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SHORT GEODESICS AND GRAVITATIONAL INSTANTONS

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1. Introduction

In [3], Cheeger proved a basic estimate on the length of the shortest closed geodesic $l(\gamma)$ in a compact Riemannian manifold M, namely, $l(\gamma) \geq c(\lambda, v, D)$, where c is a constant depending only on a lower bound on the sectional curvature $K_M \geq -\lambda$, a lower bound on the volume $\mathrm{vol}_M \geq v$ and an upper bound on the diameter $\mathrm{diam}_M \leq D$. Combined with Klingenberg's estimate [11], one obtains a lower bound on the injectivity radius i_M of M in terms of a bound on $|K_M|$, vol_M and diam_M. This gives a good control over the local geometry and topology of M, and is a basic step in the proof of Cheeger's finiteness theorem, that there are only finitely many diffeomorphism classes of compact n-manifolds such that $|K_M| \leq \Lambda$, $\mathrm{vol}_M \geq v$ and $\mathrm{diam}_M \leq D$.

Recently, Grove-Petersen [9] have proved an analogue of Cheeger's finiteness theorem assuming only a lower bound on the sectional curvature, namely, there are only finitely many homotopy types among compact n-manifolds satisfying $K_M \geq -\lambda$, $\mathrm{vol}_M \geq v$ and $\mathrm{diam}_M \leq D$. Crucial to their proof of this result is a generalization of Cheeger's estimate to critical points of the distance function $\rho\colon M \times M \to \mathbb{R}$.

If one drops the lower bound on the volume, Gromov [7] proved that all Betti numbers of M, with respect to any coefficient field, are bounded from above by a constant depending only on a lower curvature bound $K_M \geq -\lambda$ and an upper diameter bound. If $\lambda \geq 0$, then one has an absolute bound, depending only on dimension, since the diameter may be scaled to one. Note for example that the family of 3-dimensional lens spaces $L(p, q)$ gives infinitely many homotopy types of manifolds satisfying these bounds.

In this paper we consider these and related questions for the class of compact n-dimensional Riemannian manifolds such that

$$\mathrm{Ric}_M \geq -(n-1)k^2, \quad \mathrm{vol}_M \geq v, \quad \mathrm{diam}_M \leq D. \tag{1.1}$$

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We show first that a Cheeger-type estimate holds under these bounds for curves which are sufficiently nontrivial in $\pi_1(M)$. More precisely, there is an explicit lower bound

$$l(\gamma) \geq \frac{Dv}{v_k(2D)} \tag{1.2}$$

on the length of a closed loop γ in M, provided $[\gamma] \in \pi_1(M)$ satisfies $[\gamma]^p \neq 0$, $\forall p \leq N = v_k(2D)/v$. Here $v_k(r)$ is the volume of a geodesic ball of radius r in the space form of curvature $-k$. The same argument shows that the subgroup of $\pi_1(M)$ generated by the short loops, i.e., the loops not satisfying (1.2), is a finite group of order $\leq N$. Using an argument of Gromov [8], we then show that there are only finitely many possibilities for $\pi_1(M)$ among compact n-manifolds satisfying (1.1).

Is there a bound on the length of the shortest closed geodesic depending only on the bounds (1.1)? This question was raised for instance in [4]. In §3, we show that the answer is no by constructing examples based on the Eguchi-Hanson metric. In particular, it follows that there is a compact simply connected 4-manifold M_1 (the double of $T(S^2)$ or equivalently $S^2 \times S^2$) with a family of metrics of positive Ricci curvature, vol $\geq v$, diam $\leq D$, with closed geodesics of arbitrarily small length. This shows for instance that the method of proof of the finiteness results of Grove-Peterson [9] or Cheeger [3] will not generalize to handle similar questions under the bounds (1.1).

We also construct two families of Riemannian manifolds M_k and N_k based on the family of gravitational instantons of Gibbons-Hawking [6] with the following properties: Each M_k, $k = 2, 3, \cdots$, is homeomorphic to $\#_1^k S^2 \times S^2$ and has a (family of) Riemannian metrics with $\mathrm{Ric}_{M_k} \geq 0$, $\operatorname{diam} M_k \leq 1$, but $\operatorname{vol} M_k \leq c/k$. Similarly, N_k is homeomorphic to $\#_1^{2k} \mathbb{CP}^2$ and has metrics with the same properties. These examples show that one cannot improve Gromov's theorem to positive Ricci curvature in place of sectional curvature. Note however that since $\operatorname{vol} M_k \to 0$ and $\operatorname{vol} N_k \to 0$ as $k \to \infty$, these manifolds do not satisfy the bounds (1.1). Sha-Yang [13] have independently constructed different metrics on M_k and other simply connected 4-manifolds with the same properties. Previously [14], they have also constructed metrics with similar properties in dimension seven. In the final section, we list a number of further remarks and open questions along these lines.

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2. Estimates of the 1-systole

In this section, we derive some estimates on the systole of a compact manifold and discuss some applications to finiteness theorems. Recall that the 1-systole $\mathrm{sys}_1(M)$ of a compact Riemannian manifold M is the length of the shortest noncontractible curve in M. Throughout this section, M will be a compact, connected n-dimensional Riemannian manifold satisfying the bounds

$$\mathrm{Ric}_M \geq -(n-1)k^2, \quad \mathrm{vol}_M \geq v, \quad \mathrm{diam}_M \leq D. \tag{2.1}$$

The basic estimate is contained in the following result. The volume of a geodesic ball of radius r in the space form of constant curvature $-k$ will be denoted by $v_k(r)$.

Theorem 2.1. *Let M be a compact n-manifold satisfying the bounds (2.1). If γ is a curve in M with $[\gamma]^k \neq 0$ in $\pi_1(M)$ for all $k \leq N \equiv v_k(2D)/v$, then*

$$l(\gamma) \geq \frac{Dv}{v_k(2D)}. \tag{2.2}$$

Proof. We present a proof simplifying the original proof of the author, suggested by J. Cheeger. Let $\Gamma = \pi_1(M) = \pi_1(M, x_0)$ be the subgroup generated by $[\gamma]$, so that $|\Gamma| \geq N$. We let $\tilde{M}$ be the universal cover of M and let $F \subset \tilde{M}$ be a fundamental domain for the action of $\pi_1(M)$ on $\tilde{M}$. For example, one may choose F to be a Dirichlet fundamental domain, i.e.,

$$F = \bigcap_{g \in \pi_1(M)} \{x \in \tilde{M} \colon \mathrm{dist}(x, x_0) \leq \mathrm{dist}(x, gx_0)\}.$$

Let $B_{x_0}^{\tilde{M}}(r)$ (resp. $B_{x_0}^{M}(r)$) be the geodesic ball of radius r in $\tilde{M}$ (M) about x_0, where we abuse notation slightly and denote a lift of x_0 to $\tilde{M}$ by x_0 also. Then it is easily verified that $B_{x_0}^{\tilde{M}}(r) \cap F$ is mapped isometrically under the covering projection onto $B_{x_0}^{M}(r)$, modulo a set of measure zero corresponding to ∂F. In particular, $\mathrm{vol}(B_{x_0}^{\tilde{M}}(r) \cap F) = \mathrm{vol}(B_{x_0}^{M}(r))$.

Let $U(r) = \{g \in \Gamma \colon g = g_0^i, |i| \leq r\}$, where g_0 is the generator of Γ ($g_0 = [\gamma] \in \pi_1(M)$). Note that $\mathrm{dist}(g_0 x_0, x_0) \leq l(\gamma)$, so that $\mathrm{dist}(g x_0, x_0) \leq r \cdot l(\gamma)$ for all $g \in U(r)$. Choose the smallest $r = r_0$ such that $\# U(r_0) \geq N$, and consider the domain

$$\bigcup_{g \in U(r_0)} g(B_{x_0}^{\tilde{M}}(D) \cap F) \subset B_{x_0}^{\tilde{M}}(Nl(\gamma) + D). \tag{2.3}$$

Taking the volume of both sides of (2.3), we obtain,

$$N \cdot \mathrm{vol}\, M \leq \mathrm{vol}\, B_{x_0}^{\tilde{M}}(Nl(\gamma) + D) \leq v_k(Nl(\gamma) + D), \tag{2.4}$$

where the last inequality follows from the well-known Bishop comparison theorem. Now suppose (2.2) were false, i.e., $l(\gamma) < D/N$. Then (2.4) implies that

$$N \cdot v \leq N \cdot \operatorname{vol} M < v_k(2D), \tag{2.5}$$

that is,

$$N < \frac{v_k(2D)}{v},$$

which contradicts the definition of N. Thus the estimate (2.2) is established.

Remarks 2.2. (1) The result above can easily be localized. Namely, let γ be a closed curve in M, satisfying the bounds (2.1), and let $T(\varepsilon_0) = \{x \in M\colon \operatorname{dist}(x, \gamma) < \varepsilon_0\}$ be the ε_0 tubular neighborhood of γ in M. In the proof above, if we replace M by $T(\varepsilon_0)$, and define the number N as before, then the same argument gives the estimate

$$N \cdot \operatorname{vol} T(\varepsilon_0) \leq v_k(Nl(\gamma) + D). \tag{2.6}$$

Note that $\operatorname{vol} T(\varepsilon_0) \geq \operatorname{vol} B_{x_0}(\varepsilon_0) \geq v/v_k(D) \cdot v_k(\varepsilon_0)$, where the last inequality follows from the Bishop comparison theorem. Thus, from (2.6), we obtain,

$$\frac{v_k(2D)}{v_k(D)} \cdot v_k(\varepsilon_0) \leq v_k(Nl(\gamma) + D).$$

Solving this for $l(\gamma)$ gives a lower bound for $l(\gamma)$ of the form $l(\gamma) \geq \varepsilon_0 v / v_k(2D)$.

Thus, if γ is a very short curve in M, a multiple of γ must bound a disc in a small tubular neighborhood.

(2) Instead of considering subgroups generated by one element, we may also consider subgroups generated by several elements. Let Γ be the subgroup of $\pi_1(M) = \pi_1(M, x_0)$ generated by elements $\{g_i\}$, and let γ_i be loops in M, based at x_0, representing these classes. As in Theorem 2.1, we let $U(r) = \{g \in \Gamma\colon g = g_1^{i_1} \cdots g_k^{i_k}, \Sigma|i_j| \leq r\}$, so that $\operatorname{dist}(gx_0, x_0) \leq r \cdot \max l(\gamma_i)$ for $g \in U(r)$. Then exactly the same argument shows that

$$\max l(\gamma_i) \geq \frac{Dv}{v_k(2D)},$$

provided $|\Gamma| \geq v_k(2D)/v$.

Thus, the subgroup of $\pi_1(M)$ generated by 'short loops', i.e., loops of length $< Dv/v_k(2D)$, has order bounded by N.

One consequence of these results is the following.

Theorem 2.3. *In the class of compact n-dimensional Riemannian manifolds M such that* $\mathrm{Ric}_M \geq -k^2$, $\mathrm{vol}_M \geq v$ *and* $\mathrm{diam}_M \leq D$, *there are only finitely many isomorphism classes of* $\pi_1(M)$.

Proof. This follows by combining the estimates above with some arguments of Gromov [8]. Namely, Gromov [8, 5.15] shows that, given M as above, there is a system of generators $\{g_i\}$ of $\pi_1(M) = \pi_1(M, x_0)$ and representatives γ_i of g_i such that $l(\gamma_i) \leq 2D$ and all relations among the generators are of the form $g_i g_j g_k^{-1} = 1$. Clearly, it suffices to prove there is a bound on the number p of generators, since the isomorphism class of π_1 is determined by p and the set of relations in $\{1, 2, \cdots, p\}^3$.

By Remark 2.2(2) above, there is a bound $N = v_k(2D)/v$ on the number of classes $[\gamma]$ in $\pi_1(M)$ such that $l(\gamma) < \delta \equiv D/N$. Consider the geodesic balls $B_{g_k x_0}(\delta/2)$, $k = 1, \cdots, p$. If $x \in \bigcap_{i=1}^{s} B_{g_{j_i}}(\delta/2)$, then the $s(s-1)/2$ classes $g_{j_i} \cdot g_{j_k}^{-1}$ have representatives γ_{ik} with $l(\gamma_{ik}) < \delta$. Thus $s(s-1)/2 < N$, so that there is a bound on the multiplicity of intersections of balls $B_{g_k}(\delta/2)$. Since these balls are all contained in the ball $B_{x_0}(2D + \delta)$, the (relative) Bishop comparison theorem implies that the number of such balls is uniformly bounded above. q.e.d.

It is clear that an explicit upper bound, depending on n, k, v, D, can be derived for the number of isomorphism classes of $\pi_1(M)$.

3. Examples of manifolds of positive Ricci curvature

In this section, we construct several examples of compact 4-manifolds with metrics of positive Ricci curvature. These show that some of the topological and metric bounds mentioned in §1, obtained under a lower bound on the sectional curvature, do not remain valid assuming a lower bound on the Ricci curvature.

We first turn to the question of whether Cheeger's estimate on the length of the shortest closed geodesic remains valid under a lower bound on Ricci curvature. Theorem 2.1 shows that the answer is yes if the geodesic is sufficiently nontrivial in $\pi_1(M)$. On the other hand, we have the following.

Proposition 3.1. *There is a family of metrics* ds_a^2, $a \in (0, 1]$, *on* $M_1 = S^2 \times S^2$ *with* $\mathrm{Ric}_{M_1} \geq 0$, $\mathrm{vol}_{M_1} \geq \frac{1}{5}$ *and* $\mathrm{diam}_{M_1} \leq 5$, *and with closed geodesics of length* $2\pi a$.

Proof. First we note that $S^2 \times S^2$ is diffeomorphic to the double of the tangent bundle of S^2, i.e., $M_1 = TS^2 \cup_{\partial TS^2} (-TS^2)$, where $-TS^2$ denotes TS^2 with the opposite orientation. This follows from the fact that $TS^2 \cup_{\partial TS^2} (-TS^2)$ is naturally an S^2 bundle over S^2, of which there are only two isomorphism classes since $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$. Since for instance the intersection form of $TS^2 \cup_{\partial TS^2} (-TS^2)$ is congruent to that of $S^2 \times S^2$, it follows that these manifolds are diffeomorphic. The metrics on M_1 are modifications of the Eguchi-Hanson metrics [5]. This is a complete metric on the tangent bundle of S^2 given by

$$ds^2 = \frac{du^2}{(1+(a/r)^4)^2} + u^2\sigma_z^2 + r^2(\sigma_x^2 + \sigma_y^2). \tag{3.1}$$

Here $a \in \mathbb{R}^+$ is a free parameter, and $\sigma_x, \sigma_y, \sigma_z$ are the standard left-invariant coframing of $\mathbb{RP}^3$ with $d\sigma_x = 2\sigma_y \wedge \sigma_2$, etc. (Note that the dual left-invariant vector fields have half the length of the standard orthonormal framing of $S^3(1)$.) Further, $r \in [a, \infty)$ and $u = r[1-(a/r)^4]^{1/2}$, so $u = 0$ when $r = a$. Recall that the sphere bundle in TS^2 is naturally diffeomorphic to $\mathbb{RP}^3$ so that TS^2 is a family of $\mathbb{RP}^3$'s, parametrized by u, with $u = 0$ corresponding to a collapse of $\mathbb{RP}^3$ to S^2 (the zero-section). It is clear that the metric is invariant under the natural SO(3) action. The 1-form

$$dt = \frac{1}{1+(a/r)^4}\, du$$

is the unit length 1-form dual to the gradient of the distance function t to S^2.

The basic feature of this metric is that it is Ricci-flat (in fact self-dual) and locally asymptotically Euclidean. If (e^0, e^1, e^2, e^3) are the orthonormal coframing corresponding to $(du^2, \sigma_x^2, \sigma_y^2, \sigma_z^2)$, then the components of the curvature tensor are

$$\begin{aligned} R_1^0 &= -R_3^2 = -\frac{2a^4}{r^6}(e^0 \wedge e^1 - e^2 \wedge e^3), \\ R_2^0 &= -R_1^3 = -\frac{2a^4}{r^6}(e^0 \wedge e^2 - e^3 \wedge e^1), \\ R_3^0 &= -R_2^1 = \frac{4a^4}{r^6}(e^0 \wedge e^3 - e^1 \wedge e^2). \end{aligned} \tag{3.2}$$

Thus the sectional curvature decays at a rate of $1/r^6$ and outside compact sets the metric approaches the flat metric on the cone $C(\mathbb{RP}^3)$ at a rate $1/r^4$. There is a Killing field Z dual to the 1-form σ_z with $|Z| = 0$ on the zero-section. Thus the zero-section is a totally geodesic embedding of a constant curvature S^2 of curvature $1/a^2$ into TS^2. In particular, we have a family of Ricci-flat, locally asymptotically Euclidean metrics on TS^2, with arbitrarily short closed geodesics.

In order to construct compact manifolds, we will take the double of the domain $B = t^{-1}[0,1] \subset TS^2$ and thus obtain a compact 4-manifold

$M_1 = B \cup_{\partial B} (-B) = TS^2 \cup_{\mathbb{RP}} (-TS^2)$. The metric however must be rounded off in a neighborhood of ∂B in order to obtain a smooth metric on M_1.

We first show there is a conformal deformation of the metric ds^2 on B to a metric of nonnegative Ricci curvature, which near ∂B is arbitrarily close (depending on a), in the C^∞ topology, to a neighborhood of the totally geodesic embedding of $\mathbb{RP}^3$ in an $\mathbb{RP}^4$ of constant curvature $+4$. Thus, consider conformally related metrics $ds_h^2 = (1/h^2)\, ds^2$ on B, where $h = h(t)$, $t = \int du/1 + (a/r)^4$. We note that t is a convex function on (TS^2, ds^2). Thus, the zero-section $S^2 \to TS^2$ corresponding to $t^{-1}(0)$ is a soul in the terminology of Cheeger-Gromoll. In fact, the Hessian $D^2 t$ is diagonal in the framing (e^0, e^1, e^2, e^3) and one computes

$$(3.3) \qquad D^2 t(e_i, e_i) = \begin{cases} 0, & i = 0, \\ \frac{1}{r}(1 - (\frac{a}{r})^4)^{1/2}(1 + (\frac{a}{r})^4), & i = 1, 2, \\ \frac{1}{r}(1 + (\frac{a}{r})^4)^2(1 - (\frac{a}{r})^4)^{-1/2}, & i = 3. \end{cases}$$

One also computes that

$$(3.4) \qquad \mathrm{Ric}_h(x, x) = h^2 \mathrm{Ric}(x, x) + h\Delta h + 2hD^2 h(x, x) - 3|dh|^2.$$

We now assume h is a C^∞ convex function of t, with $h(t) = 1$ for $t \le \frac{1}{4}$, and $h(t) = 1 + t^2$ for t near 1. From (3.3) and (3.4), we may then estimate

$$(3.5) \qquad \mathrm{Ric}_h(x, x) \ge hh'\Delta t + hh'' - 3(h')^2 \ge \frac{3hh'}{r} - 3(h')^2 + hh''.$$

Clearly, this can be made nonnegative for an appropriate choice of h; near ∂B, Ric_h is very close (depending on a) to $12I$, where I is the identity matrix, i.e., the metric ds_h^2. If II_h and II denote the 2nd fundamental forms of ∂B in the metrics ds_h^2 and ds^2, then one calculates that

$$\frac{1}{h}\mathrm{II}_h = \mathrm{II} + \frac{h'}{h} I = D^2 t - \frac{h'}{h} I = D^2 t - I.$$

Thus, II_h is close (depending on a) to 0.

We see that the metric ds_h^2, for an appropriate choice of h as above, is of nonnegative Ricci curvature and near ∂B is C^∞ close to the neighborhood N of the standard embedding $\mathbb{R}P^3 \subset \mathbb{R}P^4(+4)$. Thus, we may perturb the metric ds_h^2 slightly to a metric $ds_a^2 = ds_h^2 + E$, where E is of compact support near ∂B, $\|E\|_{C^\infty} \le \varepsilon$, and ds_a^2 is isometric to $N \subset \mathbb{R}P^4(+4)$ near ∂B. Clearly, the metric ds_a^2 is still of nonnegative Ricci curvature on B and of positive Ricci curvature near ∂B. This metric then extends to give a smooth metric of nonnegative Ricci curvature on the double $M_1 = B \cup_{\partial B} (-B)$. It is clear from the construction that $\mathrm{diam}(M_1) < 5$ and $\mathrm{vol}(M_1) > w_4/5$, where w_4 is the volume of the unit ball in $\mathbb{R}^4$.

Remarks 3.2. (1) It is interesting to note that the metrics ds_a^2 have a pair of totally geodesic 2-spheres $S^2 \to TS^2 \subset M_2$, of constant curvature a^{-2}, and thus of arbitrarily small area. These 2-spheres generate $H_2(M_1, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and therefore there is no lower bound for the 2-systole $\mathrm{sys}_2(M) = \inf\{\mathrm{area}(\Sigma) \colon \Sigma \in H_2(M, \mathbb{Z})\}$, for compact 4-manifolds satisfying the bounds (2.1). Theorem 2.1 gives of course a lower bound for the lengths of curves $\gamma \in H_1(M, \mathbb{Z})$ which are nonzero in $H_1(M, \mathbb{R})$. Similarly, if $\Sigma \subset M^4$ is a compact minimal hypersurface, then it is well known that $\mathrm{vol}(M) \leq c(\inf \mathrm{Ric}, \mathrm{diam}) \cdot \mathrm{vol}(\Sigma)$, so that there is a lower bound for the 3-systole of nonzero classes in $H_3(M, \mathbb{R})$ in this class of manifolds.

(2) By taking for instance products with spheres $S^n(1)$, one obtains compact Riemannian manifolds in any dimension ≥ 4 with the properties of Proposition 3.1 and the Remark above.

(3) One may view $\mathbb{C}P^2$ with a ball removed as a 2-fold branched cover of $TS^2 = T(\mathbb{C}P^1)$, branched over the zero-section. The Eguchi-Hanson metric (3.1) lifts to a singular Riemannian metric on $\mathbb{C}P^2 \backslash B$. In fact, one may verify that the following metric on $\mathbb{C}P^2 \backslash B$

$$ds^2 = \frac{4}{(1 + (a/r)^4)^2}\, du^2 + u^2 \sigma_z^2 + r^2(\sigma_x^2 + \sigma_y^2)$$

is a smooth complete metric with nonnegative Ricci curvature which is (weakly) asymptotically flat; the curvature decays as c/r^2 as $r \to \infty$, but not faster. Here $\sigma_x, \sigma_y, \sigma_z$ are now the standard left-invariant coframing of $S^3(1)$ and r, u, a are as in (3.1). There is a totally geodesic constant curvature $S^2 \subset \mathbb{C}P^2 \backslash B$ in this metric, and one may alter the metric near $\partial(\mathbb{C}P^2 \backslash B)$ as above to produce complete metrics of nonnegative Ricci curvature on $\mathbb{C}P^2 \# - \mathbb{C}P^2$ with arbitrarily short closed geodesics.

In a similar fashion, one may construct metrics of positive Ricci curvature on compact 4-manifolds of higher topological type, using the gravitational multi-instantons of Gibbons-Hawking [6], Hitchin [10].

Proposition 3.3. *For each $k \geq 2$, the compact simply connected manifolds $M_k = \#_1^k S^2 \times S^2$ and $N_k = \#_1^{2k} \mathbb{CP}^2$ have a $(3k-3)$-parameter family of metrics with*

$$\mathrm{Ric}(M_k) \geq 0, \quad \mathrm{diam}(M_k) \leq 5, \quad \mathrm{vol}(M_k) \leq \frac{c}{k}. \tag{3.6}$$

and similarly for N_k.

Proof. Let $m = k+1$, let $\mathbb{Z}_m$ act on $\mathbb{C}^2$ in the standard fashion $(z_1, z_2) \to (e^{2\pi i n/m} z_1, e^{-2\pi i n/m} z_2)$, $n \in \mathbb{Z} \pmod m$, and consider the quotient $\mathbb{C}^2/\mathbb{Z}_m$. This is isometric to the cone $C(S^3/\mathbb{Z}_m)$ on the lens space $S^3/\mathbb{Z}_m$. Following

Hitchin [10], $\mathbb{C}^2/\mathbb{Z}_m$ may be viewed as the complex surface

$$xy = z^m$$

in $\mathbb{C}^3$ and the singularity at the origin may be resolved by adding on lower order terms

$$xy = \prod_{i=1}^{m}(z - a_i).$$

The resulting surface Z_k is a nonsingular complex surface, provided the discriminant of $\Pi(z - a_i)$ is nonzero. Z_k is obtained topologically by successively plumbing k copies of the tangent bundle of S^2, as prescribed by the Dynkin diagram of A_k. Thus, Z_k has the homotopy type of a wedge of k two-spheres and has intersection matrix B the Cartan matrix of A_k, i.e., $B(e_j, e_j) = -2$, $B(e_j, e_i) = 1$ if $|j-i| = 1$ and $B(e_j, e_i) = 0$ if $|j-1| > 1$, for an appropriate basis (e_j) of $H_2(Z_k, \mathbb{Z})$ (cf. [1]). Outside a compact set, Z_k has the topology of $S^3/\mathbb{Z}_m \times \mathbb{R}^+$. Gibbons-Hawking [6] and Hitchin [10] have constructed a $(3k-3)$-parameter family of complete, Ricci flat (in fact self-dual) metrics on Z_k which are asymptotic to the flat metric on $C(S^3/\mathbb{Z}_m)$ at a rate $= O(r^{-4})$.

If we fix a metric g in this family, there is an R_0 such that $Z_k \backslash B(R_0)$ is diffeomorphic, under a map F, to $S^3/\mathbb{Z}_m \times (R_0, \infty)$ and the metric has the form $g = F^*\delta + h$, where δ is the flat metric on $C(S^3/\mathbb{Z}_m)$ and $|h(x)| \leq c|x|^{-4}$. In particular, the function $\rho = \pi_2 \circ F \colon Z_k \backslash B(R_0) \to (R_0, \infty)$ is a strictly convex function, for R_0 sufficiently large, and the 2nd fundamental forms II_t of the hypersurfaces $S_t = F^{-1}(S^3/\mathbb{Z}_m \times \{t\})$ satisfy $|\mathrm{II}_t - \mathrm{I}/t| \to 0$ as $t \to \infty$.

Consider the rescaled metrics $g_r = (1/r^2)g|_{D_r}$, where D_r is the domain in Z_k with $\partial D_r = S_r$. These are Ricci flat metrics of diameter ≤ 2 on a manifold of fixed topological type, with ∂D_r having 2nd fundamental form converging to I, as $r \to \infty$. The function ρ is strictly convex in this metric, with $D^2\rho^2 \approx 2I$. One may now bend, i.e., conformally deform, the metric g_r as in Proposition 3.1 to a metric of nonnegative Ricci curvature on D_r, positive Ricci curvature near ∂D_r, and so that ∂D_r has 2nd fundamental form arbitrarily close (depending on r) to zero. One may perturb the metric near ∂D_r, preserving positive Ricci curvature, so that ∂D_r is then totally geodesic. We set $M_k = D_r \cup_{\partial D_r} (-D_r)$ for r sufficiently large, and the perturbed metric then extends smoothly over ∂D_r to a smooth metric on M_k.

It follows that $\mathrm{Ric}(M_k) \geq 0$ and $\mathrm{diam}(M_k) \leq 5$. Since $\partial D_r \subset M_k$ is totally geodesic, a standard volume comparison argument shows that the s-tubular neighborhood of ∂D_r has volume $\leq cs\,\mathrm{vol}(\partial D_r) \leq c_2 s 1/k$.

This gives the bounds (3.7). Finally, since the intersection form of M_k is congruent to that of $\#_1^k S^2 \times S^2$, M_k is homeomorphic to $\#_1^k S^2 \times S^2$, by Freedman's theory. Similarly, we may form $N_k = D_r \cup_{\partial D_r} D_r$ and produce metrics with the same properties on N_k. The intersection form of N_k is seen to be congruent to that of $\#_1^{2k} \mathbb{CP}^2$, so that the spaces are homeomorphic again by Freedman's theory.

4. Further remarks and questions

4.1. The examples in §3 begin in dimension 4, and it remains an open question what can be said in dimension 3 regarding for instance Cheeger's estimate under the bounds (2.1).

Theorem 2.1 implies there are only finitely many possibilities for $\pi_1(M)$ under the bounds (2.1). For 3-manifolds, π_1 determines a great deal of the topology and for instance, for irreducible, sufficiently large 3-manifolds, π_1 determines the homeomorphism type. Nevertheless, it is not known if there are only finitely many homotopy types or homeomorphism types of 3-manifolds satisfying (2.1).

A question related to the validity of Cheeger's estimate is the following: Does a noncompact, complete 3-manifold M of positive Ricci curvature admit a closed geodesic if the volume growth $v(r) \geq cr^3$? A result of Schoen-Yau [12] implies that M is diffeomorphic to $\mathbb{R}^3$. If the sectional curvature of M is nonnegative, then the answer is no, since a closed geodesic must be contained in the soul, and this forces M to have volume growth smaller than r^3.

4.2. The examples of §3 indicate a relation between compact n-manifolds satisfying the bounds (2.1) and complete, noncompact Riemannian manifolds with $\mathrm{Ric} \geq 0$ and $v(r) \geq cr^n$. One might expect that if there are infinitely many homotopy types of compact n-manifolds satisfying the bounds (2.1), then there should exist a complete, noncompact n-manifold with $\mathrm{Ric} \geq 0$ and $v(r) \geq cr^n$ of infinite topological type (not homeomorphic to the interior of a compact manifold with boundary). One also might expect the converse to hold. Thus we raise the following:

Question 1. Is there a complete, noncompact Riemannian manifold M of $\mathrm{Ric} \geq 0$, and $v(r) \geq cr^n$ of infinite topological type.

If one assumes the sectional curvature decays as $|K_M| \leq cr^{-2}$, then using Gromov's Lipschitz convergence theorem, one may show that M must have finite topological type (this has been done independently by A.

Kasue). If however one drops the assumption on the volume growth, then there are examples of infinite topological type due to Sha-Yang [14].

Of course, if M is of nonnegative sectional curvature, then M is of finite topological type (regardless of volume growth), by the soul theorem of Cheeger-Gromoll.

A related question is the following.

Question 2. Does there exist a constant $\varepsilon = \varepsilon(n)$ such that if M^n is a complete, noncompact manifold of $\mathrm{Ric} \geq 0$ and $v(r) \geq (w_n - \varepsilon)r^n$, then M is contractible (or diffeomorphic to $\mathbb{R}^n$)?

Here w_n is the volume of the unit n-ball in $\mathbb{R}^n$. One expects that an answer to Question 1 implies the same answer to Question 2 and conversely. An analogous question for compact manifolds is the following: Is there an $\varepsilon = \varepsilon(n)$ such that if M^n is a compact manifold with $\mathrm{Ric} \geq (n-1)$ and $\mathrm{vol}\, M \geq w_n - \varepsilon_n$, then M^n is a homotopy sphere?

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