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RIGIDITY OF COMPLETE EUCLIDEAN HYPERSURFACES

MARCOS DAJCZER & DETLEF GROMOLL

Dedicated to Manfredo do Carmo on his sixtieth birthday

Understanding the congruence classes of isometric immersions f of a connected riemannian manifold M^n into euclidean space $\mathbf{R}^{n+1}$ for $n \geq 3$ is the classical rigidity problem for hypersurfaces. Let ρ be the rank of the Gauss map $\gamma\colon M^n \to S^n$ of f. It is well known that f is rigid if $\rho \geq 3$, by the Beez-Killing Theorem, and highly deformable in the flat case $\rho \leq 1$. The situation for constant rank $\rho = 2$ is quite complex. Sbrana [12] and later É. Cartan [4] gave a detailed local analysis. The deformations are discrete, a one-parameter family, and infinite dimensional only in the ruled case, $n \geq 4$.

In this paper we deal with the rigidity problem for complete hypersurfaces. The compact case was solved in [11]. Here f is always rigid provided the totally geodesic points do not disconnect M. This is not true for complete hypersurfaces. However, we shall prove: If $n \geq 4$ and M has no euclidean factors $\mathbf{R}^{n-3}$ anywhere, then nondiscrete deformations of f are possible only along completely ruled subsets. This result extends Sacksteder's Theorem.

In §1 we discuss properties of a basic isometric invariant of immersions with constant rank ρ, the splitting tensor. This is used in §2 to give a global description of submanifolds with rank $\rho \leq 2$, in arbitrary codimension. Recall that f is a cylinder over a curve for $\rho \leq 1$, as a consequence of Hartman's Theorem [9]. We will also derive a related rigidity result for real Kähler submanifolds. Finally, §3 contains the proof of our main result and further discussions.

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1. The splitting tensor of a nullity foliation

Let M^n be a riemannian manifold with Levi-Civita connection ∇ and curvature tensor R, and let $f\colon M^n \to \mathbf{R}^{n+p}$ be an isometric immersion. The relative nullity Δ_x of f at the point $x \in M$ is the subspace of the tangent space T_xM defined by

$$\Delta_x = \{X \in T_x \mid \alpha(X, Y) = 0 \quad \text{for all } Y \in T_xM\},$$

where $\alpha\colon T_xM \oplus T_xM \to T_x^{\perp}M$ denotes the second fundamental form of f with values in the normal bundle. Throughout this section, we assume the index of relative nullity

$$\nu(x) = \dim \Delta_x = \nu$$

to be constant in an open subset U of M. It is well known that in this case the nullity distribution Δ is smooth and integrable, with leaves L_x totally geodesic in $\mathbf{R}^{n+p}$. We have an orthogonal splitting $TU = \Delta \oplus \Delta^{\perp}$ and write $X = X^v + X^h$ accordingly for any $X \in TU$. We also refer to X^v (X^h) as the vertical (horizontal) component of X.

The *splitting tensor* C of f assigns to each $T \in \Delta$ the endomorphism C_T of $\Delta^{\perp}$ given by

$$C_T X = -\overset{h}{\nabla}_X T.$$

Lemma 1.1. (i) *The distribution $\Delta^{\perp}$ is integrable iff C_T is selfadjoint for all $T \in \Delta$.*

(ii) *C vanishes identically if and only if each point in U has a product neighborhood $V^{n-\nu} \times L^{\nu}$ on which $f = f_1 \times \mathrm{id}$ splits isometrically.*

Proof. Part (i) follows immediately from the definition of C. For part (ii) observe that $C \equiv 0$ iff Δ is parallel in U and therefore also parallel along U in $\mathbf{R}^{n+p}$. q.e.d.

We now derive some basic identities involving the splitting tensor and the second fundamental form.

Lemma 1.2. *Whenever $T_1, T_2 \in \Delta$,*

$$\nabla_{T_1} C_{T_2} = C_{T_2} C_{T_1} + C_{\nabla_{T_1} T_2}.$$

So in particular $dC = [C, C]$.

Proof. We compute with $X \in \Delta^{\perp}$,

$$\begin{aligned}(\nabla_{T_1} C_{T_2})X &= \nabla_{T_1} C_{T_2} X - C_{T_2} \nabla_{T_1} X \\ &= -\nabla_{T_1} \overset{h}{\nabla}_X T_2 - C_{T_2} \nabla_{T_1} X.\end{aligned}$$

Since Δ is totally geodesic, $\overset{h}{\nabla}_{T_1}\overset{v}{\nabla}_X T_2 = 0$, and thus

$$(\nabla_{T_1} C_{T_2})X = -\overset{h}{\nabla}_{T_1}\nabla_X T_2 - C_{T_2}\nabla_{T_1}X. \tag{1.3}$$

Now $R(T_1, X)T_2 = \nabla_{T_1}\nabla_X T_2 - \nabla_X\nabla_{T_1}T_2 - \nabla_{[T_1,X]}T_2 = 0$ by the Gauss equation, and $\overset{h}{\nabla}_{[T_1,X]^v}T = 0$. Therefore,

$$\begin{aligned}-\overset{h}{\nabla}_{T_1}\nabla_X T_2 &= C_{\nabla_{T_1}T_2}X - \overset{h}{\nabla}_{\overset{h}{\nabla}_{T_1}X} T_2 + \overset{h}{\nabla}_{\overset{h}{\nabla}_X T_1} T_2 \\ &= C_{\nabla_{T_1}T_2}X + C_{T_2}\nabla_{T_1}X + C_{T_2}C_{T_1}X.\end{aligned} \tag{1.4}$$

From (1.3) and (1.4) we obtain 1.2.

Lemma 1.5. *For $X, Y \in \Delta^\perp$ and $T \in \Delta$, we have*

$$(\overset{h}{\nabla}_X C_T)Y - (\overset{h}{\nabla}_Y C_T)X = C_{\overset{v}{\nabla}_X T}Y - C_{\overset{v}{\nabla}_Y T}X.$$

Proof. We first compute

$$\begin{aligned}(\overset{h}{\nabla}_X C_T)Y &= \overset{h}{\nabla}_X C_T Y - C_T\overset{h}{\nabla}_X Y = -\overset{h}{\nabla}_X\overset{h}{\nabla}_Y T - C_T\overset{h}{\nabla}_X Y \\ &= -\overset{h}{\nabla}_X\nabla_Y T + \overset{h}{\nabla}_X\overset{v}{\nabla}_Y T + \overset{h}{\nabla}_{\overset{h}{\nabla}_X Y} T \\ &= -\overset{h}{\nabla}_X\nabla_Y T - C_{\overset{v}{\nabla}_Y T}X + \overset{h}{\nabla}_{\overset{h}{\nabla}_X Y} T.\end{aligned}$$

Thus,

$$\begin{aligned}(\overset{h}{\nabla}_X C_T)Y - (\overset{h}{\nabla}_Y C_T)X &= -R^h(X,Y)T - \overset{h}{\nabla}_{[X,Y]^v}T - C_{\overset{v}{\nabla}_Y T}X + C_{\overset{v}{\nabla}_X T}Y \\ &= -C_{\overset{v}{\nabla}_Y T}X + C_{\overset{v}{\nabla}_X T}Y,\end{aligned}$$

since $R(X,Y)T = 0$ by the Gauss equation and $\overset{h}{\nabla}_{[X,Y]^v}T = 0$. q.e.d.

Given a normal field ξ along f, the selfadjoint tensor $A_\xi\colon TM \to TM$ is defined by $\langle A_\xi X, Y\rangle = \langle \alpha(X,Y), \xi\rangle$. We denote by $\nabla^\perp$ the induced connection in the normal bundle along f.

Lemma 1.6. *If $T \in \Delta$ and $\xi \in T^\perp M$, then*

$$\nabla_T A_\xi = A_\xi C_T + A_{\nabla_T^\perp \xi} \quad \text{on } \Delta^\perp.$$

Proof. From Codazzi's equation,

$$(\nabla_T A_\xi)X - A_{\nabla_T^\perp\xi}X = (\nabla_X A_\xi)T - A_{\nabla_X^\perp\xi}T,$$

and $\nabla_X A_\xi T = 0$, $A_{\nabla_X^\perp\xi}T = 0$, we obtain

$$(\nabla_T A_\xi)X = -A_\xi\nabla_X T + A_{\nabla_T^\perp\xi}X = A_\xi C_T X + A_{\nabla_T^\perp\xi}X. \quad \text{q.e.d.}$$

The following "completeness" result for relative nullity foliations is of basic importance (cf. [3], [8]).

Proposition 1.7. *Let $\gamma\colon [0,b] \to M$ be a geodesic such that $\gamma[0,b)$ is contained in a leaf of Δ in U. Then $\nu(\gamma(b)) = \nu$ and $C_{\gamma'}$ extends smoothly to* $[0,b]$.

Lemma 1.8. *Assume the leaves of the nullity foliation Δ on U are complete. Then for any $x_0 \in U$ and $T_0 \in \Delta_{x_0}$, the only possible real eigenvalue of C_{T_0} is* 0. *Furthermore,* $\ker C_T$ *is parallel along the velocity field T of the line $x_0 + tT_0$.*

Proof. Suppose $C_0 = C_{T_0}$ has the nonzero real eigenvalues $\lambda_1, \cdots, \lambda_k$. Let $\tau^{-1} = \max|\lambda_i|$. By 1.2, the field C_T satisfies $\nabla_T C_T = C_T^2$, and since the operator $C_t = I - tC_0$ is invertible for $-\tau < t < \tau$, we have that

$$C_t = C_0(I - tC_0)^{-1}$$

is the unique solution with initial condition C_0 for $t = 0$. Now either $(\tau - t)^{-1}$ or $-(\tau + t)^{-1}$ is an eigenvalue of C_t, $-\tau < t < \tau$, which diverges as $t \to \tau$, or $t \to -\tau$. This is impossible since C_t is well defined for all t, by our completeness assumption. q.e.d.

From now on we restrict attention to the case where the Gauss map has constant rank 2.

Lemma 1.9. *If f has constant relative nullity $\nu = n - 2$ on U, and the leaves in U are complete, then the codimension of* $\ker C$ *in Δ satisfies* $\operatorname{codim}\ker C \leq 1$.

Proof. Otherwise, the image of C would contain a selfadjoint $C_T \neq 0$, for dimension reasons, contradicting 1.8.

Lemma 1.10. *Suppose $\nu = n - 2$ and* $\operatorname{codim}\ker C = 1$ *on U. Let T be a* (*local*) *unit vector field perpendicular to* $\ker C$. *If C_T is invertible, then* $\ker C$ *is a distribution on U, constant in $\mathbf{R}^{n+p}$. If in addition the leaves in U are complete, then $U = L^3 \times \mathbf{R}^{n-3}$ and f splits.*

Proof. It follows from 1.5 that for arbitrary $S \in \ker C$ and $X, Y \in \Delta^\perp$,

$$\langle \nabla_X S, T\rangle C_T Y = \langle \nabla_Y S, T\rangle C_T X.$$

Therefore,

$$\langle \langle \nabla_X S, T\rangle Y - \langle \nabla_Y S, T\rangle X, C_T^t Z\rangle = 0$$

for all $Z \in \Delta^\perp$. Since $\det C_T^t \neq 0$, we obtain $\overset{v}{\nabla}_X S \in \ker C$. But $\langle \nabla_X S, Y\rangle = -\langle C_S X, Y\rangle = 0$, so $\nabla_X S \in \ker C$. On the other hand, we have by 1.2 for any vertical R,

$$C_{\nabla_R S} = \nabla_R C_S - C_S C_R = 0,$$

and therefore $\nabla_R S \in \ker C$. This completes the argument.

2. Complete submanifolds of rank at most 2

Let N^n be a riemannian manifold, possibly with some boundary ∂N. An isometric immersion $f\colon N^n \to \mathbf{R}^{n+p}$ is called *ruled* if N admits a continuous codimension 1 foliation tangent along ∂N such that f maps each leaf (*ruling*) onto an open subset of an affine subspace of $\mathbf{R}^{n+p}$. We say that f is *completely ruled* if all rulings are complete. Observe that in this case, the leaves in each connected component of N (called a *ruled strip*) form an affine vector bundle over a curve with or without end points. We then say f is a *cylinder* if $N = L^1 \times \mathbf{R}^{n-1}$ and $f = f_1 \times \mathrm{id}$ splits. Now we state our first main result.

Proposition 2.1. *Let $f\colon M^n \to \mathbf{R}^{n+p}$, $n \geq 3$, be an isometric immersion of a complete riemannian manifold which does not contain an open set $L^3 \times \mathbf{R}^{n-3}$ with L^3 unbounded, and ρ the rank of the Gauss map. Suppose that $\rho \leq 2$ everywhere, and let M^* be the open subset of all points in M with $\rho = 2$. Then the following hold*:

(i) *M^* is a union of smoothly ruled strips.*

(ii) *If f is completely ruled on M^*, then it is completely ruled everywhere, and a cylinder on each component of the complement of the closure of M^*.*

As a consequence, if f is real analytic, then either $M = L^3 \times \mathbf{R}^{n-3}$ and $f = f_1 \times \mathrm{id}$ splits, or f is completely ruled. Before going into the proof of 2.1, we will discuss some facts about ruled immersions.

Let $f\colon N^n \to \mathbf{R}^{n+p}$ be smoothly ruled. Consider any unit speed curve $c\colon I \to N$, perpendicular to the rulings, and an orthonormal frame field $\dot{c} = T_0, T_1, \cdots, T_{n-1}, N_1, \cdots, N_p$ along c such that $T_1, \cdots, T_{n-1}$ are parallel in the bundle of rulings and $N_1, \cdots, N_p$ are parallel in the normal bundle of f, with respect to the induced connections. This frame field thus satisfies

$$(2.2) \qquad \begin{cases} \dot{T}_0 = & -\sum_i \omega_i T_i \quad + \sum_j \gamma_j N_j, \\ \dot{T}_i = \omega_i T_0 & \qquad\qquad\quad + \sum_j \beta_{ij} N_j, \\ \dot{N}_j = -\gamma_j T_0 & -\sum_i \beta_{ij} T_i, \end{cases}$$

for $i \leq i \leq n-1$ and $1 \leq j \leq p$. Here $\omega_i, \beta_{ij}, \gamma_j$ are smooth functions on I. Along c we have that $\omega = -\sum_i \omega_i T_i = \nabla_{T_0} T_0$ is the curvature vector of c in N, $\sum_j \beta_{ij} N_j = \beta(T_i) = \alpha(T_0, T_i)$, and $\sum_j \gamma_j N_j = \gamma = \alpha(T_0, T_0)$ is the mean curvature vector, since the second fundamental form α clearly satisfies $\alpha(T_i, T_j) = 0$ for $1 \leq i \leq n-1$.

We parametrize f near c by means of the normal exponential map of c in N, i.e. by the map $F\colon I \times \mathbf{R}^{n-1} \to \mathbf{R}^{n+p}$,

$$F(s,t) = c(s) + \sum_{i=1}^{n-1} t_i T_i(s), \tag{2.3}$$

restricted to a neighborhood of $I \times \{0\}$. Conversely, prescribe ω, β, γ arbitrarily on I, as smooth curves in $\mathbf{R}^{n-1}, \mathbf{R}^{n-1} \otimes \mathbf{R}^p, \mathbf{R}^p$, respectively. Then (2.2) has a solution frame field, unique up to a fixed orthogonal transformation. With $c(s) = \int_{s_0}^{s} T_0(\sigma)\, d\sigma$, equation (2.3) defines a parametrization F of a smooth ruled submanifold, uniquely up to congruence, whenever F is regular, so in particular near $I \times \{0\}$.

We now argue that F is regular as far as f is defined. In order to decide where F is singular, using (2.2) we compute

$$F_s = (1 + \langle \omega, T \rangle) T_0 + \beta(T), \qquad F_t = T \tag{2.4}$$

at a point (s,t), where $T = \sum_i t_i T_i$. Therefore, F has maximal rank iff $F_s \neq 0$, or

$$|F_s|^2 = (1 + \langle \omega, T \rangle)^2 + |\beta(T)|^2 > 0, \tag{2.5}$$

and F has precisely one singular point on each line in a direction T which is not perpendicular to ω and in the kernel of β, i.e. in the relative nullity of f along c. For each $s \in I$, the set of singular points of F in $\{s\} \times \mathbf{R}^{n-1}$ is empty or an affine hyperplane in the kernel of β.

Consider now any open neighborhood W of $I \times \{0\}$ in $I \times \mathbf{R}^{n-1}$ such that $W_s = W \cap \{s\} \times \mathbf{R}^{n-1}$ is star-shaped with respect to $s \times 0$ and F maps W_s into the ruling through $c(s)$ for all $s \in I$. Then the exponential map $F|W$ is injective in N by construction, and we claim it must have maximal rank. Let $t \in \mathbf{R}^{n-1}$ and $t \in W_s$ on some open interval $I_0 \subset I$. The field $T = F_t$ is parallel along c in the bundle of rulings, and $\tilde{c}(s) = c(s) + T(s)$ is the reparametrization $\tilde{c} = c_1 \circ \varphi$ of the unit speed trajectory $c_1\colon I_0 \to N$, orthogonal to the rulings, $c_1(s_0) = \tilde{c}(s_0)$ for some $s_0 \in I_0$, where φ is the C^1 arc length function of $\tilde{c}$ on I_0, measured from s_0. If T_1 is parallel in the bundle of rulings along c_1, $T_1(s_0) = -T(s_0)$, then $T_1 \circ \varphi = -T$ on I_0. Since $c = c_1 \circ \varphi + T_1 \circ \varphi$ is regular, we conclude $\varphi' \neq 0$. But this means $F_s \neq 0$ in (2.4), and F is regular. This proves our claim.

We will also need that the rank of the Gauss map of f is constant along a ruling if it is at least 2 somewhere. It suffices to consider the parametrization (2.3). Let α_T be the second fundamental form at $F(s,t) = c(s) + T(s)$, and U a parallel vector field tangent to the rulings. Then by (2.2) or (2.4),

$$\alpha_T(F_s, U) = (\langle \omega, U \rangle T_0 + \beta(U))^{\perp}. \tag{2.6}$$

Here the component, taken at $F(s,t)$, is simply obtained by subtracting the projection in direction F_s. With $L(U) = \langle \omega, U \rangle T_0 + \beta(U)$, F_s is in the image of L iff $\operatorname{rk} L = \operatorname{rk} \beta + 1$. Therefore,

$$\operatorname{rk} \alpha_T(F_s, \ ) = \operatorname{rk} \beta$$

is constant on the ruling. Now $\ker \beta$ is the relative nullity of f exactly where β is not identically 0, i.e. the rank of the Gauss map of f is at least 2.

In this context, we also conclude from (2.5) and (2.6) that for any $T \in \mathbf{R}^{n-1}$ with $\beta(T) \neq 0$,

$$\alpha_{tT}(F_s/|F_s|, U) \to 0 \quad \text{as } t \to \infty. \tag{2.7}$$

We will make use of two particular applications of the preceding discussion.

Lemma 2.8. *Let $f\colon N^n \to \mathbf{R}^{n+p}$ be smoothly ruled such that $\rho \equiv k \geq 0$ everywhere for the rank ρ of the Gauss map of f. Suppose the leaves of the relative nullity foliation Δ are complete. Then every point in N has an open neighborhood W such that $f|W$ extends uniquely to a smoothly ruled strip. Furthermore, for this extension $\rho \equiv k$. If N is simply connected, then f extends to a ruled strip globally.*

Remark 2.9. Let $f\colon N^n \to \mathbf{R}^{n+p}$ be a simply connected ruled strip. Then any other ruled immersion $\tilde{f}\colon N^n \to \mathbf{R}^{n+p}$ with the same rulings in N is given by (2.2) and (2.3) (after fixing an orthogonal trajectory to the rulings in N), where $\tilde{\omega} = \omega$, $\tilde{\beta} = Q\beta$, and $\tilde{\gamma}$ is arbitrary. Here Q is any smooth curve in the orthogonal group $\mathrm{O}(n-1)$ of $\mathbf{R}^{n-1}$, parametrized on I. Clearly, f and $\tilde{f}$ are congruent iff

$$\tilde{\beta} = \beta \quad \text{and} \quad \tilde{\gamma} = \gamma,$$

up to a constant orthogonal transformation. This is an immediate consequence of (2.4) and (2.5).

Proof of 2.1. (i) According to the degeneracy of the splitting tensor C of f, we have a disjoint decomposition $M^* = M_0 \cup M_1 \cup M_2$ such that M_0 is the (in M^*) closed set of points where $C \equiv 0$, and M_2 is the open set of points where rank $C_T = 2$ (cf. 1.9 and 1.10). By 1.8, these three sets are saturated, i.e. they are unions of (complete) leaves of Δ.

Let $V = \operatorname{int} M_0 \cup M_2$. It follows from 1.1 that any connected component of $\operatorname{int} M_0$ is a product $L^2 \times \mathbf{R}^{n-2}$ on which f splits. By 1.10, any component of M_2 is a product $L^3 \times \mathbf{R}^{n-3}$ where f splits. We conclude that $M_2 = \varnothing$, M_1 is open, and $\operatorname{int} M_0 = \varnothing$.

Now we claim that the bundle $\Delta \oplus \ker C_T$ is smooth and involutive on M_1, with leaves totally geodesic in $\mathbf{R}^{n+p}$, i.e. M_1 is smoothly ruled. To see this, let $\ker C_T$ be spanned locally by a unit vector field X. Since $C_T X = 0$, we have $\overset{h}{\nabla}_X T = 0$. Using 1.8, we obtain $\nabla_T X = 0$. Thus $[X, T]$ is vertical and $\Delta \oplus \ker C_T$ involutive. By 1.6, for any normal field ξ parallel along the leaves of the nullity foliation,

$$\nabla_T A_\xi = A_\xi C_T,$$

so

$$A_\xi C_T = C_T^t A_\xi. \tag{2.10}$$

The last relation yields $C_T^t A_\xi X = 0$. On the other hand, both eigenvalues of C_T are 0 by 1.8, so C_T and thus C_T^t are nilpotent. Therefore,

$$\ker C_T^t = \operatorname{im} C_T^t. \tag{2.11}$$

We conclude from (2.10) and (2.11) that

$$\langle A_\xi X, X\rangle = 0 \tag{2.12}$$

for any $\xi \in T^\perp M$, i.e., the leaves of $\Delta \oplus \ker C_T$ are totally geodesic in $\mathbf{R}^{n+p}$.

Our next step is to show that the rulings in M_1 are complete. Consider orthonormal basis fields X, Y of $\Delta^\perp$. Since C_T is nilpotent, we have $C_T Y = \mu X$, where μ is a smooth function. On the other hand, by 1.5,

$$(\nabla_X C_T)Y = (\nabla_Y C_T)X,$$

or equivalently,

$$X\mu = \langle \nabla_Y Y, X\rangle \mu. \tag{2.13}$$

The leaves of the relative nullity foliation are complete, and thus parallel hyperplanes in each ruling. Therefore, any integral curve of X is a line segment in $\mathbf{R}^{n+p}$. It suffices to show: If $\gamma\colon [0, b] \to M$ is the segment in $\mathbf{R}^{n+p}$ whose restriction to $[0, b)$ is an integral curve of X, then $\gamma(b) \in M_1$. It follows from 2.8 that $\gamma(b) \in M^*$ and the linear differential equation (2.13) extends smoothly to the point $\gamma(b)$. Now $\mu \neq 0$ on $[0, b]$, and $\gamma(b) \in M_1$, which is sufficient since M_1 is open.

The closure $\overline{N}$ in M of a connected component N of M_1 is a smooth submanifold, with possibly nonempty boundary, on which f is a ruled strip. Let $x \in \overline{N}$ and $x_j \in N$ be a sequence, $x_j \to x$. Now the rulings L_j through x_j must converge to a complete totally geodesic euclidean space $L \cong \mathbf{R}^{n-1}$ through x in $\overline{N}$. Otherwise, we would find subsequences L_j' and L_j'' converging to limits L' and L'' in $\overline{N}$ which intersect transversally at x. But then almost all L_j' and L_j'' would intersect transversally near x, which is impossible for leaves of a foliation. It follows that $L \subset \partial\overline{N}$, and $\overline{N}$ is a continuous affine vector bundle over a connected 1-dimensional manifold with or without boundary, and the last claim is immediate.

It remains to show $\overline{N} \cap M^*$ is a smoothly ruled strip. If $x \in L \cap M^*$ we will conclude that $L \subset M^*$. The field X spanning $\ker C_T$ near x in N extends smoothly to a field $\overline{X}$ in a neighborhood of x in M. To see this take a normal field ξ near x so that A_ξ is invertible on $\Delta^\perp$. By (2.12) we may choose $\overline{X}$ to be the unique smooth isotropic unit vector field of the nondegenerate bilinear form

$$(Y, Z) \to \langle A_\xi Y, Z \rangle$$

on $\Delta^\perp$ which extends X. Now $\overline{N} \cap M_1$ is smoothly ruled, and $L \subset M_1$ by 2.8. Notice also that if N_1, N_2 are two such completely ruled strips, closed in M^*, $N_1 \cap N_2 \neq \varnothing$, then $N_1 \cup N_2$ is again a smoothly ruled strip.

(ii) First consider the subset M^{**} of all points in M with $\rho = 1$. We claim all leaves of the relative nullity foliation in the interior of M^{**} are complete. Otherwise, there is a geodesic $\gamma\colon [0, b] \to M$ such that $\gamma[0, b)$ is contained in a leaf, but $\gamma(b)$ is not. Since $\rho = 1$ at $\gamma(b)$, this point lies in the closure of M^*, which is completely ruled by assumption. But the relative nullity space at $\gamma(b)$ is contained in the limit ruling transversal to $\dot{\gamma}(b)$. This is a contradiction. By 1.1 and 1.8, f is now a cylinder on each connected component of $\operatorname{int} M^{**}$. The remaining arguments are straightforward. q.e.d.

Although ruled strips have been completely described earlier in this section, it is sometimes useful to have more explicit examples:

(i) Suppose the curve c has a Frenet frame $\dot{c} = e_1, \cdots, e_{n+p}$. Then

$$F(s, t) = c(s) + \sum_{j=1}^{n-1} t_j e_{1+j+p} \tag{2.14}$$

parametrizes a smoothly ruled strip, $t \in \mathbf{R}^{n-1}$.

(ii) If c is a straight line in $\mathbf{R}^{n+1}$ and ξ is a nowhere parallel unit normal field, then the orthogonal complements of $\dot{c}$ and ξ are the rulings of a complete hypersurface with $\rho \equiv 2$.

(iii) We give an example of how a ruled strip can be "attached" to a reducible immersion of a nonruled $L^2 \times \mathbf{R}^{n-2}$ as a complete hypersurface in $\mathbf{R}^{n+1}$. In the situation of (i), let $e_1, \cdots, e_{n+1}$ be the Frenet frame with curvatures $\tau_1, \cdots, \tau_n$. Assume $\tau_1, \tau_2 \neq 0$ everywhere, and $\tau_k \equiv 0$ exactly on some interval $[a, b]$ for $3 \leq k \leq n$. Then F in (2.14) parametrizes a completely ruled hypersurface which splits as a product $N^2 \times \mathbf{R}^{n-2}$ over $[a, b]$. Clearly, N^2 can be replaced smoothly by a nonruled surface L^2 over (a, b).

We conclude this section with a result on complete real Kähler submanifolds of euclidean spaces.

Theorem 2.15. *Let $f\colon M^{2m} \to \mathbf{R}^{2m+p}$ be an isometric immersion of a complete Kähler manifold, with $\rho = 2$ on an open dense connected subset of M. Then $M^{2m} = L^2 \times \mathbf{R}^{2m-2}$ and $f = f_1 \times \mathrm{id}$ splits.*

Remark 2.16. If M^{2m} is not everywhere flat, the splitting $M^{2m} = L^2 \times \mathbf{R}^{2m-2} \cong L^2 \times \mathbf{C}^{m-1}$ is Kähler. Moreover, if f is real analytic, the conclusion of the theorem holds under the assumption $\rho \le 2$ everywhere.

Special cases of 2.15 were obtained in [1], [2], and [10] (cf. also [6] for the classification of (noncomplete) real Kähler hypersurfaces).

Proof. We claim $C \equiv 0$ in the open dense and saturated subset M^* where $\rho = 2$. To see this, let $x \in M^*$. First consider the case where M is not flat at x. It follows from the relation $R \circ J = J \circ R$ and the Gauss equation that the relative nullity distribution Δ is invariant under the Kähler structure J of M, in a neighborhood of x. Then, we have $C_{JS} = JC_S$ for all vertical S. It is an immediate consequence of the last relation and 1.9 that $C = 0$ at x.

Now let $x \in M^*$ be a flat point of M. By the Gauss equation, for any orthonormal basis $\xi_1, \cdots, \xi_p$ of $T_x^\perp M$,

$$\sum_{i=1}^{p} \det A_{\xi_i}|\Delta_X^\perp = 0.$$

Then we can choose a particular basis such that all the summands are zero, or equivalently,

$$\operatorname{rk} A_{\xi_i} \le 1 \quad \text{for } 1 \le i \le p.$$

Simply observe that if $\det A_{\xi_i}|\Delta_x^\perp > 0$ and $\det A_{\xi_j}|\Delta_x^\perp < 0$, then there exists a unit vector $\xi \in \operatorname{span}\{\xi_i, \xi_j\}$ such that $\det A_\xi|\Delta_x^\perp = 0$. Replacing ξ_i, ξ_j by $\xi, \xi^\perp$ where $\xi^\perp \in \operatorname{span}\{\xi_i, \xi_j\}$ is a unit vector orthogonal to ξ, we continue this process and obtain the desired basis. Now if $Z \in \ker A_{\xi_i}|\Delta_x^\perp$, we conclude $C_T Z \in \ker A_{\xi_i}|\Delta_x^\perp$ using the relation $A_{\xi_i} C_T = C_T^t A_{\xi_i}$. It follows then from 1.8 that $C_T Z = 0$ if $\operatorname{rk} A_{\xi_i} = 1$. Since all $A_{\xi_i}|\Delta_x^\perp$ cannot have a common nontrivial kernel, we have again $C = 0$ at x.

Since $C \equiv 0$, as a consequence of 1.1, f is the identity on a euclidean factor $\mathbf{R}^{2m-2}$, first in the open dense and connected set M^*, and then in M.

As to Remark 2.16, if M is not flat at some x, then $J\Delta_x = \Delta_x$ by the above, and the euclidean factor $\mathbf{R}^{2m-2}$ is everywhere J-invariant, since J is parallel. For the real analytic case, observe that the subset M^* is open and dense if nonempty, although not necessarily connected. We omit the straightforward argument. The case $\rho \leq 1$ is similar.

3. Rigidity of hypersurfaces

We begin this section with a discussion of isometric deformations of ruled hypersurfaces.

Proposition 3.1. *Let $f\colon N^n \to \mathbf{R}^{n+1}$, $n \geq 3$, be a ruled strip with $\rho = 2$ everywhere. Suppose N is simply connected and does not contain an open subset $L^2 \times \mathbf{R}^{n-2}$. Then all isometric immersions of N into $\mathbf{R}^{n+1}$ are smoothly ruled, with the same rulings in N, and in one-to-one correspondence with the differentiable functions on an open interval.*

Proof. We refer to notation and some facts in the proof of 2.1. There are smooth orthonormal horizontal fields X, Y such that Y is orthogonal to the rulings, globally defined since N is simply connected. Therefore,

$$\langle AX, X\rangle = 0 \quad \text{and} \quad C_S X = 0$$

for all vertical S. Let $\tilde{f}\colon N \to \mathbf{R}^{n+1}$ be another isometric immersion. Since $C \neq 0$ on a dense subset, it follows from the proof of 2.1 that $\tilde{f}$ is also ruled, with the same rulings in N. In particular, $\langle \tilde{A}X, X\rangle = 0$. Thus by the Gauss equation,

$$\tilde{A} = A + \begin{pmatrix} 0 & 0 \\ 0 & \phi \end{pmatrix}$$

on $\Delta^{\perp}$ relative to the basis X, Y. Compare also our Remark 2.9 in this context. The differentiable function ϕ on N is constant along leaves of the nullity foliation, and the Codazzi equation for f is equivalent to the intrinsic linear differential equation

$$X\phi = \langle \nabla_Y Y, X\rangle \phi. \tag{3.2}$$

Now the claim follows by choosing initial conditions along a fixed maximal orthogonal trajectory to the rulings. q.e.d.

The last result is clearly false for $N = L^2 \times \mathbf{R}^{n-2}$. Consider for example the helicoid L^2 in $\mathbf{R}^3$ whose associated family of minimal surfaces is not ruled.

Corollary 3.3. *Let $f\colon N^n \to \mathbf{R}^{n+1}$, $n \geq 3$, be completely ruled and suppose N does not contain an open subset $L^2 \times \mathbf{R}^{n-2}$. Then any other isometric immersion of N into $\mathbf{R}^{n+1}$ is completely ruled, with the same rulings in N.*

Proof. Observe that $\rho \leq 2$. On the open set N_2 of points with $\rho = 2$ the leaves of the relative nullity foliation are contained in the rulings and thus complete. By the proof of 2.1, we conclude that the integrability tensor C_T has rank 1 on an open dense saturated subset N_2' of N_2. Now each connected component of N_2' satisfies the assumptions of 3.1, and therefore any isometric immersion $f: N \to \mathbf{R}^{n+1}$ is smoothly completely ruled on N_2'. Our claim then follows by continuity.

We now state our main result.

Theorem 3.4. *Let $f: M^n \to \mathbf{R}^{n+1}$, $n \geq 3$, be an isometric immersion of a complete riemannian manifold which does not contain an open set $L^3 \times \mathbf{R}^{n-3}$ with L^3 unbounded. Then f admits (nondiscrete) isometric deformations only along ruled strips. Furthermore, if f is nowhere completely ruled and the set of totally geodesic points does not disconnect M, then f is rigid.*

Remark 3.5. By 3.3, all isometric deformations of f preserve rulings. If M is simply connected, then any ruled strip gives rise to global isometric deformations, according to 3.1. Recall that a closed ruled strip is the closure of smoothly ruled strips. Notice also that our proof shows the assumption that L^3 is unbounded can be weakened to that L^3 is not foliated by complete totally geodesic lines. For the existence of such (deformable) L^3 compare [5, in particular pp. 8–9]. As far as discrete deformations are concerned, they sometimes do exist. We will analyze this question further at the end of the section.

Proof of 3.4. Let $\tilde{f}: M^n \to \mathbf{R}^{n+1}$ be any other isometric immersion. For $k \geq 0$, we define the open subsets $U_k, \tilde{U}_k$ of M where the ranks $\rho, \tilde{\rho}$ of $A, \tilde{A}$ are $\geq k$. By the Gauss equation, $\tilde{U}_k = U_k$ whenever $k \geq 2$, and by the classical Beez-Killing rigidity theorem, $\tilde{A} \equiv \pm A$ on each connected component of $\tilde{U}_3 = U_3$.

We consider first the open set $W = U_2 - \overline{U}_3$. Through any point $p \in W$ we have $\Delta_p = \tilde{\Delta}_p$ for the leaves of the relative nullity foliations, again by the Gauss equation. Let $\gamma: [0, a] \to M$ be a geodesic with $\gamma(0) = p$, $\gamma[0, a) \subset \Delta_p$, and $\gamma(a) \notin W$. Using 1.7, $\gamma(a) \in \overline{U}_3$ and thus $\tilde{A} = \pm A$ at $\gamma(a)$. Now by 1.6,

$$\nabla_{\dot{\gamma}} A = C_{\dot{\gamma}}^t A \tag{3.6}$$

on $[0, a)$, and (3.6) extends smoothly to $[0, a]$ according to 1.7. Therefore, $\tilde{A}(p) = \pm A(p)$. Consider the open saturated subset

$$V = \{q \in W | \tilde{A}(q) \neq \pm A(q)\}. \tag{3.7}$$

By the above, all leaves in V are complete. Since V does not contain an open set $L^3 \times \mathbf{R}^{n-3}$, our proof of 2.1 shows that f and $\tilde{f}$ are smoothly ruled on V. We will argue next that these rulings must be complete in V.

Suppose there is an incomplete ruling L in V. Then we find a geodesic $\delta\colon [0,b] \to M$ such that $\delta[0,b) \subset L$, $\delta(b) \notin V$, and $\dot{\delta} \in \Delta^{\perp}$. Now we apply 2.8 to conclude that $\delta(b) \in W$ and thus $\tilde{A} = A$ at $\delta(b)$, after possibly changing the local orientation of $\tilde{f}$. Moreover, the differential equation (3.6), with $X = \dot{\delta}$, extends smoothly to $[0,b]$. Since ϕ in the proof of 3.1 vanishes at $\delta(b)$, we have $\phi \equiv 0$, so $\tilde{A} = A$ at $\delta(0)$, and this is a contradiction. Now f and $\tilde{f}$ are smoothly completely ruled on V. The closure of a connected component V_ι of V is a ruled strip. We record

$$\tilde{A} = \pm A \quad \text{on } U_2 - \overline{V}. \tag{3.8}$$

In our next step we deal with the open subset $W' = U_1 \cap \tilde{U}_1 - \overline{U}_2$. We first claim $\Delta = \tilde{\Delta}$ on W'. Otherwise, consider the open set $V^* = \{q \in W' | \Delta_q = \tilde{\Delta}_q\}$ and the smooth $(n-2)$-dimensional foliation $q \to \Gamma_q = \Delta_q \cap \tilde{\Delta}_q$. The leaves are in fact complete affine subspaces. To see this, let $\varepsilon\colon [0,c] \to M$ be a geodesic, $\varepsilon(0) = q \in V^*$, $\varepsilon[0,c) \subset \Gamma_q$, $\varepsilon(c) \notin W'$. We conclude $\varepsilon(c) \in \overline{U}_2$ and $\tilde{A} = \pm A$ at $\varepsilon(c)$. Otherwise, $\varepsilon(c) \in \partial V_{\iota_0}$ for some ι_0, and $\rho = 1$ at $\varepsilon(c)$ by 1.7. This is a contradiction since $L_{\varepsilon(c)} = \Delta_{\varepsilon(c)}$ and ε is transversal to $L_{\varepsilon(c)}$. Now, in particular, $\tilde{A} = \pm A$ have the same kernel at $\varepsilon(c)$, and then at $\varepsilon(0) = q$, contradicting $q \in V^*$. The complete leaves of Γ must be parallel both in the leaves of Δ and $\tilde{\Delta}$. Therefore, they are parallel in M, and then in $\mathbf{R}^{n+1}$, along V^*. This means V^* contains a product $L^2 \times \mathbf{R}^{n-2}$, which we had excluded. Hence our claim is proved. Now it follows that $\tilde{A} = \pm A$ on W'. The argument is analogous to the one applied to W, using also the above transversality. In particular, we have

$$\tilde{A} = \pm A \quad \text{on } U_1 \cap \tilde{U}_1 - \overline{V}. \tag{3.9}$$

Finally, the open set $W'' = U_1 - \overline{\tilde{U}_1}$ must be empty, and the same applies to $\tilde{U}_1 - \overline{U}_1$. Otherwise, there exist a point $p \in W''$ and a geodesic $\eta\colon [0,d] \to M$ such that $\eta(0) = p$, $\eta[0,d] \subset \Delta_p$, but $\eta(d) \notin W''$. Here we use that the relative nullity foliation Δ cannot be complete on an open subset of W'' by our assumption. According to 1.7, $\rho = 1$ and $\tilde{\rho} = 0$ at $\eta(d)$. Again we apply the transversality argument to obtain first $\eta(d) \notin \overline{V}$. If $\eta(d) \notin U_1$, then $\eta(d) \in \overline{U}_2 - \overline{V}$. If $\eta(d) \in U_1$, then $\eta(d) \in U_1 \cap \overline{\tilde{U}}_1 - \overline{V} \subset \overline{U_1 \cap \tilde{U}_1 - \overline{V}}$. Now (3.8) or (3.9) implies the contradiction $\rho = \tilde{\rho}$ at $\eta(d)$. We have therefore shown that $U_1 \cap \tilde{U}_1$ is dense both in U_1 and $\tilde{U}_1$, and this together with (3.9) yields $\tilde{A} = \pm A$ on $U_1 - \overline{V} = \tilde{U}_1 - \overline{V}$, and thus on $M - \overline{V}$.

If V is empty and the set of totally geodesic points $M - U_1 = M - \tilde{U}_1$ does not disconnect M, then $\tilde{A} \equiv A$ or $\tilde{A} \equiv -A$ on M. This completes our proof.

Corollary 3.10. *Let $f\colon M^n \to \mathbf{R}^{n+1}$ be a complete irreducible real analytic hypersurface, $n \geq 4$. Then unless f is completely ruled, f is rigid in the category of analytic isometric immersions, and f is also rigid in the C^∞-category if the set of totally geodesic points does not disconnect M.*

Theorem 3.11. *Let $f\colon M^n \to \mathbf{R}^{n+1}$, $n \geq 3$, be an isometric immersion of a complete riemannian manifold which does not contain an open set $L^2 \times \mathbf{R}^{n-2}$. Suppose the scalar curvature s of M satisfies either $s > 0$ or $s \leq \varepsilon < 0$ everywhere. Then f is rigid.*

Proof. We argue first that f is nowhere completely ruled. Otherwise, on a ruled strip, the scalar curvature $s = o(t^4)$ approaches zero (from below) along any line in the rulings transversal to the nullity Δ, according to (2.7), which we had excluded. It is interesting to observe that s is constant along leaves of the nullity foliation, which follows also from (2.7) or (3.12), since the splitting tensor is nilpotent.

Next, suppose f has constant relative nullity $n-2$ on some open set U. Along any unit speed line $x_0 + tT_0$ in the leaf of the relative nullity foliation, $A(I - tC_0)$ is parallel, $C_0 = C_{T_0}$. This follows immediately from 1.6 and 1.8. In particular, we have for the scalar curvature

$$s \cdot \det(I - tC_0) = s_0. \tag{3.12}$$

Thus if the line is complete, $s \to 0$ as $t \to \pm\infty$, unless C_0 is nilpotent. On the other hand, in case C_0 has nonreal eigenvalues, an easy argument for 2×2 matrices shows $s = \det A < 0$, since $AC = A'$ is selfadjoint on $\Delta^\perp$. If all leaves in U are complete, it follows with our assumptions from 1.8 and the above that C_S is nilpotent for all $S \in \Delta$. Furthermore, $C \neq 0$ on an open dense (saturated) subset U' of U (cf. 1.1). As we had shown in 2.1, this implies f is smoothly ruled on U'.

Finally by one of the crucial arguments in 3.4, the immersion f can be deformed along a connected ruled subset V with relative nullity $n-2$ only if V is contained in a smoothly ruled strip. But such strips cannot exist according to the first part of this proof, and thus f is rigid.

Remark 3.13. It is clear that 3.11 holds as well for $s \geq 0$ provided the set of totally geodesic points does not separate M. Furthermore, the conditions need only be satisfied outside some compact set.

We finally discuss to what extent isometric deformations may exist if the set of totally geodesic points separates the hypersurface in 3.4. The following fact actually holds for arbitrary codimension.

Lemma 3.14. *Let $f\colon M^n \to \mathbf{R}^{n+1}$ be any isometric immersion, and S a connected component of the subset of totally geodesic points. Then $f(S)$ is* *contained in an n-dimensional affine subspace of* $\mathbf{R}^{n+1}$ *tangent to* f *along* S.

Proof. Note that any smooth function $\varphi\colon M^n \to \mathbf{R}$ with $d\varphi \equiv 0$ on S must be constant on S. This is because $\varphi(S)$ is an interval in $\mathbf{R}$, which must contain regular values of φ unless it is a point. Let $x_0 \in S$ and $f(x_0) = 0$. Now $\varphi_1 = \langle N, a\rangle$ satisfies $d\varphi_1 = 0$ on S if N is a unit normal and a is any constant vector field in $\mathbf{R}^{n+1}$. Thus N is constant in $\mathbf{R}^{n+1}$ along S by the above. Furthermore, we also have $d\varphi_2 = 0$ for the support function $\varphi_2 = \langle f, N\rangle$, and thus $\varphi_2 = 0$ along S. This completes the argument.

Consider a complete hypersurface $f\colon M^n \to \mathbf{R}^{n+1}$ without ruled strips and open sets $L^3 \times \mathbf{R}^{n-3}$ as in 3.4. Looking back at the proof it is now easy to see that the set of all isometric deformations of f is discrete and can be obtained by reflecting components of the complement of a connected separating set S of totally geodesic points in the common affine tangent space along S in $\mathbf{R}^{n+1}$.

In this paper we have restricted attention to isometric deformations of hypersurfaces M^n in euclidean space. If the ambient space is a euclidean sphere, the local rigidity problem becomes a special aspect of the euclidean case by considering cones, as was already observed in [12, p. 44]. Complete M^n are always rigid for $n \geq 4$ (D. Ferus, unpublished), but not for $n = 3$ (cf. the discussion in [5, p. 9]). Even the local situation in hyperbolic space is not completely clear as yet; the claims in [7] are incorrect. We expect that there are more types of complete deformable hypersurfaces. Another important question in euclidean space $\mathbf{R}^4$ concerns the existence of complete nonruled deformable hypersurfaces M^3 with rank $\rho = 2$ almost everywhere. This problem arose already in [5] for minimal immersions.

References

[1] K. Abe, *A complex analogue of Hartman-Nirenberg cylinder theorem*, J. Differential Geometry **7** (1972) 453–460.

[2] ——, *Applications of a riccati type differential equation to riemannian manifolds with totally geodesic distributions*, Tôhoku Math. J. **25** (1973) 425–444.

[3] M. do Carmo & M. Dajczer, *A rigidity theorem for higher codimensions*, Math. Ann. **274** (1986) 577–583.

[4] É. Cartan, *La déformation des hypersurfaces dans l'espace euclidien réel a n dimensions*, Bull. Soc. Math. France **44** (1916) 65–99.

[5] M. Dajczer & D. Gromoll, *Gauss parametrizations and rigidity aspects of submanifolds*, J. Differential Geometry **22** (1985) 1–12.

[6] ——, *Real Kaehler submanifolds and uniqueness of the Gauss map*, J. Differential Geometry **22** (1985) 13–28.

[7] S. Dolbeault-Lemoine, *Sur la déformabilité des variétés plongées dans espace de Riemann*, Ann. Sci. École Norm. Sup. (3) **73** (1956) 357–438.

[8] D. Ferus, *On the completeness of nullity foliations*, Michigan Math. J. **18** (1971) 61–64.
[9] P. Hartman, *on the isometric immersions in euclidean space of manifolds with nonnegative sectional curvature.* II, Trans. Amer. Math. Soc. **147** (1970) 529–540.
[10] A. Rosenthal, *Kaehler manifolds of constant nullity*, Michigan Math. J. **15** (1968) 433–440.
[11] R. Sacksteder, *The rigidity of hypersurfaces*, J. Mech. **11** (1962) 929–940.
[12] U. Sbrana, *Sulle varietá ad $n - 1$ dimensione deformabili nello spazio euclideo ad n dimenione*, Rend. Circ. Mat. Palermo **27** (1909) 1–45.

INSTITUTO DE MATEMATICA PURA E APLICADA, RIO DE JANEIRO
STATE UNIVERSITY OF NEW YORK, STONY BROOK