e_\kappa). \tag{6.1}$$

It is thus easy to show that if $\mathscr{E}$ is trivial and if X_Γ is endowed with the natural orientation induced by its complex structure,

$$-\tfrac{1}{2} \operatorname{signature}(\tilde{\tau} D_n) = \operatorname{signature}(*_v L), \tag{6.2}$$

where $*_v L$ is viewed as an endomorphism of $\Lambda^{n-1, n-1} \mathfrak{v}^*$. Moreover, $*_v L$ anticommutes with the action of $\mathfrak{t}$. Hence, arguing as in Lemma (4.6), signature $(*_v L) = \operatorname{signature}_0(*_v L)$, where $\operatorname{signature}_0(*_v L)$ denotes the signature of the restriction of $*_v L$ to $\Lambda_0^{n-1, n-1}$, the zero weight space of $\Lambda^{n-1, n-1} \mathfrak{v}^*$ with respect to $\mathfrak{t}$. Decompose $\Lambda_0^{n-1, n-1} \mathfrak{v}$ as $\bigoplus_{j=0}^{n-1} L^j P_j$, where P_j denotes $\operatorname{Ker} L^* \cap \Lambda_0^{n-1-j, n-1-j}$. According to the Hodge-Riemann bilinear relations (see [10, p. 123]), $*_v L$ has the sign $(-1)^j$ on each

$L^j P_j$. Thus,

$$\operatorname{signature}(*_v L) = \sum_{j=0}^{n-1}(-1)^{n-j-1}\dim P_{n-1-j}$$

$$= 1 + \sum_{j=0}^{n-2}(-1)^{n-j-1}\left[\binom{2n-1}{n-1-j} - \binom{2n-1}{n-2-j}\right]. \tag{6.3}$$

Hence, for $\mathscr{E}$ trivial,

$$\eta_P(0) = 1 + \sum_{j=0}^{n-2}(-1)^{n-j-1}\left[\binom{2n-1}{n-1-j} - \binom{2n-1}{n-2-j}\right]. \tag{6.4}$$

In particular, we readily see that this need not vanish for $n > 1$, contrary to [6, Theorem 7.6]. To account for this discrepancy, we observe that [6, Proposition 6.2] is slightly incorrect. The distribution T_1' (defined in [6]) is incorrectly identified in [17] with the distribution $T(\mathrm{id})$ of [1]. In fact, fix a maximal torus H_M in M (notation as in §1), and let Φ_M denote the set of positive roots of H_M in M (for an appropriate ordering). Then

$$T_1' = \frac{\lim_{a\to \mathrm{id}} T(a)}{\prod_{\alpha\in\Phi_M}(a^{\alpha/2} - a^{-\alpha/2})}.$$

Thus $T(\mathrm{id})$ differs from T_1' by a factor $\prod_{\alpha\in\Phi_M}(a^{\alpha/2} - a^{-\alpha/2})_{a=\mathrm{id}}$. We see that this factor vanishes for all $\mathbf{R}$-rank one groups except $\mathrm{Sl}_2(\mathbf{R})$ and $SU(2,1)$. The inclusion of this factor also occurs in [8]. Substituting $\lim_{a\to\mathrm{id}} T(a)/\prod_{\alpha\in\Phi_M}(a^{\alpha/2} - a^{-\alpha/2})$ for $\lim_{a\to\mathrm{id}} T(a)$, in the computation of T_1', we see that the expression $\{\sum_{w\in W_K}\det(w)\operatorname{sgnk}(w\tau)\}$ arising in [8, Theorem 6] and in [6, Proposition 6.5, Theorem 7.1, and Theorem 7.6] should be replaced by

$$\left\{\sum_{w\in W_M\backslash W_K}\det(w)\operatorname{sgnk}(w\tau)\, d_{w\tau}\right\}. \tag{6.5}$$

Here W_M is the Weyl group of M, and $d_{w\tau}$ denotes the degree of the representation of M with highest weight $\varepsilon w\tau|_{H_M} - \delta_M$, where $\varepsilon \in W_M$ is chosen (depending on $w\tau$) so that $\varepsilon w\tau$ is strictly dominant. All unexplained notation is as in [8].

In order to discuss the computation of the zeta function Z_P defined in [20, (5.2.14)] (specialized to the $\mathbf{R}$-rank one case), we need more notation. Let $d_{E^\pm}$ denote the dimension of $\mathscr{E}^\pm$. Let Γ_W denote Clifford multiplication by $\|4e_\kappa\|^{-1}\omega$. (This corresponds to the term ${}^L\Gamma_W$ defined in [20, (4.4.5)], which enters [20, (5.2.4)] as the maximal mass ([20, §4.4]) component of Γ_W.) Let κ_0 denote $\mathrm{Vol}(\Gamma \cap U \backslash U)^{-1}$.

From [20, (5.1.18)], the zeta function vanishes when $d_u \neq 1$. In particular, in the cases we are considering, it is nonzero only if $G = \mathrm{SU}(2n, 1)$. For $\mathrm{SU}(2n, 1)$, $d_v = 4n - 2$. Inserting these numbers into [20, 5.2.14], we obtain the following,

$$Z_P = (d_E + -d_E-) 4\zeta(2n)\pi^{-2n}\kappa_0^{2n} \mathrm{Vol}(\Gamma \cap N\backslash N)\, \mathrm{tr}_{\Lambda^\cdot v}\, \tau_V(\Gamma_W)^{2n-1},$$

where $\zeta(s)$ denotes the Riemann zeta function $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$.

We now evaluate $\mathrm{tr}_{\Lambda^\cdot v}\, \tau_v(\Gamma_W)^{2n-1} = \mathrm{tr}_{\Lambda^\cdot v}\, \tau_v(\|4e_\kappa\|^{-1}\omega)^{2n-1}$. ω^{2n-1} equals Clifford multiplication by $(2n-1)!\ \mathrm{vol}_v$. Hence

$$\begin{aligned}\mathrm{tr}_{\Lambda^\cdot v}\, \tau_V(\Gamma_W)^{2n-1} &= (2n-1)!\|4e_\kappa\|^{1-2n}\, \mathrm{tr}_{\Lambda^\cdot v}\, \tau_V^2 \\ &= -(2n-1)!\|e_\kappa\|^{1-2n}.\end{aligned}$$

We thus obtain

$$(6.6)\quad Z_P = -(d_{E^+} - d_{E^-})\zeta(2n)\pi^{-2n}\kappa_0^{2n} \mathrm{Vol}(\Gamma \cap N\backslash N)(2n-1)!4\|e_\kappa\|^{1-2n}.$$

We sec that, up to volume normalizations (and factors of 2), this agrees with the contribution of the unipotent term to the index computed in [6].

Combining these results and retaining our earlier notation, we obtain the following proposition.

(6.7) **Theorem.** *Let $X_\Gamma = \Gamma\backslash G/K$ be a real rank one, equal rank, locally symmetric space of finite volume. Let κ_Γ denote the number of cusps of X_Γ. Then the L^2-signature of X_Γ with coefficients in $\mathscr{E}$ is equal to*

$$2^n \int_{X_\Gamma} [\mathrm{Ch}\,\mathscr{E}^+ - \mathrm{Ch}\,\mathscr{E}^-] L\, dx + \tfrac{1}{2}\kappa_\Gamma\ \textit{signature}(\tilde{\tau} D_n)$$

for $G \neq \mathrm{SU}(2n, 1)$. For $G = \mathrm{SU}(2n, 1)$, the L^2-signature of X_Γ with coefficients in $\mathscr{E}$ is equal to

$$\begin{aligned}2^n \int_{X_\Gamma} L\, dx + \kappa_\Gamma\{(d_{E^+} - d_{E^-})\zeta(2n)\pi^{-2n}\kappa_0^{2n} \mathrm{Vol}(\Gamma \cap N\backslash N) \\ \times (2n-1)!4\|e_\kappa\|^{1-2n} + \tfrac{1}{2}\ \textit{signature}(\tilde{\tau} D_n)\}.\end{aligned}$$

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