J. DIFFERENTIAL GEOMETRY
31 (1990) 163-184

A PROOF OF THE SPLITTING CONJECTURE OF S.-T. YAU

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Abstract

According to a conjecture of Yau, a geodesically complete space-time satisfying the timelike convergence condition and admitting a timelike line is isometric to the product of that line and a spacelike hypersurface. A proof for globally hyperbolic space-times has recently been provided by Eschenberg. The present paper shows how Eschenburg's arguments may be refined so as to yield a complete proof of Yau's conjecture.

1. Introduction

The Cheeger-Gromoll [4] splitting theorem of Riemannian geometry states that a complete Riemannian $(n+1)$-manifold of nonnegative Ricci curvature admitting an absolutely maximizing geodesic is isometric to the product of that geodesic and a complete Riemannian n-manifold. The simplest known proof of this result is due to Eschenburg and Heintze [6]. The following conjecture of Yau proposed that there should be an analogue of the Cheeger-Gromoll theorem for Lorentzian manifolds.

Conjecture(*Yau* [10]). *A geodesically complete Lorentzian* 4*-manifold of nonnegative Ricci curvature in the timelike direction which contains an absolutely maximizing timelike geodesic is isometrically the cross product of that geodesic and a spacelike hypersurface.*

Various restricted results pertaining to this conjecture were first established by Galloway [7] and by Beem et al. [2], [3]. However the central development came with a recent paper of Eschenberg [5]. This showed that the elementary proof of the Cheeger-Gromoll theorem given in [6] could be directly translated to the Lorentzian case to give a proof of the Yau conjecture, subject to an additional hypothesis of global hyperbolicity. A subsequent paper of Galloway [8] employed maximal surface techniques to obtain a more natural and more powerful approach to the key step in Eschenberg's theorem. This led to a demonstration that, if global hyperbolicity does hold, then the hypothesis of timelike geodesic completeness

Received January 28, 1988.

in Eschenberg's result is redundant, provided the given absolutely maximizing timelike geodesic is complete.

It is the purpose of this paper to show that Eschenberg's arguments may be refined in a different direction so as to dispense with the hypothesis of global hyperbolicity and thus yield a complete proof of Yau's conjecture. No essentially new results are required. Even the principal step (Lemma 3.9) in overcoming the lack of a hypothesis of global hyperbolicity is achieved by an adaptation of the techniques of the proof of Lemma 3.2 in [5]. Galloway's minimal surface techniques are employed for the sake of elegance.

Throughout the subsequent sections, $(M, \mathbf{g})$ will denote a C^∞ connected time-orientable Lorentzian $(n+1)$-manifold, $n \geq 1$. All notation and terminology will be as in Eschenberg [5] and Penrose [9]. All causal curves are future directed except where stated otherwise.

2. Busemann functions

For the purposes of this section let $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ be any unit speed C^1 timelike curve.

For each $p \in J^-(|\alpha|)$ let $A^+_{\alpha,p} := \alpha^{-1}(J^+(p)) \subset A_\alpha$ and $t^+_{\alpha,p} := \inf\{t \in A^+_{\alpha,p}\}$. Similarly for each $p \in J^+(|\alpha|)$ let $A^-_{\alpha,p} := \alpha^{-1}(J^-(p)) \subset A_\alpha$ and $t^-_{\alpha,p} := \sup\{t \in A^-_{\alpha,p}\}$. If the chronology condition holds at a point $p \in J^+(|\alpha|) \cap J^-(|\alpha|)$ then $t^-_{\alpha,p} \leq t^+_{\alpha,p}$. For any $q, r \in J^+(|\alpha|) \cap J^-(|\alpha|)$ such that $r \in J^+(q)$ one has $t^-_{\alpha,q} \leq t^-_{\alpha,r}$ and $t^+_{\alpha,q} \leq t^+_{\alpha,r}$.

The following definition is fundamental.

Definition 2.1. For each $t \in A_\alpha$ the pre-Busemann functions $b^\pm_{\alpha,t}\colon J^\mp(\alpha(t)) \to \mathbb{R} \cup \{\mp\infty\}$ are defined by

$$b^+_{\alpha,t}(p) := t - d(p, \alpha(t)), \qquad b^-_{\alpha,t}(q) := t + d(\alpha(t), q).$$

The Busemann functions $b^\pm_\alpha\colon J^\mp(|\alpha|) \to \mathbb{R} \cup \{\mp\infty\}$ are defined by

$$b^+_\alpha(p) := \inf_{t \in A^+_{\alpha,p}} b^+_{\alpha,t}(p), \qquad b^-_\alpha(q) := \sup_{t \in A^-_{\alpha,q}} b^-_{\alpha,t}(q).$$

Note that certain sign conventions relating to the functions $b^-_{\alpha,t}$ and b^-_α differ from those of Eschenberg [5] and Galloway [8].

The following result may be obtained by repeated applications of the reverse triangle inequality.

Proposition 2.2. (I) *For any $t \in A_\alpha$ one has $b^+_{\alpha,u}(\alpha(t)) \leq t \leq b^-_{\alpha,s}(\alpha(t))$ for all $s, u \in A_\alpha$ such that $s \leq t \leq u$;*

(II) *for any $p \in J^\mp(|\alpha|)$ the functions $A^\pm_{\alpha,p} \to \mathbb{R}$ defined by $t \mapsto b^\pm_{\alpha,t}(p)$ are monotonically decreasing;*

(III) (a) *for any* $p, q \in J^-(|\alpha|)$ *such that* $q \in J^+(p)$ *one has* $b^+_{\alpha,t}(q) \geq b^+_{\alpha,t}(p) + d(p,q)$ *for all* $t \in A^+_{\alpha,q}$;

(b) *for any* $p, q \in J^+(|\alpha|)$ *such that* $q \in J^+(p)$ *one has* $b^-_{\alpha,t}(q) \geq b^-_{\alpha,t}(p) + d(p,q)$ *for all* $t \in A^-_{\alpha,p}$.

Corollary. *The sets* $\{b^{\pm}_{\alpha,t} = \text{const}\}$ *are achronal.*

Proposition 2.3. (I) *For each* $t \in A_\alpha$ *one has* $b^+_\alpha(\alpha(t)) \leq t \leq b^-_\alpha(\alpha(t))$;

(II) *for all* $p, q \in J^{\mp}(|\alpha|)$ *such that* $q \in J^+(p)$ *one has* $b^{\pm}_\alpha(q) \geq b^{\pm}_\alpha(p) + d(p,q)$.

Corollary. *The sets* $\{b^{\pm}_\alpha = \text{const}\}$ *are achronal.*

3. Lines and line segments

Given any unit speed timelike C^1 curve, the associated Busemann and pre-Busemann functions defined in the previous section have few general properties other than those established by Propositions 2.2 and 2.3. However much progress can be made for timelike curves that satisfy the following.

Definition 3.1. A causal curve $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ satisfying $d(\alpha(t_1), \alpha(t_2)) = L(\alpha|[t_1, t_2])$ for all $t_1, t_2 \in A_\alpha$ such that $t_2 \geq t_1$ is a line segment. A future (respectively past) endless line segment from (to) a point $p \in M$ is a future (past) ray from (to) p. An endless line segment is a line.

Clearly any line segment is a geodesic segment.

Remark 3.2. Suppose strong causality holds at a point $p \in M$. Let $\mathscr{U}$ be a local causality neighborhood of p. Then every causal geodesic segment in $\mathscr{U}$ is a line segment in M.

Proposition 3.3. *If* $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ *is a line segment, then the chronology condition holds at each point of* $J^+(|\alpha|) \cap J^-(|\alpha|)$.

Proof. Suppose the chronology condition is violated at some point $p \in J^+(|\alpha|) \cap J^-(|\alpha|)$. Then there exists a timelike curve λ from p to p. Let $t_1 \in A_\alpha$ be such that there exists a causal curve μ_1 from $\alpha(t_1)$ to p, and $t_2 \in A_\alpha \cap [t_1, \infty)$ such that there exists a causal curve μ_2 from p to $\alpha(t_2)$. Since λ is timelike, and accordingly has strictly positive length, there exists an integer $m > 0$ such that $mL(\lambda) > d(\alpha(t_1), \alpha(t_2))$. The concatenation of μ_1, m copies of λ, and μ_2 gives a causal curve ν from $\alpha(t_1)$ to $\alpha(t_2)$ of length $L(\nu) \geq mL(\lambda) > d(\alpha(t_1), \alpha(t_2))$. This is impossible.

Corollary. *For each* $p \in J^+(|\alpha|) \cap J^-(|\alpha|)$ *one has* $t^-_{\alpha,p} \leq t^+_{\alpha,p}$.

Proposition 3.4. *Let* $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ *be a timelike line segment. Then for each* $t \in A_\alpha$ *the causality condition holds at* $\alpha(t) \in M$.

Proof. Suppose there exists $t \in A_\alpha$ such that there is a nondegenerate causal curve γ from $\alpha(t)$ to $\alpha(t)$. By Proposition 3.3 one has that γ is a null line segment. Suppose there exists $t_- \in A_\alpha \cap (-\infty, t)$. Then the concatenation of $\alpha|[t_-, t]$ and γ is a nongeodesic causal curve ν from $\alpha(t_-)$ to $\alpha(t)$ of length $L(\nu) = L(\alpha[t_-, t]) + L(\gamma) = t - t_-$. Hence there should exist a timelike curve from $\alpha(t_-)$ to $\alpha(t)$ of length strictly greater than $t - t_-$, and this is impossible. If t_- does not exist then there must exist $t_+ \in A_\alpha \cap (t, \infty)$, and one is again led to a contradiction. q.e.d.

If $t \in A_\alpha$ is such that $\alpha(t)$ is not an endpoint of α, then the conclusion of the preceding result may be strengthened as follows.

Proposition 3.5. *If $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ is a timelike line segment, then strong causality holds at $\alpha(t)$ for all $t \in \mathring{A}_\alpha$.*

Proof. Suppose there exists $t \in \mathring{A}_\alpha$ such that strong causality fails at $\alpha(t)$. Let $t_- \in A_\alpha \cap (-\infty, t)$ and $t_+ \in A_\alpha \cap (t, \infty)$, and let $\mathscr{N}_0 \subset I^+(\alpha(t_-)) \cap I^-(\alpha(t_+))$ be a compact neighborhood of $\alpha(t)$ which does not totally future imprison any future endless causal curve. There exist a neighborhood $\mathscr{N} \subset \mathscr{N}_0$ of $\alpha(t)$ and a decreasing sequence of neighborhoods $\mathscr{N}_i \subset \mathscr{N}$ of $\alpha(t)$ converging to $\{\alpha(t)\}$ such that for each $i > 0$ there exists, from some $p_i \in \mathscr{N}_i$ to some $r_i \in \mathscr{N}_i$, a causal curve λ_i which cuts $M - \mathscr{N}$. The λ_i admit a nondegenerate causal cluster curve λ from $\alpha(t)$ which is either future endless in M or has a future endpoint at $\alpha(t)$. Since the latter possibility is excluded by Proposition 3.4 there exists $q \in |\lambda| - \mathscr{N}_0$. If λ cut $I^+(\alpha(t))$ then, for any $t' \in A_\alpha \cap (t, \infty)$ such that $\alpha(t') \in I^-(|\lambda|)$, there would exist $j > 0$ such that λ_j cut $I^+(\alpha(t'))$, with $r_j \in \mathscr{N}_j \subset I^-(\alpha(t'))$. Since this would imply chronology violation at $\alpha(t')$, contrary to Proposition 3.3, λ cannot therefore cut $I^+(\alpha(t))$ and so must be a null geodesic. The concatenation of $\alpha|[t_-, t]$ and the segment of λ from $\alpha(t)$ to q is therefore a nongeodesic causal curve from $\alpha(t_-)$ to q of length $t - t_-$. Hence there exist $\varepsilon > 0$ and a neighborhood $\mathscr{V}_q \subset (M - \mathscr{N}_0) \cap I^+(\alpha(t_-))$ of q such that $d(\alpha(t_-), q') > t - t_- + 2\varepsilon$ for all $q' \in \mathscr{V}_q$. There also exists $I > 0$ such that $d(r', \alpha(t_+)) > t - t_+ - \varepsilon$ for all $r' \in \mathscr{N}_I \subset I^-(\alpha(t_+))$. Choose $j > I$ such that there exists $q_j \in |\lambda_j| \cap \mathscr{V}_q$. There exists a timelike curve σ_j^- from $\alpha(t_-)$ to q_j of length $L(\sigma_j^-) > t - t_- + 2\varepsilon$ and a timelike curve σ_j^+ from $r_j \in \mathscr{N}_j \subset \mathscr{N}_I$ to $\alpha(t_+)$ of length $L(\sigma_j^+) > t_+ - t - \varepsilon$. The concatenation of σ_j^-, the segment of λ_j from q_j to r_j, and σ_j^+ is a causal curve σ_j from $\alpha(t_-)$ to $\alpha(t_+)$ of length $L(\sigma_j) \geq L(\sigma_j^-) + L(\sigma_j^+) > t_+ - t_- + \varepsilon$. This is impossible. q.e.d.

If $(M, \mathbf{g})$ is not globally hyperbolic, there may exist causal curves of arbitrarily great length between fixed points $p, q \in M$. According to the next result this does not occur if $p, q \in J^+(|\alpha|) \cap J^-(|\alpha|)$ for a timelike line segment α.

Proposition 3.6. *Let $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ be a unit speed timelike line segment. Then for any $p, q \in J^+(|\alpha|) \cap J^-(|\alpha|)$ such that $q \in J^+(p)$ one has $d(p,q) \leq t^+_{\alpha,q} - t^-_{\alpha,p}$.*

The following result and its corollary imply that the pre-Busemann and Busemann functions associated with a timelike line segment α are finite wherever they are defined in $J^+(|\alpha|) \cap J^-(|\alpha|)$.

Proposition 3.7. *Let $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ be a unit speed timelike line segment. Then:*

(I) *for each $t \in A_\alpha$ one has $b^+_{\alpha,u}(\alpha(t)) = b^-_{\alpha,s}(\alpha(t)) = t$ for all $s \in A_\alpha \cap (-\infty, t]$ and all $u \in A_\alpha \cap [t, \infty)$;*

(II) *for each $p \in J^+(|\alpha|) \cap J^-(|\alpha|)$ one has $t^-_{\alpha,p} \leq b^-_{\alpha,s}(p) \leq b^+_{\alpha,u}(p) \leq t^+_{\alpha,p}$ for all $s \in A^-_{\alpha,p}$ and all $u \in A^+_{\alpha,p}$.*

Corollary. (I) *$b^+_\alpha(\alpha(t)) = b^-_\alpha(\alpha(t)) = t$ for all $t \in A_\alpha$;*

(II) *$t^-_{\alpha,p} \leq b^-_\alpha(p) \leq b^+_\alpha(p) \leq t^+_{\alpha,p}$ for all $p \in J^+(|\alpha|) \cap J^-(|\alpha|)$.*

Let $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ be a unit speed timelike line segment, and let $\{\mathbf{e}_\mu\colon \mu = 0, \cdots, n\}$ be a parallelly propagated orthonormal tetrad along α such that $\mathbf{e}_0 = \dot\alpha$. There exists an open neighborhood $\hat{\mathscr{A}}_\alpha$ of $A_\alpha \times \{0\}$ in $A_\alpha \times \mathbb{R}^n$ admitting a diffeomorphism

$$\Phi\colon \mathbb{R} \times \mathbb{R}^n \supset \hat{\mathscr{A}}_\alpha \to \mathscr{A}_\alpha \subset M$$

of the form

$$(x^0; x^1, \cdots, x^n) \to \exp_{\alpha(x^0)}\left(\textstyle\sum_{i=1}^n x^i \mathbf{e}_i\right).$$

Let $\hat{\mathscr{A}}_\alpha$ be chosen sufficiently small such that the hypersurfaces $\{x^0 = \text{const}\}$ of $\mathscr{A}_\alpha$ are spacelike, and such that the hypersurfaces $\{r = \text{const}\}$ are timelike. If A_α is open in $\mathbb{R}$, then $\mathscr{A}_\alpha$ is open in M.

The coordinates $(x^0; x^1, \cdots, x^n)$ on $\mathscr{A}_\alpha$ will be of use in the expression of estimates near α. Components of tensors at points of $\mathscr{A}_\alpha$ will be expressed solely with respect to the basis $\{\partial_\mu\}$. For any vector $\mathbf{v} = \sum_{\lambda=0}^n v^\lambda \partial_\lambda$ at any point of $\mathscr{A}_\alpha$ it is convenient to define $\underline{\mathbf{v}} := \sum_{i=1}^n v^i \partial_i$. Norms of tensors on $\mathscr{A}_\alpha$ will be evaluated with respect to the positive definite metric $\overline{\mathbf{g}}$ defined on $\mathscr{A}_\alpha$ by $\overline{\mathbf{g}}(\mathbf{X}, \mathbf{Y}) = \sum_{\lambda=0}^n X^\lambda Y^\lambda$. Use will also be made of the flat Lorentzian metric on $\mathscr{A}_\alpha$ defined by $\boldsymbol{\eta} := \sum_{\mu,\nu=0}^n \eta_{\mu\nu} \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$ where $\eta_{\mu\nu} := \operatorname{diag}(-1, 1, \cdots, 1)$.

For each $t \in A_\alpha$, $T > 0$ and $R > 0$, set $\hat U(T, R; t) := [t - T, t + T] \times B^n_R \subset \mathbb{R} \times \mathbb{R}^n$, where B^n_R is the closed n-ball of radius R centered on the origin in $\mathbb{R}^n$. Let $U(T, R; t) := \Phi(\hat U(T, R; t) \cap \hat{\mathscr{A}}_\alpha)$. If $t \in \overset{\circ}{A}_\alpha$, then $U(T, R; t)$ is a neighborhood of $\alpha(t)$ in M. For any $t \in \overset{\circ}{A}_\alpha$ it is possible to choose T

and R sufficiently small to give $\hat{U}(T,R;t) \subset \hat{\mathscr{A}}_\alpha$, in which case $U(T,R;t)$ is compact.

For the remainder of this section let $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ be a unit speed timelike line segment and let $t \in \overset{\circ}{A}_\alpha$. If $(M,\mathbf{g})$ is globally hyperbolic then, for any $s \in A_\alpha \cap (-\infty,t)$ and $u \in A_\alpha \cap (t,\infty)$, the functions $b^+_{\alpha,u} = u - d(\cdot,\alpha(u))$ and $b^-_{\alpha,s} = s + d(\alpha(s),\cdot)$ are smooth in a neighborhood of $\alpha(t)$. This leads to estimates for $b^+_{\alpha,u}$ and $b^-_{\alpha,s}$ near $\alpha(t)$ as in Lemma 3.1 of Eschenberg [5]. It will now be shown that the strong causality at $\alpha(t)$ suffices for such estimates.

Lemma 3.8. *For any $t_- \in A_\alpha \cap (-\infty,t)$ there exist a neighborhood $\mathscr{N}_- \subset I^+(\alpha(t_-))$ of $\alpha(t)$ in M, and a constant $C_- > 0$ such that*

$$|b^-_{\alpha,s} - x^0| \le C_- r^2 \tag{1a}$$

on $\mathscr{N}_- \cap \mathscr{A}_\alpha$ for all $s \in A_\alpha \cap (-\infty,t_-]$. For any $t_+ \in A_\alpha \cap (t,\infty)$ there exist a neighborhood $\mathscr{N}_+ \subset I^-(\alpha(t_+))$ of $\alpha(t)$ in M and a constant $C_+ > 0$ such that

$$|b^+_{\alpha,u} - x^0| \le C_+ r^2 \tag{1b}$$

on $\mathscr{N}_+ \cap \mathscr{A}_\alpha$ for all $u \in A_\alpha \cap [t_+,\infty)$.

Proof. Let $\mathscr{U} \subset A_\alpha$ be a local causality neighborhood of $\alpha(t)$ in M, and let $t'_+ \in (t,t_+] \cap \alpha^{-1}(\mathscr{U})$, $t'_- \in [t_-,t) \cap \alpha^{-1}(\mathscr{U})$. Choose $T,R > 0$ such that $U(T,R;t)$ is compact and contained in $I^+(\alpha(t'_-)) \cap I^-(\alpha(t'_+)) \subset \mathscr{U}$. For each $p \in I^-(\alpha(t'_+)) \cap \mathscr{U}$ one has $d(p,\alpha(t'_+)) = d(p,\alpha(t'_+);\mathscr{U})$. Hence the function $d(\cdot,\alpha(t'_+))$ is smooth on $I^-(\alpha(t'_+);\mathscr{U})$, and those of its level surfaces which meet α in $I^-(\alpha(t'_+);\mathscr{U})$ do so orthogonally. Thus $b^+_{\alpha,t'_+} = t'_+ - d(\cdot,\alpha(t'_+))$ is smooth on $U(T,R;t)$ and satisfies

$$b^+_{\alpha,t'_+} - x^0 = \partial_i b^+_{\alpha,t'_+} = 0, \qquad i = 1,\cdots,n \tag{2}$$

on $|\alpha| \cap U(T,R;t)$. Since $U(T,R;t)$ is compact, there exists a constant $C'_+ > 0$ such that

$$|\partial_i\partial_j b^+_{\alpha,t'_+}| \le 2C'_+, \qquad i,j = 1,\cdots,n \tag{3}$$

on $U(T,R;t)$. From (2) and (3) we obtain

$$|b^+_{\alpha,t'_+} - x^0| \le C'_+ r^2 \tag{4}$$

on $U(T,R;t)$. Similarly there exists a constant $C'_- > 0$ such that

$$|b^-_{\alpha,t'_-} - x^0| \le C'_- r^2 \tag{5}$$

on $U(T,R;t)$. Since one has $s \le t_- \le t'_- < t'_+ \le t_+ \le u$, the inequalities (4) and (5) give

$$x^0 - C'_- r^2 \le b^-_{\alpha,t'_-} \le b^-_{\alpha,s} \le b^+_{\alpha,u} \le b^+_{\alpha,t'_+} \le x^0 + C'_+ r^2$$

on $U(T, R; t)$. Setting $C := \max\{C'_+, C'_-\}$ one obtains

$$x^0 - Cr^2 \leq b^-_{\alpha,s} \leq b^+_{\alpha,u} \leq x^0 + Cr^2 \tag{6}$$

on $U(T, R; t)$. This implies both (1a) and (1b) for $\mathscr{N}_+ = \mathscr{N}_- = U(T, R; t)$.

Corollary. *There exists a neighborhood* $\mathscr{N} \subset I^+(|\alpha|) \cap I^-(|\alpha|) \cap \mathscr{A}_\alpha$ *of* $\alpha(t)$ *admitting a constant* $C > 0$ *such that* $|b^+_\alpha - x^0| \leq Cr^2$ *and* $|b^-_\alpha - x^0| \leq Cr^2$ *on* $\mathscr{N}$.

Let $p, q \in M$ be such that $q \in I^+(p)$. If $(M, \mathbf{g})$ is globally hyperbolic, then one is assured of the existence of a timelike line segment from p to q. In general however there will only be a line segment from p to q under special conditions. In the case where $(M, \mathbf{g})$ is either globally hyperbolic or timelike geodesically complete, the following result furnishes a method of construction for the required line segment, provided p and q are suitably restricted in relation to the given line segment α.

Lemma 3.9. *Suppose* $(M, \mathbf{g})$ *is either globally hyperbolic or timelike geodesically complete. Let* $t_+ \in A_\alpha \cap (t, \infty)$. *Then there exists a neighborhood* $\mathscr{N} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+))$ *of* $\alpha(t)$ *such that, for any* $p \in \mathscr{N}$ *and* $u \in A_\alpha \cap [t_+, \infty)$, *and any sequence of causal curves* λ_i *to* $\alpha(u)$ *with past endpoints* $p_i \in \mathscr{N}$ *converging to* p *and lengths* $L(\lambda_i) > d(p, \alpha(u)) - 2^{-i}$, *every maximal causal cluster curve of the* λ_i *to* $\alpha(u)$ *is a timelike line segment from* p *to* $\alpha(u)$.

Proof. If $(M, \mathbf{g})$ is globally hyperbolic, the conclusion follows immediately. Suppose henceforth that $(M, \mathbf{g})$ is timelike geodesically complete but not necessarily globally hyperbolic.

Let $\mathscr{U} \subset I^-(\alpha(t_+)) \cap A_\alpha$ be a local causality neighborhood of $\alpha(t)$. Choose $T_1, R_1 > 0$ such that $U(T_1, R_1; t)$ is a compact subset of $\mathscr{U}$ and admits a constant $C > 0$ such that $|b^+_{\alpha,u} - x^0| \leq Cr^2$ on $U(T_1, R_1; t)$ for all $u \in A_\alpha \cap [t_+, \infty)$. Since the hypersurfaces $\{x^0 = \text{const}\}$ are smooth and spacelike, there exists a constant $k > 0$ such that, for any causal vector $\mathbf{X}$ at any point of $U(T_1, R_1; t)$, one has $\|\underline{\mathbf{X}}\| \leq k|X^0|$. Without loss of generality, assume $R_1 \geq 2kT_1$. Choose any $\delta \in (0, 1)$ such that $\delta T_1 < 1/(4k^2C)$. Let $T := (1-\delta)T_1$ and $R := \min\{R_1 - (2-\delta)kT_1, 1/(32kC), \delta k T_1\}$. Let $\mathscr{N} \subset \mathscr{U}(T, R; t)$ be a local causality neighborhood of $\alpha(t)$.

Let $p \in \mathscr{N}$ and $u \in A_\alpha \cap [t_+, \infty)$. Let λ_i be a sequence of causal curves to $\alpha(u)$ from points $p_i \in \mathscr{N}$ converging to p with lengths $L(\lambda_i) > d(p, \alpha(u)) - 2^{-i}$. Let σ^+ be a maximal causal cluster curve of the λ_i to $\alpha(u)$, and let σ_i be a subsequence of the λ_i having σ^+ as a limit curve. Let σ^- be a maximal causal cluster curve of the σ_i from p. One may assume, by passing to subsequences if necessary, that σ^- is a limit curve of the σ_i.

Suppose σ^- is not a geodesic. Let $\sigma^-_{p,q}$ be a nongeodesic segment of σ^- from p to some point q. Then there exists a timelike curve ν' from p to q of length $L(\nu') > L(\sigma^-_{p,q}) + 4\varepsilon_1$ for some $\varepsilon_1 > 0$. Let $\mathscr{V}_q \subset I^+(p)$ be an open neighborhood of q such that $d(p,q') > L(\nu') - \varepsilon_1 > L(\sigma^-_{p,q}) + 3\varepsilon_1$ for all $q' \in \mathscr{V}_q$. Clearly $\sigma^-_{p,q}$ is a limit curve of a sequence of segments ν_i of the corresponding σ_i from the p_i to points q_i converging to q. One may, by passing to subsequences if necessary, assume $L(\nu_i) < L(\sigma^-_{p,q}) + \varepsilon_1$ and $q_i \in \mathscr{V}_q$ for all i. For each i, there exists a timelike curve ν'_i from p to q_i of length $L(\nu'_i) > L(\sigma^-_{p,q}) + 2\varepsilon_1 > L(\nu_i) + \varepsilon_1$. For each i, the concatenation of ν'_i and the segment of σ_i from q_i to $\alpha(u)$ is a causal curve σ'_i from p to $\alpha(u)$ of length $L(\sigma'_i) \geq L(\sigma_i) + \varepsilon_1 > d(p,\alpha(u)) - 2^{-i} + \varepsilon_1$. This gives a contradiction for i sufficiently large. Hence σ^- is a causal geodesic.

Suppose σ^- is a future endless timelike geodesic. Then σ^- is future complete and so admits a segment $\sigma^-_{p,r}$ from p to some point r such that $L(\sigma^-_{p,r}) > d(p,\alpha(u)) + 3\varepsilon_2$ for some $\varepsilon_2 > 0$. Let $\mathscr{V}_r \subset I^+(p)$ be an open neighborhood of r such that $d(p,r') > L(\sigma^-_{p,r}) - \varepsilon_2 > d(p,\alpha(u)) + 2\varepsilon_2$ for all $r' \in \mathscr{V}_r$. Choose any integer $j > 0$ such that there exists $r_j \in |\sigma_j| \cap \mathscr{V}_r$. Let μ_j be a timelike curve from p to r_j of length $L(\mu_j) > d(p,\alpha(u)) + \varepsilon_2$. The concatenation of μ_j and the segment of σ_j from r_j to $\alpha(u)$ is then a causal curve from p to $\alpha(u)$ of length strictly greater than $d(p,\alpha(u)) + \varepsilon_2$. This is impossible.

Now suppose σ^- is a null geodesic, either future endless or with a future endpoint at $\alpha(u) \notin \mathscr{U}$. The inequality $R_1 - R \geq (2-\delta)kT_1 = k(T_1+T)$ ensures that σ^- leaves $U(T_1,R_1;t)$ in the future direction through the future boundary thereof. Hence, with σ^- parametrized such that $x^0(\sigma^-(v)) - x^0(p) = v$ for all $v \in (\sigma^-)^{-1}(\mathscr{U})$, $\sigma^-(v)$ must exist and lie in $\mathring{U}(T_1,R_1;t)$ for all $v \in [0,v_0]$, where $v_0 := \delta T_1 = T_1 - T$. Let $s := \sigma^-(v_0) \in \mathring{U}(T_1,R_1;t)$. Then $x^0(s) - x^0(p) = v_0$, $|r(s) - r(p)| \leq kv_0$ and $d(p,s) = d(p,s;\mathscr{U}) = 0$. Hence there exist open neighborhoods $\mathscr{V}_p \subset \mathring{U}(T,R;t)$ and $\mathscr{V}_s \subset \mathring{U}(T_1,R_1;t)$ of p and s respectively such that $|x^0(s') - x^0(s)| < v_0/16$, $|r^2(s') - r^2(s)| < v_0/16C$ and $d(p',s') < v_0/4$ for all $p' \in \mathscr{V}_p$ and $s' \in \mathscr{V}_s$ such that $s' \in J^+(p';\mathscr{U})$. Choose any integer $J > 0$ such that there exist $p_J \in |\sigma_J| \cap \mathscr{V}_p$ and $s_J \in |\sigma_J| \cap \mathscr{V}_s$, and such that $L(\sigma_J) > d(p,\alpha(u)) - v_0/4$. The segment of σ^- from $p_J \in \mathscr{V}_p$ to $s_J \in \mathscr{V}_s$ has length strictly less than $v_0/4$. Thus the segment of σ^- from s_J to $\alpha(u)$ has length strictly greater than $d(p,\alpha(u)) - v_0/2$, and therefore

$$b^+_{\alpha,u}(s_J) - b^+_{\alpha,u}(p) < v_0/2. \tag{7}$$

However one also has

$$\begin{aligned} b^+_{\alpha,u}(s_J) - b^+_{\alpha,u}(p) &\geq x^0(s_J) - x^0(p) - C(r^2(p) + r^2(s_J)) \\ &\geq v_0 - C(r^2(p) + r^2(s)) - v_0/8 \\ &\geq (1 - k^2 C v_0) v_0 - 2Cr(p)\{r(p) + kv_0\} - v_0/8. \end{aligned}$$

The use of the inequalities $k^2 C v_0 = k^2 C \delta T_1 < 1/4$ and $r(p) \leq R \leq \min\{1/(32kC), kv_0\}$ herein gives

$$b^+_{\alpha,u}(s_J) - b^+_{\alpha,u}(p) > v_0/2. \tag{8}$$

Clearly (7) and (8) are contradictory. Hence σ^- cannot be a null geodesic.

The preceding arguments yield that σ^- is a timelike geodesic from p to $\alpha(u)$. Since causality holds at $p \in \mathscr{N}$ and $\alpha(u)$, it follows that σ^- and σ^+ are coincident. Let $\sigma := \sigma^+ = \sigma^-$.

For each $i > 0$ let $\mathscr{U}_i$ be a local causality neighborhood of p such that no causal curve in $\mathscr{U}_i$ has length strictly greater than e^{-i}. Since σ is timelike, for each $i > 0$ there exists $j(i) > i$ such that $p_j \in \mathscr{U}_i$ and such that there exists $p_j^+ \in |\sigma_j| \cap I^+(p, \mathscr{U}_i)$. Let γ_j be a causal curve in $\mathscr{U}_i$ from p to p_j^+, and let σ_j' be the concatenation of γ_j and the segment of σ_j from p_j^+ to $\alpha(u)$. Then σ_j' is a causal curve from p to $\alpha(u)$ of length $L(\sigma_j') \geq L(\sigma_j) - e^{-i}$. Since one has $|\gamma_j| \dot{\subset} \mathscr{U}_i$ for all i, with $\bigcap_i \mathscr{U}_i = \{p\}$, the geodesic σ, being a limit curve of the σ_i, must also be a limit curve of the σ_j'. Hence

$$\begin{aligned} L(\sigma) &\geq \limsup_{j\to\infty} L(\sigma_j') \geq \limsup_{i\to\infty} (L(\sigma_{j(i)}) - e^{-i}) \\ &= \limsup_{i\to\infty} L(\sigma_{j(i)}) \geq d(p, \alpha(u)). \end{aligned}$$

In fact one must have $L(\sigma) = d(p, \alpha(u))$, which shows that σ is a line segment.

Corollary. *For any $u \in A_\alpha \cap [t_+, \infty)$ and any $p \in \mathscr{N} \subset I^-(\alpha(u))$ there exists at least one timelike line segment from p to $\alpha(u)$.*

The next result concerns the uniqueness of the timelike line segments referred to in the previous corollary. For the purposes of this and subsequent results, for each $p \in M$ let $U_p^+ \subset T_pM$ denote the space of all future-directed unit timelike vectors at p.

Lemma 3.10. *Suppose $(M, \mathbf{g})$ is either globally hyperbolic or timelike geodesically complete. Let $u \in A_\alpha \cap (t, \infty)$. Then there exist an open neighborhood $\mathscr{N}_u$ of $\alpha(t)$, an open neighborhood $\mathscr{X}_u$ of $\dot\alpha(u)$ in $U^+_{\alpha(u)}$, and $\varepsilon > 0$ such that*

(I) *the mapping $\mathscr{X}_u \times (u - t - \varepsilon, u - t + \varepsilon) \to M$ defined by $(\mathbf{X}, s) \mapsto \exp_{\alpha(u)}(-s\mathbf{X})$ is a diffeomorphism onto $\mathscr{N}_u$;*

(II) *for each* $\mathbf{X} \in \mathscr{X}_u$ *and each* $\hat{s} \in (u-t-\varepsilon, u-t+\varepsilon)$ *the curve* $[0,\hat{s}] \to M$ *given by* $s \mapsto \exp_{\alpha(u)}(-(\hat{s}-s)\mathbf{X})$ *is the unique line segment in* M *from* $\exp_{\alpha(u)}(-\hat{s}\mathbf{X})$ *to* $\alpha(u)$.

Proof. Let $\mathscr{V}$ be a neighborhood of $\alpha(t)$ as in Lemma 3.9 (with u here taking the place of t_+). For each $p \in \mathscr{V}$ there exists at least one timelike line segment from p to $\alpha(u)$. Since $t \in \overset{\circ}{A}_\alpha$ and $u \in A_\alpha$ cannot be conjugate values for α, there exist an open neighborhood $\mathscr{X}'_u$ of $\dot{\alpha}(u)$ in $U^+_{\alpha(u)}$ and $\varepsilon' > 0$ such that the mapping $\Theta\colon \mathscr{X}'_u \times (u-t-\varepsilon', u-t+\varepsilon') \to M$ defined by $(\mathbf{X}, s) \mapsto \exp_{\alpha(u)}(-s\mathbf{X})$ is a diffeomorphism onto an open neighborhood $\mathscr{Y} \subset \mathscr{V}$ of $\alpha(t)$. Let $\mathscr{X}'_u$ be such that there exists $\delta > 0$ such that $\|\mathbf{X}\| < 1+\delta$ for all $\mathbf{X} \in \mathscr{X}'_u$. Since strong causality holds at $\alpha(t)$, one may suppose that $\mathscr{X}'_u$ and $\varepsilon' > 0$ are sufficiently small such that, for each $\mathbf{X} \in \mathscr{X}'_u$, there exists no $s > u-t+\varepsilon'$ such that $\exp_{\alpha(u)}(-s\mathbf{X}) \in \mathscr{Y}$.

Let $\varepsilon_i \in (0, \varepsilon')$ and $\mathscr{X}_i \subset \mathscr{X}'_u$ be decreasing sequences, with the ε_i converging to zero and the $\mathscr{X}_i$ neighborhoods of $\dot{\alpha}(u)$ in $U^+_{\alpha(u)}$ converging to $\{\dot{\alpha}(u)\}$. Then the $\mathscr{N}_i := \{\exp_{\alpha(u)}(-s\mathbf{X})\colon \mathbf{X} \in \mathscr{X}_i,\ s \in (u-t-\varepsilon_i, u-t+\varepsilon_i)\} \subset \mathscr{Y}$ are a decreasing sequence of neighborhoods of $\alpha(t)$ converging to $\{\alpha(t)\}$. Suppose that for each i there exists, from some $p_i \in \mathscr{N}_i \subset \mathscr{V} \subset I^-(\alpha(u))$ to $\alpha(u)$, a unit speed timelike line segment $\sigma_i\colon [0, s_i] \to M$, $s_i := d(p_i, \alpha(u))$, not of the form $s \mapsto \exp_{\alpha(u)}(-(\hat{s}-s)\mathbf{X})$ for any $(\mathbf{X}, \hat{s}) \in \mathscr{X}'_u \times (u-t-\varepsilon', u-t+\varepsilon')$. One must have $\dot{\sigma}_i(s_i) \notin \mathscr{X}'_u$ and hence $\|\dot{\sigma}_i(s_i)\| \geq 1+\delta$ for each i. Clearly the p_i converge to $\alpha(t)$. Moreover, for each i, there exists $j(i) > i$ such that $p_j \in I^-(\alpha(t+\min\{2^{-i}, u-t\}))$, and hence such that $L(\sigma_j) > u-t-2^{-i}$. One may therefore, by passing to subsequences if necessary, assume $L(\sigma_i) > u-t-2^{-i}$ for all i. Thus the σ_i admit a causal cluster curve $\sigma\colon [0, u-t] \to M$ which is a timelike line segment from $\alpha(t)$ to $\alpha(u)$. One must have $\|\dot{\sigma}(u-t)\| \geq 1+\delta$ and hence $\dot{\sigma}(u-t) \neq \dot{\alpha}(u)$. Let $t_- \in A_\alpha \cap (-\infty, t)$. Then the concatenation of $\alpha|[t_-, t]$ and σ is a broken timelike geodesic segment from $\alpha(t_-)$ to $\alpha(u)$ of length $u-t_-$. There follows $d(\alpha(t_-), \alpha(u)) > u-t_-$ which gives a contradiction.

It is now evident that there exists an integer $i > 0$ such that, for each $(\mathbf{X}, \hat{s}) \in \mathscr{X}_i \times (u-t-\varepsilon_i, u-t+\varepsilon_i)$, the geodesic $[0,\hat{s}] \to M$ defined by $s \mapsto \exp_{\alpha(u)}(-(\hat{s}-s)\mathbf{X})$ is the unique unit speed timelike line segment from $\exp_{\alpha(u)}(-\hat{s}\mathbf{X}) \in \mathscr{N}_i$ to $\alpha(u)$. Moreover $\Theta|\mathscr{X}_i \times (u-t-\varepsilon_i, u-t+\varepsilon_i)$ is a diffeomorphism onto $\mathscr{N}_i$.

Corollary. *The function* $b^+_{\alpha,u}|\mathscr{N}_u$ *is smooth and is given by*

$$b^+_{\alpha,u}(\exp_{\alpha(u)}(-s\mathbf{X})) = u - s$$

for all $(\mathbf{X}, s) \in \mathscr{X}_u \times (u-t-\varepsilon, u-t+\varepsilon)$.

The next result is a refinement of Lemma 3.2 of [5] and generalizes the Proposition of the appendix of [8]. It provides estimates for the initial directions of timelike line segments which originate near and terminate on $|\alpha|$.

Lemma 3.11. *Suppose* $t \in \mathring{A}_\alpha$ *and let* $t_+ \in A_\alpha \cap (t, \infty)$. *Then there exists a neighborhood* $\mathcal{N} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+)) \cap \mathscr{A}_\alpha$ *of* $\alpha(t)$ *admitting a constant* $B > 0$ *such that, for any* $p \in \mathcal{N}$, *any* $u \in A_\alpha \cap [t_+, \infty)$ *and any unit speed timelike line segment* $\sigma\colon [0, d(p, \alpha(u))] \to M$ *from* p *to* $\alpha(u)$, *one has* $\|\dot{\sigma}(0)\| \leq 1 + Br(p)$ *and* $\|\underline{\dot{\sigma}}(0)\| \leq Br^{1/2}(p)$.

Proof. Let $t_- \in A_\alpha \cap (-\infty, t)$, and let $\mathscr{U} \subset\subset I^+(\alpha(t_-)) \cap I^-(\alpha(t_+)) \cap \mathscr{A}_\alpha$ be a local causality neighborhood of $\alpha(t)$. Choose $T_1, R_1 > 0$ such that $U(T_1, R_1; t)$ is a compact subset of $\mathscr{U}$ and admits a constant $C > 0$ such that $|b^+_{\alpha,u} - x^0| \leq Cr^2$ on $U(T_1, R_1; t)$ for all $u \in A_\alpha \cap [t_+, \infty)$. Since the hypersurfaces $\{x^0 = \text{const}\}$ are smooth and spacelike, there exists a constant $k > 0$ such that, for every $q \in U(T_1, R_1; t)$, every causal vector $\mathbf{X} \in T_qM$ satisfies $\|\underline{\mathbf{X}}\| \leq k|X^0|$. Without loss of generality one may assume $R_1 \geq 2kT_1$. Set $\mathbf{h} := \mathbf{g} - \boldsymbol{\eta}$ on $\mathscr{A}_\alpha$. Then $\mathbf{h}$, $\nabla\mathbf{h}$ and the Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ all vanish on $|\alpha|$. Hence there exist constants $P_1, P_2 > 0$ such that $\|\mathbf{h}\| \leq P_1 r^2$ and $\|\Gamma\| \leq P_2$ on $U(T_1, R_1; t)$.

Choose $T, R > 0$ such that $T < T_1$ and $R < \min\{R_1 - k(T_1 + T), k(T_1 - T), 1/(20kC), 5kC/4P_1, k/10P_2\}$. Let $p \in U(T, R; t)$, $u \in A_\alpha \cap [t_+, \infty)$ and suppose there exists a unit speed timelike line segment $\sigma\colon [0, v_2] \to M$ from p to $\alpha(u)$. If $p \in |\alpha|$ then, since α is a line segment and cannot have a past endpoint at $p \in U(T, R; t) \subset I^+(\alpha(t_-))$, σ is a segment of α and must satisfy $\|\dot{\sigma}(0)\| = 1$ and $\|\underline{\dot{\sigma}}(0)\| = 0$ as required. Suppose henceforth $p \notin |\alpha|$ and therefore $r(p) > 0$.

Since one has $\mathscr{U} \subset\subset I^-(\alpha(u)) \cap \mathscr{A}_\alpha$, the unit speed timelike line segment $\sigma\colon [0, v_2] \to M$ may be reparametrized to give a timelike line segment $\tilde{\sigma}\colon [0, \tilde{v}_2] \to M$ from $p \in U(T, R; t)$ to $\alpha(u) \notin \overline{\mathscr{U}}$ such that $x^0(\tilde{\sigma}(\tilde{v})) - x^0(p) = \tilde{v}$ for all $\tilde{v} \in \tilde{\sigma}^{-1}(\mathscr{U})$. The inequality $R_1 - R \geq k(T_1 + T)$ ensures that $\tilde{\sigma}$ can only leave $U(T_1, R_1; t)$ through the future boundary thereof, and so one must have $\tilde{v}_2 \geq T_1 - T$. Let $\tilde{v}_1 := k^{-1}r(p) \leq k^{-1}R \leq T_1 - T \leq \tilde{v}_2$. Then $\tilde{\sigma}|[0, \tilde{v}_1]$ is a timelike line segment in $U(T_1, R_1; t)$. Setting $v_1 := \sigma^{-1} \circ \tilde{\sigma}(\tilde{v}_1)$ one has

$$v_1 = b^+_{\alpha,u}(\tilde{\sigma}(\tilde{v}_1)) - b^+_{\alpha,u}(p), \tag{9}$$

and therefore

$$|v_1 - x^0(\tilde{\sigma}(\tilde{v}_1)) + x^0(p)| \leq C\{r^2(p) + r^2(\tilde{\sigma}(\tilde{v}_1))\}.$$

Since one has $r(\tilde{\sigma}(\tilde{v})) \leq r(p) + k\tilde{v} \leq r(p) + k\tilde{v}_1 \leq 2r(p)$ for all $\tilde{v} \in [0, \tilde{v}_1]$, there follows

$$|v_1 - \tilde{v}_1| \leq 5Cr^2(p) = 5kCr(p)\tilde{v}_1. \tag{10}$$

One now has

$$x^0(\tilde{\sigma}(\tilde{v}_1)) - x^0(p) = \tilde{v}_1 \leq (1 - 5kCr(p))^{-1} v_1, \tag{11}$$

$$v_1 < 5\tilde{v}_1/4 = 5r(p)/4k \tag{12}$$

by means of $r(p) \leq R < 1/(20kC)$. In conjunction with the intermediate value theorem, inequality (11) implies that there exists $v_0 \in [0, v_1]$ such that $\dot{\sigma}^0(v_0) \leq (1 - 5kCr(p))^{-1}$. By means of the equation

$$-1 = g_{\mu\nu}\dot{\sigma}^\mu\dot{\sigma}^\nu = (\eta_{\mu\nu} + h_{\mu\nu})\dot{\sigma}^\mu\dot{\sigma}^\nu \tag{13}$$

one has

$$(1 - \|\mathbf{h}(v_0)\|)\|\dot{\sigma}(v_0)\|^2 \leq 2|\dot{\sigma}^0(v_0)|^2 - 1 \leq \left\{\frac{1 + 5kCr(p)}{1 - 5kCr(p)}\right\}^2.$$

Moreover, since the inequalities

$$r(\sigma(v_0)) \leq 2r(p) \leq 2R \leq \min\{5kC/2P_1, 1/(10kC)\}$$

imply

$$\|\mathbf{h}(\sigma(v_0))\| \leq P_1 r^2(\sigma(v_0)) \leq 4P_1 r^2(p) \leq 4P_1 R r(p) \leq 5kCr(p) \leq 1/4,$$

there follows

$$\|\dot{\sigma}(v_0)\| \leq \frac{1 + 5kCr(p)}{(1 - 5kCr(p))^{3/2}} \leq \left\{1 - \frac{25kCr(p)}{2}\right\}^{-1}. \tag{14}$$

The geodesic equation for σ gives

$$-\frac{d}{dv}\|\dot{\sigma}(v)\| \leq \|\Gamma\|\|\dot{\sigma}(v)\|^2 \leq P_2\|\dot{\sigma}(v)\|^2,$$

integration of which yields

$$\|\dot{\sigma}(v_0)\|^{-1} - \|\dot{\sigma}(0)\|^{-1} \leq P_2 v_0 \leq P_2 v_1 \leq 5P_2 r(p)/4k.$$

Thus (14) implies

$$\|\dot{\sigma}(0)\|^{-1} \geq 1 - \tfrac{1}{4}B_0 r(p), \tag{15}$$

where $B_0 := 5(10kC + P_2/k)$. Since one has $r(p) \leq R < \min\{1/(20kC), k/10P_2\}$ and hence $B_0 r(p) \leq 3$, the inequality (15) may be rearranged to give

$$\|\dot{\sigma}(0)\| - 1 \leq B_0 r(p) \leq 3. \tag{16}$$

Moreover by (13) and (16) one has

$$\begin{aligned}\|\dot{\underline{\sigma}}(0)\|^2 &= \tfrac{1}{2}\{\|\dot{\sigma}(0)\|^2 - 1 - \mathbf{h}(\dot{\sigma}(0), \dot{\sigma}(0))\} \\ &\leq \tfrac{1}{2}\{(1 + \|\mathbf{h}(p)\|)\|\dot{\sigma}(0)\|^2 - 1\} \leq B_1^2 r(p),\end{aligned} \tag{17}$$

where $B_1 := (8P_1R + 5B_0/2)^{1/2}$. Setting $B := \max\{B_0, B_1\}$ one obtains the result. q.e.d.

It is now possible to establish a result which will lead directly to conditions for the continuity of b_α^+ near $\alpha(t)$.

Proposition 3.12. *Suppose $(M, \mathbf{g})$ is either globally hyperbolic or timelike geodesically complete. Suppose $t \in \overset{\circ}{A}_\alpha$ and let $t_+ \in A_\alpha \cap (t, \infty)$. Then there exists a neighborhood $\mathcal{N} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+)) \cap \mathcal{A}_\alpha$ of $\alpha(t)$ admitting a constant $L > 0$ such that $b_{\alpha,u}^+$ is Lipschitz L-continuous on $\mathcal{N}$ for all $u \in A_\alpha \cap [t_+, \infty)$.*

Proof. Let $\mathscr{U} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+)) \cap A_\alpha$ be a local causality neighborhood of $\alpha(t)$. Choose $T, R > 0$ such that $U(T, R; t)$ is a compact subset of $\mathscr{U}$, and is a neighborhood of $\alpha(t)$ having the property described in Lemma 3.9 and admitting a constant $B > 0$ as in Lemma 3.11. There exists a constant $P > 0$ such that $\|\mathbf{g}\| \leq P$ on $U(T, R; t)$.

Let $p \in U(T, R; t)$. Let $u \in A_\alpha \cap [t_+, \infty)$ and set $\tau := d(p, \alpha(u))$. The choice of $T, R > 0$ ensures that there exists a unit speed timelike line segment $\sigma \colon [u - \tau, u] \to M$ from $p \in U(T, R; t)$ to $\alpha(u)$. Let $t'_+ \in (u - \tau, u) \cap \sigma^{-1}(\mathscr{U})$. Then, on the neighborhood $I^-(\sigma(t'_+); \mathscr{U})$ of p, the function b_{σ,t'_+}^+ is smooth and satisfies $b_{\sigma,t'_+}^+ \geq b_{\sigma,u}^+ = b_{\alpha,u}^+$ with $b_{\sigma,t'_+}^+(p) = u - \tau = b_{\alpha,u}^+$. Moreover from

$$\nabla b_{\sigma,t'_+}^+(p) = (\dot{\sigma}(u - \tau))^\flat$$

one has

$$\|\nabla b_{\sigma,t'_+}^+(p)\| \leq \|\mathbf{g}(p)\| \|\dot{\sigma}(u - \tau)\| \leq L,$$

where $L := P(1 + BR)$. This shows that, for any $p \in U(T, R; t)$ there exists, on a neighborhood of p, a real-valued function f_p which smoothly supports $b_{\alpha,u}^+$ from above at p, and satisfies $\|\nabla f_p(p)\| \leq L$ where $L > 0$ is a constant independent of p. It follows [5, Appendix] that $b_{\alpha,u}^+$ is Lipschitz L-continuous on $U(T, R; t)$ for all $u \in A_\alpha \cap [t_+, \infty)$.

Corollary. *b_α^+ is Lipschitz L-continuous on $\mathcal{N}$.*

There remains the possibility that b_α^+ is not C^1 at $\alpha(t)$. However the next result shows that, near $\alpha(t)$, b_α^+ admits a class of smooth upper support functions, the second derivatives of which are readily estimated.

Lemma 3.13. *Let $(M, \mathbf{g})$ satisfy the timelike convergence condition ($\mathbf{Ricc}(\mathbf{v}, \mathbf{v}) \geq 0$ for all timelike $\mathbf{v}$) and be either globally hyperbolic or timelike* *geodesically complete. Suppose α is future complete, with $t \in \overset{\circ}{A}_\alpha$, and let $t_+ \in (t, \infty) \subset A_\alpha$. Then there exists a neighborhood $\mathcal{N} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+))$ of $\alpha(t)$ admitting a constant $D > 0$ such that, for any $u \in [t_+, \infty)$, there exists a dense subset $\mathcal{N}'$ of $\mathcal{N}$ which intersects every spacelike hypersurface of $\overset{\circ}{\mathcal{N}}$ in a relatively dense set, and is such that for each $p \in \mathcal{N}'$ there is a unit speed timelike line segment $\sigma\colon [s_{p,u}, u] \to M$, $s_{p,u} := u - d(p, \alpha(u))$, from p to $\alpha(u)$, and $\hat{s}_{p,u} \in (s_{p,u}, u)$ such that*

(I) *$b^+_{\sigma,s}$ smoothly supports $b^+_{\alpha,u}$ from above at p;*

(II) $\operatorname{Hess} b^+_{\sigma,s}(\mathbf{X}, \mathbf{X}) \le D\|\mathbf{X}\|^2$ *for all* $\mathbf{X} \in T_pM$;

(III) $\Box b^+_{\sigma,s}(p) \le n/(s - s_{p,u})$,

for all $s \in [\hat{s}_{p,u}, u]$.

Proof. Let $\mathcal{U} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+)) \cap A_\alpha$ be a local causality neighborhood of $\alpha(t)$ such that $d(q, \alpha(t_+)) > (t_+ - t)/2$ for all $q \in \mathcal{U}$. Let $T, R > 0$ be such that $U(T, R; t)$ is a compact subset of $\mathcal{U}$ having the property described in the corollary to Lemma 3.9 and admitting a constant $B > 0$ as in Lemma 3.11. Let $\mathcal{N} := U(T, R; t)$. Choose $u \in [t_+, \infty)$ and, for each $q \in \mathcal{N}$, let $s_q := u - d(q, \alpha(u))$ and $\hat{s}_q := s_q + (t_+ - t)/2 \in (s_q, u)$. The timelike line segments to $\alpha(u)$ of the form $\mathbf{R} \supset (a, b] \to M$ cut a dense subset $\mathcal{N}'$ of $\mathcal{N}$ having a relatively dense intersection with every spacelike hypersurface of $\overset{\circ}{\mathcal{N}}$. Choose $p \in \mathcal{N}'$ and let $\sigma\colon [s_p, u] \to M$ be a timelike line segment from p to $\alpha(u)$.

(I) Let $s \in [\hat{s}_p, u]$. Since $\sigma\colon [s_p, u] \to M$ is a line segment from p to $\alpha(u)$, one has $b^+_{\sigma,s}(p) = s_p = u - d(p, \alpha(u)) = b^+_{\alpha,u}(p)$. Proposition 2.2(II) gives $b^+_{\sigma,s} \ge b^+_{\sigma,u} = b^+_{\alpha,u}$ on $J^-(\sigma(s))$, and Lemma 3.10 gives that $b^+_{\sigma,s}$ is smooth near p. Thus $b^+_{\sigma,s}$ smoothly supports $b^+_{\alpha,u}$ from above at p.

(II) For each $q \in \mathcal{N}$ let $U^+_{q,B} := \{\mathbf{v} \in U^+_q \colon \|\underline{\mathbf{v}}\| \le Br^{1/2}\}$. For each $q \in \mathcal{N}$ and each $\mathbf{Y} \in U^+_q$ let $\mu_\mathbf{Y}\colon A_{\mu_\mathbf{Y}} \to M$ be the maximal unit speed timelike line segment from q in M such that $\dot{\mu}_\mathbf{Y}(s_q) = \mathbf{Y}$. Since $\mathcal{N}$ is a compact subset of the open set $\mathcal{U}$, there exists $\delta \in (0, (t_+ - t)/2)$ such that $\mu_\mathbf{Y}(s) \in \mathcal{U}$ for all $\mathbf{Y} \in U^+_{q,B}$, all $q \in \mathcal{N}$ and all $s \in [s_q, \check{s}_q]$, where $\check{s}_q := s_q + \delta \in (s_q, \hat{s}_q)$. For each $q \in \mathcal{N}$ the mapping $T_qM \times U^+_{q,B} \to \mathbb{R}$ defined by $(\mathbf{X}, \mathbf{Y}) \mapsto \operatorname{Hess} b^+_{\mu_\mathbf{Y}, \check{s}_q}(\mathbf{X}, \mathbf{X})$ is continuous. Since $U^+_{q,B}$ is compact it follows that, for each $q \in \mathcal{N}$, there exists a constant $D_q > 0$ such that $\operatorname{Hess} b^+_{\mu_\mathbf{Y}, \check{s}_q}(\mathbf{X}, \mathbf{X}) \le D_q\|\mathbf{X}\|^2$ for all $(\mathbf{X}, \mathbf{Y}) \in T_qM \times U^+_{q,B}$. The continuity of $\operatorname{Hess} b^+_{\mu_\mathbf{Y}, \check{s}_q}(\mathbf{X}, \mathbf{X})$ as a function of $(\mathbf{X}, \mathbf{Y})$ on the compact space $\cup_{q \in \mathcal{N}} T_qM \times U^+_{q,B}$ ensures that $D := \sup_{q \in \mathcal{N}} D_q$ exists.

For any $\mathbf{Y} \in U^+_{p,B}$ and any $s \in A_{\mu_\mathbf{Y}} \cap [\check{s}_p, \infty)$, the function $b^+_{\mu_\mathbf{Y},s}$ smoothly supports $b^+_{\mu_\mathbf{Y},\check{s}}$ from below at p. One therefore has $\operatorname{Hess} b^+_{\mu_\mathbf{Y},s}(\mathbf{X},\mathbf{X}) \leq \operatorname{Hess} b^+_{\mu_\mathbf{Y},\check{s}_p}(\mathbf{X},\mathbf{X}) \leq D\|\mathbf{X}\|^2$ for all $(\mathbf{X},\mathbf{Y}) \in T_pM \times U^+_{p,B}$ and all $s \in A_{\mu_\mathbf{Y}} \cap [\check{s}_p, \infty)$. The unit speed timelike line segment $\sigma\colon [s_p, u] \to M$ from p to $\alpha(u)$ is of the form $\mu_\mathbf{Z}|[s_p, u]$ for some $\mathbf{Z} \in U^+_p$. The choice of $T, R > 0$ guarantees $\mathbf{Z} \in U^+_{p,B}$ and the result follows.

(III) This is an easy consequence of the timelike convergence condition and the Raychaudhuri equation.

The next result is essentially the same as Lemma 2.4 of Galloway [8].

Lemma 3.14. *Suppose $(M, \mathbf{g})$ satisfies the timelike convergence condition and is either globally hyperbolic or timelike geodesically complete. Suppose also that α is future complete. Let $t \in \mathring{A}_\alpha$. Then there exists a neighborhood $\mathscr{N} \subset I^+(|\alpha|) \cap I^-(|\alpha|)$ of $\alpha(t)$ such that every smooth connected embedded maximal spacelike hypersurface $\mathscr{S}$ in $\mathring{\mathscr{N}}$ satisfies the following.*

(I) *If $b^+_\alpha|\mathscr{S}$ attains its infimum then it is constant.*

(II) *If $\operatorname{edge}(\mathscr{S}) \neq \emptyset$, then $\inf_\mathscr{S} b^+_\alpha = \inf_{\operatorname{edge}(\mathscr{S})} b^+_\alpha$.*

Proof. Let $t_+ \in (t, \infty) \subset A_\alpha$ and $t_- \in A_\alpha \cap (-\infty, t)$. Let $\mathscr{N} \subset I^+(\alpha(t_-)) \cap I^-(\alpha(t_+))$ be a compact neighborhood of $\alpha(t)$ having the properties described in Lemma 3.13 and such that $b^+_\alpha|\mathscr{N}$ is continuous. Let $\mathscr{S}$ be a smooth connected embedded maximal spacelike hypersurface of $(M, \mathbf{g})$ in $\mathring{\mathscr{N}}$.

Suppose $b^+_\alpha|\mathscr{S}$ does attain its infimum at some point $p \in \mathscr{S}$ but is not constant. Then $\Lambda := \{p' \in \mathscr{S} \colon b^+_\alpha(p') = b^+_\alpha(p)\}$ is a proper subset of $\mathscr{S}$, and there exists $q \in \dot{\Lambda}$. Let $\mathscr{B}$ be a smoothly embedded closed 3-ball in $\mathscr{S}$ such that $q \in \mathring{\mathscr{B}}$ and $\dot{\mathscr{B}} \not\subset \Lambda$. One can construct a smooth function $\varphi\colon \mathscr{S} \to \mathbb{R}$ which vanishes at q, has nowhere-vanishing gradient on $\mathscr{B}$ and is strictly positive on the proper subset $\dot{\mathscr{B}} \cap \Lambda$ of $\dot{\mathscr{B}}$. For sufficiently large $\alpha > 0$ there exists a constant $E_1 > 0$ such that the function $\psi := 1 - e^{-\alpha\varphi}$ on $\mathscr{S}$ satisfies

(a) $\psi(q) = 0$;

(b) $\Delta_\mathscr{S} \psi < -E_1$ on $\mathscr{B}$;

(c) $\psi|\dot{\mathscr{B}} \cap \Lambda > 0$,

where $\Delta_\mathscr{S}$ is the Laplacian on $\mathscr{S}$. Extend $\psi\colon \mathring{\mathscr{N}} \supset \mathscr{S} \to \mathbb{R}$ to a smooth function $\Psi\colon \mathring{\mathscr{N}} \supset \mathscr{V} \to \mathbb{R}$ on a neighborhood $\mathscr{V}$ of $\mathscr{S}$ in $\mathring{\mathscr{N}}$ such that

(d) $\nabla_\mathbf{N} \Psi = 0$ on $\mathscr{B}$,

where $\mathbf{N}$ is the unit future-directed normal to $\mathscr{S}$. There exists a constant $E_2 > 0$ such that $\|(\nabla\Psi)^\#\| < E_2$ at every point of $\mathscr{B}$. Since the infimum of b^+_α on the compact subset $\{q' \in \mathscr{B} \colon \psi(q') \leq 0\}$ of $\dot{\mathscr{B}} - \Lambda = \{q' \in \dot{\mathscr{B}} \colon b^+_\alpha(q') > b^+_\alpha(q)\}$ is strictly greater than $b^+_\alpha(q)$, there exists

$\varepsilon \in (0, E_1/4DE_2^2)$ such that the function $\beta^\varepsilon := b_\alpha^+ + \varepsilon\Psi$ on $\mathscr{V}$ satisfies $\beta^\varepsilon|\dot{\mathscr{B}} > \beta^\varepsilon(q)$. Clearly $\beta^\varepsilon|\mathscr{B}$ must attain its infimum at some point $r \in \mathring{\mathscr{B}}$. Relaxing condition (a) if necessary, one may perturb Ψ to achieve $r \in \mathscr{N}' \cap \mathring{\mathscr{B}}$.

Choose any $u \in (t_+ + 2n/\varepsilon E_1, \infty) \subset A_\alpha$ and let $\sigma\colon [s_{r,u}, u] \to M$ be a timelike line segment from r to $\alpha(u)$, as in Lemma 3.13. Since one has $u - s_{r,u} = d(r, \alpha(u)) \geq d(\alpha(t_+), \alpha(u)) = u - t_+ > 2n/\varepsilon E_1$, there exists $s \in [\hat{s}_{r,u}, u)$ such that $s - s_{r,u} > 2n/\varepsilon E_1$. The function $\beta_s^\varepsilon := b_{\sigma,s}^+ + \varepsilon\Psi$ on $\mathscr{V}$ smoothly supports $\beta^\varepsilon = b_\alpha^+ + \varepsilon\Psi$ from above at r and so $\beta_s^\varepsilon|\mathscr{S}$ attains a local minimum at r. Hence there exists $\lambda \in \mathbb{R}$ such that

$$\nabla\beta_s^\varepsilon(r) = \nabla b_{\sigma,s}^+(r) + \varepsilon\nabla\Psi(r) = -\lambda\mathbf{N}^\flat(r). \tag{18}$$

In fact by (d) one must have $\lambda = \nabla_{\mathbf{N}} b_{\sigma,s}^+ = -\mathbf{g}(\mathbf{N}, \dot{\sigma}(s_{r,u})) \geq 1$. From first principles one has

$$\Delta_{\mathscr{S}}\beta_s^\varepsilon = \Box b_{\sigma,s}^+ - H_{\mathscr{S}}\nabla_{\mathbf{N}} b_{\sigma,s}^+ + \operatorname{Hess} b_{\sigma,s}^+(\mathbf{N}, \mathbf{N}) + \varepsilon\Delta_{\mathscr{S}}\psi,$$

where, by hypothesis, the mean curvature $H_{\mathscr{S}}$ of $\mathscr{S}$ is zero. By (18) and the fact that $\nabla b_{\sigma,s}^+$ is smooth and unit timelike near r, there follows

$$\Delta_{\mathscr{S}}\beta_s^\varepsilon(r) = \Box b_{\sigma,s}^+ + (\varepsilon/\lambda)^2 \operatorname{Hess} b_{\sigma,s}^+((\nabla\Psi)^\#, (\nabla\Psi)^\#) + \varepsilon\Delta_{\mathscr{S}}\psi,$$

where the right-hand side is to be evaluated at r. One thus has

$$\begin{aligned}\Delta_{\mathscr{S}}\beta_s^\varepsilon(r) &\leq n/(s - s_{r,u}) + (\varepsilon/\lambda)^2 D\|(\nabla\Psi(r))^\#\|^2 - \varepsilon E_1 \\ &\leq \tfrac{1}{2}\varepsilon E_1 + \varepsilon^2 DE_2^2 - \varepsilon E_1 \leq -\tfrac{1}{4}\varepsilon E_1,\end{aligned}$$

which is incompatible with $\beta_s^\varepsilon|\mathscr{S}$ having a local minimum at r. This establishes a contradiction. Hence if $b_\alpha^+|\mathscr{S}$ attains its infimum, then it is constant.

Suppose $\operatorname{edge}(\mathscr{S}) \neq \varnothing$. Since $\overline{\mathscr{S}} = \mathscr{S} \cup \operatorname{edge}(\mathscr{S}) \subset \mathscr{N}$ is compact, $b_\alpha^+|\overline{\mathscr{S}}$ must attain its infimum at some point $x \in \overline{\mathscr{S}}$. If $x \in \operatorname{edge}(\mathscr{S})$ then clearly $\inf_{\mathscr{S}} b_\alpha^+ = \inf_{\operatorname{edge}(\mathscr{S})} b_\alpha^+$. However if $x \in \mathscr{S}$ then $b_\alpha^+|\mathscr{S}$ is constant, so is $b_\alpha^+|\overline{\mathscr{S}}$, and one again has $\inf_{\mathscr{S}} b_\alpha^+ = \inf_{\operatorname{edge}(\mathscr{S})} b_\alpha^+$.

4. Corays and colines

For the purposes of this section let $\alpha\colon \mathbb{R} \supset A_\alpha \to M$ be a line. If $\beta\colon \mathbb{R} \supset A_\beta \to M$ is a causal curve in $J^+(|\alpha|) \cap J^-(|\alpha|)$ satisfying $b_\alpha^+(\beta(v)) = b_\alpha^+(\beta(u)) + d(\beta(u), \beta(v))$ for all $u, v \in A_\beta$ such that $v \geq u$ then β must be a line segment. This observation motivates the following.

Definition 4.1. Let $p \in J^+(|\alpha|) \cap J^-(|\alpha|)$. A future/past ray $\mu_\pm \colon \mathbb{R} \supset A_{\mu_\pm} \to M$ from/to p satisfying

(a) $|\mu_\pm| \subset J^+(|\alpha|) \cap J^-(|\alpha|)$,

($\mathrm{b}_\pm$) $b_\alpha^\pm(\mu_\pm(v)) = b_\alpha^\pm(\mu_\pm(u)) + d(\mu_\pm(u), \mu_\pm(v))$ for all $u, v \in A_{\mu_\pm}$ such that $v \geq u$,

is a future/past coray of α. A line $\beta \colon \mathbb{R} \supset A_\beta \to M$ satisfying

(I) $|\beta| \subset J^+(|\alpha|) \cap J^-(|\alpha|)$,

(II) $b_\alpha^+(\beta(v)) = b_\alpha^-(\beta(v)) = b_\alpha^+(\beta(u)) + d(\beta(u), \beta(v))$ for all $u, v \in A_\beta$ such that $v \geq u$,

is a coline of α.

Note that this definition differs in details from corresponding definitions of Eschenberg [5] and Galloway [8].

Remark 4.2. Conditions ($\mathrm{b}_\pm$) in Definition 4.1 are equivalent to:

(b'_+) $b_\alpha^+(\mu_+(v)) = b_\alpha^+(p) + d(p, \mu_+(v))$ for all $v \in A_{\mu_+}$;

(b'_-) $b_\alpha^-(\mu_-(v)) = b_\alpha^-(p) - d(\mu_-(v), p)$ for all $v \in A_{\mu_-}$.

Condition (II) is equivalent to

(II$'$) there exists $v \in A_\beta$ such that $b_\alpha^+(\beta(w)) = b_\alpha^-(\beta(w)) = b_\alpha^+(\beta(v)) + d(\beta(v), \beta(w))$ for all $w \in A_\beta \cap [v, \infty)$ and such that $b_\alpha^-(\beta(u)) = b_\alpha^+(\beta(u)) = b_\alpha^-(\beta(v)) - d(\beta(u), \beta(v))$ for all $u \in A_\beta \cap (-\infty, v]$.

Proposition 4.3. *Let $p \in J^+(|\alpha|) \cap J^-(|\alpha|)$ and suppose α admits a past coray μ to p and a future coray ν from p. Then the concatenation of μ and ν is a coline of α iff $b_\alpha^+(p) = b_\alpha^-(p)$.*

Proof. The 'only if' claim is clear. For the converse, suppose $b_\alpha^+(p) = b_\alpha^-(p)$. For all $v \in A_\nu$ one has

$$\begin{aligned} b_\alpha^+(\nu(v)) &= b_\alpha^+(p) + d(p, \nu(v)) = b_\alpha^-(p) + d(p, \nu(v)) \\ &\leq b_\alpha^-(\nu(v)) \leq b_\alpha^+(\nu(v)) \end{aligned}$$

and hence

$$b_\alpha^-(\nu(v)) = b_\alpha^+(\nu(v)) = b_\alpha^-(p) + d(p, \nu(v)).$$

Similarly, for all $u \in A_\mu$ one has

$$b_\alpha^+(\mu(u)) = b_\alpha^-(\mu(u)) = b_\alpha^+(p) - d(\mu(u), p).$$

For all $u \in A_\mu$ and all $v \in A_\nu$ one therefore has

$$\begin{aligned} d(\mu(u), p) + d(p, \nu(v)) &\leq d(\mu(u), \nu(v)) \leq b_\alpha^+(\nu(v)) - b_\alpha^+(\mu(u)) \\ &= d(\mu(u), p) + d(p, \nu(v)) \end{aligned}$$

and hence

$$d(\mu(u), \nu(v)) = d(\mu(u), p) + d(p, \nu(v)).$$

The concatenation of μ and ν is thus a coline of α.

Corollary. *If $b_\alpha^+(p) = b_\alpha^-(p)$, then there exists at most one coline of α through p, modulo reparametrization.*

Remark 4.4. Let β be a unit speed timelike line such that $|\beta| \subset J^+(|\alpha|) \cap J^-(|\alpha|)$ and $|\alpha| \subset J^+(|\beta|) \cap J^-(|\beta|)$. If β is a coline of α, then α need not be a coline of β.

Suppose henceforth that α is a unit speed timelike line. The following result then establishes conditions for the existence of corays of α.

Proposition 4.5. *Suppose that $(M, \mathbf{g})$ is either globally hyperbolic or timelike geodesically complete. Let $t \in A_\alpha$. Then there exists a neighborhood $\mathcal{N} \subset I^+(|\alpha|) \cap I^-(|\alpha|)$ of $\alpha(t)$ such that, for each $p \in \mathcal{N}$, there exists a timelike coray of α from p.*

Proof. Let $t_+ \in A_\alpha \cap (t, \infty)$. Choose $T, R > 0$ such that $U(T, R; t)$ is a compact subset of $\mathscr{A}_\alpha \cap I^+(|\alpha|) \cap I^-(\alpha(t_+))$, has the property of the neighborhood of $\alpha(t)$ described in the corollary to Lemma 3.9, and admits a constant $B > 0$ as in Lemma 3.11. Let $p \in \bar{U}(T, R; t)$.

Let $t_i \in A_\alpha \cap [t_+, \infty)$ be an increasing sequence without cluster point. For each i there exists a unit speed timelike line segment $\sigma_i : [0, v_i] \to M$ from p to $\alpha(t_i)$. Since one has $\|\dot{\sigma}_i(0)\| \leq 1 + BR$ for all i one may, by passing to subsequences if necessary, assume that the $\dot{\sigma}_i(0) \in T_pM$ converge to a future-directed unit timelike vector $\mathbf{X} \in T_pM$.

Let $\sigma : [0, v) \to M$, $v \in (0, \infty]$, be the maximal unit speed timelike geodesic from p such that $\dot{\sigma}(0) = \mathbf{X}$. Let $\sigma' : [0, v') \to M$, $v' \in (0, \infty]$, be a maximal causal cluster curve of the σ_i from p. Since the $\dot{\sigma}_i(0)$ converge to $\dot{\sigma}(0) = \mathbf{X}$, σ' must be a segment of σ. If the $\alpha(t_i)$ have no cluster point in M, then σ' is future endless in M and coincides with σ. Suppose however that the $\alpha(t_i)$ do admit at least one cluster point. Then $(M, \mathbf{g})$ cannot be globally hyperbolic and so must be timelike geodesically complete. In this case σ' is either future endless in M and coincident with σ, or has a future endpoint at some cluster point q of the $\alpha(t_i)$. Suppose the latter. Let $\delta > 0$. The future completeness of α implies that there exists $I > 0$ such that $d(p, \alpha(t_i)) = L(\sigma_i) > L(\sigma') + \delta$ for all $i > I$. Since the $\dot{\sigma}_i(0) \in T_pM$ converge to the timelike vector $\dot{\sigma}(0) \in T_pM$, it follows that there exists $j > I$ such that $\sigma(v_j) = \alpha(t_j) \in I^+(q)$, and hence such that $q \in I^-(\alpha(t_j))$. Since q is a cluster point of the $\alpha(t_i)$, there must exist $k > j$ such that $\alpha(t_k) \in I^-(\alpha(t_j))$. But this implies that the chronology condition fails at $\alpha(t_j)$, contrary to Proposition 3.3. In general therefore the $\sigma_i : [0, v_i] \to M$ converge to $\sigma : [0, v) \to M$.

Let $u \in (0, v)$. There exists $I > 0$ such that $\min\{v_i, v\} > u + 2^{-I}$ for all $i > I$. For each $i > I$ there exists $j(i) > i$ such that there exists $u_j \in$

$(0, \min\{u+2^{-i}, v_j\}] = (0, u+2^{-i}]$ satisfying $\sigma_j(u_j) \in I^+(\sigma(u))$. Otherwise there would exist $i > I$ such that $\sigma_j(u') \in M - I^+(\sigma(u))$ for all $u' \in (0, u+2^{-i}]$ and all $j > i$, and this would imply $\sigma(u'') \in M - I^+(\sigma(u))$ for all $u'' \in (0, u+2^{-i}] \cap [0, v) \supset (u, u+2^{-i}]$. For each $i > I$ one has

$$\begin{aligned} b^+_{\alpha,t_j}(\sigma(u)) = t_j - d(\sigma(u), \sigma_j(v_j)) &\leq t_j - \{d(\sigma(u), \sigma_j(u_j)) + v_j - u_j\} \\ &\leq t_j - v_j + u_j \leq b^+_{\alpha,t_j}(p) + u + 2^{-i}. \end{aligned}$$

There follows $b^+_\alpha(\sigma(u)) \leq b^+_\alpha(p) + u \leq b^+_\alpha(p) + d(p, \sigma(u)) \leq b^+_\alpha(\sigma(u))$ and hence $d(p, \sigma(u)) = u$ and $b^+_\alpha(\sigma(u)) = b^+_\alpha(p) + d(p, \sigma(u))$. Thus σ is a future coray of α from p. q.e.d.

The existence of corays can be most useful, as in the proof of the following result.

Proposition 4.6. *Let $(M, \mathbf{g})$ be either globally hyperbolic or timelike geodesically complete. Let $t \in A_\alpha$. Then for any $t_+ \in A_\alpha \cap (t, \infty)$ there exists a neighborhood $\mathscr{N} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+))$ of $\alpha(t)$ such that*

(I) *for any $u \in A_\alpha \cap [t_+, \infty)$ the sets $\{b^+_{\alpha,u} = \text{const}\} \cap \mathscr{N}$ are acausal in M;*

(II) *the sets $\{b^+_\alpha = \text{const}\} \cap \mathscr{N}$ are acausal in M.*

Proof. Let $\mathscr{N} \subset I^+(|\alpha|) \cap I^-(\alpha(t_+))$ be a neighborhood of $\alpha(t)$ as in Proposition 4.5 and the corollary to Lemma 3.9.

(I) Suppose there exist $p, q \in \{b^+_{\alpha,u} = \text{const}\} \cap \mathscr{N}$ such that there exists a nondegenerate causal curve μ in M from p to q. From the inequality $b^+_\alpha(q) \geq b^+_\alpha(p) + d(p,q)$ one has $d(p,q) = 0$ and hence that μ is a null geodesic. Let $\sigma_{q,\alpha(u)}$ be a timelike line segment from q to $\alpha(u)$. The concatenation of μ and $\sigma_{q,\alpha(u)}$ is a broken causal geodesic γ from p to $\alpha(u)$ of length $L(\gamma) = L(\sigma_{q,\alpha(u)}) = d(q, \alpha(u))$. One therefore has $d(q, \alpha(u)) < d(p, \alpha(u))$. This implies $b^+_{\alpha,u}(p) < b^+_{\alpha,u}(q)$ which gives a contradiction.

(II) Suppose there exist $p, q \in \{b^+_\alpha = \text{const}\} \cap \mathscr{N}$ such that there exists a nondegenerate causal curve μ in M from p to q. From the inequality $b^+_\alpha(q) \geq b^+_\alpha(p) + d(p,q)$ one has $d(p,q) = 0$ and hence that μ is a null geodesic. Let $\nu\colon \mathbb{R} \supset A_\nu \to M$ be a timelike future coray of α from q. Choose any $v \in A_\nu$ such that $\nu(v) \neq q$. The concatenation of μ and the segment of ν from q to $\nu(v)$ is a broken causal geodesic from p to $\nu(v)$. One therefore has $d(q, \nu(v)) < d(p, \nu(v))$ and hence $b^+_\alpha(p) + d(q, \nu(v)) < b^+_\alpha(p) + d(p, \nu(v)) \leq b^+_\alpha(\nu(v)) = b^+_\alpha(q) + d(q, \nu(v))$. This implies $b^+_\alpha(p) < b^+_\alpha(q)$ which gives a contradiction.

5. The splitting theorem

The results of the previous sections will be brought together here to prove the splitting conjecture of Yau [10] quoted in the Introduction. A local splitting theorem will first be established with methods which closely follow Galloway [8].

Theorem 5.1. *Suppose $(M, \mathbf{g})$ is timelike geodesically complete and satisfies the null convergence condition. If $(M, \mathbf{g})$ admits a unit speed timelike line $\alpha: \mathbb{R} \to M$, then there exists an open neighborhood $\mathscr{N}$ of $|\alpha|$ such that:*

(I) $\mathscr{S} := \{b_\alpha^+ = b_\alpha^- = 0\} \cap \mathscr{N}$ *is a smooth embedded acausal hypersurface of* $(M, \mathbf{g})$;

(II) $(\mathscr{N}, \mathbf{g}|\mathscr{N})$ *is isometric to* $(\mathbb{R} \times \mathscr{S}, -\mathbf{d}t^2 \oplus \mathbf{h})$, *where* $\mathbf{h}$ *is the induced metric on* $\mathscr{S}$;

(III) *for each* $p \in \mathscr{S}$ *the curve* $\mathbb{R} \to \mathscr{N}$ *defined by* $t \mapsto (t, p)$ *is a unit speed timelike coline of* α.

Proof. By Propositions 3.5, 4.5 and 4.6, the corollary to Proposition 3.12, and the time-reversed analogues of these results, there exists a local causality neighborhood $\mathscr{U} \subset I^+(|\alpha|) \cap I^-(|\alpha|)$ of $\alpha(0)$ such that $b_\alpha^+|\mathscr{U}$ and $b_\alpha^-|\mathscr{U}$ are continuous, the set $\{b_\alpha^+ = 0\} \cap \mathscr{U}$ is acausal, and for each $p \in \mathscr{U}$, there exists a timelike future coray of α from p and a timelike past coray of α to p. One may suppose that $\mathscr{U}$ is also a neighborhood of $\alpha(0)$ as in Lemma 3.14 and in the corresponding time-reversed result.

Since $b_\alpha^+|\mathscr{U}$ is continuous, the set $\{b_\alpha^+ = 0\} \cap \mathscr{U}$ is relatively edgeless in $\mathscr{U}$, and there exists a relative open neighborhood $\hat{\mathscr{S}}$ of $\alpha(0)$ in $\{b_\alpha^+ = 0\} \cap \mathscr{U}$ such that $\hat{\mathscr{S}} \subset\subset \mathscr{U}$, $\operatorname{edge}(\hat{\mathscr{S}}) \neq \varnothing$, $D(\hat{\mathscr{S}}) \subset \mathscr{U}$. Since $\operatorname{edge}(\hat{\mathscr{S}}) \subset \{b_\alpha^+ = 0\} \cap \mathscr{U}$ is acausal, results of Bartnik [1] give that there exists a smooth maximal hypersurface $\mathscr{S}$ of $D(\hat{\mathscr{S}})$ such that

(a) $\operatorname{edge}(\mathscr{S}) = \operatorname{edge}(\hat{\mathscr{S}}) \subset \{b_\alpha^+ = 0\} \cap \mathscr{U}$;

(b) $D(\mathscr{S}) = D(\hat{\mathscr{S}})$.

The line α, which cuts $\hat{\mathscr{S}} \subset D(\hat{\mathscr{S}}) = D(\mathscr{S})$ at $\alpha(0)$, must cut $\mathscr{S}$ at $\alpha(t)$ for some $t \in \mathbb{R}$. One has

$$t = b_\alpha^-(\alpha(t)) \leq \sup_{\mathscr{S}} b_\alpha^- = \sup_{\operatorname{edge}(\mathscr{S})} b_\alpha^- \leq \sup_{\operatorname{edge}(\mathscr{S})} b_\alpha^+ = 0,$$
$$0 = \inf_{\operatorname{edge}(\mathscr{S})} b_\alpha^+ = \inf_{\mathscr{S}} b_\alpha^+ \leq b_\alpha^+(\alpha(t)) = t,$$

and hence $\inf_{\mathscr{S}} b_\alpha^+ = b_\alpha^+(\alpha(t)) = 0$ and $\sup_{\mathscr{S}} b_\alpha^- = b_\alpha^-(\alpha(t)) = 0$. There follows $b_\alpha^+|\mathscr{S} = b_\alpha^-|\mathscr{S} = 0$.

By Proposition 4.3, for each $p \in \mathscr{S}$ there exists a unique unit speed timelike coline $\beta_p: \mathbb{R} \to M$ of α such that $\beta_p(0) = p$. One has $b_\alpha^+(\beta_p(t)) = b_\alpha^-(\beta_p(t))$ for all $p \in \mathscr{S}$ and all $t \in \mathbb{R}$. For any $p \in \mathscr{S}$ and $t \geq 0$ one must have $d(\mathscr{S}, \beta_p(t)) = t$; otherwise there would exist $p' \in \mathscr{S}$ such that $t < d(p', \beta_p(t)) \le b_\alpha^+(\beta_p(t)) - b_\alpha^+(p') = t$ which is impossible. Similarly, for any $p \in \mathscr{S}$ and $t \le 0$ one must have $d(\beta_p(t), \mathscr{S}) = |t|$. Hence each β_p meets $\mathscr{S}$ orthogonally and does not admit any value conjugate to $\mathscr{S}$. The mapping $\Sigma\colon \mathbb{R} \times \mathscr{S} \to M$ defined by $(t, p) \mapsto \beta_p(t)$ is therefore a local diffeomorphism. Moreover, since there can be at most one coline of α through any point of M, Σ must be an injection and consequently a diffeomorphism onto its image. This image is an open neighborhood $\mathscr{N}$ of $|\alpha|$ in M. Henceforth let $\mathscr{N}$ be identified with $\mathbb{R} \times \mathscr{S}$ via Σ.

Define a unit timelike geodesic vector field $\mathbf{V}$ on $\mathscr{N}$ by $(t, p) \mapsto \dot\beta_p(t) \in T_{(t,p)}M$. Since each maximal integral curve of $\mathbf{V}$ is complete without conjugate values, and $(M, \mathbf{g})$ satisfies the timelike convergence condition, $\mathbf{V}$ must have zero expansion and shear. It follows that $\mathbf{V}$, being unit timelike and geodesic, is covariantly constant and therefore a Killing field. Hence the flow of $\mathbf{V}$ is a one-parameter family of isometries of $(\mathscr{N}, \mathbf{g}|\mathscr{N})$. Since the integral curves of $\mathbf{V}$ meet the hypersurfaces $\{t = \text{const}\}$ orthogonally, and $\mathbf{d}t$ is a unit timelike one-form on $\mathscr{N}$, one has that $(\mathscr{N}, \mathbf{g}|\mathscr{N})$ is isometric to $(\mathbb{R} \times \mathscr{S}, -\mathbf{d}t^2 \oplus \mathbf{h})$, where $\mathbf{h}$ is the induced metric on metric on $\mathscr{S}$. q.e.d.

This local splitting theorem may be globalized by precisely the same arguments as used by Eschenburg [5] in the globally hyperbolic case. This yields the final result.

Theorem 5.2. *Suppose $(M, \mathbf{g})$ is timelike geodesically complete, satisfies the timelike convergence condition and admits a unit speed timelike line $\alpha\colon \mathbb{R} \to M$. Then $(M, \mathbf{g})$ is isometric to $(\mathbb{R} \times \mathscr{S}, -\mathbf{d}t^2 \oplus \mathbf{h})$, where $(\mathscr{S}, \mathbf{h})$ is a complete Riemannian 3-manifold, and $\alpha\colon \mathbb{R} \to M$ is given by $t \mapsto (t, p)$ for some $p \in \mathscr{S}$.*

The work of Galloway [8] establishes that the conclusions of Theorem 5.2 are maintained if the hypothesis of timelike geodesic completeness is replaced by a hypothesis of global hypobilicity. Thus Theorem 5.2 complements Galloway's result, and each generalizes the theorem of Eschenberg [5] which establishes the same conclusions in the case where both hypotheses hold.

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