J. DIFFERENTIAL GEOMETRY
30 (1989) 689-759

THE GAUSS MAP OF A GENERIC HYPERSURFACE IN $\mathbb{P}^4$

CLINT McCRORY, THEODORE SHIFRIN & ROBERT VARLEY

Table of Contents

0.	Introduction	689
1.	The parabolic surface	694
2.	The cusp hierarchy	698
3.	Geometry of the cusp curve	704
4.	The cubic threefold	716
5.	Gauss-stable hypersurfaces	723
6.	The transversality package	726
7.	Classification of singularities	733
8.	Genericity and deformations	736
9.	Genericity and jet transversality	752

0. Introduction

The study of singularities of Gauss maps has played an important role in the development of algebraic geometry. Two beautiful examples of such work are the analysis of duals of cubic surfaces in projective 3-space $\mathbb{P}^3$ undertaken by Cayley and others in the nineteenth century, and Fano's investigation of the cubic threefold in the early part of this century. More recently, Andreotti exploited the Gauss map of the theta divisor to give a new solution to the Torelli problem for curves; Clemens-Griffiths, using a similar approach, proved that the cubic threefold is irrational.

Differential geometry and singularity theory are well adapted to the study of Gauss maps. Cartan's moving frames provide a dynamic tool which complements the static arguments of classical projective geometry. Arnold's theory of Lagrange and Legendre singularities has roots not only in classical mechanics, but also in projective geometry, for the Gauss map is the prototype of a Legendre map. Singularity theory provides a rich supply of analogies among singularities in different geometric contexts,

---

Received October 19, 1987. The first author was supported in part by National Science Foundation grant DMS-8603085, and the third author was supported in part by National Science Foundation grant DMS-8603281.

including striking parallel classifications of singularities that arise in very different ways. Using the deformation theory of singularities, we can make precise the notion of a "generic" singularity, a concept which was sometimes abused by classical geometers.

This paper is an outgrowth of two of the authors' work [29] on the singularities of the Gauss map of a smooth complex surface in $\mathbb{P}^3$. In the present work we analyze the Thom-Boardman singularity hierarchy of the Gauss map $\gamma\colon M \to \mathbb{P}^{4*}$ of a generic (Gauss-stable) complex hypersurface $M \subset \mathbb{P}^4$ of arbitrary degree. If M has degree $d \geq 2$, we compute the following numerical invariants:

(0.1)

locus	Thom-Boardman symbol	Arnold symbol	degree
Π: parabolic surface	$\Sigma^1(\gamma)$	A_2	$5d(d-2)$
Π': singular points of Π	$\Sigma^2(\gamma)$	D_4	$20d(d-2)^3$
C: cusp curve	$\Sigma^{1,1}(\gamma)$	A_3	$5d(d-2)(7d-15)$
Λ: second-order cusp points	$\Sigma^{1,1,1}(\gamma)$	A_4	$5d(d-2)(3d-7)(17d-36)$

Each of these loci is the *closure* of the set of points with the corresponding symbol. The Arnold symbol at $x \in M$ is the singularity type of the tangent hyperplane section of M at x.

From the standpoint of singularity theory, the Gauss map arises from the study of the incidence projection

$$(0.2)\qquad p\colon \Gamma \to \mathbb{P}^{4*}, \qquad \Gamma = \{(x,H)\colon x \in H\} \subset M \times \mathbb{P}^{4*}.$$

The singular locus $\Sigma(p)$ is canonically identified with M, and $p|\Sigma(p)$ then gives the Gauss map of M. We say M is *Gauss-stable* if the germ of p at (x,H) is stable for all $(x,H) \in \Gamma$. We prove that Gauss stability is a generic property:

(0.3) Theorem. *For $d \geq 2$ the set of Gauss-stable hypersurfaces is a nonempty Zariski-open subset of the space $\mathscr{M}_d$ of smooth hypersurfaces of degree d in $\mathbb{P}^4$.*

We begin in §1 by discussing the relation between the Gauss map and the projective second fundamental form of M. Thus the parabolic surface arises as in the classical differential-geometric theory of hypersurfaces. We complete the first section by considering the global issue: we determine the rational equivalence classes of Π and Π'; by Gauss stability, Π' consists of ordinary double points of Π.

In §2 we begin to study the cusp curve and the second-order cusp locus Λ. To compute their classes, we must first resolve the singularities of Π;

then C and Λ are expressible in terms of the Chern-Mather classes of Π. The proper transform of C is defined as the divisor of a section of a line bundle, generalizing the approach of [29], and then Λ expressed in terms of its intrinsic derivative. Gauss stability implies that C has triple points at points of Π', is smooth otherwise, and that $\Lambda \cap \Pi' = \varnothing$.

It should be noted that the computation of the number of second-order cusp points, say, is too complicated to be amenable to standard topological Thom polynomial techniques, for $\Lambda = \Sigma^{3,1,1,1}(p)$ is a fourth-order Thom-Boardman singularity of the stable map p. It is the inherent differential geometric structure which makes these singularities accessible.

We turn next in §3 to a detailed study of the geometry of the cusp curve C, which is the closure of the set of points of M at which the tangent hyperplane section has an A_3-singularity. We begin by reinterpreting this condition in terms of *contact with lines*, motivated by the work of Kulikov [22] and by one of the central results in [29], both dealing with surfaces. We find the following

(0.4) Theorem. $C = \{x \in \Pi | \textit{some line in } \mathbb{P}^4 \textit{ has order of contact at least } 3 \textit{ with } M \textit{ at } x\}.$

It is most inviting, then, to conjecture that the second-order cusp locus Λ corresponds to those points where the order of contact escalates to 4. The most convincing way to check that this is wrong is to count the latter set by a synthetic geometric argument. For M belonging to a Zariski-open subset of $\mathscr{M}_d$ we have the following:

$$\begin{aligned} &\operatorname{degree}\{x \in C |\ \text{some line in } \mathbb{P}^4 \text{ has order of contact at least 4 with } M \text{ at } x\} \\ &\quad = 5d(d-2)(39d^2 - 179d + 204), \qquad d \geq 4. \end{aligned} \tag{0.5}$$

There are further loci associated to the projective geometry along C (and lying away from the triple points):

$$\kappa = \{x \in C - \Pi' | T_x\Pi \text{ is II-null}\},$$
$$\mu = \{x \in C - \Pi' | \ker d\gamma_x \text{ is an asymptotic direction in } \Pi\}.$$

Using Chern class methods, both these loci can be counted and we find that

$$\deg \kappa = \deg \mu = \deg \Lambda = 5d(d-2)(3d-7)(17d-36). \tag{0.6}$$

This is suggestive indeed. When $d = 3$, these three loci coincide, but when $d \geq 4$ there is a projective invariant ρ associated to the tangent hyperplane section of M at x which distinguishes among these loci. It follows that for M belonging to a nonempty Zariski-open subset $\mathscr{M}_d'$ of $\mathscr{M}_d$ these zero cycles in M are disjoint. In fact, we establish the following.

(0.7) Theorem. *For $d \geq 4$ and $M \in \mathscr{M}_d'$, ρ is a rational function on the normalization of C with degree $\rho = 5d(d-2)(3d-7)(17d-36)$.*

The modulus ρ provides a fascinating invariant; ρ takes the value ∞ at the points corresponding to Π', as well as at the locus defined in (0.5). Thus these points are put into the same pencil of divisors as Λ, κ, and μ.

We turn in §4 to the study of the cubic threefold. While our treatment is self-contained and based on the results of §§1–3, the works of Fano [11], Clemens-Griffiths [9], and the classic literature on the classification of cubic surfaces (especially [32] and [8]) were useful to us. Our point of departure is the two-parameter family of lines lying on the cubic threefold, called the *Fano surface*; we distinguish the divisor $\mathscr{D}$ of *special lines* along which every tangent hyperplane to M is bitangent. The lines of $\mathscr{D}$ generate a developable ruled surface Σ and we identify our cast of characters in terms of Σ and its curve of striction σ. The culminating result is Fano's identity:

(0.8) Theorem. $\sigma \cdot \Pi = 3\Lambda$.

Our policy in §§1–4 is to prove all the results possible under the basic assumption of Gauss stability. The latter half of the paper contains a careful analysis of the ramifications and consequences of Gauss stability. Gauss stability of $M \subset \mathbb{P}^4$ can be characterized by a list of transversality conditions on the Gauss map γ, which we dub the *transversality package* (6.1). To discern more subtle properties of the singularities of γ, we use the local analytic classification of these singularities; this is accomplished by means of a list of *normal forms* (7.3). Each time we use (6.1) or (7.3) we give a specific reference.

The main result of §§5–9 is the genericity of Gauss stability (0.3). We give two proofs of this theorem. The first proof (§8) uses the theory of versal deformations of singularities of projective varieties; the second (§9) uses unfoldings of singularities of maps and jet transversality. These two points of view are closely related but complementary. The deformation-theoretic proof is more computational and yields a method for determining whether a given polynomial equation defines a Gauss-stable hypersurface. Given a point of a smooth hypersurface in $\mathbb{P}^4$, we first determine whether the singularity of the tangent hyperplane section (an isolated, 2-dimensional hypersurface singularity) is (a) one of the types A_1, A_2, A_3, A_4, D_4 or (b) worse, and then, in case (a), we determine whether the singularity is versally deformed by the nearby hyperplane sections of the *same* hypersurface. The methods introduced here are applied in several instances in §§2–3.

The second proof of (0.3) is broader in scope; it can be adapted to the study of contact of k-planes with hypersurfaces in $\mathbb{P}^n$. Both proofs employ the universal degree d hypersurface in $\mathbb{P}^4$. This classical device has been used recently and in the present context by Bruce [7] and Ronga [30], [31]. Our proof, while similar to theirs, has the advantage that it deals explicitly with the Zariski topology on the space of smooth hypersurfaces in $\mathbb{P}^4$.

The remainder of the second part of our work provides an interpretation, in the geometric context of the Gauss map, of results in the theory of complex analytic map germs. In §5 we give three equivalent characterizations of Gauss stability at a point x of a smooth hypersurface in $\mathbb{P}^4$:

(0.9)
(a) The family of hyperplane section germs is versal.
(b) The incidence projection germ (0.2) is stable.
(c) The Gauss map is a stable Legendre map germ.

The equivalence follows from the work of Mather and Arnold.

In §6 we characterize Gauss stability by the *transversality package*. The proof of this result is based on Boardman's description of Thom-Boardman singularities in terms of intrinsic derivatives [6]. In §7 we present a classification of singularities from the three viewpoints corresponding to (a), (b), and (c) in (0.9), following the proof of the equivalence to derive normal forms for the incidence projection p (0.2) and the Gauss map γ from Arnold's classification of simple hypersurface singularities. The transversality package can be checked directly from the normal forms for the Gauss map exhibited in this classification. From §6 it follows that, conversely, the package (6.1) guarantees that the singularities of the Gauss map have the stated normal forms.

Related results have been obtained by several authors. Prior to the work of Bruce and Ronga mentioned above, Roberts (unpublished) and Vainsencher [35] studied the rank singularities $\overline{\Sigma}^i(\gamma)$ for smooth subvarieties of $\mathbb{P}^n$, computing the degree of the image of $\overline{\Sigma}^i(\gamma)$ in $\mathbb{P}^{n*}$. A natural context for the study of such degeneracy loci in geometry has been developed and applied by Harris, Tu, and others (cf. [34]).

The extrinsic geometry of surfaces in $\mathbb{P}^3$ has been studied deeply by the Russian school. Platonova and Landis have obtained a projective classification of the points of a generic surface in 3-space (cf. [2, Chapter VI]). It would be very interesting to extend their results to higher dimensions.

The methods of Kulikov [22] do extend to hypersurfaces of $\mathbb{P}^n$; in fact, we have verified several of our numerical results using his techniques.

There are many questions left to explore. We plan next to investigate multiple points of the Gauss map of a smooth hypersurface in $\mathbb{P}^4$, following Ronga's computations for surfaces in $\mathbb{P}^3$. We would also like, as we did in the case of surfaces [29], to interpret our results for nongeneric projective hypersurfaces.

Preliminary reports on this work were given in April 1986 at the University of North Carolina's singularities year, and at the special session on singularities in algebraic geometry at the October 1986 meeting of the American Mathematical Society in Charlotte. We thank Gary Kennedy for helpful remarks on Chern-Mather classes, and Felice Ronga for sending us his paper [31] prior to its publication.

Guide to the reader. The geometric content of §§1–4 rests on the singularity theoretic foundations of §§5–9. We have chosen to put the geometric results first in order to make the paper accessible to the reader not versed in singularity theory; thus the geometric applications should provide the reader a more motivated and tractable entree to the technicalities of the singularity theory. On the other hand, an understanding of our basic premise, Gauss stability, sufficient for our geometric purposes can be obtained from the statements of the transversality package (6.1) for the second fundamental form and the normal forms (7.3) for the Gauss map.

We have attempted to provide complete and accurate references for all singularity theoretic results. Further background may be found in the excellent expository works of Arnold, et al. [3] and Martinet [25].

1. The parabolic surface

Let M be a smooth hypersurface in $\mathbb{P}^4$, with Gauss map $\gamma\colon M \to \mathbb{P}^{4*}$. Let

$$\Pi = \mathrm{Cl}\{x \in M \mid \dim\ker(d\gamma_x) = 1\} = \overline{\Sigma}^1(\gamma),$$
$$\Pi' = \mathrm{Cl}\{x \in M \mid \dim\ker(d\gamma_x) = 2\} = \overline{\Sigma}^2(\gamma),$$

where Cl and bar denote closure. (For an introduction to the Thom-Boardman singularities Σ^I, cf. [3, §2].) If M is Gauss-stable of degree ≥ 3 then the transversality package (6.1) implies that $\dim \Pi = 2$, $\dim \Pi' = 0$ and $\Pi^0 = \Pi - \Pi'$ is smooth. We call Π the *parabolic surface* in M.

We wish first to relate γ, or, more precisely, its derivative and the projective second fundamental form of M. Let $\pi\colon \mathbf{C}^5 - \{0\} \to \mathbb{P}^4$ be the canonical projection. We say that a basis $Z_0, Z_1, \cdots, Z_4$ for $\mathbf{C}^5$ is a *frame* at $x \in \mathbb{P}^4$ if $\pi(Z_0) = x$. A frame on an open set $\tilde{U} \subset \mathbb{P}^4$ is by definition a holomorphic map $\tilde{U} \to \mathrm{GL}(5, \mathbf{C})$, $x \to (Z_i(x))$, $i = 0, 1, \cdots, 4$, such that $\pi(Z_0(x)) = x$. Given a hypersurface $M \subset \mathbb{P}^4$, an *adapted frame* on an open subset $U \subset M$ is a frame on U satisfying the further condition that $\pi(Z_0 \wedge Z_1 \wedge Z_2 \wedge Z_3) = T_xM \subset T_x\mathbb{P}^4$. We denote by $\tilde{T}_xM$ the three-plane $Z_0 \wedge Z_1 \wedge Z_2 \wedge Z_3$ in $\mathbb{P}^4$ tangent to M; this is of course the Gauss image $\gamma(x)$.

Given an adapted frame field $Z_0, \cdots, Z_4$ on $U \subset M$, define one-forms ω_i^j, $0 \leq i, j \leq 4$, on M by

$$dZ_i = \sum_{j=0}^{4} \omega_i^j Z_j. \tag{1.1}$$

Setting $\omega_0^j = \omega^j$, one sees that $\omega^1, \omega^2, \omega^3$ give a basis for the (holomorphic) one-forms on U. Differentiating (1.1), we obtain the structure equations

$$d\omega_i^j = \sum_{k=0}^{4} \omega_i^k \wedge \omega_k^j; \tag{1.2}$$

since $\omega^4 = 0$ on M, we infer that $d\omega^4 = \sum_{\alpha=1}^{3} \omega^\alpha \wedge \omega_\alpha^4 = 0$, and hence, by the Cartan lemma,

$$\omega_\alpha^4 = \sum_{\beta=1}^{3} h_{\alpha\beta}\omega^\beta, \qquad h_{\alpha\beta} = h_{\beta\alpha}. \tag{1.3}$$

It is straightforward (cf. [29]) to verify that the tensor $\mathrm{II} = \sum h_{\alpha\beta}\omega^\alpha \otimes \omega^\beta \otimes Z_4$ is well defined. II is a section of the bundle $\mathrm{Sym}^2(T^*M) \otimes N_M$ and is called the *(projective) second fundamental form of M*.

On the other hand, the derivative of the Gauss map

$$\begin{aligned} d\gamma_x \colon T_xM &\to T_{\gamma(x)}\mathbb{P}^{4*} \cong \mathrm{Hom}(\tilde{T}_xM, \mathbb{C}^5/\tilde{T}_xM), \\ v &\to \sum_{j=0}^{3} \omega_j^4(v)\omega^j \otimes Z_4 \end{aligned} \tag{1.4}$$

may be interpreted as a bilinear map

$$T_xM \otimes \tilde{T}_xM \to \mathbb{C}^5/\tilde{T}_xM. \tag{1.5}$$

From the Euler sequence

$$\begin{cases} 0 \to \mathcal{O}_M(-1) \to \tilde{T}M \to TM \otimes \mathcal{O}_M(-1) \to 0, \\ 0 \to \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathbb{C}^5 \to T\mathbb{P}^4 \otimes \mathcal{O}_{\mathbb{P}^4}(-1) \to 0, \end{cases} \tag{1.6}$$

it follows that $\mathbb{C}^5/\tilde{T}M \cong N_M \otimes \mathscr{O}_M(-1)$. Moreover, the restriction of the map (1.5) to $T_xM \otimes \operatorname{Span}(x)$ is trivial since $\omega_0^4 = 0$ on M. Therefore, the map (1.5) induces by (1.6) a bundle map

$$d\gamma\colon TM \otimes TM \otimes \mathscr{O}_M(-1) \to N_M \otimes \mathscr{O}_M(-1);$$

twisting the bundles by $\mathscr{O}_M(1)$, it follows from (1.4) that this map is none other than the second fundamental form.

For future reference we remark that if M is presented locally as a graph $\{x_4 = f(x_1, x_2, x_3)\} \subset \mathbb{C}^4$, we may then take

$$Z_1 = (1,0,0,f_1), \quad Z_2 = (0,1,0,f_2), \quad Z_3 = (0,0,1,f_3), \quad Z_4 = (0,0,0,1),$$
$$\omega^\alpha = dx_\alpha, \qquad \alpha = 1,2,3.$$

Then, writing $f_\alpha = \partial f/\partial x_\alpha$, $f_{\alpha\beta} = \partial^2 f/\partial x_\alpha \partial x_\beta$,

$$\mathrm{II} = (\Sigma f_{\alpha\beta}\, dx_\alpha \otimes dx_\beta) \otimes Z_4. \tag{1.7}$$

As a geometric consequence, it follows that the conic $\{\mathrm{II}(v,v) = 0\} \subset \mathbb{P}T_xM$ of asymptotic directions at x is the tangent cone of $M \cap \tilde{T}_xM$. (Proof: take $x = 0$, $T_xM = \{x_4 = 0\}$; then $M \cap T_xM$ is defined in $\mathbb{C}^3$ by $f(x_1, x_2, x_3) = \Sigma f_{\alpha\beta}(0) x_\alpha x_\beta + \cdots$.)

The parabolic surface arises from the algebro-geometric viewpoint (e.g., in [11]) by way of the Hessian. Suppose $M^3 \subset \mathbb{P}^4$ is defined as the zero locus of a homogeneous polynomial F of degree d in five variables. Let $\operatorname{Hess}(F) = \det(F_{ij})$ be the Hessian determinant of F.

(1.8) Proposition. *The parabolic surface* Π *is cut out on* M *by the equation* $\operatorname{Hess}(F) = 0$.

Proof. Let $A = [F_{ij}(x)]$. The function F_i is homogeneous of degree $d-1$ and so Euler's theorem implies that $Ax = (d-1)\nabla F(x)$. Suppose now that $v \in \ker A$, so that $v^TA = 0$. Then $0 = v^TAx = (d-1)\nabla F(x) \cdot v$, so $v \in (\nabla F(x))^\perp$, whence $v \in \tilde{T}_xM$. But from the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^5 - \{0\} & \xrightarrow{\nabla F} & \mathbb{C}^{5*} - \{0\} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \mathbb{P}^4 \supset M & \xrightarrow{\gamma} & \mathbb{P}^{4*} \end{array}$$

it follows that $d\pi(v) \in \ker d\gamma_x$. Provided that $d\pi(v) \neq 0$, it will be the case that $x \in \Pi$. But $d\pi(v) = 0$ if and only if v is a multiple of x, and $Ax \neq 0$ since M is smooth. q.e.d.

In fact, the proof shows more. The Hessian matrix A and the second fundamental form II have isomorphic kernels, the isomorphism being given by $d\pi$. Using this observation, we now prove the stronger result:

(1.9) Proposition. *If M is Gauss-stable and $p \in \Pi^0$, then* $\operatorname{Hess}(F)$ *has nonzero differential at p. Hence* $\operatorname{Hess}(F) = 0$ *defines the (reduced) scheme $\Pi \subset M$.*

Proof. Choose coordinates $(x_0, x_1, \cdots, x_4)$ in $\mathbb{P}^4$ so that $p = (1, 0, \cdots, 0)$, $\nabla F(p) = (0,0,0,0,1)$, and $F_{1j}(p) = 0$. Locally $M = \{x_4 = f(x_1, x_2, x_3)\}$ for some analytic function f near $0 \in \mathbb{C}^3$. Setting $x_0 = 1$, $x = (x_1, x_2, x_3)$, and working in affine coördinates, we have

$$\begin{aligned} &F(1, x, f(x)) = 0 \\ &\quad \Rightarrow F_{\alpha\beta} + F_{4\beta} f_\alpha + F_4 f_{\alpha\beta} = 0, \qquad 1 \le \alpha, \beta \le 3, \\ &\quad \Rightarrow f_{11} = -(F_{11} + F_{41} f_1)/F_4 \\ &\quad \Rightarrow df_{11}(0) = -dF_{11}(p). \end{aligned}$$

Using the expression (1.7) for the second fundamental form we find that since $f_{1\beta}(0) = 0$, the transversality condition (6.1.1) may be expressed in the form $df_{11}(0) \neq 0$. On the other hand, expanding $\operatorname{Hess}(F)$ in cofactors along row 1,

$$d \operatorname{Hess}(F)(p) = c\, dF_{11}(p);$$

$c \neq 0$ since $\operatorname{rank} A = 4$, and so Gauss stability implies $d \operatorname{Hess}(F)(p) \neq 0$. q.e.d.

Let H denote the divisor class of a hyperplane section of M. We now deduce easily from (1.8) the following.

(1.10) Proposition. *If M is a hypersurface of degree d in $\mathbb{P}^4$, then Π is an element of the linear system $|5(d-2)H|$.*

Proof. $\operatorname{Hess}(f)$ is a homogeneous polynomial of degree $5(d-2)$. q.e.d.

In the case of a surface in $\mathbb{P}^3$, the parabolic set is generally a smooth curve [29]. However, here the parabolic surface Π will generally have zero-dimensional singular locus as we remarked at the outset, and as one would suspect from a naive dimension count. It is a straightforward application of results of [19], [17] to compute the degree of the singular locus Π'.

(1.11) Proposition. *If M is a Gauss-stable threefold of degree $d \geq 3$ in $\mathbb{P}^4$, then the singular locus Π' of the parabolic surface Π is a reduced zero-cycle of degree $20d(d-2)^3$ and consists of ordinary double points (i.e., the tangent cone of Π at $x \in \Pi'$ is a rank three quadric).*

Proof. Let E be a bundle or rank 3, and L a line bundle on M. According to [17], the "general" symmetric bundle map $\Phi\colon E \to E^* \otimes L$ has rank ≤ 1 on a subvariety whose rational equivalence class is given by the formal determinant

$$4 \begin{vmatrix} c_2 & c_3 \\ c_0 & c_1 \end{vmatrix},$$

where $c_i = c_i(E^* \otimes \sqrt{L})$. We apply this with $E = TM$, $L = N_M \cong \mathcal{O}_M(d)$, taking φ to be the second fundamental form and interpreting $\sqrt{L}$ formally. The necessary transversality condition is guaranteed by (6.1.1) and (6.1.2).

Now $c_1(E) = (5-d)H$, $c_2(E) = (10-5d+d^2)H^2$, and $c_3(E) = (10-10d+5d^2-d^3)H^3$. Thus $c_1 = (-5+\frac{5}{2}d)H$, $c_2 = (10-10d+\frac{11}{4}d^2)H^2$, and $c_3 = (-10+15d-\frac{35}{4}d^2+\frac{15}{8}d^3)H^3$, from which one obtains

$$4(c_1c_2 - c_3) = 4(-40+60d-30d^2+5d^3)H^3 = 20(d-2)^3H^3,$$

whose degree is $20d(d-2)^3$. The final assertion of the proposition may be checked directly from (6.1.2) (cf. the proof of (3.3)), or, equivalently, may be deduced from the normal form (7.3) (cf. (2.9)).

2. The cusp hierarchy

For the smooth hypersurface $M \subset \mathbb{P}^4$, let

$$C = \overline{\Sigma}^{1,1}(\gamma), \qquad \Lambda = \overline{\Sigma}^{1,1,1}(\gamma)$$

be the closures of the Thom-Boardman loci $\Sigma^{1,1}(\gamma)$, $\Sigma^{1,1,1}(\gamma)$ respectively. If $\Pi^0 = \Sigma^1(\gamma)$ is smooth, then

$$\Sigma^{1,1}(\gamma) = \{x \in \Sigma^1(\gamma) \mid \dim\ker d(\gamma|\Sigma^1(\gamma))_x = 1\};$$

and if $\Sigma^{1,1}(\gamma)$ is smooth, then

$$\Sigma^{1,1,1}(\gamma) = \{x \in \Sigma^{1,1}(\gamma) \mid \dim\ker d(\gamma|\Sigma^{1,1}(\gamma))_x = 1\}.$$

If M is Gauss-stable, we infer from the transversality package (6.1) that $\Sigma^1(\gamma)$ and $\Sigma^{1,1}(\gamma)$ are smooth, $\dim C = 1$ and $\dim \Lambda = 0$. We call C the *cusp curve* and Λ the *second-order cusp locus*. Furthermore, $C = \Sigma^{1,1}(\gamma) \cup \Sigma^2(\gamma)$, and the singular locus of C is $\Sigma^2(\gamma) = \Pi'$, consisting, as we shall see, of ordinary triple points of C. In order to define C and Λ by vector bundle methods, we are obliged to resolve the singularities of Π and it is to this matter we turn first.

The normal form (7.3) for the Gauss map of a Gauss-stable hypersurface at a point of Π' is

(2.1) $$\gamma(x,y,z) = (2x^3+2xy^2+x^2z, 3x^2+y^2+2xz, 2xy, z).$$

The resulting local analytic equation of Π in M is

(2.2) $$3x^2 - y^2 + xz = 0,$$

which defines a quadric cone of maximal rank.

If we blow up the origin in $\mathbb{C}^3$, the proper transform of the quadric cone $\{x^2+y^2+z^2=0\}$ (for simplicity) is a smooth surface with a smooth plane conic lying over the origin. Thus it follows from (2.2) that if we blow up our threefold M at the singular points of Π, then the proper transform $\tilde{\Pi}$ of Π is a smooth surface with smooth conics as exceptional fibers. Denote by $\sigma\colon \tilde{M} \to M$ the blow-up, and for $p \in \Pi'$, let $\tilde{Z}_p = (\sigma|\tilde{\Pi})^{-1}(p)$ be the conic over p; put $\tilde{Z} = \bigcup_{p\in\Pi'} \tilde{Z}_p$. Now consider the map $\sigma|\tilde{\Pi}\colon \tilde{\Pi} \to \Pi$. Off $\tilde{Z}$ it is an isomorphism; if $\xi \in \tilde{Z}_p \subset \mathbb{P}(T_pM)$, however, then

$$\text{(2.3)} \qquad \operatorname{im}(d\sigma)_\xi = \operatorname{span}(\xi) \subset T_pM.$$

Next we consider the *Nash blow-up* $\hat{\Pi}$ of Π defined as follows:

$$\hat{\Pi} = \operatorname{closure}\{(x, T_x\Pi)\colon x \in \Pi^0\} \subset \mathbb{P}(T^*M)|\Pi.$$

Since a smooth plane conic is self-dual, it is immediate from (2.2) that $\hat{\Pi} \cong \tilde{\Pi}$; let $\hat{Z}$ correspondingly denote the fibers of $\hat{\Pi}$ over Π'. Now $\hat{\Pi}$ supports a naturally defined bundle $\hat{T}\Pi$, called the *Nash tangent bundle* of Π, obtained by restricting the tautological bundle of rank 2 on $\mathbb{P}(T^*M)$ to $\hat{\Pi}$. Clearly,

$$(\hat{T}\Pi)|\hat{\Pi} - \hat{Z} \cong (T\hat{\Pi})|\hat{\Pi} - \hat{Z},$$

but the relation between the two bundles along $\hat{Z}$ will prove crucial.

Considering the projection $\sigma\colon \tilde{\Pi} \to \Pi \subset M$, its derivative provides by continuity a bundle map

$$\text{(2.4)} \qquad d\sigma\colon T\tilde{\Pi} \to \hat{T}\Pi$$

on $\tilde{\Pi} \cong \hat{\Pi}$, which by (2.3) drops rank along $\tilde{Z}$. Work in local coordinates, with $\Pi = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3$, $\tilde{\Pi} = \{(x, v, w) \subset \mathbb{C}^3 | v^2 + w^2 + 1 = 0\}$ and

$$\sigma\colon \tilde{\mathbb{C}}^3 \to \mathbb{C}^3, \qquad \sigma(x, v, w) = (x, xv, xw).$$

Using the framings

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w},$$
$$f_1 = \frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \qquad f_2 = w\frac{\partial}{\partial y} - v\frac{\partial}{\partial z}$$

for $T\tilde{\Pi}$ and $\hat{T}\Pi$ respectively, the map (2.4) is given by

$$d\sigma(e_1) = f_1, \qquad d\sigma(e_2) = xf_2.$$

From this computation we draw two conclusions:

(2.5) Viewing (2.4) as a sheaf homomorphism of locally free sheaves, the cokernel $\mathbf{K}$ is a locally free $\mathcal{O}_{\tilde{Z}}$-module of rank one, i.e., a line bundle supported on $\tilde{Z}$.

(2.6) The bundle map $(d\sigma)^{-1}$ defined on $\tilde{\Pi} - \tilde{Z}$ extends to give a homomorphism $\psi: \hat{T}\Pi \otimes \mathscr{O}(-\tilde{Z}) \to T\tilde{\Pi}$.

Pursuing the former remark, we have the following

(2.7) Lemma. $\mathbf{K} \cong \mathscr{O}_{\tilde{Z}}$.

Proof. First of all, it follows from (2.3) that, restricting to $\tilde{Z}$, $\mathbf{K} \cong \hat{T}\Pi/\mathscr{O}_{\tilde{Z}}(-1)$, where $\mathscr{O}_{\tilde{Z}}(-1)$ is the restriction to $\tilde{Z}$ of $\mathscr{O}_{\mathbf{P}(TM)}(-1)$. Working in $\mathbf{P}^2 = \mathbf{P}(T_pM)$, let C be a smooth conic and let $g: C \to \mathbf{P}^{2*}$ be its Gauss map. Denote by $L = \mathscr{O}_C(-1)$ and by E the pullback under g of the tautological rank 2 bundle on $\mathbf{P}^{2*}$. Then $c_1(E/L) = 0$: $c_1(L)\cdot C = -H\cdot C = -2$, while $c_1(E)\cdot C = -H^*\cdot g(C) = -2$, where H^* is the "hyperplane" class in $\mathbf{P}^{2*}$. Since $C \cong \mathbf{P}^1$, E/L is the trivial line bundle on C. q.e.d.

We therefore have the exact sequence of coherent sheaves

$$0 \to T\tilde{\Pi} \to \hat{T}\Pi \to \mathscr{O}_Z \to 0 \tag{2.8}$$

on $\tilde{\Pi}$; from this it is straightforward to compare the Chern classes of the bundles $T\tilde{\Pi}$ and $\hat{T}\Pi$.

(2.9) Proposition. *Let $M \subset \mathbf{P}^4$ be a Gauss-stable hypersurface of degree $d \geq 3$. Then:*

(a) *the canonical bundle $K_{\tilde{\Pi}}$ of $\tilde{\Pi}$ is cut out on $\tilde{\Pi}$ by the linear system $|(6d-15)H|$ (here we omit the σ^*),*

(b) $c_1(\hat{T}\Pi) = c_1(T\tilde{\Pi}) + [\tilde{Z}]$, $c_2(\hat{T}\Pi) = c_2(T\tilde{\Pi}) + [\tilde{Z}]^2$.

Proof. (a) Let $\sigma: \tilde{M} \to M$ be the blow-up of M at Π', and let $E = \sigma^{-1}(\Pi')$ be the exceptional divisor. Apply the adjunction formula (cf. [15, p. 147]) to the smooth surface $\tilde{\Pi} \subset \tilde{M}$, using $K_{\tilde{M}} = \sigma^* K_M + 2E$ [15, p. 187]. Since Π' consists of ordinary double points, one checks easily in local coordinates that

$$[\tilde{\Pi}] = \sigma^*[\Pi] - 2E, \tag{2.10}$$

and so

$$\begin{aligned} K_{\tilde{\Pi}} &= (K_{\tilde{M}} + [\tilde{\Pi}])|\tilde{\Pi} = \sigma^*(K_M + [\Pi])|\tilde{\Pi} \\ &= \sigma^*((d-5)H + 5(d-2)H) \quad \text{by (1.10)} \\ &= \sigma^*(6d-15)H. \end{aligned}$$

(b) We can use (2.8) to compare $c_*(T\tilde{\Pi})$ and $c_*(\hat{T}\Pi)$ once we know the Chern classes of the coherent sheaf $\mathscr{O}_{\tilde{Z}}$ on $\tilde{\Pi}$ (cf. [18]). From the obvious resolution

$$0 \to \mathscr{O}_{\tilde{\Pi}}(-\tilde{Z}) \to \mathscr{O}_{\tilde{\Pi}} \to \mathscr{O}_{\tilde{Z}} \to 0,$$

it follows that $\mathrm{ch}(\mathscr{O}_{\tilde{Z}}) = \mathrm{ch}(\mathscr{O}_{\tilde{\Pi}}) - \mathrm{ch}(\mathscr{O}_{\tilde{\Pi}}(-\tilde{Z})) = [\tilde{Z}] - \frac{1}{2}[\tilde{Z}]^2$, whence $c_1(\mathscr{O}_{\tilde{Z}}) = [\tilde{Z}]$ and $c_2(\mathscr{O}_{\tilde{Z}}) = [\tilde{Z}]^2$. The result now is immediate, since $c_1(T\tilde{\Pi}) \cdot [\tilde{Z}] = -K_{\tilde{\Pi}} \cdot [\tilde{Z}] = 0$ by (a). q.e.d.

There is yet another resolution of Π, which we shall dub the *kernel blow-up* $\Pi^{\#}$ of Π, defined as follows:

$$\Pi^{\#} = \{(x,\xi) \in \mathbb{P}(TM)|\Pi \mid \xi \in \ker d\gamma_x\}.$$

Insofar as $\dim\ker d\gamma_x = 1$ when $x \in \Pi^0$, $\Pi^{\#}$ is birational to Π; on the other hand, since $\dim\ker d\gamma_x = 2$ when $x \in \Pi'$, the exceptional fibers of $\Pi^{\#}$ are $\mathbb{P}^1$'s. Computing with the normal form (2.1) we find that (locally) $\Pi^{\#} \subset \mathbb{C}^5 \subset \mathbb{C}^3 \times \mathbb{P}^2$ is given by the equations $3x + z - xv^2 = 0$, $y + xv = 0$, $w = 0$, and hence is a smooth surface.

It follows from the structure theory of birational maps of surfaces (cf. [15, p. 510]) that the three resolutions $\tilde{\Pi}$, $\hat{\Pi}$ and $\Pi^{\#}$ of Π must all be isomorphic. It is also easy to exhibit the isomorphism explicitly using (2.1), viz.;

$$(2.11)\qquad \boxed{\begin{array}{c}\tilde{\Pi}\\ \text{limiting secant}\\ [(x,y,z)]\end{array}} \leftrightarrow \boxed{\begin{array}{c}\hat{\Pi}\\ \text{limiting tangent plane}\\ [(6x+z,-2y,x)]\end{array}} \leftrightarrow \boxed{\begin{array}{c}\Pi^{\#}\\ \text{limiting kernel direction}\\ \begin{cases}[(x,-y,0)] & (x,y)\neq 0\\ [(0,1,0)] & (x,y)=0\end{cases}\end{array}}\ .$$

(The latter arrow is reversible since z is determined whenever $(x,y) \neq 0$ by the equation (2.2) of Π.)

We are at last in a position to address the geometry of the cusp curve C. Let $x \in \Sigma^{1,1}(\gamma)$. Then $\ker d\gamma_x \subset T_x\Pi$, so that under the projection

$$T_xM \to T_xM/T_x\Pi = (N_{\Pi/M})_x$$

the kernel direction $\operatorname{span}(\ker d\gamma_x)$ maps to zero. The advantage of this point of view is that it is amenable to globalization on $\tilde{\Pi}$. Let $\mathscr{L}$ be the tautological line bundle on $\tilde{\Pi}$ obtained by restricting $\mathscr{O}_{\mathbb{P}(TM)}(-1)$ to $\Pi^{\#}$. Since $\mathscr{L} \subset \sigma^*TM$, we may consider on $\tilde{\Pi}$ the homomorphism of line bundles

$$\operatorname{proj}\colon \mathscr{L} \to \sigma^*TM/\hat{T}\Pi \underset{\text{def}}{=} \hat{N}_{\Pi}.$$

Changing notation somewhat, let s be this section of the homomorphism bundle $\operatorname{Hom}(\mathscr{L}, \hat{N}_{\Pi})$, and set $\hat{C} = (s)$, the zero divisor of s; then $C = \sigma_*\hat{C}$.

If M is Gauss-stable, it follows from (6.1.4) that s is transverse to the zero section away from $\tilde{Z}$ and hence that $\hat{C}$ is a smooth (reduced) divisor on $\tilde{\Pi}$ away from $\tilde{Z}$. It follows from (2.11) that the limiting kernel direction $\mathscr{L}_\xi$ at $\xi \in \tilde{Z}$ is contained in $\hat{T}_\xi\Pi \Leftrightarrow \xi = [(0,0,1)]$, $[(1,i,-4)]$, or $[(1,-i,-4)]$; i.e., $\hat{C}$ meets each component of $\tilde{Z}$ in three points, and hence C has an ordinary triple point at each point of Π'. Moreover, it is an easy calculation to check from (2.1) that the three branches of $\hat{C}$ at $\tilde{Z}$ are given in these local coordinates by the three lines

$$(2.12)\qquad \hat{C}_1\colon \{t(0,0,1)\}, \quad \hat{C}_2\colon \{t(1,i,-4)\}, \quad \hat{C}_3\colon \{t(1,-i,-4)\}.$$

Thus $\hat{C}$ is smooth.

To compute the divisor class of $\hat{C}$ on $\hat{\Pi}$, we use (2.9):

$$\begin{aligned}[\hat{C}] &= c_1(\mathscr{L}^* \otimes \sigma^* TM/\hat{T}\Pi) = -c_1(\mathscr{L}) - \sigma^* K_M - (c_1(T\tilde{\Pi}) + [\tilde{Z}]) \\ &= -c_1(\mathscr{L}) - (d-5)H + (6d-15)H - [\tilde{Z}] \\ &= 5(d-2)H - c_1(\mathscr{L}) - [\tilde{Z}].\end{aligned}$$

(2.13) Lemma. *If M is a Gauss-stable hypersurface of degree $d \geq 3$, then we have (on $\tilde{\Pi}$) $c_1(\mathscr{L}) = (-2d+5)H + \frac{1}{2}[\tilde{Z}]$.*

Proof. Let $\mathscr{E} \subset T^*M$ be the annihilator of $\mathscr{L}$; then

$$\mathscr{E} \cong (TM/\mathscr{L})^*. \tag{2.14}$$

If we interpret the second fundamental form of M as a section of $\mathrm{Hom}(TM, T^*M \otimes N_M)$, then we obtain the exact sequence of bundles on Π^0:

$$0 \to \mathscr{L} \to TM \xrightarrow{\mathrm{II}} \mathscr{E} \otimes N_M \to 0. \tag{2.15}$$

That is, we have on Π the induced bundle homomorphism

$$TM/\mathscr{L} \xrightarrow{\overline{\mathrm{II}}} \mathscr{E} \otimes N_M, \tag{2.16}$$

which is an isomorphism on $\tilde{\Pi} - \tilde{Z}$ and fails to be injective on $\tilde{Z}$. Condition (6.1.2) of the transversality package implies that $\overline{\mathrm{II}}$ is transverse to the stratum of elements of $\mathrm{Hom}(TM/\mathscr{L}, \mathscr{E} \otimes N_M)$ of rank one. Hence, by Thom-Porteous [12], the scheme along which $\overline{\mathrm{II}}$ drops rank is given by the divisor $c_1(\mathscr{E} \otimes N_M - TM/\mathscr{L})$, and so

$$\begin{aligned}[\tilde{Z}] &= c_1(\mathscr{E} \otimes N_M) - c_1(TM/\mathscr{L}) \\ &= -2c_1(TM/\mathscr{L}) + 2c_1(N_M) \quad \text{by (2.14)} \\ &= 2(2d-5)H + 2c_1(\mathscr{L}).\end{aligned}$$

Therefore $c_1(\mathscr{L}) = (-2d+5)H + \frac{1}{2}[\tilde{Z}]$, as required. q.e.d.

We are now in a position to conclude

(2.17) Proposition. $[\hat{C}] = (7d-15)H - \frac{3}{2}[\tilde{Z}]$ *on $\tilde{\Pi}$, and hence $[C] = \sigma_*[\hat{C}] = (7d-15)H$, as a rational equivalence class on H. As a consequence,* $\deg C = 5d(d-2)(7d-15)$.

We turn next to the second-order cusp locus Λ. On Π^0 this locus is described by the condition that the kernel direction becomes tangent to C, i.e., that the projection $\mathscr{L} \to N_{C/\Pi}$ vanishes. We observe that $\Lambda \cap \Pi' = \varnothing$: the kernel plane $z = 0$ (cf. (2.11)) is certainly transverse to the three branches (2.12) of C. Nevertheless, globalizing our description of Λ on $\tilde{\Pi}$ is subtle, for the "obvious" bundle map

$$\nu : \mathscr{L} \to N_{\hat{C}/\hat{\Pi}} = \mathrm{Hom}(\mathscr{L}, \hat{N}_\Pi),$$

while having the requisite zeros at the lifts of Λ points, is not holomorphic across $\tilde{Z}$!

To remedy the situation, we begin by observing (cf. [6] and §6) that ν is the intrinsic derivative Ds of the section s defining C. Recall that if L is a line bundle on X, and $Y \subset X$ is the zero divisor of a section s of L, then the intrinsic derivative $Ds\colon (TX)|Y \to L$ should be interpreted as a one-form on X, defined along Y, with values in L. (Moreover, if s is transverse to the zero section, we have $L|Y \cong N_{Y/X}$.) So in our case, still restricting our attention to Π^0, the projection ν is obtained as follows:

$$(2.18)\qquad \begin{tikzcd} s\colon & \mathscr{L} \arrow[r] & TM \arrow[r] & N_{\Pi/M} \\ Ds\colon & (T\Pi)|C \arrow[r] & \operatorname{Hom}(\mathscr{L}, N_{\Pi/M}) \cong N_{C/\Pi}. \\ & \mathscr{L} \arrow[u, phantom, "\cup"] \arrow[ur, "\nu"'] \end{tikzcd}$$

To globalize this construction on $\tilde{\Pi}$, we must utilize (2.6). On $\tilde{\Pi}$ we have the section s of $\operatorname{Hom}(\mathscr{L}, \hat{N}_\Pi)$ defining $\hat{C}$. Hence its intrinsic derivative gives the map

$$Ds\colon (T\tilde{\Pi})|\hat{C} \to \operatorname{Hom}(\mathscr{L}, \hat{N}_\Pi) \cong N_{\hat{C}/\tilde{\Pi}}.$$

Unfortunately, $\mathscr{L}$ is a subbundle of $\hat{T}\Pi$, not of $T\tilde{\Pi}$. Applying (2.6), there is an injection of $\mathscr{L} \otimes \mathscr{O}(-\tilde{Z})$ into $T\tilde{\Pi}$, and so we define the bundle map $\tilde{\nu}$ in the obvious way:

$$\begin{tikzcd} Ds\colon (T\tilde{\Pi})|\hat{C} \arrow[r] & N_{\hat{C}/\tilde{\Pi}} \\ \mathscr{L} \otimes \mathscr{O}(-\tilde{Z}) \arrow[u, "\psi"] \arrow[ur, "\tilde{\nu}"] \end{tikzcd}$$

Of course, away from $\tilde{Z}$, $\tilde{\nu}$ is equal to the map ν already discussed, and so we define $\hat{\Lambda} = (\tilde{\nu})$, $\Lambda = \sigma_*\hat{\Lambda}$.

(2.19) Proposition. *If M is Gauss-stable of degree $d \geq 3$, then Λ is a reduced zero-cycle of degree $5d(d-2)(3d-7)(17d-36)$.*

Proof. Let ξ denote the bundle $\operatorname{Hom}(\mathscr{L} \otimes \mathscr{O}(-\tilde{Z}), \operatorname{Hom}(\mathscr{L}, \hat{N}_\Pi))$ defined on $\tilde{\Pi}$. We compute its Chern class:

$$\begin{aligned} c_1(\xi) &= -2c_1(\mathscr{L}) + [\tilde{Z}] + 5(d-2)H - [\tilde{Z}] \\ &= (9d-20)H - [\tilde{Z}] \quad \text{by (2.13)}. \end{aligned}$$

Applying (2.17), we find

$$\begin{aligned} c_1(\xi)\cdot[\hat{C}] &= ((9d-20)H - [\tilde{Z}])\cdot((7d-15)H - \tfrac{3}{2}[\tilde{Z}]) \\ &= (9d-20)(7d-15)(\deg \Pi) + \tfrac{3}{2}[\tilde{Z}]^2 \\ &= (9d-20)(7d-15)5d(d-2) + \tfrac{3}{2}(-2)(\deg \Pi') \\ &= 5d(d-2)((9d-20)(7d-15) - 12(d-2)^2) \\ &= 5d(d-2)(3d-7)(17d-36); \end{aligned}$$

note that $[\tilde{Z}_p]^2 = -2$ for $p \in \Pi'$, since $\tilde{Z}_p$ is a conic in the exceptional $\mathbb{P}^2$.

Now $\tilde{\nu}$ is transverse to the zero section of ξ as a consequence of (6.1.5) (cf. also ((6.2g)). Therefore, if M is Gauss-stable, $\hat{\Lambda}$ is a reduced zero-dimensional subscheme of $\hat{C}$; since $\Lambda \cap \Pi' = \varnothing$, it is immediate that the same is true of Λ in C. q.e.d.

Query. It is not hard to see that C cannot be cut out on Π by a divisor class on M. For example, at $p \in \Pi'$, the fiber of $\tilde{\Pi}$ is a conic and that of $\hat{C}$ is *three* points on the conic. But is some multiple of C a scheme-theoretic complete intersection? (In the case of surfaces [29], twice the cusp locus is cut out on the smooth parabolic curve by the asymptotic flex curve $\mathscr{F}$.) Similarly, is Λ or some multiple thereof a complete intersection of C and some other curve in Π?

3. Geometry of the cusp curve

We begin by recalling the notion of *order of contact* of a hypersurface $M \subset \mathbb{P}^4$ with a line L: we say M has kth *order contact* with L at $x \in M$ if their intersection multiplicity $\mathscr{I}(M \cdot L)_x$ at x equals $k+1$. In affine coordinates, if $x = 0 \in \mathbb{C}^4$, L is the x_1-axis, and $M = \{x_4 = f(x_1, x_2, x_3)\}$ locally, then M has kth order contact with L at 0 provided

$$(3.1) \qquad \frac{\partial^i f}{\partial x_1^i}(0) = 0, \quad i = 0, \ldots, k, \qquad \frac{\partial^{k+1} f}{\partial x_1^{k+1}}(0) \neq 0.$$

For example, M has at least zeroth order contact with L at x if $x \in M \cap L$, at least first order contact if L is tangent to M at x, and at least second order contact if L is an asymptotic direction in M at x.

Consider the sequence of incidence correspondences [22]

$$\mathscr{Z}_0 \supset \mathscr{Z}_1 \supset \mathscr{Z}_2 \supset \cdots \supset \mathscr{Z}_k \supset \cdots,$$
$$\mathscr{Z}_k = \{(x, L) \in M \times \mathbb{G}(1,4) \mid M \text{ has at least } kth \text{ order contact with } L \text{ at } x\}.$$

Given $d \geq 3$ and $k \leq d$, for generic $M \subset \mathbb{P}^4$ of degree d, $\mathscr{Z}_k$ is a submanifold of $\mathscr{Z}_0$ of codimension k (cf. (9.6) and [5]). With $M = \{x_4 = f(x_1, x_2, x_3)\} \subset \mathbb{C}^4$, we use $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$ as a framing for TM with corresponding coordinates p_1, p_2, p_3. Then it follows from (3.1) that, for $k \geq 2$, $\mathscr{Z}_k$ is cut out in $\mathscr{Z}_{k-1}$ by the equation (homogeneous of degree k in p)

$$(3.2) \qquad \sum_{|I|=k} \frac{\partial^{|I|} f}{\partial x^I} p^I = 0.$$

We infer from (1.7) that

$$\mathscr{Z}_2 \cong \{(x,v) \in \mathbb{P}TM | \mathrm{II}_x(v,v) = 0\}.$$

It will be useful to introduce the higher forms

$$\mathrm{III} = \sum_{|I|=3} \frac{\partial^3 f}{\partial x^I} dx_{i_1} \otimes dx_{i_2} \otimes dx_{i_3}, \qquad 1 \le i_1, i_2, i_3 \le 3,$$
$$\mathrm{IV} = \sum_{|I|=4} \frac{\partial^4 f}{\partial x^I} dx_{i_1} \otimes dx_{i_2} \otimes dx_{i_3} \otimes dx_{i_4}, \qquad 1 \le i_1, i_2, i_3, i_4 \le 3.$$

While the cubic form III, for example, is not intrinsically defined, the subschemes of $\mathbb{P}T_xM$

$$\{\mathrm{II}_x = 0\} \supset \{\mathrm{II}_x = 0\} \cdot \{\mathrm{III}_x = 0\}$$

represent the fibers of $\mathscr{Z}_2$ and $\mathscr{Z}_3$, respectively, over x.

It is enlightening to consider what these fibers look like at points of the various loci which we have been studying. A few sketches below represent study yet to follow.

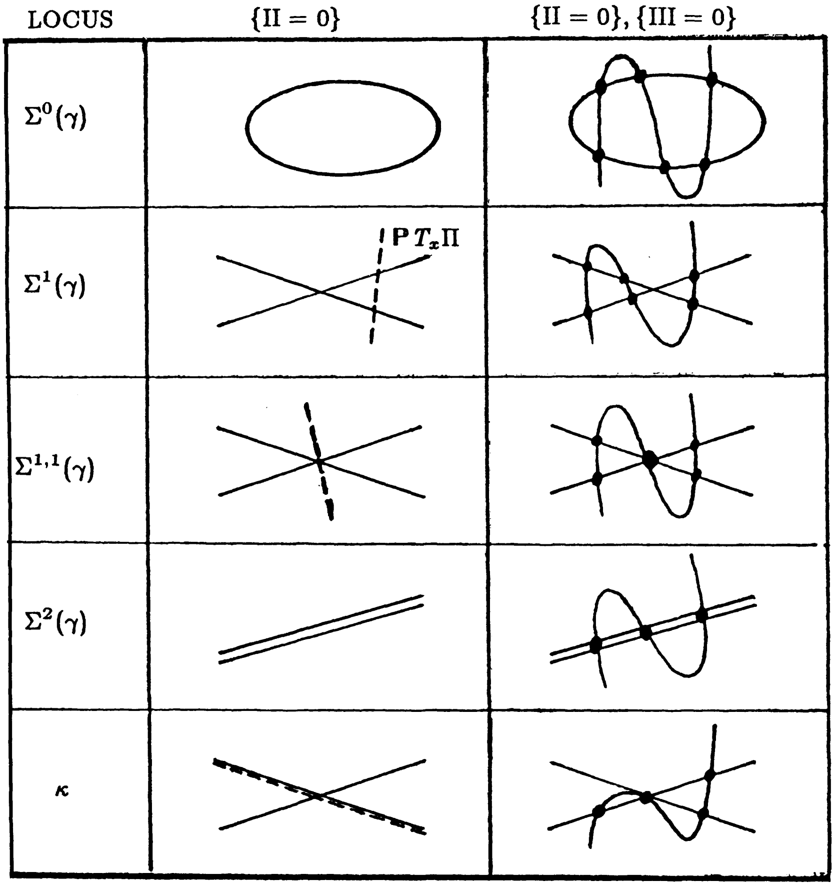
The first result on our agenda is the characterization of the cusp locus by the behavior of the cubic $\{\mathrm{III} = 0\}$, as pictured in (3.3): the curve C is the locus of points of Π at which the cubic passes through the vertex of the conic $\{\mathrm{II} = 0\}$. This generalizes the result of [29] for surfaces. But the picture suggests more: the tangent line to the cubic at the vertex of the conic (which is intrinsically defined) coincides with $\mathbb{P}(T_x\Pi)$. We are also led to consider the locus κ of *Cayley points* x with the property that the tangent space $T_x\Pi$ becomes II-null; κ is then distinguished by the pictures above. Another locus contained in C is $\mu = \{x \in C | \ker d\gamma_x$ is an asymptotic direction in $\Pi\}$. The last locus we shall consider is the set Γ of points $x \in C$ with the property that the quartic $\{\mathrm{IV}_x = 0\}$ as well passes through the vertex of the conic $\{\mathrm{II}_x = 0\}$. The rest of this section is devoted to finding the divisor classes of the zero-cycles Λ, κ, μ, Γ, and Π' in C and to describing any geometric relations among them.

We begin by enunciating the following fact which supplements the results at the end of §9.

(3.4) Lemma. *Let M be a Gauss-stable hypersurface of degree $d \ge 2$. Then $\mathscr{Z}_2$ is a smooth hypersurface in $\mathscr{Z}_1 \cong \mathbb{P}TM$.*

Sketch of proof. A careful analysis of the usual proof of Sylvester's law leads to the following parametric version. Let $x \in M$; then there is an

(3.3)



open set $U \subset M$ containing x and a frame for TM over U so that

$$\mathrm{II} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \gamma \end{bmatrix}$$

(where α, β, γ are analytic functions on U), and so that, furthermore,

$$\begin{aligned} x \in \Sigma^0 &\Rightarrow \text{we may take } \alpha = \gamma = 1, \beta = 0; \\ x \in \Sigma^1 &\Rightarrow \text{we may take } \alpha = 1, \beta = 0, \text{and} \\ &\qquad \gamma \text{ a local defining function for } \Pi; \\ x \in \Sigma^2 &\Rightarrow \alpha\gamma - \beta^2 = 0 \text{ on } \Pi, \alpha = \beta = \gamma = 0 \text{ at } x, \\ &\qquad \text{but } d\alpha \wedge d\beta \wedge d\gamma \neq 0. \end{aligned}$$

Now the equation of $\mathcal{Z}_2$ is $p_1^2 + \alpha p_2^2 + 2\beta p_2 p_3 + \gamma p_3^2 = 0$, whose differential in each of the three cases, as is easily checked, is nonzero. q.e.d.

Next, to understand the behavior of the cubic form III at points of C, we investigate the intersection $\mathcal{Z}_3 \cap \Pi^\# \subset \mathcal{Z}_2$. We remark that, for a Zariski-open set of hypersurfaces $M \in \mathcal{M}_d$, $d \geq 3$, $\mathcal{Z}_3$ is a smooth three-dimensional variety (9.6).

(3.5) Proposition. *Let M be a Gauss-stable hypersurface of degree $d \geq 3$, and assume $\mathcal{Z}_3$ is smooth and of dimension three. Then $\mathcal{Z}_3$ is transverse to $\Pi^\#$ in $\mathcal{Z}_2$, and $\mathcal{Z}_3 \cdot \Pi^\# = C^\# \cong \hat{C}$.*

Proof. We first prove that the set-theoretic intersection of $\mathcal{Z}_3$ and $\Pi^\#$ is isomorphic to $\hat{C}$. Let $x \in \Pi^0$; represent M as a graph $x_4 = f(x_1, x_2, x_3)$, as usual, with x at the origin and choosing linear coordinates on $\mathbb{C}^3$ so that $\partial/\partial x_1$ spans the kernel direction at x. Then II is given by the Hessian matrix $H = [f_{ij}]$ and Π by $\det H = 0$. Let α_{ij} be the ij-cofactor of H; thus

$$\Pi = \{f_{11}\alpha_{11} + f_{12}\alpha_{12} + f_{13}\alpha_{13} = 0\},$$
$$\mathcal{Z}_3 = \{\Sigma f_{ijk} p_i p_j p_k = 0\}.$$

Now $x \in C - \Pi' \Leftrightarrow \partial/\partial x_1$ is tangent to Π at $x \Leftrightarrow f_{111}\alpha_{11} = 0$ at $0 \Leftrightarrow f_{111}(0) = 0$. That is, $x \in C - \Pi' \Leftrightarrow (x, \partial/\partial x_1) \in \mathcal{Z}_3 \cap (\Pi^\# - Z^\#)$. Provided $\mathcal{Z}_3$ contains no exceptional fiber of $\Pi^\#$, it then follows that $C^\# \cong \hat{C} - \tilde{Z} = \hat{C}$. If the fiber $Z_x^\#$ over $x \in \Pi'$ is given by $p_3 = 0$ and $Z_x^\# \subset \mathcal{Z}_3$, then $f_{ijk}(x) = 0$, $1 \leq i, j, k \leq 2$, whence the tangent hyperplane section of M at x is of the form $z^2 + zq(x, y, z) + \cdots$, where q is quadratic and $\cdots$ denotes terms of degree ≥ 4. It is easy to check, blowing up the origin, that x could not then be a D_4-singularity; cf. (8.14ii) and (8.20).

Lastly we must check that the transversality package (6.1) implies the intersection is transverse. We work near $x \in \Pi^0$. The section s of $\mathrm{Hom}(\mathcal{L}, N_\Pi)$ defining C may be given as follows, using the coordinates defined earlier: if $\Pi = \{\lambda = 0\}$, then $s = \partial\lambda/\partial x_1$ and $s \pitchfork 0 \Leftrightarrow d(\partial\lambda/\partial x_1) \neq 0$ when $\partial\lambda/\partial x_1 = 0$. The transversality condition (6.1.4) translates to the condition $df_{111} \neq 0$. Now to check that $\mathcal{Z}_3 \pitchfork \Pi^\#$, it suffices to verify that $N^*_{\Pi^\#/\mathcal{Z}_1} + N^*_{\mathcal{Z}_3/\mathcal{Z}_2}$ is a four-dimensional subbundle (locally) of $T^*\mathcal{Z}_1|C^\#$. Let $g_i = \Sigma f_{ij} p_j = 0$, $i = 1, 2, 3$, define $\Pi^\#$ in $\mathcal{Z}_1$; then we must establish that

$$d\lambda \wedge dg_1 \wedge dg_2 \wedge dg_3 \neq 0 \quad \text{on } C^\#. \tag{3.6}$$

Set $p_1 = 1$ and compute in local coordinates x_1, x_2, x_3, p_2, p_3. Then $dg_1 \wedge dg_2 \wedge dg_3 = \alpha_{11} df_{11} \wedge dp_2 \wedge dp_3$; since $d\lambda \equiv df_{111} \bmod(dp_2, dp_3)$, (3.6) is equivalent to

$$df_{11} \wedge df_{111} \neq 0.$$

Finally, since df_{11} spans $N^*_{\Pi/M}$, the latter is in turn equivalent to the condition that $df_{111}|\Pi \neq 0$.

(3.7) Corollary. *Let $x \in C$, and let $L \in \mathbb{P}T_xM$ denote the kernel direction. Then $\mathbb{P}T_x\Pi = \tilde{T}_L\{\mathrm{III}_x = 0\}$.*

Proof. Using the notation of the proof of (3.5), $T_x\Pi$ is cut out by the equation

$$\begin{aligned} 0 = df_{11} &= f_{111}dx_1 + f_{112}dx_2 + f_{113}dx_3 \\ &= f_{112}dx_2 + f_{113}dx_3. \end{aligned}$$

But in terms of affine coordinates p_2, p_3 on $\mathbb{P}T_xM$,

$$\{\mathrm{III}_x = 0\} = \{f_{112}p_2 + f_{113}p_3 + \sum f_{1ij}p_ip_j + \sum f_{ijk}p_ip_jp_k = 0\}.$$

So $T_L\{\mathrm{III}_x = 0\} = \{f_{112}p_2 + f_{113}p_3 = 0\} = T_x\Pi$. q.e.d.

We turn next to a brief investigation of the projective differential geometric invariants at a second-order cusp point. The following result will be useful in our analysis of the cubic threefold in §4.

(3.8) Proposition. *Let M be Gauss-stable, and let $x \in \Lambda$. Then the following are equivalent.*

(a) *$T_x\Pi$ is* II*-null.*

(b) *The kernel direction L at x is asymptotic in Π, i.e., is null for the second fundamental form of $\Pi \subset \mathbb{P}^4$.*

Proof. Choose two frame fields $Z_0, \cdots, Z_4$ and $\overline{Z}_0, \cdots, \overline{Z}_4$ near x on Π so that

(i) $Z_i = \overline{Z}_i$ for $i \neq 1$, and $Z_1 = \overline{Z}_1$ along C;
(ii) $\overline{Z}_1, Z_2$ span $T\Pi$, $\overline{Z}_1, Z_2, Z_3$ span TM; and
(iii) Z_1 spans the kernel direction.

Since $\omega_1^4 = 0$ identically on Π, we infer from (1.2) that

$$0 = d\omega_1^4 = (\omega_1^1 - \omega_4^4) \wedge \omega_1^4 + \omega_1^2 \wedge \omega_2^4 + \omega_1^3 \wedge \omega_3^4;$$

evaluating on (Z_1, Z_2) at $x \in C$, we find

$$0 = h_{22}\omega_1^2(Z_1) + h_{32}\omega_1^3(Z_1) - h_{21}\omega_1^2(Z_2) - h_{31}\omega_1^3(Z_2).$$

Therefore

$$\omega_1^2(Z_1)h_{22} + \omega_1^3(Z_1)h_{23} = 0. \tag{3.9}$$

To compute the second fundamental form II_Π of Π, we use the barred frame field:

$$\mathrm{II}_\Pi = (\overline{\omega}_1^3 \otimes \overline{\omega}^1 + \overline{\omega}_2^3 \otimes \overline{\omega}^2) \otimes Z_3 + (\overline{\omega}_1^4 \otimes \overline{\omega}^1 + \overline{\omega}_2^4 \otimes \overline{\omega}^2) \otimes Z_4.$$

Hence, if $x \in C$, then

$$\mathrm{II}_\Pi(Z_1, Z_1) = \overline{\omega}_1^3(Z_1)Z_3 + \overline{\omega}_1^4(Z_1)Z_4.$$

Imposing the further condition that $x \in \Lambda$, since $Z_1 = \overline{Z}_1$ along C and Z_1 is tangent to C at x, we have $\overline{\omega}_1^j(Z_1) = \omega_1^j(Z_1)$ at x, $j = 3, 4$, and so we conclude that L is asymptotic at $x \Leftrightarrow \omega_1^3(Z_1) = 0$.

On the other hand, since $h_{1j} = 0$, $T_x\Pi$ is II-null $\Leftrightarrow h_{22} = 0$. Thus from (3.9) it follows that (a) $\Rightarrow$ (b), as rank II $= 2$ implies $h_{23} \neq 0$. Conversely, if $\omega_1^3(Z_1) = 0$, then by (6.1.6) we cannot have $dZ_1(Z_1) \equiv 0 \bmod (Z_1)$, whence $\omega_1^2(Z_1) \neq 0$ and $h_{22} = 0$, establishing (b) $\Rightarrow$ (a). q.e.d.

We turn next to the task of understanding the loci $\kappa = \{x \in C \mid T_x\Pi$ is II-null$\}$ and $\mu = \{x \in C \mid \ker d\gamma_x$ is asymptotic in $\Pi\}$ as divisors of C. For the moment, we assume they are zero-dimensional loci, disjoint from Π', and use bundle-theoretic methods to compute their degrees. We will see later in this section that for a nonempty Zariski-open subset of the Gauss-stable hypersurfaces, those loci are in fact reduced zero-cycles.

(3.10) Proposition. *The loci $\hat{\kappa} = \sigma^{-1}(\kappa)$ and $\hat{\mu} = \sigma^{-1}(\mu)$ are cut out on $\hat{C}$ by the linear system $|(9d - 20)H - [\tilde{Z}]|$, and both have degree $5d(d-2)(3d-7)(17d-36)$.*

Proof. On $\hat{C}$, the "kernel bundle" $\mathscr{L}$ is a subbundle of $\hat{T}\Pi$, and the second fundamental form II induces a bundle homomorphism

$$\overline{\mathrm{II}}\colon \mathrm{Sym}^2(\hat{T}\Pi/\mathscr{L}) \to N_M$$

defining the divisor $\hat{\kappa}$. Using (2.9) and (2.13), we compute

$$\begin{aligned} c_1(\mathrm{Hom}((\hat{T}\Pi/\mathscr{L})^2, N_M)) &= 2c_1(\mathscr{L}) - 2c_1(\hat{T}\Pi) + c_1(N_M) \\ &= (9d-20)H - [\tilde{Z}]. \end{aligned}$$

To define $\hat{\mu}$, it is more convenient to work on the surface $\tilde{\Pi} \subset \tilde{M}$ and to use its second fundamental form

$$\mathrm{II}_{\tilde{\Pi}}\colon \mathrm{Sym}^2(T\tilde{\Pi}) \to N_{\tilde{\Pi}/\tilde{M}}.$$

Using (2.6), consider the bundle map

$$(\mathscr{L} \otimes \mathscr{O}(-\tilde{Z}))^2 \to \mathrm{Sym}^2(T\tilde{\Pi}) \xrightarrow{\mathrm{II}_{\tilde{\Pi}}} N_{\tilde{\Pi}/\tilde{M}}.$$

To compute its divisor, we note that by (2.10), $c_1(N_{\tilde{\Pi}/\tilde{M}}) = \sigma^*[\Pi] - 2[\tilde{Z}]$; thus $\hat{\mu}$ is defined by the divisor class

$$\begin{aligned} c_1(\mathrm{Hom}(\mathscr{L}(-\tilde{Z})^2, N_{\tilde{\Pi}/\tilde{M}})) &= -2c_1(\mathscr{L}) + 2[\tilde{Z}] + \sigma^*[\Pi] - 2[\tilde{Z}] \\ &= (9d-20)H - [\tilde{Z}]. \end{aligned}$$

Comparing with the proof of (2.19), we see that the same linear system cuts out $\hat{\Lambda}$ on $\hat{C}$, and so

$$\deg \hat{\kappa} = \deg \hat{\mu} = \deg \hat{\Lambda} = 5d(d-2)(3d-7)(17d-36),$$

as required. q.e.d.

Now we relate the loci κ, μ, and Λ more carefully. Before doing so, we complete the cast of characters with the locus $\Gamma = \{x \in C | M$ has at least fourth order contact at x with the line spanned by $\ker d\gamma_x\} = \{x \in C|$ the quartic $\{\mathrm{IV}_x = 0\}$ passes through the vertex of $\{\mathrm{II}_x = 0\}\}$. To study Γ, we introduce the ruled surface $S \subset \mathbf{P}^4$ generated by the kernel lines along the cusp curve. More precisely, let E be the rank 2 bundle on $\hat{C}$ corresponding to the line bundle $\mathscr{L}$ according to the Euler sequence

$$(3.11) \qquad 0 \to \mathscr{O}(-1) \to E \to \mathscr{L} \otimes \mathscr{O}(-1) \to 0.$$

Provided that there are only finitely many lines $L \subset \mathbf{P}^4$ having (at least) third order contact with M at each of two distinct points x, y, the map $\mathbf{P}(E) \to S$ is birational. We now check that if $d \geq 3$, this is a generic property.

(3.12) Lemma. *Let $d \geq 4$. There is a nonempty Zariski-open subset of the moduli space $\mathscr{M}_d$ of smooth hypersurfaces of degree d for which the map $\mathbf{P}(E) \to S$ is birational.*

Proof. (i) $d \geq 8$. For $x, y \in \mathbf{P}^4$ fixed and $L = \overline{xy}$, generically eight conditions are imposed on $M \in \mathscr{M}_d$ if M is to have intersection multiplicity four with L at x and at y. Now a dimension count shows that the Zariski-closed set $\{M \in \mathscr{M}_d | M$ has intersection multiplicity 4 with infinitely many lines at more than one point$\}$ is a proper subset of $\mathscr{M}_d$.

(ii) If $4 \leq d \leq 7$ and M has intersection multiplicity 8 with a line L, then $L \subset M$. Now a dimension count again shows that if $d \geq 6$, the generic hypersurface contains no line and if $d = 5$ it contains only finitely many.

(iii) $d = 4$. Suppose $L \subset M$ and M has third order contact with L at x and at y, $x \neq y$. Then restricting the Gauss map γ of M to L gives a linear system $|\mathscr{E}|$ of degree 3 on $L \cong \mathbf{P}^1$ and of dimension at most 2. The points x, y are ramification points of $\gamma|L$, and a plane cubic curve cannot have two cusps. On the other hand, if $\dim|\mathscr{E}| = 1$, $\gamma|L$ gives a threefold cover $\mathbf{P}^1 \to \mathbf{P}^1$ branched at four points; but then L must be a singular point of the Fano variety of M, and by [5] for generic M this variety is smooth. q.e.d.

Assuming the map $\mathbb{P}(E) \to S$ is birational, we may compute the degree of the ruled surface S (cf., for example, [15, p. 410]):

$$\begin{aligned}\deg S &= -c_1(E)\cdot[\hat{C}] = (2H - c_1(\mathscr{L}))\cdot[\hat{C}] \quad \text{by (3.11)}\\ &= ((2d-3)H - \tfrac{1}{2}[\tilde{Z}])\cdot((7d-15)H - \tfrac{3}{2}[\tilde{Z}]) \quad \text{by (2.13), (2.17),}\end{aligned}$$

and so

$$\deg S = 5d(d-2)(8d^2 - 27d + 21). \tag{3.13}$$

(3.14) Proposition. *Suppose M is Gauss-stable of degree $d \geq 4$, $\mathscr{Z}_4$ is smooth, Γ is a zero-cycle disjoint from Π', and the map $\mathbb{P}(E) \to S$ is birational. Then*

$$\deg\Gamma = 5d(d-2)(39d^2 - 179d + 204).$$

Remark. It follows from (8.1), (9.6), (8.21) and (3.12) that these conditions are generic, and from (3.22) that for a nonempty Zariski-open subset of $\mathscr{M}_d$ the zero-cycle is reduced and of the appropriate degree (cf. (3.22)).

Proof. We work with the ruled surface S: by (3.5) M has intersection multiplicity at least four with each kernel line at the corresponding point of C. Therefore

$$S\cdot M = 4C + \tau, \tag{3.15}$$

where the residual cycle τ is either a certain number, β, of lines contained in M (if $d = 4$) or a multi-section of the ruled surface S (if $d \geq 5$). We then have, respectively,

$$\begin{aligned}\beta &= \#(\mathscr{Z}_4\cdot\Pi^\#) \qquad (d=4),\\ \tau\cdot C &= \#(\mathscr{Z}_4\cdot\Pi^\#) \qquad (d\geq 5).\end{aligned}$$

The curve C and a ruling L generate the homology of S subject to the intersection properties

$$\begin{aligned}C\cdot L &= 1,\\ L\cdot L &= 0,\\ C\cdot C &= c_1(L)\cdot\hat{C}.\end{aligned}$$

Therefore $S\cdot H \sim C + nL$, where

$$n = \deg S - \deg C = 5d(d-2)(8d^2 - 34d + 36).$$

Since $M \sim dH$, $S\cdot M \sim dC + (nd)L$; juxtaposing with (3.15), we find that

$$\tau \sim (d-4)C + (nd)L,$$

whence

$$\begin{aligned} \tau \cdot C &= nd + (d-4)C \cdot C \\ &= 5d(d-2)(39d^2 - 179d + 204). \quad \text{q.e.d.} \end{aligned}$$

We now devote the rest of this section to an analysis of the relation among the five divisors Λ, κ, μ, Γ, and Π' on the cusp curve C. At this point it is quite telling to work with the equation of a tangent hyperplane section of M.

(3.16) Proposition. *Let $p \in C - \Pi'$ and taking the equation in affine coordinates of the hyperplane section $M \cap T_pM$ to be of the form*

$$0 = x^2 + y^2 + \underbrace{(\alpha x + \beta y)z^2 + \cdots}_{\text{cubic}} + \underbrace{\gamma z^4 + \cdots}_{\text{quartic}} + \cdots, \tag{3.17}$$

then

$$\begin{aligned} &p \in \Lambda \Leftrightarrow \alpha^2 + \beta^2 = 4\gamma, \\ &p \in \kappa \Leftrightarrow \alpha^2 + \beta^2 = 0, \\ &p \in \mu \Leftrightarrow \alpha^2 + \beta^2 = 6\gamma, \\ &p \in \Gamma \Leftrightarrow \gamma = 0. \end{aligned}$$

Proof. Note first of all that the kernel direction is $\partial/\partial z$ and so the coefficient of z^3 in (3.17) vanishes by (3.5); likewise, $p \in \Gamma \Leftrightarrow \gamma = 0$. Now the tangent line to the cubic $\{\mathrm{III} = 0\} \subset \mathbb{P}T_pM$ at $(0,0,1)$ is $\alpha x + \beta y = 0$; this line is II-null $\Leftrightarrow \alpha^2 + \beta^2 = 0$. By (3.7), this then is the condition for $T_p\Pi$ to be II-null. The condition characterizing the locus Λ is given by (8.19).

Lastly, to distinguish the locus μ, we make the following computation. Working at the origin in $\mathbb{C}^4$ with $T_0M = \{(x,y,z,w) | w = 0\}$, we write M as a graph $w = f(x,y,z)$, f analytic. Then

$$w = f(x,y,z) = u(x,y,z)(x^2 + y^2 + (\alpha x + \beta y)z^2 + \gamma z^4 + \cdots),$$

where u is a unit in $\mathscr{O}_{\mathbb{C}^3,0}$, say with $u(0) = 1$. Using this equation for M, we can calculate that the equation for Π is

$$x + \beta y + (6\gamma - \alpha^2 - \beta^2)z^2 + \cdots = 0,$$

where $\cdots$ denotes other terms of degree ≥ 2. Thus the kernel direction $\partial/\partial z$ is asymptotic in $\Pi \Leftrightarrow 6\gamma - \alpha^2 - \beta^2 = 0$.

Remark. From the equation of Π given above, we can also deduce the characterization of κ without referring to (3.7); in addition, (3.8) follows directly, albeit less geometrically, from (3.16).

The foregoing proposition leads us to consider the quotient $\rho = (\alpha^2+\beta^2)/\gamma$. Let $G \subset \mathbf{P}\,\mathrm{GL}(5)$ denote the subgroup stabilizing the form of equation (3.17), i.e., fixing the point $(1,0,0,0,0)$, the hyperplane $x_4 = 0$ and the quadratic form x^2+y^2. Then G consists of matrices $\mathbf{a} = [a_{ij}]$ of the form

$$\mathbf{a} = \begin{bmatrix} 1 & a_{01} & a_{02} & a_{03} & a_{04} \\ 0 & a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{21} & a_{22} & 0 & a_{24} \\ 0 & a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 & a_{44} \end{bmatrix},$$

where $a_{33}, a_{44} \neq 0$, and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in O(2,\mathbf{C}).$$

Under the action of G, α, β, γ transform to

$$\alpha' = (a_{11}\alpha + a_{21}\beta)a_{33}^2, \quad \beta' = (a_{12}\alpha + a_{22}\beta)a_{33}^2, \quad \gamma' = a_{33}^4\gamma, \tag{3.18}$$

respectively, so that the quotient $(a^2+\beta^2)/\gamma$ is indeed well-defined and thus a projective invariant associated to the point $p \in C - \Pi'$.

To investigate its behavior as we approach a point of Π', we consider the family of equations (3.17) with parameter $t \to 0$:

$$x^2 + ty^2 + (\alpha x + \beta y)z^2 + \cdots + \gamma z^4 + \cdots. \tag{3.19}$$

Passing to a branched double cover, we replace y by $y/\sqrt{t}$ and then we find

$$\rho_t = \frac{t\alpha^2 + \beta^2}{t\gamma}. \tag{3.20}$$

Provided β does not approach 0 as $t \to 0$, we have $\rho_t \to \infty$ as $t \to 0$. It follows from (8.20) that $\beta_0 = \lim_{t\to 0}\beta$ cannot be zero at a D_4-singularity (if the root (0,1) of the cubic $\varphi(y,z) = \beta_0 yz^2 + \cdots$ is to be simple), and hence ρ does indeed take the value ∞ at the points Π' of a Gauss-stable hypersurface.

Let $\mathscr{M}_d' = \{M \in \mathscr{M}_d \mid M$ is a Gauss-stable hypersurface with $\Lambda \cap \Gamma = \varnothing\}$, $d \geq 4$. Using (3.16) it is easy to exhibit an A_4-singularity on a surface in $\mathbf{P}^3$ which is not a Γ-point; hence by (8.21) the condition $\Lambda \cap \Gamma \neq \varnothing$ defines a proper Zariski-closed subset of $\mathscr{M}_d$. Thus $\mathscr{M}_d'$ is a nonempty Zariski-open subset of $\mathscr{M}_d$. We now come to the culminating result on the geometry of the cusp curve.

(3.21) Theorem. *If $M \in \mathscr{M}_d'$, $d \geq 4$, the invariant ρ defines a rational function on $\hat{C}$ of degree $D = 5d(d-2)(3d-7)(17d-36)$.*

Remarks. We have seen (cf. (2.19) and (3.10)) that $\hat{\Lambda}$, $\hat{\kappa}$, and $\hat{\mu}$ are cut out on $\hat{C}$ by the same linear system; this theorem puts them all in a particular pencil. On the other hand, the locus Γ enters with the value $\rho = \infty$, as do the $3 \deg \Pi'$ points of $\hat{\Pi}' = \hat{C} \cap \tilde{Z}$ (by our preceding discussion). Hence $\Gamma + \hat{\Pi}'$ is another divisor in this pencil; this explains the relation $\deg \Gamma = \deg \Lambda - 3 \deg \Pi'$.

Proof. We use notation and an approach similar to that of §8. Let V be the set of $(x, H, M) \in \mathbf{P}^4 \times \mathbf{P}^{4*} \times \mathscr{M}_d'$ such that x is an A_3-, A_4-, or D_4-singularity of $M \cap H$ which is versally deformed by the nearby hyperplane sections of M, and let $\pi \colon V \to \mathscr{M}_d'$ be the projection. The fiber $\pi^{-1}(M)$ is the cusp curve of M. Let W be the subset of V consisting of these (x, H, M) such that x is an A_3- or A_4-singularity of $M \cap H$. The fiber of W over $M \in \mathscr{M}_d'$ is the curve $C^0 = C - \Pi'$; clearly $V = \overline{W}$, and W is the nonsingular locus of V. Define

$$\hat{V} = \text{closure}\{(x, H, M; \xi) \in W \times \mathbf{P}TM \mid \xi = \ker d(\gamma_M)_x\}.$$

Then $\hat{V}$ is smooth and the fiber of $\hat{V}$ over M is the curve $\hat{C}$.

We begin by showing that there is a well-defined rational function $\boldsymbol{\rho} \colon \hat{V} \to \mathbf{P}^1$ whose restriction to each fiber is the invariant $\rho \colon \hat{C} \to \mathbf{P}^1$. By (8.11) the projection

$$W \to \mathscr{I} = \{(x, H) \in \mathbf{P}^4 \times \mathbf{P}^{4*} \mid x \in H\}$$

displays W as a Zariski fiber bundle with fiber $W_0 = \{M \in \mathscr{M}_d' \mid (1,0,0,0,0)$ is an A_3- or A_4-singularity of $M \cap \{x_4 = 0\}$ versally deformed by nearby hyperplane sections$\}$. Let X be the space of polynomials $f \colon \mathbf{C}^3 \to \mathbf{C}$ of degree d, mod scalars, so that $f^{-1}(0)$ has a versally deformed A_3- or A_4-singularity (obtained from W_0 by dehomogenization). Let Y be the set of polynomials $f \colon \mathbf{C}^3 \to \mathbf{C}$ of degree d of the form

$$f(x, y, z) = x^2 + y^2 + 0z^3 + (\alpha x + \beta y)z^2 + \cdots + \gamma z^4 + \cdots,$$
$$(\alpha^2 + \beta^2, \gamma) \neq (0, 0).$$

Let $H \subset \mathrm{GL}(3)$ consist of matrices of the form

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{O(2, \mathbf{C})} & \begin{matrix}0\\0\end{matrix} \\ \hline * & * & * \end{array}\right].$$

Let $U \subset \mathscr{I}$ be an open set over which the bundle $W \to \mathscr{I}$ is trivial. We obtain a map $\eta_U \colon W_U \to Y/H$ as the composition

$$W_U \to W_0 \to X \to Y/H.$$

It is shown in (8.18) that an A_3-singularity of the form $f(x,y,z) = 0$ as above is versally deformed if and only if α and β are not both 0. The condition that $\Lambda \cap \Gamma = \varnothing$ if $M \in \mathscr{M}_d'$ translates to $(\alpha^2 + \beta^2, \gamma) \neq (0,0)$ by (3.16), and so the map η_U makes sense.

Now $\rho_0 = (\alpha^2 + \beta^2)/\gamma$ is a rational function on Y; the calculation (3.18) shows that ρ_0 descends to a function on Y/H. It also follows from (3.18) that if U, U' are two open subsets of $\mathscr{I}$ over which W is trivial, then $\rho_0 \circ \eta_U = \rho_0 \circ \eta_{U'}$ on $U \cap U'$; thus ρ_0 pulls back to a globally defined function $\boldsymbol{\rho}_W$ on W whose restriction to any fiber of W over $\mathscr{M}_d'$ is the invariant $\rho\colon C^0 \to \mathbb{P}^1$. To see that $\boldsymbol{\rho}_W\colon W \to \mathbb{P}^1$ is analytic, note that there is a local analytic map $X \xrightarrow{\psi} Y$ (Sylvester's law with parameters; cf. also (8.12)) making a commutative diagram:

$$\begin{array}{ccccccc} & & & & Y & \xrightarrow{\rho_0} & \mathbb{P}^1 \\ & & & \overset{\psi}{\nearrow} & \downarrow & & \\ W_U & \to & W_0 & \to X \to & Y/H & & \end{array}$$

Now we wish to show that $\boldsymbol{\rho}_W$ extends to an analytic map, hence rational function, $\boldsymbol{\rho}\colon \hat{V} \to \mathbb{P}^1$. Given a D_4-singularity $x \in M$, by a linear change of coordinates in $\mathbb{P}^4$ we may take the defining equation of $M \cap H$ to be $x^2 + \varphi(x,y,z) = 0$ (where φ has degree ≥ 3); if we consider the family of equations $x^2 + ty^2 + \varphi(x,y,z) = 0$, it follows from (8.13) that there is a corresponding family of hypersurfaces $M_t \in \mathscr{M}_d'$ so that (x, H, M_t) is a curve in V which lifts canonically to $\hat{V}$. Now the calculation (3.20) shows that $\boldsymbol{\rho}$ has the value ∞ at the corresponding point $\hat{x}$ over x. (Since any one-parameter family Q_t of quadratic forms with rank $Q_0 = 1$ and rank $Q_t = 2$, $t \neq 0$, is locally analytically equivalent to $x^2 + t^\nu y^2$ (some $\nu \in \mathbb{N}$), one can see that $\boldsymbol{\rho}$ is in fact well defined at $\hat{x} \in \hat{V}$.)

The differential of $\rho_0\colon Y \to \mathbb{P}^1$ is

$$d\rho_0 = (2\alpha\, d\alpha + 2\beta\, d\beta - \rho_0 d\gamma)/\gamma$$

and is everywhere nonzero. Since the map $W_U \to W_0 \to X$ is clearly a submersion and the locally defined map $\psi\colon X \to Y$ is as well, it follows that $\boldsymbol{\rho}_W$ is a submersion. For $s \in \mathbb{P}^1$ fixed, we obtain a submanifold W_s of W, $W_s = \boldsymbol{\rho}_W^{-1}(s)$. Since $\pi\colon V \to \mathscr{M}_d'$ is proper, $\pi|W_s\colon W_s \to \mathscr{M}_d'$ is proper, at least for $s \neq \infty$. Since $\dim W_s = \dim \mathscr{M}_d'$, the branch locus of $\pi|W_s$ is a subvariety of $\mathscr{M}_d'$ of codimension at least one, and so we obtain a nonempty Zariski-open subset $\mathscr{M}_d(s) \subset \mathscr{M}_d'$ with the property that s is a regular value of $\rho\colon C^0 \to \mathbb{P}^1$ if $M \in \mathscr{M}_d(s)$.

Since Λ is the fiber of $\pi|W_4$ by (3.16) and consists of D distinct points for all Gauss-stable M by (2.19), it follows that $\pi|W_4$ is unramified, i.e., that 4 is a regular value of $\rho\colon C^0 \to \mathbb{P}^1$ for all $M \in \mathscr{M}_d'$. Therefore 4 is a regular value of $\rho\colon \hat{C} \to \mathbb{P}^1$, and degree $\rho = D$ for all $M \in \mathscr{M}_d'$.

(3.22) Corollary. *For M in a nonempty Zariski-open subset of $\mathscr{M}_d$, $d \geq 4$, the zero-cycles Λ, κ, μ, Γ, and Π' are disjoint and reduced, with* $\deg\Lambda = \deg\kappa = \deg\mu = \deg\Gamma + 3\deg\Pi'$.

Proof. For $M \in \mathscr{M}_d'$ by (3.16) the cycles Λ, κ, μ, and Γ are disjoint, and Π' is disjoint from Λ, κ, and μ since $\rho(\hat{\Pi}') = \infty$. By (8.21) the condition $\Gamma \cap \Pi' = \varnothing$ is also generic. For Gauss-stable M, Λ and Π' are reduced and Λ has degree D (cf. (2.19) and (1.11)). If we assume now that $M \in \mathscr{M}_d(0) \cap \mathscr{M}_d(6)$, then κ and μ each consist of D distinct points, hence are reduced by (3.10).

Referring to the proof of (3.21), consider the function $\rho_t = (t\alpha^2 + \beta^2)/(t\gamma)$ obtained in (3.20) by evaluating $\boldsymbol{\rho}$ on the curve (x, H, M_t) through $\hat{x} \in \hat{V}$. Then $d\rho_t/dt|_{t=0} = \gamma/\beta^2$. For $\gamma \neq 0$, then $d\boldsymbol{\rho}_{\hat{x}} \neq 0$. By (8.21), this condition is generic. Thus for a Zariski-open subset $\mathscr{M}_d(\infty) \subset \mathscr{M}_d'$, ∞ will be a regular value of $\rho\colon \hat{C} \to \mathbb{P}^1$ as well, and so by (3.14) Γ will be a reduced zero-cycle of degree $D - 3\deg\Pi'$.

Remark. It follows that if we relax the condition $\Lambda \cap \Gamma = \varnothing$, the linear system on $\hat{V}$ defined by $\boldsymbol{\rho}$ acquires base points, and so the degree of ρ will decrease on a Gauss-stable M with $\Lambda \cap \Gamma \neq \varnothing$.

4. The cubic threefold

For the duration of this section, M will be a Gauss-stable cubic hypersurface in $\mathbb{P}^4$, defined by the homogeneous polynomial $F(x_0, x_1, \cdots, x_4) = 0$. The crucial observation with which we begin is this: since $\deg M = 3$, a line L has contact order at least three with M at x if and only if L is contained in M; and so we may identify

$$\mathscr{Z}_3 \cong \{(x, L) \in M \times \mathbb{G}(1,4) | x \in L \text{ and } L \subset M\}.$$

(4.1) Lemma. *The projection $\mathscr{Z}_3 \to M$ has degree six; i.e., counting multiplicities, through each point of M pass six lines contained in M.*

Proof. The fiber of $\mathscr{Z}_3$ over $x \in M$ is given by $\{\mathrm{II}_x = 0\} \cdot \{\mathrm{III}_x = 0\} \subset \mathbb{P}T_xM$. q.e.d.

Projecting to the other factor, the image of $\mathscr{Z}_3$ in $\mathbb{G}(1,4)$ is a surface called the *Fano surface S* associated to M. The Fano surface is smooth (cf. [5], [9]). There are two types of lines $L \subset M$, distinguished by the behavior of the Gauss map γ of M along L; since γ is a linear system of conics on M, its restriction to $L = \mathbb{P}^1$ is a subsystem of $|\mathscr{O}_{\mathbb{P}^1}(2)|$ of dimension one or two, for $\gamma\colon M \to \mathbb{P}^{4*}$ has finite fibers. We say the line $L \subset M$ is

(i) *general* if $\gamma|L\colon L \to \mathbb{P}^{4*}$ embeds L as a smooth plane conic in $\mathbb{P}^{2*} \subset \mathbb{P}^{4*}$ ($\Leftrightarrow N_{L/M} \cong \mathscr{O} \oplus \mathscr{O}$);

(ii) *special* if $\gamma|L\colon L \to \mathbb{P}^{4*}$ is a branched two-fold covering of a line $\mathbb{P}^{1*} \subset \mathbb{P}^{4*}$ ($\Leftrightarrow N_{L/M} \cong \mathscr{O}(-1) \oplus \mathscr{O}(1)$).

(4.2) Proposition. *The special lines constitute a divisor $\mathscr{D}$ in S.*

Remark. Fano proves moreover that the divisor is bicanonical.

Proof. L is a special line $\Leftrightarrow$ there are two points P_1, $P_2 \in L$ so that $d(\gamma|L)_{P_i} = 0$, $i = 1, 2 \Leftrightarrow L$ is the kernel direction for $d\gamma_{P_i}$, $i = 1, 2 \Leftrightarrow (P_i, L) \in \mathscr{Z}_3 \cap \Pi^{\#}$, $i = 1, 2 \Leftrightarrow P_1, P_2 \in C$ (by (3.5)). Since there is a two-to-one correspondence between C and $\mathscr{D}$, $\mathscr{D} \subset S$ is a curve (cf. also (4.10) below). q.e.d.

At a point of C the kernel direction is tangent to Π; hence we infer from the proof just given that each special line L is bitangent to Π (cf. Proposition (2.7) in [29] for the analogous statement for the case of the cubic surface). On the other hand, since by (1.8) the parabolic surface is cut out on M by the Hessian of F, which is a quintic, any line on M must intersect Π five times. That is,

$$L \cdot \Pi = 2P_1 + 2P_2 + R, \tag{4.3}$$

where R is a residual point.

It is natural to consider next the ruled surface $\Sigma = \bigcup_{s \in \mathscr{D}} L_s$ generated by the special lines. Its crudest invariant is, of course, its degree.

(4.4) Lemma. $\deg \Sigma = 90$.

Proof. We alluded in the proof of (4.2) to the correspondence between C and $\mathscr{D}$. Each ruling of Σ intersects the cusp curve C twice; when it does, it is the kernel direction. Comparing with the discussion of (3.12), in this case we see that the map $\mathbb{P}(E) \to \Sigma$ has degree 2, and hence by (3.13) $\deg \sigma = \frac{1}{2}(180) = 90$. q.e.d.

Now since γ maps a special line to a line in $\mathbb{P}^{4*}$, we obtain by projective duality a $\mathbb{P}^2(L) \subset \mathbb{P}^4$ which is everywhere tangent to M along L. It seems plausible then that Σ is the envelope of these $\mathbb{P}^2(L_s)$, $s \in \mathscr{D}$, and hence is a developable surface. Rather than make this reasoning precise, we take a more analytic tack.

We begin with a quick review of ruled surfaces. A ruled surface $\Sigma \subset \mathbb{P}^n$ is a curve $\Gamma \subset \mathbb{G}(1, n)$, i.e., a one-parameter family of lines (*rulings*) in $\mathbb{P}^n$. We say Σ is *developable* if its tangent plane is constant along the rulings. Recalling that

$$T_L\mathbb{G}(1, n) \cong \operatorname{Hom}(\tilde{L}, \mathbb{C}^{n+1}/\tilde{L}),$$

we give a criterion for the ruled surface Σ to be developable.

(4.5) Lemma. *Let $\xi(s)$ be a tangent vector to Γ at $L(s)$; if $\operatorname{rank}\xi(s) \leq 1$ for all $L(s) \in \Gamma$, then Σ is developable. Indeed, if $\operatorname{rank}\xi(s) = 1$ for all s, let $Z(s) \in \mathbb{P}^n$ span $\ker\xi(s)$; then the curve $\{Z(s)|L(s) \in \Gamma\}$ is called the* ***curve of striction*** *σ of Σ, and the tangent line to σ at $Z(s)$ is the ruling of Σ through that point.*

Remarks. In the more graphic classical style, $\xi \in T_L\Gamma$ is called "the infinitely near ruling of Σ," and Σ is developable precisely when each ruling meets the infinitely near ruling. Moreover, their point of intersection lies on the curve of striction.

Proof. Choose a frame field $Z_0(s)$, $Z_1(s), \cdots, Z_n(s)$ along Γ so that $Z_0(s)$ spans $\ker\xi(s)$, $Z_0(s) \wedge Z_1(s) = L(s)$, and so that $Z_1'(s) \equiv \alpha(s)Z_2(s) \bmod L(s)$. Now the ruled surface Σ is given parametrically by $X(s,t) = Z_0(s) + tZ_1(s)$; its tangent plane at $X(s,t)$ is

$$\begin{aligned}\Lambda(s,t) &= X(s,t) \wedge \partial X/\partial s \wedge \partial X/\partial t \\ &= -t\alpha(s)Z_0(s) \wedge Z_1(s) \wedge Z_2(s).\end{aligned}$$

It is now patent that $\partial\Lambda/\partial t \equiv 0 \bmod \Lambda$, i.e., that Σ is developable. The tangent line to σ at $Z_0(s)$ is $Z_0(s) \wedge Z_0'(s) = L(s)$, since $Z_0' \equiv 0 \bmod L$ (if we had $Z_0' \equiv 0 \bmod Z_0$, then Σ would be a cone and we would have $\operatorname{rank}\xi = 0$). q.e.d.

To apply (4.5) to our case of the special lines on the cubic threefold, we need to study the geometry of the Fano surface and the divisor $\mathscr{D}$ near a special line L_0. To this end, we work in affine coordinates in $\mathbb{C}^4$ with L_0 the x_1-axis and $\pi = \{x_3 = x_4 = 0\}$ the 2-plane tangent to M along L_0. We take γ to be given along L_0 by $\gamma(x_1, 0, 0, 0) = (0, 0, x_1 + x_1^2, 1)$.

We also assume that 0 is not a parabolic point; by a linear change of coordinates, then, we may take as the defining function of M

$$\begin{aligned}F(1, x_1, x_2, x_3, x_4) = x_4 &+ (x_1x_3 + x_2^2) \\ &+ (x_1^2x_3 + \delta x_1x_2^2 + \mu x_2^3 + \nu x_1x_2x_3 + \cdots),\end{aligned} \tag{4.6}$$

where $\cdots$ denotes either terms which are cubic in (x_2, x_3, x_4) or those of the form x_1 times quadratic in (x_2, x_3, x_4), other than the terms already included. We shall soon see the importance of the parameters δ, μ, and ν.

We specify coordinates near L_0 on the Grassmannian of affine lines in $\mathbb{C}^4$ as follows: let $\alpha = (\alpha_2, \alpha_3, \alpha_4)$, $\beta = (\beta_2, \beta_3, \beta_4) \in \mathbb{C}^3$, and put $L_{(\alpha,\beta)} = \{(0, \alpha_2, \alpha_3, \alpha_4) + t(1, \beta_2, \beta_3, \beta_4)\}$. L_0 is, of course, the line $L_{(0,0)}$. Now if we impose the condition that $L_{(\alpha,\beta)}$ lie on M, we find a two-parameter family

$\{L(u,v)\}$ given as follows:
(4.7)

$$\begin{aligned} \alpha_2 &= u, & \beta_2 &= v, \\ \alpha_3 &= (\delta-1)v^2 - 2\delta uv + \cdots, & \beta_3 &= -\delta v^2 + \cdots, \\ \alpha_4 &= -u^2 + \cdots, & \beta_4 &= -\delta u^2 + (\delta-1)(2uv - v^2) + \cdots; \end{aligned}$$

these Taylor expansions are valid in a neighborhood of (0,0), and $\cdots$ denotes terms of degree ≥ 3. The first conclusion we draw from (4.7) is that $T_{L_0}S$ is spanned by $\partial/\partial u$ and $\partial/\partial v$; that is,

(4.8) Proposition [9]. *Let $L_0 \in S$ be a special line, and let π be the $\mathbb{P}^2$ tangent to M along L_0. Then*

$$T_{L_0}S \cong \{l \in \mathbb{G}(1,4) | l \subset \pi\}.$$

(4.9) Corollary. *The ruled surface Σ of special lines in M is developable.*

Proof. Any infinitely near ruling lies in $T_{L_0}S$, hence is represented by a line contained in π, which must perforce intersect L_0. Alternately a tangent vector $\xi \in T_{L_0}S$ is represented by a matrix of the form

$$\begin{bmatrix} * & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which has rank ≤ 1. q.e.d.

Next we restrict γ to the line $L(u,v)$; that restriction ramifies if and only if $L(u,v)$ is special. A somewhat tedious calculation shows that $\mathscr{D}$ is given in these coordinates by

$$u = \frac{\delta - 1}{\delta} v + \cdots, \tag{4.10}$$

where, as usual, $\cdots$ denotes terms of degree ≥ 2 in v. In particular, the tangent space $T_{L_0}\mathscr{D}$ to the divisor of special lines is spanned by

$$\xi = \frac{\delta - 1}{\delta}\frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$

from which follows the

(4.11) Lemma. *The origin lies on the curve of striction σ of the developable surface Σ if and only if $\delta = 1$ in* (4.6).

Proof. Applying (4.5) again, we see that the infinitely near ruling ξ is coplanar with L_0 and hence intersects it. If $\delta = 1$, the infinitely near ruling ξ is a line through the origin, whence the origin lies on the curve of striction. q.e.d.

We saw at the outset that through the general point of M pass six lines contained in M. In this light, we may characterize the general point $x \in \Sigma$

by the coalescence of *two* of these six lines, and the general point $x \in \sigma$ by that of *three*. To do so, we will make use of the basic

(4.12) Observation. The cubic form III_x on $\mathbb{P}T_xM$ differs from the restriction $F|\mathbb{P}T_xM$ by the product of a linear form and the quadratic form II_x.

As a consequence, we have

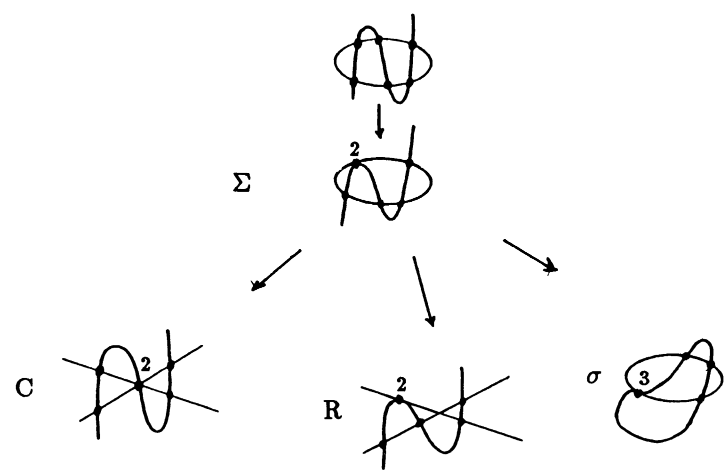
(4.13) The intersection multiplicity at $L_0 \in \mathbb{P}T_xM$ of II_x and III_x equals the intersection multiplicity at L_0 of II_x and $F|\mathbb{P}T_xM$.

But the latter is easy to compute using (4.6): setting $x_4 = 0$, $x_1 = 1$, $x_2 = y, x_3 = z$, we must in turn compute

$$\mathscr{I}(z + y^2, z + \delta y^2 + \mu y^3 + \nu yz + \cdots)_0$$

in $\mathbb{C}^2$. Here the significance of the parameter δ emerges again: if $\delta \neq 1$ (so that by (4.11) we are at a general point of Σ) this intersection multiplicity is 2; if $\delta = 1$ (so that we are at a point of σ), we see it is ≥ 3—provided $\mu - \nu \neq 0$, it is equal to 3.

Assembling this information in pictorial fashion, we have sketches analogous to (3.3):

(4.14) 

The natural question remaining is this: how do we characterize $\sigma \cap \Pi$? We need the following.

(4.15) Lemma [11, §§11,12]. *Let $L \subset M$ be a special line. Let Q be the unique point of σ on L and let R be the unique point with $\gamma(Q) = \gamma(R)$. Then R is the residual point of intersection of L and* Π (*cf.* (4.3)).

Proof. We use the defining equation (4.6) for M with $\delta = 1$. By (4.11) the origin is the point Q of σ lying on the special line L, the x_1-axis. One may easily check that $\gamma(Q) = (0,0,0,1) = \gamma(-1,0,0,0)$. We must therefore show that $R = (-1,0,0,0)$ lies on Π but *not* on C. Centering coordinates at R, i.e., replacing x_1 by $x_1 + 1$, we have

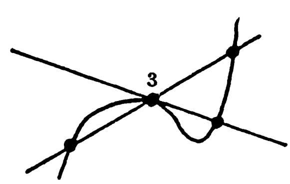
$$F(1,x_1,x_2,x_3,x_4) = x_4 - \underbrace{(x_1 + \nu x_2)x_3}_{\mathrm{II}} + \underbrace{(x_1^2 x_3 + x_1 x_2^2 + \mu x_2^3 + \nu x_1 x_2 x_3}_{\mathrm{III}} + \cdots).$$

Since II has rank 2, $R \in \Pi$. Now we apply the criterion (3.5): the kernel direction is spanned by $(-\nu, 1, 0)$ and the cubic III $= 0$ does not pass through this point of $\mathbb{P}T_R M$ provided $\mu - \nu \neq 0$. Hence $R \in \Pi - C$. (In the event that $\mu - \nu = 0$, Q is a cusp point of σ and the residual point R lies on C—but with kernel direction distinct from L.)

Remark. Fano speaks of Q and R as being harmonic conjugates on L with respect to the cusp points P_1, P_2. That is, $\gamma|L$ gives a branched two-fold covering with branch points P_1, P_2 and with $Q + R$ a fiber. This can be seen directly from the calculations above: on the x_1-axis, $P_1 = -\frac{1}{2}$, $P_2 = \infty$, $Q = 0$, and $R = -1$, the harmonic conjugacy being established by $x \to (x + \frac{1}{2})^2$.

Now we can provide a picture at a point of $\sigma \cap \Pi$. If the point Q belongs to Π, then R must coincide with Q (or else L would meet Π six times!), and hence $Q \in C$. Thus we complete (4.14) with

$\sigma \cap \Pi$



and this leads to the set-theoretic statement

(4.16) Proposition. $\sigma \cap \Pi = \Lambda$.

Proof. If $x \in \sigma \cap \Pi$, then by (3.7), $T_x\Pi$ is II-null, i.e., $x \in \kappa$. Applying (3.16), we see that $x \in \Lambda$, since $\gamma = 0$ on a cubic. Conversely, if $x \in \Lambda = \kappa$, $\mathscr{I}(\mathrm{II}_x \cdot \mathrm{III}_x)_L \geq 3$, and hence $x \in \sigma$. q.e.d.

It is by far more interesting to investigate the cycle-theoretic intersection $\sigma \cdot \Pi$. First of all, we claim that $\sigma \cdot \Pi \supset 3\Lambda$: if $x \in \sigma \cap \Pi = \Lambda$, then from the proof of (4.16) and (3.8) we infer that the kernel direction L at x is asymptotic in Π. Since L is tangent to σ at x, this means that the osculating plane of σ at x equals $T_x\Pi$, *unless* x is a flex point of σ. That is, $\mathcal{I}(\sigma\cdot\Pi)_x \geq 3$, as required. Armed with this information, we now prove a much more detailed statement.

(4.17) Theorem. *The curve of striction σ has genus $g = 136$, has $\beta =$ 720 cusps, and no flexes. Hence $\sigma \cdot \Pi = 3\Lambda$.*

Proof. (1) From the adjunction formula [15, p. 471], (2.9) and (2.17) it follows that

$$\begin{aligned} g(\hat{C}) &= 1 + \tfrac{1}{2}(\hat{C}\cdot\hat{C} + K_{\tilde{\Pi}}\cdot\hat{C}) \\ &= 1 + \tfrac{1}{2}(6H - \tfrac{3}{2}[\tilde{Z}] + 3H)\cdot(6H - \tfrac{3}{2}[\tilde{Z}]) = 271. \end{aligned}$$

Since $\hat{C}$ is an *unbranched* two-fold cover of σ, the genus of σ is $g = \frac{1}{2}(g(\hat{C}) + 1) = 136$.

(2) To count the cusps of σ, we use the Plücker formulas [15, p. 270]. Let $f_i\colon \sigma \to \mathbb{G}(i,4)$, $i = 0,1,2$, denote the ith associated curve of σ, $d_i =$ $\deg f_i(\sigma)$, and $\beta_i =$ the number of ramification points of f_i. Then

$$\beta = \beta_0 = 2d_0 - d_1 + 2g - 2.$$

Now $d_0 = \deg\sigma \geq 270$, since $\deg\sigma\cdot\Pi \geq 3\deg\Lambda = 3\cdot 450$ by (2.19) and therefore $\deg\sigma \geq \frac{3}{5}\cdot 450 = 270$. $d_1 = \deg\Sigma = 90$ by (4.4). Therefore $\beta \geq 2\cdot 270 - 90 + 270 = 720$. Next we wish to compute $\beta_1 =$ the number of flexes of σ.

$$\beta_1 = 2d_1 - d_0 - d_2 + 2g - 2.$$

(3) We turn to the calculation of d_2, the degree of the Gauss image of Σ in $\mathbb{G}(2,4)$. Then d_2 is equal to the number of intersections of a generic line $l \subset \mathbb{P}^4$ with the one-parameter family of two-planes $\pi = \mathbb{P}^2(L)$ tangent to Σ. Suppose l and one such plane π intersect; then their span is a hyperplane which is bitangent to M (for contained in π is the special line L). Conversely, every bitangent hyperplane must arise in this manner: the line joining the two points of tangency has intersection multiplicity at least four with M, and hence must lie in M. Thus d_2 is equal to the number of hyperplanes bitangent to M and containing a generic line l (cf. also [11, §10]).

Rephrasing this slightly, let

$$B = \bigcup_{s\in\mathscr{D}} \gamma(L_s) \subset \mathbb{P}^{4*};$$

then $d_2 = \deg B$. But B is a ruled surface in $\mathbb{P}^{4*}$, and $\gamma|\Sigma\colon \Sigma \to B$ is of degree 2. Since $\gamma\colon M \to \mathbb{P}^{4*}$ is given by a linear system of quadrics, γ pulls the hyperplane class H^* on $\mathbb{P}^{4*}$ back to twice the hyperplane class H on M. Thus $2B\cdot(H^*)^2 = \Sigma\cdot(\gamma^*H^*)^2 = \Sigma\cdot 4H^2$, whence $d_2 = \deg B = 2\deg\Sigma = 180$.

(4) Assembling these data, we have

$$\beta_1 \leq 2 \cdot 90 - 270 - 180 + 270 = 0.$$

Since $\beta_1 \geq 0$, we must in fact have $\beta_1 = 0$, $\beta = 720$, and $d_0 = 270$, and we conclude *a fortiori* that $\sigma \cdot \Pi = 3\Lambda$.

5. Gauss-stable hypersurfaces

Our study of the Gauss map is based on the geometry of the hyperplane incidence correspondence. Let $M \subset \mathbb{P}^4$ be a nonsingular algebraic hypersurface, with Gauss map $\gamma: M \to \mathbb{P}^{4*}$. Let $\Gamma \subset M \times \mathbb{P}^{4*}$ be the incidence correspondence, $\Gamma = \{(x, H) | x \in H\}$, and let $p: \Gamma \to \mathbb{P}^{4*}$ be the incidence projection, $p(x, H) = H$. Let $\Sigma(p)$ be the singular locus of p. Then $\Sigma(p) = \{(x, H) | H$ is tangent to M at $x\}$, so M is isomorphic with $\Sigma(p)$, and γ is the composition of p with this isomorphism:

$$\begin{tikzcd} & \Gamma \arrow[dl] \arrow[dr, "p"] & \\ M \arrow[rr, bend right, "\gamma"'] & \Sigma(p) \arrow[l, "\approx"'] \arrow[u, phantom, "\cup"] \arrow[r] & \mathbb{P}^{4*} \end{tikzcd}$$

One can interpret the incidence projection p in two ways—as a family of complex analytic spaces, or as an analytic map of complex manifolds.

(a) The incidence projection is a family. The fibers of p form a family of complex hypersurfaces of M, namely the family of hyperplane sections of M in $\mathbb{P}^4$. From this viewpoint we apply the theory of deformations of complex spaces and their singularities (cf. [20], [24]).

(b) The incidence projection is a map. From this viewpoint we apply the theory of singularities of complex analytic map germs, which is parallel to the Thom-Mather theory of singularities of real C^∞ maps (cf. [25], [26]).

A third, more basic approach is that of Arnold [1], [2]:

(c) The Gauss map is a Legendre map. This point of view brings into play the contact structure on the big incidence correspondence $\mathcal{I} = \{(x, H) \in \mathbb{P}^n \times \mathbb{P}^{n*} | x \in H\}$. A *Legendre map* is the composition of the inclusion of a Legendre submanifold with the projection of a Legendre bundle. The incidence projection $\mathbf{p}: \mathcal{I} \to \mathbb{P}^{4*}$ is a Legendre bundle, and the canonical inclusion $i: M \to \mathcal{I}$ is a Legendre submanifold, so $\gamma = \mathbf{p} \circ i$ is a Legendre map. For details, see [2, pp. 108–110]. (The affine version of the Gauss map is the *Legendre transform.*)

(5.1) Theorem. *Let $M \subset \mathbb{P}^4$ be a smooth hypersurface, $x \in M$, and $H = \gamma(x)$. The following are equivalent*:

(a) *The map* $p\colon (\Gamma,(x,H)) \to (\mathbb{P}^{4*},H)$ *of local complex spaces is a versal deformation of the local complex space* $(M\cap H,x)$.

(b) *The germ at* (x,H) *of the incidence projection* p *is a stable analytic map germ.*

(c) *The germ at* x *of the Gauss map* γ *is a stable analytic Legendre map germ.*

Definition. A smooth hypersurface $M\subset \mathbb{P}^4$ is *Gauss-stable* if (5.1) (a)(b)(c) hold for all $x\in M$.

Proof of (5.1). (a) $\Leftrightarrow$ (b). For the theory of deformations of complex spaces we refer to [20]. A *deformation* of a local complex space X_0 is a flat map $\pi\colon (X,x_0)\to(T,t_0)$ of local complex spaces for which $\pi^{-1}(t_0)=X_0$. The deformation π is *versal* if every other deformation $W\to S$ of X_0 is induced from π by a map $S\to T$. Now suppose X_0 is the hypersurface $f(x_1,\cdots,x_n)=0$ in $\mathbb{C}^n$, the base space T is $\mathbb{C}^k$, X is the smooth hypersurface $F(x_1,\cdots,x_n,t_1,\cdots,t_k)=0$ in $\mathbb{C}^{n+k}$, and π is the restriction of the projection $\mathbb{C}^{n+k}\to\mathbb{C}^k$. Then π is a versal deformation if and only if the functions $\partial F/\partial t_i|_{t=0}$, $i=1,\cdots,k$, span the complex space

$$T^1_f=\mathcal{O}_{\mathbb{C}^n,0}\Big/\left(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\right),$$

which is called the space of *first order deformations* of X_0. Here $\mathcal{O}_{\mathbb{C}^n,0}$ is the local ring of germs at 0 of analytic functions on $\mathbb{C}^n$. Note that π is flat since X_0 is a hypersurface and X is smooth. This criterion for versality is identical to the statement that the map germ $F\colon(\mathbb{C}^n\times\mathbb{C}^k,0)\to(\mathbb{C},0)$ is an infinitesimally V-versal deformation of the map germ $f\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$, which is equivalent to the statement that F is a V-versal deformation of f (cf. [25, Theorem 4.4, p. 20]).

To apply this discussion to the family of hyperplane sections of M, we introduce the following *standard coordinates*, which will also be useful in §§6, 8, and 9.

Coordinates for $\mathbb{P}^4\times\mathbb{P}^{4*}$ at $(a,B)\in\mathcal{I}$ are defined as follows. Let $\mathbf{x}=(\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)$ be homogeneous coordinates for $\mathbb{P}^4$, so that a has coordinates $(1,0,0,0,0)$. Let $\mathbf{t}=(\mathbf{t}_0,\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4)$ be dual coordinates on $\mathbb{P}^{4*}$, so $(\mathbf{x},\mathbf{t})\in\mathcal{I}$ if and only if $\mathbf{x}\cdot\mathbf{t}=0$, with $B=(0,0,0,0,1)$. Let $x_i=\mathbf{x}_i/\mathbf{x}_0$, $t_j=\mathbf{t}_j/\mathbf{t}_4$ be the corresponding affine coordinates with origins at a,B.

Analytic coordinates for Γ at $(a,B)\in\Gamma$ are defined as follows. Express the germ of M at a as the graph of an analytic function in affine coordinates

$M = \{(x_1, x_2, x_3, x_4) | x_4 = f(x_1, x_2, x_3)\}$, with

$$f(0,0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_1}(0) = \frac{\partial f}{\partial x_2}(0) = \frac{\partial f}{\partial x_3}(0) = 0.$$

The incidence correspondence $\Gamma \subset M \times \mathbb{P}^{4*}$ is defined in these coordinates by

$$(5.2) \quad F(x_1, x_2, x_3, t_1, t_2, t_3) = f(x_1, x_2, x_3) - (t_0 + t_1 x_1 + t_2 x_2 + t_3 x_3) = 0;$$

thus $(x_1, x_2, x_3, t_1, t_2, t_3)$ are coordinates for Γ at (a, B), and the incidence projection $p \colon \Gamma \to \mathbb{P}^{4*}$ is given by

$$(5.3) \quad p(x_1, x_2, x_3, t_1, t_2, t_3) = (-f(x_1, x_2, x_3) - t_1 x_1 - t_2 x_2 - t_3 x_3, t_1, t_2, t_3).$$

Now we return to the proof of (5.1). Let $X_0 = (M \cap H, x)$, $X = (\Gamma, (x, H))$, $T = (\mathbb{P}^{4*}, H)$, and $\pi = p$. We have shown so far that $\pi \colon X \to T$ is a versal deformation of X_0 if and only if, in standard coordinates (5.2), F is a V-versal deformation of f.

The relation between V-versality and stability is due to Mather. We follow the description of this relation given in [25]. A map germ is *stable* if all its unfoldings are trivial. Stability of a germ is equivalent to the V-versality of an associated deformation [25, pp. 26–27]: Let $x = (x_1, \cdots, x_n)$, $y = t_0$, and $t = (t_1, \cdots, t_l)$. If F is a deformation of the form $F(x, y, t) = g(x, t) - y$, then F is versal if and only if the map germ $h(x, t) = (g(x, t), t)$ is stable. Applying this criterion to the incidence correspondence, in standard coordinates (5.4), we conclude that F is a V-versal deformation if and only if the map

$$h(x_1, x_2, x_3, t_1, t_2, t_3) = (f(x_1, x_2, x_3) - (t_1 x_1 + t_2 x_2 + t_2 x_2), t_1, t_2, t_3)$$

is stable. But this is just the incidence projection p in local coordinates (5.3).

(a) $\Leftrightarrow$ (c). A Legendre map is *Legendre stable* if every nearby Legendre map is Legendre equivalent to it. Stability of a Legendre map germ is defined similarly [3]. Arnold proves that if a Legendre map germ is given by a generating family of hypersurfaces $F(x, t) = 0$ with parameter t, then the map germ is Legendre stable if and only if $F(x, t)$ is a V-versal deformation of $f(x) = F(x, 0)$ [3, p. 333]. So we just need to see that the Gauss map γ is generated by the family of hyperplane sections of M (taking F to be the local equation of the incidence correspondence given in the proof (a) $\Leftrightarrow$ (b)). Arnold's definition of a generating family of hypersurfaces [3, pp. 319, 322] adapted to our case uses the following diagram:

$$\begin{array}{ccc} M \times \mathbb{P}^{4*} & \longleftarrow & \mathbb{P}T^*(M \times \mathbb{P}^{4*}) \\ \Big\downarrow & & \cup \\ & & PA \\ & & \downarrow \\ \mathbb{P}^{4*} & \longleftarrow & \mathbb{P}T^*\mathbb{P}^{4*} \end{array}$$

The projection $M \times \mathbb{P}^{4*} \to \mathbb{P}^{4*}$ is the *auxiliary bundle*, with *big space* $M \times \mathbb{P}^{4*}$ and *base* $\mathbb{P}^{4*}$. The *mixed space* PA is the subset of $\mathbb{P}T^*(M \times \mathbb{P}^{4*})$ consisting of contact elements (hyperplanes in $T(M \times \mathbb{P}^{4*})$) which contain TM. The projection $PA \to \mathbb{P}T^*\mathbb{P}^{4*}$ assigns to such a hyperplane the corresponding hyperplane of $T\mathbb{P}^{4*}$. Now the incidence correspondence $\Gamma \subset M \times \mathbb{P}^{4*}$ is the *generating hypersurface* of the Legendre submanifold $X \subset \mathbb{P}T^*(M \times \mathbb{P}^{4*})$ consisting of all the contact elements tangent to Γ. Let Y be the projection of $X \cap PA$ to $\mathbb{P}T^*\mathbb{P}^{4*}$. Y is a Legendre submanifold of $\mathbb{P}T^*\mathbb{P}^{4*}$, which is a Legendre bundle over $\mathbb{P}^{4*}$, and Γ is by definition the *generating family* of the Legendre map $l\colon Y \to \mathbb{P}^{4*}$. But $\mathbb{P}T^*\mathbb{P}^{4*}$ can be identified with the incidence correspondence $\mathscr{I} \subset \mathbb{P}^4 \times \mathbb{P}^{4*}$, and then $Y = \{(x, H) | x \in M \text{ and } H \text{ is tangent to } M \text{ at } x\}$, so l is the Gauss map γ. This completes the proof of Theorem (5.1).

6. The transversality package

We show that the hypersurface $M \subset \mathbb{P}^4$ is Gauss-stable if and only if its second fundamental form II (the derivative of the Gauss map γ) satisfies certain transversality conditions, viz., first, that II be transverse to the rank stratification of quadratic forms, and second, that the line field Ker(II) have nondegenerate contact with the parabolic surface.

Let $\mathbf{Q} = \mathrm{Sym}^2(T^*M) \otimes N$ be the bundle of quadratic forms on TM with values in the normal bundle N of M in $\mathbb{P}^4$. The second fundamental form II is a section of the bundle $\mathbf{Q}$ (see §1). Let $\mathbf{Q}^i \subset \mathbf{Q}$ be the forms of corank i. The *rank stratification* $\mathbf{Q} = \mathbf{Q}^0 \cup \mathbf{Q}^1 \cup \mathbf{Q}^2 \cup \mathbf{Q}^3$ is an algebraic Whitney stratification of $\mathbf{Q}$, with $\operatorname{codim} \mathbf{Q}^i = \frac{1}{2}i(i+1)$. Let $\Sigma^i(\mathrm{II}) = \mathrm{II}^{-1}(\mathbf{Q}^i) = \{x \in M | \operatorname{corank} \mathrm{II}_x = i\}$. Thus $\Sigma^i(\mathrm{II}) = \Sigma^i(\gamma)$, so $\overline{\Sigma}^1(\mathrm{II}) = \Pi$ and $\overline{\Sigma}^2(\mathrm{II}) = \Pi'$.

For $k = 1, 2, 3, \cdots$, let $\Sigma^{1_k}(\mathrm{II}) = \Sigma^{1_k}(\gamma)$. Thus $\overline{\Sigma}_1^{1^1}(\mathrm{II}) = \Pi$, and if $\Sigma^{1_k}(\mathrm{II})$ is smooth, then $\Sigma_1^{1_{k+1}}(\mathrm{II})$ is the set of $x \in \Sigma^{1_k}(\mathrm{II})$ such that $\operatorname{Ker} \mathrm{II}_x$ is tangent to $\Sigma^{1_k}(\mathrm{II})$.

If Y is a submanifold of X and K is a field of lines in $TX|Y$, we say K has *nondegenerate tangency* with Y in X if the section of $\mathbb{P}(TX|Y)$ defined by K is transverse to the subbundle $\mathbb{P}(TY)$.

(6.1) The transversality package for $M \subset \mathbb{P}^4$. The second fundamental form II is transverse to the rank stratification of $\mathbf{Q}$:

(1) $\mathrm{II} \pitchfork \mathbf{Q}^1$,

(2) $\mathrm{II} \pitchfork \mathbf{Q}^2$,

(3) $\mathrm{II} \cap \mathbf{Q}^3 = \varnothing$.

Ker II has nondegenerate contact with the parabolic surface:

(4) Ker II has nondegenerate tangency with $\Sigma^1(\mathrm{II})$ in M,

(5) Ker II has nondegenerate tangency with $\Sigma^{1,1}(\mathrm{II})$ in $\Sigma^1(\mathrm{II})$.

We will show in the proof of Theorem (6.2), part (g), that (6.1)(5) implies $\Sigma^{1,1,1,1}(\mathrm{II}) = \varnothing$.

Recall that $\Pi = \overline{\Sigma}^1(\gamma)$, $\Pi' = \overline{\Sigma}^2(\gamma)$, $C = \overline{\Sigma}^{1,1}(\gamma)$, and $\Lambda = \overline{\Sigma}^{1,1,1}(\gamma)$. So (6.1) implies that Π is a surface, Π' is 0-dimensional, $\Pi^0 = \Pi - \Pi'$ is smooth, C is a curve, Λ is 0-dimensional, and $C^0 = C - \Pi'$ is smooth.

(6.2) Theorem. *The transversality package* (6.1) *holds for M if and only if M is Gauss-stable.*

There are two steps in the proof. First, we show stability of the incidence projection p is equivalent to the transversality of the jet map Jp to a certain stratification of the jet space. Second, this transversality condition on Jp is equivalent to the transversality package (6.1) by a theorem of Boardman on intrinsic derivatives.

(6.3) Stability and jet transversality. The hypersurface M is Gauss-stable if and only if the germ at z of $p\colon \Gamma \to \mathbb{P}^{4*}$ is stable for all $z \in \Gamma$. Mather proved that the (complex analytic) map germ $f\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ is stable if and only if it is infinitesimally stable (cf. [25, pp. 10–11], [3, p. 141]). He also proved that f is infinitesimally stable if and only if the n-jet extension $J^n f\colon (\mathbb{C}^m, 0) \to J^n(m, n)$ is transverse to the orbit of $J^n f(0)$, the n-jet of f at 0 (cf. [3, p. 140]). Here $J^n(m, n)$ is the space of n-jets at 0 of maps $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, with orbits under the action of the group of isomorphism germs of the source and target (left-right equivalence).

If X^m and Y^n are manifolds, the space of n-jets $J^n(X, Y)$ is a bundle over $X \times Y$ with fiber $J^n(m, n)$. If $\Sigma \subset J^n(m, n)$ is invariant under change of coordinates in the source and target, then there is a corresponding subset $\Sigma(X, Y)$ of $J^n(X, Y)$. For a map $f\colon X \to Y$, the germ of f at x is stable if and only if the jet map $J^n f\colon X \to J^n(X, Y)$ is transverse to $\Sigma(X, Y)$, where $\Sigma \subset J^n(m, n)$ is the orbit of the n-jet extension of f at x. We conclude that M is Gauss-stable if and only if $J^4p\colon \Gamma \to J^4(\Gamma, \mathbf{P}^{4*})$ is transverse to $\Sigma(\Gamma, \mathbf{P}^{4*})$ for all orbits $\Sigma \subset J^4(6,4)$.

(6.4) The Thom-Boardman stratification. For an introduction to the Thom-Boardman singularities, see [3, §2]. For a pair of manifolds X, Y, and a multiindex $I = (i_1, \cdots, i_k)$, Boardman [6] defined a submanifold Σ^I of the infinite jet space $J(X,Y)$ with the property that if $f\colon X \to Y$, then $\Sigma^I(f) = (Jf)^{-1}\Sigma^I$. $J(X,Y)$ is the inverse limit of the l-jet spaces $J^l(X,Y)$, and Σ^I is the preimage of a set $\Sigma^I(k) \subset J^k(X,Y)$ [6, 2.20]. If $l > k$, let $\Sigma^I(l) \subset J^l(X,Y)$ be the preimage of $\Sigma^I(k)$ by the projection $J^l(X,Y) \to J^k(X,Y)$. Given $l \geq k$, $Jf \pitchfork \Sigma^I$ if and only if $J^lf \pitchfork \Sigma^I(l)$. Thus we can project to $J^l(X,Y)$ to detect transversality of Jf to all Σ^I of order $\leq l$. In the following discussion we will abbreviate $\Sigma^I(l)$ to Σ^I.

The orbits of $J^4(6,4)$ of codimension ≤ 6 are the following Thom-Boardman sets. (This follows from the techniques of Mather, cf. [26, XVII] and §7 below.) Arrows denote incidence: $\Sigma^I \leftarrow \Sigma^J$ means $\overline{\Sigma}^I \supset \Sigma^J$. Subscripts denote codimension.

$$\begin{tikzcd}[column sep=small] & & & & \Sigma^{3,1,1,1}_{(6)} \arrow[dl] \\ \Sigma^2_{(0)} & \Sigma^{3,0}_{(3)} \arrow[l] & \Sigma^{3,1,0}_{(4)} \arrow[l] & \Sigma^{3,1,1,0}_{(5)} \arrow[l] & \\ & & & & \Sigma^{3,2,0}_{(6)} \arrow[ul] \end{tikzcd}$$

The union Ω of the orbits of codimension > 6 equals the union of the Thom-Boardman loci $\Sigma^4_{(8)}$, $\Sigma^{3,2,1}_{(8)}$, and $\Sigma^{3,3}_{(9)}$. Furthermore, $\Sigma^3 \supset \Sigma^{3,1} \supset \Sigma^{3,1,1} \supset \Sigma^{3,1,1,1}$ are smooth submanifolds of $J^4(6,4)$, with $\Sigma^3 - \Sigma^{3,1} = \Sigma^{3,0}$, $\Sigma^{3,1} - \Sigma^{3,1,1} = \Sigma^{3,1,0}$, and $\Sigma^{3,1,1} - \Sigma^{3,1,1,1} = \Sigma^{3,1,1,0}$. All the loci σ^I are smooth, Zariski locally closed subsets of $J^4(6,4)$ [28], and the above Thom-Boardman decomposition is a Whitney stratification of $J^4(6,4) - \Omega$ (cf. [27, (9.30)]). This means that each pair of strata is Whitney regular and satisfies the frontier condition (cf. [13]). That the Thom-Boardman decomposition of $J^4(6,4) - \Omega$ is a Whitney stratification can be checked directly using the normal forms (7.2).

(6.5) The intrinsic derivative construction of Σ^I. Boardman [6, §7] gave a description of the loci Σ^I using Porteous' intrinsic derivative. He also showed that transversality of the jet map to Σ^I is equivalent to a sequence of intrinsic derivative conditions. We summarize Boardman's setup, with notation slightly different from his:

Given $f\colon V \to W$ and $I = (i_1, \cdots, i_k)$, let $\Sigma_r = \Sigma^{i_1, \cdots, i_r}(f)$, $1 \leq r \leq k$. We inductively define Σ_r and give a condition $(\imath_r)$ such that

$$Jf \pitchfork \Sigma_{r-1} \Rightarrow (Jf \pitchfork \Sigma_r \Leftrightarrow (\imath_r)).$$

Start with $D_1 = df\colon TV \to f^*TW$.

Inductive step: Given $D_r\colon K_{r-1} \to P_{r-1}$ a bundle homomorphism over a submanifold Σ_{r-1} of V, let $\Sigma_r = \{x \in \Sigma_{r-1} \mid \dim \operatorname{Ker}(D_r)_x = i_r\}$, $K_r = \operatorname{Ker} D_r$, $Q_r = \operatorname{Coker} D_r$.

Construction of D_{r+1}: (1) Take the intrinsic derivative of D_r:

$$D(D_r)\colon T\Sigma_{r-1}|\Sigma_r \to \operatorname{Hom}(K_r, Q_r).$$

(2) Restrict the target to the symmetric subbundle:

$$d_{r+1}\colon T\Sigma_{r-1}|\Sigma_r \to P_r.$$

The condition (ι_r) is that d_{r+1} is surjective on every fiber.

(3) Restrict the source to the kernel subbundle:

$$D_{r+1}\colon K_r \to P_r.$$

We have omitted two essential features of Boardman's setup: the definition of P_r and the exact inductive hypotheses needed. If K_r or Q_r has rank 1, then $P_r = \operatorname{Hom}(K_r, Q_r)$. We need only one case when $P_r \neq \operatorname{Hom}(K_r, Q_r)$, and there we give a concrete description of P_r. The condition (ι_r) assures that the induction can continue. If (ι_t) holds for $1 \leq t < r$, then $D(D_r)$ does factor through P_r, and K_r is contained in $T\Sigma_{r-1}|\Sigma_r$. If (ι_r) holds as well, then Σ_r is a manifold. For further details, see [6, §7].

(6.6) Properties of the incidence correspondence. To apply Boardman's analysis to the incidence correspondence p, we first need to derive some geometric properties of p. Recall from §5 the basic incidence diagram:

$$\begin{tikzcd}
 & M \times \mathbb{P}^{4*} \arrow[dl, "Q"'] \arrow[dr, "P"] & \\
M & \Gamma \arrow[l, "q"'] \arrow[r, "p"] \arrow[u, phantom, "\cup"] & \mathbb{P}^{4*} \\
 & \Sigma \arrow[ul, "q'"] \arrow[ur, "p'"'] \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

Here Σ is the singular set of p, and $\Sigma = \{(x,H) | H$ tangent to M at $x\}$. The map $q'\colon \Sigma \to M$ is an isomorphism, and its differential dq' is an isomorphism of $T\Sigma$ with $(q')^*TM$. Since $p = P|\Gamma$, $\dim \operatorname{Ker}(dp) \leq 3$, and $(x,H) \notin \Sigma \Leftrightarrow \dim \operatorname{Ker}(dp)_{(x,H)} = 2$, $(x,H) \in \Sigma \Rightarrow \dim \operatorname{Ker}(dp)_{(x,H)} = 3$. Now $T\Gamma \subset T(M \times \mathbb{P}^{4*}) = Q^*TM \oplus P^*T\mathbb{P}^{4*}$, and $\Sigma = \{(x,H) | T_{(x,H)}\Gamma \supset Q^*TM\}$, so $\operatorname{Ker}(dp) = (q')^*TM$.

Let N be the normal bundle of M in $\mathbb{P}^4$, $N = T\mathbb{P}^4/TM$, and let C be the cokernel bundle of $dp|\Sigma$, $C = T\mathbb{P}^{4*}/\operatorname{Im}(dp|\Sigma)$.

(6.7) Proposition. *There is a canonical isomorphism* $(q')^*N \cong (p')^*C$.

Proof. Let $\mathcal{I} \subset \mathbb{P}^4 \times \mathbb{P}^{4*}$ be the big incidence correspondence, $\mathcal{I} = \{(x,H) | x \in H\}$. Let η be the line bundle on $\mathcal{I}$ with fiber over (x,H)

equal to $T_x\mathbb{P}^4/T_xH$. Then $(q')^*N \cong \eta|\Sigma$. Let $E \subset T\mathcal{I}$ be the subbundle of contact hyperplanes with $v \in E_{(x,H)} \Leftrightarrow$ the projection of v to $T\mathbb{P}^4$ lies in T_xH. Then $\eta = T\mathcal{I}/E$.

Dually, let ξ be the line bundle on $\mathcal{I}$ with fiber over (x, H) equal to $T_H\mathbb{P}^{4*}/T_HX$, where X is the hyperplane of $\mathbb{P}^{4*}$ corresponding to x. Then $(p')^*C \cong \xi|\Sigma$. Let $F \subset T\mathcal{I}$ be the subbundle of contact hyperplanes with $v \in F_{(x,H)} \Leftrightarrow$ the projection of v to $T\mathbb{P}^{4*}$ lies in T_HX. Then $\xi = T\mathcal{I}/F$.

But these two contact structures on $\mathcal{I}$ are equal (cf. [2, p. 109, Example 1]), so $E = F$, which implies $\eta = \xi$, and hence $(q')^*N \cong (p')^*C$.

Remark. Using Euler sequences, one can show that $\eta \cong \mathcal{O}_{\mathbb{P}^4}(1)\otimes\mathcal{O}_{\mathbb{P}^{4*}}(1) \cong \xi$ to prove the proposition without using the contact structures.

Let $D(dp)$ be the intrinsic derivative of $dp\colon T\Gamma \to T\mathbb{P}^{4*}$:

$$D(dp)\colon T\Gamma|\Sigma \to \operatorname{Hom}(\operatorname{Ker}(dp), \operatorname{Coker}(dp)).$$

(6.8) Proposition. *Under the canonical identifications* $\operatorname{Ker}(dp) = TM$, *and* $\operatorname{Coker}(dp) = N$ *given above, we have* $D(dp)|\operatorname{Ker}(dp) = \mathrm{II}'\colon TM \to \operatorname{Hom}(TM, N)$, *where* II' *is the homomorphism defined by the second fundamental form* II.

Proof. In standard coordinates (5.3) dp has matrix

$$\begin{pmatrix} -\frac{\partial f}{\partial x_1} - t_1 & -\frac{\partial f}{\partial x_2} - t_2 & -\frac{\partial f}{\partial x_3} - t_3 & -x_1 & -x_2 & -x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $D(dp)$ is the derivative of

$$\left(-\frac{\partial f}{\partial x_1} - t_1, -\frac{\partial f}{\partial x_2} - t_2, -\frac{\partial f}{\partial x_3} - t_3\right),$$

and $D(dp)|\operatorname{Ker}(dp)$ is the derivative of

$$\left(-\frac{\partial f}{\partial x_1}, -\frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_3}\right).$$

But this is also the local coordinate description of the second fundamental form II' (cf. (1.8)).

Proof of Theorem (6.2). We show that J^4p is transverse to the Thom-Boardman strata if and only if the package (6.1) holds. Specifically, we show that $J^1p \cap \Sigma^4 = \varnothing$, $J^1p \pitchfork \Sigma^3$, and that the remaining jet transversality conditions are equivalent to parts (1)–(5) of (6.1).

(a) $J^1p \cap \Sigma^4 = \varnothing$. As noted above, $p = P|\Gamma$ implies $\dim \operatorname{Ker}(dp) \leq 3$.

(b) $J^1p \pitchfork \Sigma^3$. According to Boardman, this is equivalent to the condition (ι_1) that

$$d_2 = D(dp)\colon T\Gamma|\Sigma^3(p) \to \operatorname{Hom}(\operatorname{Ker}(dp), \operatorname{Coker}(dp))$$

is surjective on fibers. We saw in the proof of (6.8) that $D(dp)$ is the derivative of $(-\partial f/\partial x_1 - t_1, -\partial f/\partial x_2 - t_2, -\partial f/\partial x_3 - t_3)$, so $D(dp)$ has rank 3 everywhere.

(c) $\mathrm{II} \pitchfork \mathbf{Q}^1 \Leftrightarrow J^2p \pitchfork \Sigma^{3,1}$. Now $\Pi = \overline{\Sigma}^{3,1}(p)$ and $\Pi^0 = \Sigma^{3,1}(p) = \{z \in \Sigma^3(p) \mid \dim \mathrm{Ker}(D_2)_z = 1\}$. We have seen in (6.8) that D_2 can be identified with $\Pi'\colon TM \to \mathrm{Hom}(TM, N)$. With this identification, Boardman's condition (ι_2) that $J^2p \pitchfork \Sigma^{3,1}$ is that the intrinsic derivative

$$D(\mathrm{II}')\colon TM|\Sigma^{3,1}(p) \to \mathrm{Hom}(\mathrm{Ker}\,\mathrm{II}', \mathrm{Coker}\,\mathrm{II}')$$

is surjective on fibers. So (c) follows from the existence of a commutative diagram of bundle homomorphisms over $\Sigma^{3,1}(p) = \mathrm{II}^{-1}(\mathbf{Q}^1)$:

$$\begin{array}{ccc} TM & \xrightarrow{\quad D(\mathrm{II}) \quad} & T\mathbf{Q} \\ {\scriptstyle D(\mathrm{II}')}\downarrow & & \downarrow \\ \mathrm{Hom}(K, C) & \xrightarrow{\quad \approx \quad} & T\mathbf{Q}/T\mathbf{Q}^1 \end{array}$$

where $K = \mathrm{Ker}\,\mathrm{II}'$ and $C = \mathrm{Coker}\,\mathrm{II}'$. To establish this, let

$$\begin{aligned} \mathbf{R} = \mathrm{Hom}(TM, \mathrm{Hom}(TM, N)) &\cong T^*M \otimes T^*M \otimes N \\ &\supset \mathrm{Sym}^2(T^*M) \otimes N = \mathbf{Q}, \end{aligned}$$

and let $\mathbf{R}^1 \subset \mathbf{R}$ be the subbundle of homomorphisms of corank 1. Then the normal bundle of $\mathbf{R}^1$ in $\mathbf{R}$ is isomorphic with $\mathrm{Hom}(K, C)$, where K and C are the kernel and cokernel bundles (cf. [14, Corollary 3.4, p. 150]). If s is the section of $\mathbf{R}$ corresponding to II', then $D(\mathrm{II}')$ is the composition $TM \xrightarrow{ds} T\mathbf{R} \to T\mathbf{R}/T\mathbf{R}^1 \cong \mathrm{Hom}(K, C)$ (cf. [14, Proposition 3.6, p. 151]). Now $\mathbf{Q}^1$ is the transverse intersection of $\mathbf{Q}$ and $\mathbf{R}^1$, and $s = i \circ \mathrm{II}$, $i\colon \mathbf{Q} \to \mathbf{R}$ the inclusion, so $D(\mathrm{II}')$ is the composition $TM \xrightarrow{d(\mathrm{II})} T\mathbf{Q} \to T\mathbf{Q}/T\mathbf{Q}^1 \cong \mathrm{Hom}(K, C)$, as desired.

(d) $\mathrm{II} \pitchfork \mathbf{Q}^2 \Leftrightarrow J^2p \pitchfork \Sigma^{3,2}$, and as a consequence $J^3p \pitchfork \Sigma^{3,2,1} = \varnothing$. Recall that $\Pi' = \overline{\Sigma}^{3,2}(p)$ and $\Sigma^{3,2}(p) = \{z \in \Sigma^3(p) \mid \dim \mathrm{Ker}(D_2)_z = 2\}$. Identifying D_2 with II' (6.8), Boardman's condition (ι_2) for $J^2p \pitchfork \Sigma^{3,2}$ is that d_3 is surjective on fibers over $\Sigma^{3,2}(p)$, where d_3 is the target restriction of $D(D_2)$ to the symmetric subbundle P_2 of $\mathrm{Hom}(K_2, Q_2)$, and K_2 and Q_2 are the kernel and cokernel bundles of II' on $\Sigma^{3,2}(p)$ (so K_2 and Q_2 have rank 2):

$$\begin{array}{ccc} TM & \xrightarrow{\quad D(\mathrm{II}') \quad} & \mathrm{Hom}(K_2, Q_2) \\ & {\scriptstyle d_3}\searrow & \cup \\ & & P_2 \end{array}$$

It follows from [14, Corollary 3.4, p. 150] and the definition of P_2 [6, 7.8] that P_2 is isomorphic to the normal bundle of $\mathbf{Q}^2$ in $\mathbf{Q}$, and from [14, Proposition 3.6, p. 151] that over $\Sigma^{3,2}(p) = \mathrm{II}^{-1}(\mathbf{Q}^2)$ there is a commutative diagram of bundle homomorphisms:

$$\begin{array}{ccc} TM & \xrightarrow{d(\mathrm{II})} & T\mathbf{Q} \\ {\scriptstyle d_3}\downarrow & & \downarrow \\ P_2 & \xrightarrow{\approx} & T\mathbf{Q}/T\mathbf{Q}^2 \end{array}$$

Therefore $\mathrm{II} \pitchfork \mathbf{Q}^2 \Leftrightarrow J^2p \pitchfork \Sigma^{3,2}$, as desired.

Note that if $J^2p \pitchfork \Sigma^{3,2}$, then d_3 is an isomorphism, since TM and P_2 both have rank 3. By definition we have

$$\Sigma^{3,2,1}(p) = \{z \in \Sigma^{3,2}(p) | \dim \operatorname{Ker} D_3 = 1\},$$

where $D_3 = d_3|K_2$, so $J^2p \pitchfork \dot{\Sigma}^{3,2} \Rightarrow J^3p \cap \Sigma^{3,2,1} = \varnothing$.

(e) $\mathrm{II} \cap \mathbf{Q}^3 = \varnothing \Leftrightarrow J^2p \cap \Sigma^{3,3} = \varnothing$. $\Sigma^{3,3}(p) = \{z \in \Sigma^3(p) | \dim \operatorname{Ker}(D_2)_z = 3\} = \{x \in M | \dim \operatorname{Ker} \mathrm{II}_x = 3\} = \mathrm{II}^{-1}(\mathbf{Q}^3)$.

(f) If $J^2p \pitchfork \Sigma^{3,1}$, then $\operatorname{Ker}\mathrm{II}$ has nondegenerate tangency with $\Pi^0 = \Sigma^{3,1}(p)$ in $M \Leftrightarrow J^3p \pitchfork \Sigma^{3,1,1}$. Now $C = \overline{\Sigma}^{3,1,1}(p)$ and $C^0 = \Sigma^{3,1,1}(p) = \{z \in \Sigma^{3,1}(p) | \dim \operatorname{Ker}(D_3)_z = 1\}$, where $D_3 = d_3|K_2 = D(\mathrm{II}')|\operatorname{Ker}\mathrm{II}'$. Let $K = K_2 = \operatorname{Ker}\mathrm{II}'$ and $Q = Q_2 = \operatorname{Coker}\mathrm{II}'$; K and Q are line bundles on Π^0. Boardman proves that if $J^2p \pitchfork \Sigma^{3,1}$ then $J^3p \pitchfork \sigma^{3,1,1} \Leftrightarrow (\iota_3)$ holds, where (ι_3) is the condition that

$$D_2(\mathrm{II}') : T\Pi^0 \to \operatorname{Hom}(K, \operatorname{Hom}(K, Q))$$

is surjective on fibers over $\Sigma^{3,1,1}(p)$, and $D_2(\mathrm{II}') = d_4$ is the intrinsic derivative of $D(\mathrm{II}')|K \colon K \to \operatorname{Hom}(K, Q)$. Since $J^2p \pitchfork \Sigma^{3,1}$, we have $\operatorname{Hom}(K, Q) \cong TM/T\Pi^0 = N_{\Pi^0}$, the normal bundle of Π^0 in M, and $D \overset{D(\mathrm{II}')}{\to} \operatorname{Hom}(K, Q) \cong N_{\Pi^0}$ is the composition $K \subset TM \to N_{\Pi^0}$. Let $S \colon \Pi^0 \to \operatorname{Hom}(K, N_{\Pi^0})$ be the corresponding section. By [14, Proposition 3.6, p. 151], the intrinsic derivative of $D(\mathrm{II}')$ is the composition $T\Pi^0 \overset{dS}{\to} T\operatorname{Hom}(K, N_{\Pi^0}) \to \operatorname{Hom}(K, N_{\Pi^0})$, completing the proof of (f).

(g) If $J^2p \pitchfork \Sigma^{3,1}$ and $J^3p \pitchfork \Sigma^{3,1,1}$, then $\ker \mathrm{II}$ has nondegenerate tangency with C^0 in $\Pi^0 \Leftrightarrow J^4p \pitchfork \Sigma^{3,1,1,1}$ and as a consequence $J^5p \cap \Sigma^{3,1,1,1,1} = \varnothing$. Now $\Lambda = \overline{\Sigma}^{3,1,1,1}(p)$, and $\Sigma^{3,1,1,1}(p) = \{z \in \Sigma^{3,1,1}(p) | \dim \operatorname{Ker}(D_4)_z = 1\}$, where $D_4 = d_4|K_3 = D_2(\mathrm{II}')|K \colon K \to \operatorname{Hom}(K, \operatorname{Hom}(K, Q))$, $K = \operatorname{Ker}\mathrm{II}'$, $Q = \operatorname{Coker}\mathrm{II}'$. Boardman proves that if $J^2p \pitchfork \Sigma^{3,1}$ and $J^3p \pitchfork \Sigma^{3,1,1}$, then $J^4p \pitchfork \Sigma^{3,1,1,1} \Leftrightarrow (\iota_4)$ holds, where (ι_4) is the condition that

$$d_5 = D_3(\mathrm{II}') \colon TC^0 \to \operatorname{Hom}(K, \operatorname{Hom}(K, \operatorname{Hom}(K, Q)))$$

is surjective on fibers over $\Sigma^{3,1,1,1}(p)$. Now $\mathrm{Hom}(K, \mathrm{Hom}(K,Q)) \cong T\Pi^0/TC^0 = N_{C^0}$, the normal bundle of C^0 in Π^0, and $K \overset{D_2(\mathrm{II}')}{\to} \mathrm{Hom}(K, \mathrm{Hom}(K,Q)) \cong N_{C^0}$ is the composition $K \subset T\Pi^0 \to N_{C^0}$. Let $S\colon C^0 \to \mathrm{Hom}(K, N_{C^0})$ be the corresponding section. The intrinsic derivative of $D_2(\mathrm{II}')$ is the composition $TC \overset{dS}{\to} T\,\mathrm{Hom}(K, N_{C^0}) \to \mathrm{Hom}(K, N_{C^0})$, so $(\iota_4) \Leftrightarrow K$ has nondegenerate tangency with C^0 in Π^0. If $J^4 p \pitchfork \Sigma^{3,1,1,1}$ then d_5 is a bundle isomorphism. By definition, $\Sigma^{3,1,1,1,1}(p) = \{z \in \Sigma^{3,1,1,1}(p) \mid \dim \mathrm{Ker}(D_5)_z = 1\}$, where $D_5 = d_5|K$. So $J^4 p \pitchfork \Sigma^{3,1,1,1} \Rightarrow J^5 p \cap \Sigma^{3,1,1,1,1} = \varnothing$.

7. Classification of singularities

For a Gauss-stable hypersurface M in $\mathbb{P}^4$, we classify the singularities of the family of hyperplane sections $M \cap H$, of the incidence projection p, and of the Gauss map γ. For each of these three parallel classifications, which correspond to the three parts of Theorem (5.1), we give normal forms, i.e., explicit equations displaying the singularity types.

Remark. Although the singularities of $M \cap H$, p, and γ are projective invariants of M in $\mathbb{P}^4$, the classification of these singularities is much coarser than that induced by projective transformations of $\mathbb{P}^4$. Thus, the germ of the hypersurface M at a point x can be recovered up to projective equivalence from neither the singularity type of the tangent hyperplane section of M at x, nor the singularity type of the Gauss map at x.

(a) The classification of singularities of hyperplane sections is up to analytic isomorphism of the local complex space $(M \cap H, x)$. There are five singular types, with the following Arnold symbols and normal forms:

$$(7.1)\qquad \begin{array}{ll} A_1 & x^2+y^2+z^2 \\ A_2 & x^2+y^2+z^3 \\ A_3 & x^2+y^2+z^4 \\ A_4 & x^2+y^2+z^5 \\ D_4 & x^2+y^3+yz^2 \end{array}$$

(b) The classification of singularities of incidence projection germs is up to analytic changes of coordinates in the source and target (left-right equivalence). The five types correspond to versal unfoldings of the hyperplane section types (7.1). The Thom-Boardman symbols and normal forms are the following:

$$
(7.2)\qquad
\begin{array}{ll}
\Sigma^{3,0}(p) & (x^2+y^2+z^2,a,b,c)\\
\Sigma^{3,1,0}(p) & (x^2+y^2+z^3+az,a,b,c)\\
\Sigma^{3,1,1,0}(p) & (x^2+y^2+z^4+az^2+bz,a,b,c)\\
\Sigma^{3,1,1,1,0}(p) & (x^2+y^2+z^5+az^3+bz^2+cz,a,b,c)\\
\Sigma^{3,2,0}(p) & (x^2+y^3+yz^2+ay^2+by+cz,a,b,c)
\end{array}
$$

(c) The classification of singularities of Gauss map germs is also up to analytic changes of coordinates in the source and target. The five types have the following Thom-Boardman symbols and normal forms (the first type is nonsingular):

$$
(7.3)\qquad
\begin{array}{ll}
\Sigma^{0}(\gamma) & (x,y,z,0)\\
\Sigma^{1,0}(\gamma) & (3x^2,2x^3,y,z)\\
\Sigma^{1,1,0}(\gamma) & (3x^4+x^2y,4x^3+2xy,y,z)\\
\Sigma^{1,1,1,0}(\gamma) & (4x^5+2x^3y+x^2z,5x^4+3x^2y+2xz,y,z)\\
\Sigma^{2,0}(\gamma) & (2x^3+2xy^2+x^2z,3x^2+y^2+2xz,2xy,z)
\end{array}
$$

Given the normal forms (7.3) for the Gauss map, the transversality package (6.1) can be verified by direct computation.

The normal forms (7.2) for p follow from the normal forms (7.1) for $M\cap H$ by versal unfolding, as in the proof (a) $\Leftrightarrow$ (b) of Theorem (5.1). Given an analytic hypersurface germ $f(x_1,\cdots,x_n)=0$, consider the space of first order deformations

$$T_f^1=\mathscr{O}_{\mathbf{C}^n,0}/(f,\partial f/\partial x_1,\cdots,\partial f/\partial x_n).$$

The *Tjurina number* τ of the hypersurface germ is the dimension of T_f^1 as a complex vector space. (For A_k or D_k, $\tau=k$.) If τ is finite, let $g_1,\cdots,g_\tau$ be a basis for T_f^1. Then

$$F(x_1,\cdots,x_n,t_1,\cdots,t_\tau)=f(x)+t_1g_1(x)+\cdots+t_\tau g_\tau(x)$$

is a versal deformation of $f(x_1,\cdots,x_n)=0$.

Suppose $g_\tau=1$. Then the unfolding

$$h\colon \mathbf{C}^n\times\mathbf{C}^{\tau-1}\to\mathbf{C}^\tau,$$
$$h(x,t_1,\cdots,t_{\tau-1})=(f(x)+t_1g_1(x)+\cdots+\tau_{\tau-1}g_{\tau-1}(x),t_1,\cdots,t_{\tau-1})$$

is the stable map germ corresponding to the hypersurface $f(x)=0$.

The normal forms (7.3) for γ are obtained from the normal forms (7.2) for p by restriction to the singular locus, since the Gauss map is the composition of p with the canonical isomorphism $M\to\Sigma(p)$.

Example. Consider the D_4-singularity $f(x,y,z) = x^2 + y^3 + yz^2$. Then

$$\begin{aligned} T_f^1 &= \mathscr{O}_{\mathbb{C}^3,0}/(x^2 + y^3 + yz^2, 2x, 3y^2 + z^2, 2yz) \\ &\cong \mathbb{C}[[x,y,z]]_0/(x, yz, 3y^2 + z^2), \end{aligned}$$

with basis $y^2, y, z, 1$. Thus

$$h(x,y,z) = (x^2 + y^3 + yz^2 + ay^2 + by + cz, a, b, c)$$

is a normal form for $\Sigma^{3,2,0}(p)$. The singular locus of h is given by $2x = 0$, $3y^2 + z^2 + 2ay + b = 0$, $2yz + c = 0$, so $\Sigma(h)$ is parametrized by (y,z,a) and $h|\Sigma(h)$ is

$$(y,z,a) \to (-2y^3 - ay^2 - 2yz^2, a, -3y^2 - z^2 - 2ay, -2yz).$$

Changing signs, permuting the target coordinates, and renaming the variables, we obtain the stated normal form for $\Sigma^{2,0}(\gamma)$:

$$(2x^3 + x^2z + 2xy^2, 3x^2 + y^2 + 2xz, 2xy, z).$$

This description is a special case of Arnold's recipe for finding normal forms of stable Legendre singularities [3, 21.6].

Proof of the classifications. The classification (a) of hyperplane section singularities of a Gauss-stable hypersurface follows from Arnold's classification of simple hypersurface singularities (cf. [1, p. 254], [10, p. 158], [24, p. 132]). The Tjurina number of a local complex space is the minimum number of parameters in a versal deformation. Since the family of hyperplane sections of $M \subset \mathbb{P}^4$ is parametrized by $\mathbb{P}^{4*}$, the only hypersurface singularities which can occur as hyperplane sections of a Gauss-stable hypersurface in $\mathbb{P}^4$ are those of Tjurina number at most 4. According to Arnold, any singular local complex space with $\tau \leq 4$ is one of the five types A_1, A_2, A_3, A_4, D_4, all of which are simple singularity types. (It follows from the numerical formulas of §§1 and 2 that each of these types actually occurs for all Gauss-stable hypersurfaces of degree $d \geq 3$.)

Now let X_0 and X_0' be hypersurface germs at the origin in $\mathbb{C}^n$, and let $X \to T$ and $X' \to T'$ be k-parameter versal deformations of X_0 and X_0', respectively. Let h, h': $(\mathbb{C}^n \times \mathbb{C}^{k-1}, 0) \to (\mathbb{C}^k, 0)$ be the stable map germs associated to $X \to T$ and $X' \to T'$ as in the proof of Theorem (5.1), following [25, pp. 26–27]. Then X_0 is analytically equivalent to X_0' if and only if h is analytically equivalent to h' [25, Proposition (1.3), p. 27]. (This is a key result in Mather's classification of stable map germs.) Thus the classification (a) of singularities of hyperplane sections gives the classification (b) of singularities of incidence projections.

If h and h' are map germs with $\Sigma(h)$ and $\Sigma(h')$ smooth, and h is equivalent to h', then $h|\Sigma(h)$ is equivalent to $h'|\Sigma(h')$. Thus the classification

(c) of singularities of Gauss maps follows from the classification (b) of singularities of incidence projections.

8. Genericity and deformations

Let $\mathcal{M}_d$ be the space of nonsingular hypersurfaces of degree d in $\mathbb{P}^4$.

(8.1) Theorem. *The set of Gauss-stable hypersurfaces is a nonempty Zariski-open subset of $\mathcal{M}_d$ for all $d \geq 2$.*

We give two independent proofs:

(a) In this section, we consider the set of hypersurfaces such that at every point the family of hyperplane section germs is versal; we prove that it is Zariski open and nonempty. The proof is a direct analysis of the first order deformations of a hyperplane section.

(b) In the next section, we consider the set of hypersurfaces such that the incidence projection is stable; we prove that it is Zariski open and nonempty. This proof is a jet transversality argument, valid for $d \geq 4$. (The case $d = 2$ is easy, and a separate argument can be given for the case $d = 3$.) This second proof is essentially due to Ronga for surfaces in $\mathbb{P}^3$ [30], [31]. A similar but less general method was developed first by Bruce [7].

Theorem (8.1) is false for hypersurfaces $M^n \subset \mathbb{P}^{n+1}$ for large n. The appearance of nonsimple singularities prevents the versality of the family of hyperplane sections for $n \geq 6$. However, it seems likely that the theorem is true for $n \leq 5$. (The case $n = 2$ is proved in [29, 3.1] and [31].)

We turn now to the first proof of Theorem (8.1). Consider a singularity of a hyperplane section of $M^3 \subset \mathbb{P}^4$ and the 4-parameter deformation of it obtained by varying the hyperplane section. First we prove that the versality of this deformation for all local hyperplane sections of M is a Zariski open condition on $M \in \mathcal{M}_d$. Then we analyze how the various expected generic phenomena could fail. In particular, we give local methods for studying this situation and find, by dimension count, no failure on the generic hypersurface.

We will carry out the local analysis in the following standardized notation (cf. §5). By projective change of coordinates, we place the point in question at (1,0,0,0,0) and make $x_4 = 0$ the tangent hyperplane there. Then we dehomogenize: where $x_0 \neq 0$, let (x, y, z, u) be affine coordinates. Thus, we consider a polynomial $f(x, y, z, u)$ such that the affine hypersurface $f = 0$ passes through the origin (0,0,0,0) and is smooth there with tangent hyperplane $u = 0$.

Expanding in powers of u,

$$f(x,y,z,u) = f_0(x,y,z) + f_1(x,y,z)u + \cdots,$$

where $f_0(x,y,z)$ = quadratic + higher order terms, and $f_1(0,0,0) \neq 0$. We are concerned with the singularity at the origin of the (affine) tangent hyperplane section $f_0 = 0$ in $\mathbb{C}^3$. Its space of first order deformations is

$$T^1_{f_0} = \mathbb{C}[[x,y,z]]/(f_0, \partial f_0/\partial x, \partial f_0/\partial y, \partial f_0/\partial z).$$

(8.2) Lemma. *The deformation of $f_0 = 0$ at the origin by the hyperplane sections of $f = f_0 + f_1 u + \cdots = 0$ is versal if and only if f_1, xf_1, yf_1, zf_1 span $T^1_{f_0}$.*

Proof. In the original homogeneous coordinates we need to look at sections of M by hyperplanes H_α: $x_4 = \alpha_0 x_0 + \cdots + \alpha_3 x_3$ near $x_4 = 0$. Then, in our affine coordinates, $u = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z$; substituting this into $f = 0$ realizes the deformation of $f_0 = 0$ inside $\mathbb{C}^3$ as

$$f_0 + (\alpha_0 f_1 + \alpha_1 x f_1 + \alpha_2 y f_1 + \alpha_3 z f_1) + \cdots = 0,$$

where the omitted terms are quadratic or higher in the parameters $\alpha_0, \cdots, \alpha_3$. To finish we use the fact that a deformation of a hypersurface singularity over a smooth base is versal if and only if the linear terms of the deformation span $T^1_{f_0}$ (cf. [20]).

Example. Suppose the tangent hyperplane section of M is $f_0(x,y,z) = x^2 + y^2 + z^4 = 0$. The hyperplane sections of M cannot versally deform this singularity since $T^1_{f_0}$ is 3-dimensional but xf_1 and yf_1 are zero in $T^1_{f_0}$. The point is that this equation, which is an analytic normal form for an A_3 surface singularity (cf. (7.1)), is *projectively* special. Later we will see A_3 hyperplane sections which are versally deformed.

Now we consider the *universal family* of hyperplane sections of nonsingular hypersurfaces of degree $d > 1$ in $\mathbb{P}^4$,

$$p: \mathscr{X} \to \mathscr{Y},$$

where $\mathscr{X} = \{(x,H,M) | M \in \mathscr{M}_d,\ H \in \mathbb{P}^{4*},\ x \in M \cap H\}$, $\mathscr{Y} = \mathbb{P}^{4*} \times \mathscr{M}_d$, and $p(x,H,M) = (H,M)$. The *universal critical locus* is $\mathscr{C} = \{(x,H,M) | x$ is singular on $M \cap H\}$: its image $p(\mathscr{C})$, the *discriminant locus* of p, consists of pairs (H,M) such that H is a tangent hyperplane of M. For $(x,H,M) \in \mathscr{C}$ we are interested in whether the singular germ $(M \cap H, x)$ is versally deformed by the hyperplane sections of M. Let $V = \{(x,H,M) \in \mathscr{C} |$ the deformation of $(M \cap H, x)$ parametrized by the germ $(\mathbb{P}^{4*}, H)$ is versal$\}$, and let $\mathscr{G}$ be the complement of the projection of $\mathscr{C} - V$ to $\mathscr{M}_d$. Thus $\mathscr{G} = \{M \in \mathscr{M}_d |$ for all (x,H) with H tangent to M at x, the deformation of $(M \cap H, x)$ parametrized by the germ $(\mathbb{P}^{4*}, H)$ is versal}. By (5.1), $\mathscr{G}$ is the set of Gauss-stable hypersurfaces of $\mathbb{P}^4$. To prove (8.1), we will show that $\mathscr{G}$ is a nonempty Zariski open subset of $\mathscr{M}_d$.

The case $d = 2$, quadric hypersurfaces of $\mathbb{P}^4$, is very elementary; a tangent hyperplane section has an A_1-singularity and this is (versally) smoothed by a generic 1-parameter variation of the hyperplane section (cf. the proof of (8.8.1) below). From now on in this section we assume $d \geq 3$.

(8.3) Proposition. *V is Zariski open in $\mathscr{C}$.*

This implies immediately that $\mathscr{G}$ is Zariski open in $\mathscr{M}_d$. For if $B = \mathscr{C} - V$ is Zariski closed in $\mathscr{C}$, it is Zariski closed in $\mathscr{X}$, so that projection of B to $\mathscr{M}_d$ is Zariski closed.

Proof of (8.3). Associated to our family $p\colon \mathscr{X} \to \mathscr{Y}$ of hyperplane sections of nonsingular hypersurfaces of degree $d > 2$ in $\mathbb{P}^4$, we have a Kodaira-Spencer map (cf. [23], [24, Chapter 6B], [33, pp. 14–15]) for deformations of singularities: $\mathscr{T}_{\mathscr{Y}} \to p_*(\mathscr{T}^1_{\mathscr{X}/\mathscr{Y}})$, a homomorphism of coherent sheaves on $\mathscr{Y}$. Let q and r denote the projections of $\mathscr{Y} = \mathbb{P}^{4*} \times \mathscr{M}_d$ to the respective factors; then $\mathscr{T}_{\mathscr{Y}} \cong q^*(\mathscr{T}_{\mathbb{P}^{4*}}) \oplus r^*(\mathscr{T}_{\mathscr{M}_d})$. We are interested in the map $\varphi\colon q^*(\mathscr{T}_{\mathbb{P}^{4*}}) \to p_*(\mathscr{T}^1_{\mathscr{X}/\mathscr{Y}})$, governing the (first order) variation of singularities as the hyperplane H varies but the hypersurface M stays fixed. At a point $y = (H, M) \in \mathscr{Y}$, the homomorphism $\varphi|_y$ on fibers is $T_H(\mathbb{P}^{4*}) \to H^0(\mathscr{T}^1_{M \cap H})$, the natural map associated to the deformation of $M \cap H$ obtained by varying H. Notice that this map $\varphi|_y$ measures the *simultaneous* effect of varying H upon the singularities of $M \cap H$. Our definition of V measures the effect of varying H upon the *individual* singularities, one at a time. Since $\mathscr{T}^1_{\mathscr{X}/\mathscr{Y}}$ is an $\mathscr{O}_{\mathscr{C}}$-module, $p_*(\mathscr{T}^1_{\mathscr{X}/\mathscr{Y}}) = \eta_*(\mathscr{T}^1_{\mathscr{X}/\mathscr{Y}})$, where $\eta = p|_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{Y}$, so we can equally well express $\varphi\colon q^*(\mathscr{T}_{\mathbb{P}^{4*}}) \to \eta_*(\mathscr{T}^1_{\mathscr{X}/\mathscr{Y}})$ on $\mathscr{Y}$.

(8.4) Lemma. *The restriction of $p\colon \mathscr{X} \to \mathscr{Y}$ to $\mathscr{C}$ is a finite mapping.*

Proof. Since $p|\mathscr{C}$ is a proper mapping, it suffices to check that it has finite fibers. This follows from the fact that the Gauss map of a nonsingular hypersurface has finite fibers. q.e.d.

Now we are in the following situation. We have a finite morphism $\eta\colon \mathscr{C} \to \mathscr{Y}$, coherent sheaves $\mathscr{E}$, $\mathscr{F}$ on $\mathscr{Y}$, $\mathscr{C}$ resp., and a sheaf homomorphism $\varphi\colon \mathscr{E} \to \eta_*(\mathscr{F})$. For each $y \in \mathscr{Y}$, the fiber $\eta_*(\mathscr{F})|_y$ decomposes into a direct sum of vector spaces $H_x = T^1_{(M \cap H, x)}$, indexed by the points x of $\eta^{-1}(y)$. We want to show that $\{y \in \mathscr{Y} \mid$ for each $x \in \eta^{-1}(y)$, the map $\varphi_x\colon \mathscr{E}|_y \to H_x$ is surjective$\}$ is Zariski open in $\mathscr{Y}$.

(8.5) Lemma. *There exists a canonical surjection* $\eta^*\eta_*(\mathscr{F}) \twoheadrightarrow \mathscr{H}$ *of coherent sheaves on* $\mathscr{C}$, *such that for each* $x \in \mathscr{C}$, *the induced pointwise surjection* $(\eta^*\eta_*(\mathscr{F}))|_x \twoheadrightarrow \mathscr{H}|_x$ *sends the summand* H_x *isomorphically to* $\mathscr{H}|_x$.

Applying the lemma, we get the composite homomorphism Φ:

$$\eta^*(\mathscr{E}) \to \eta^*\eta_*(\mathscr{F}) \to \mathscr{H},$$

which on the fiber at a point $x \in \mathscr{C}$ is simply φ_x. Thus, $\{x \in \mathscr{C} \mid$ the map $\varphi_x \colon \eta^*(\mathscr{E})|_x \to H_x$ is not surjective$\}=\operatorname{Supp}(\operatorname{coker}\Phi)$, hence is closed in $\mathscr{C}$. Take the image under η and the complement in $\mathscr{Y}$ to complete the proof of Proposition (8.3).

Proof of (8.5). To get the quotient sheaf $\mathscr{H}$ of $\eta^*\eta_*(\mathscr{F})$ on $\mathscr{C}$, consider the fiber product $\Pi = \mathscr{C} \times_{\mathscr{Y}} \mathscr{C}$, the projections η_1, η_2 to the two factors, and the relative diagonal $\Delta \subset \Pi$, whose role here is to provide a section of η_1. Let $\mathscr{I} \subset \mathscr{O}_\Pi$ be the ideal sheaf of Δ. We claim that $\mathscr{H} = \eta_{1*}(\eta_2^*(\mathscr{F})/\mathscr{I}^\nu \cdot \eta_2^*(\mathscr{F}))$ will work for sufficiently large ν. It suffices to replace $\mathscr{Y}$ by an affine open subset $\mathscr{Y}_0$ and $\mathscr{C}$ by the preimage $\mathscr{C}_0$ of $\mathscr{Y}_0$. Since η is a finite morphism, $\mathscr{C}_0$ is also affine.

Let us restate the problem in terms of commutative algebra. We have an affine algebra A and a finite A-algebra B; these are the coordinate rings of $\mathscr{Y}_0$ and $\mathscr{C}_0$, respectively. Also we have a finitely generated B-module N, the sections of $\mathscr{F}$ on $\mathscr{C}_0$. Now $\eta^*\eta_*(\mathscr{F})$ is given by the B-module $B \otimes_A N$, where B acts on the B-factor in the tensor product. Modulo a maximal ideal m of B, this module becomes $(B/m) \otimes_A N \cong N/\mu N$, where μ is the maximal ideal $m \cap A$ of A. But $N/\mu N \cong (B/\mu B) \otimes_B N$ and $B/\mu B$ decomposes into direct summands, one for each maximal ideal m over μ (cf. [4, Chapter 8]). We want to construct the quotient of $B \otimes_A N$ which induces, for each m, the corresponding summand of $N/\mu N$. It suffices to carry out the quotient construction for the case $N = B$; the general case follows by tensoring. Thus, if I is the kernel of the multiplication homomorphism $B \otimes_A B \to B$, consider $(B \otimes_A B)/I^\nu \cdot (B \otimes_A B)$ as a B-module, where B acts on $B \otimes_A B$ via the first factor. Modulo m, we get

$$(B/m \otimes_A B)/I^\nu \cdot (B/m \otimes_A B) \cong (B/\mu B)/m^\nu \cdot (B/\mu B),$$

and this last quotient is certainly isomorphic to the desired summand of $B/\mu B$ for large ν. All we have left to do is choose ν large enough to work for all maximal ideals m of B, say $\nu > \dim_{\mathbf{C}}(B/\mu B)$, which is bounded by the number of generators for B as A-module. q.e.d.

The proof of Proposition (8.3) has the following consequence, which we use later.

(8.6) Corollary. *For each nonnegative integer k, $\{(x, H, M) \in \mathscr{C} \mid \dim(T^1_{(M\cap H,x)}) \geq k\}$ is Zariski closed in $\mathscr{C}$.*

We contrast the openness of $\mathscr{G}$ with the following easier global version.

(8.7) Proposition. *The set of M in $\mathscr{M}_d$ such that, for all H, the deformation of the singularities of $M \cap H$ by hyperplane sections of M is simultaneously versal (i.e., $\varphi|_y$ is surjective), is Zariski open in $\mathscr{M}_d$.*

Proof. The set $\{y \in \mathscr{Y} \mid \varphi|_y$ is not surjective$\}$ is the support of the (coherent) cokernel sheaf of φ; hence it is closed in $\mathscr{Y}$. Therefore, its projection to $\mathscr{M}_d$ is Zariski closed so the complement is open.

Remark. We do not address the nonemptiness of the Zariski-open subset (8.7) of $\mathscr{M}_d$, but it seems that an extension of the methods of this section should apply.

Example. Consider a hyperplane section of $M \subset \mathbb{P}^4$ with five ordinary double points (A_1's). The simultaneous deformation of the singularities cannot be versal since it takes place over the 4-dimensional base space $\mathbb{P}^{4*}$, but the deformation of the individual singularities is versal (cf. (8.8)(1)).

The rest of this section is devoted to showing that $\mathscr{G}$ is nonempty. The following theorem summarizes our strategy.

Recall that $\mathscr{C}$ is the universal critical locus, $V \subset \mathscr{C}$ is the locus of versally deformed singularities, and $B = \mathscr{C} - V$. Given an (analytic isomorphism type of) singularity $\tau = (X, 0)$, let $\mathscr{C}(\tau) = \{(x, H, M) \in \mathscr{C} \mid (M \cap H, x) = \tau\}$. Also, let $V(\tau) = \mathscr{C}(\tau) \cap V$ and $B(\tau) = \mathscr{C}(\tau) - V(\tau)$, the "bad" singularities of type τ.

(8.8) Theorem. (0) *$\mathscr{C}$ is irreducible of dimension* $= \dim(\mathbb{P}^{4*} \times \mathscr{M}_d) - 1$.

(1) *$\mathscr{C}(A_1)$ is Zariski open in $\mathscr{C}$; $B(A_1)$ is empty.*

(2) *$\mathscr{C}(A_2)$ is an irreducible, codimension* 1, *Zariski locally closed subset of $\mathscr{C}$; $B(A_2)$ is empty.*

(3) *$\mathscr{C}(A_3)$ is an irreducible, codimension* 2, *Zariski locally closed subset of $\mathscr{C}$; $B(A_3)$ has codimension ≥ 2 in $\mathscr{C}(A_3)$.*

(4) *$\mathscr{C}(A_4)$ and $\mathscr{C}(D_4)$ are irreducible, codimension* 3, *Zariski locally closed subsets of $\mathscr{C}$; $B(A_4)$ and $B(D_4)$ have codimension ≥ 1 in $\mathscr{C}(A_4)$ and $\mathscr{C}(D_4)$, respectively.*

(5) *Let $W = \mathscr{C} - (\mathscr{C}(A_1) \cup \mathscr{C}(A_2) \cup \mathscr{C}(A_3) \cup \mathscr{C}(A_4) \cup \mathscr{C}(D_4))$. Then W is Zariski closed in $\mathscr{C}$, of codimension ≥ 4. Moreover,*

$$W = \{(x, H, M) \in \mathscr{C} \mid \dim(T^1_{(M\cap H,x)}) \geq 5\}.$$

(8.9) Corollary. *$B(A_1) \cup B(A_2) \cup B(A_3) \cup B(A_4) \cup B(D_4) \cup W$ is Zariski closed in $\mathscr{C}$, of codimension ≥ 4, and coincides with $B = \mathscr{C} - V$.*

Proof. Recall from (8.3) that $B = \mathscr{C} - V$ is Zariski closed in $\mathscr{C}$. By (8.8), the union in the statement of (8.9) coincides with B and has codimension ≥ 4 in $\mathscr{C}$. q.e.d.

Now we can deduce the following more precise version of Theorem (8.1).

(8.10) Corollary. *Let $\mathscr{G}$ be the set of Gauss-stable hypersurfaces of degree $d \geq 3$ in $\mathbb{P}^4$. Then $\mathscr{G} = \{M \in \mathscr{M}_d |$ all tangent hyperplane sections have only A_1-, A_2-, A_3-, A_4-, or D_4-singularities, and each singularity is versally deformed by the hyperplane sections of $M\}$, and $\mathscr{G}$ is Zariski open in $\mathscr{M}_d$ and nonempty.*

Proof. By (8.9), together with (8.4) and (8.8)(0),

$$p(B) = p(B(A_1) \cup B(A_2) \cup B(A_3) \cup B(A_4) \cup B(D_4) \cup W)$$

is Zariski closed in $\mathbb{P}^{4*} \times \mathscr{M}_d$, of codimension ≥ 5. Therefore, the projection to $\mathscr{M}_d$ is a proper, Zariski closed subset. Now take the complement. q.e.d.

Before proceeding to the proof of (8.8), we need some elementary lemmas and background on surface singularities.

Consider the projective group $G = \mathrm{GL}(5)/\mathbb{C}^*$. This is a connected algebraic group, hence irreducible, and the natural action on $\mathbb{P}^4 = (\mathbb{C}^5 - \{0\})/\mathbb{C}^*$ is algebraic. There are induced algebraic actions of G on $\mathscr{M}_d$, $\mathbb{P}^{4*}$ $(= \mathscr{M}_1)$, and products; the action on $\mathbb{P}^4 \times \mathbb{P}^{4*} \times \mathscr{M}_d$ preserves $\mathscr{X}$ so that the projection p to $\mathbb{P}^{4*} \times \mathscr{M}_d$ is equivariant and $\mathscr{C}$ is preserved. Consider the incidence correspondence $\mathscr{I} = \{(x, H) \in \mathbb{P}^4 \times \mathbb{P}^{4*} | x \in H\}$. The action of G on $\mathbb{P}^4 \times \mathbb{P}^{4*}$ preserves $\mathscr{I}$, and the projection $\pi: \mathscr{C} \to \mathscr{I}$ is an equivariant Zariski fiber bundle.

(8.11) Lemma. *If S is a G-invariant subset of $\mathscr{C}$, then S is a Zariski fiber bundle over $\mathscr{I}$ with fibers $S_{x,H} = S \cap (\{x\} \times \{H\} \times \mathscr{M}_d)$, $x \in H$. More precisely, the restriction of π to S induces the local triviality of S over $\mathscr{I}$.*

Proof. First we show that S is a Zariski fiber bundle over $\mathbb{P}^4$ with fibers $S_x = S \cap (\{x\} \times \mathbb{P}^{4*} \times \mathscr{M}_d)$, $x \in \mathbb{P}^4$. For each $i = 0, \cdots, 4$, restrict the projection $q: \mathscr{C} \to \mathbb{P}^4$ to $U_i: x_i \neq 0$, and identify U_i with $\mathbb{C}^4$ as usual. Then translation to the origin 0 in $\mathbb{C}^4$ induces an algebraic isomorphism $q^{-1}(U_i) \cong U_i \times \mathscr{C}_0$ compatible with projection onto U_i. Now if $S \subset \mathscr{C}$ is a G-invariant subset, then since q is G-equivariant and translation to the origin 0 in $\mathbb{C}^4 = U_i$ can be induced by elements of G, the trivialization of $\mathscr{C}$ over U_i restricts to a trivialization $q^{-1}(U_i) \cap S \cong U_i \times S_0$ of S over U_i.

Next we fix $x \in \mathbb{P}^4$ and show that S_x is a Zariski fiber bundle over $\mathscr{I}_x$ with fibers $S_{x,H} = S_x \cap (\{x\} \times \{H\} \times \mathscr{M}_d)$, $H \in \mathscr{I}_x$. For each $j = 0, \cdots, 4$, identify $V_j: \alpha_j \neq 0$ with $\mathbb{C}^4$ and restrict $r: \mathscr{C}_x \to \mathscr{I}_x$ to $V_j \cap \mathscr{I}_x$. Now notice that translations in $\mathbb{C}^4$ preserving the hyperplane $V_j \cap \mathscr{I}_x$ of V_j can be induced by elements of G_x, the stabilizer in G of x. Hence, translation to the origin in $\mathbb{C}^4$ induces an algebraic trivialization of $\mathscr{C}_x$ over $V_j \cap \mathscr{I}_x$ and restricts to one of S_x.

Now we know that S and $\mathscr{I}$ are Zariski fiber bundles over $\mathbb{P}^4$, and that for every $x \in \mathbb{P}^4$ the fiber S_x is a Zariski fiber bundle over the fiber $\mathscr{I}_x$:

$$\begin{array}{ccc} S_x \rightarrow S & \searrow & \\ \downarrow \qquad \downarrow & & \mathbb{P}^4 \\ \mathscr{I}_x \rightarrow \mathscr{I} & \nearrow & \end{array}$$

We want to conclude that $S \to \mathscr{I}$ is a Zariski fiber bundle. Indeed, we can trivialize $S \to \mathbb{P}^4$ over $U_i = \mathbb{C}^4$ by translation to the origin as above, and we can trivialize $\mathscr{I} \to \mathbb{P}^4$ over U_i in the same way. Therefore $S|U_i \to \mathscr{I}|U_i$ is isomorphic to a *constant* family $U_i \times S_0 \to U_i \times \mathscr{I}_0$ over U_i, so the local triviality of $S_0 \to \mathscr{I}_0$ implies the local triviality of $S \to \mathscr{I}$. q.e.d.

Therefore, for any G-invariant subset S of $\mathscr{C}$ we have: S is irreducible, Zariski locally closed (resp., is Zariski closed) of codimension $\le r$ in $\mathscr{C}$ if and only if, for some $(x, H) \in \mathscr{I}$, the same holds for $S_{x,H}$ in $\mathscr{C}_{x,H} = \mathscr{C} \cap (\{x\} \times \{H\} \times \mathscr{M}_d)$. Therefore, to prove Theorem (8.8), we may fix $x = (1, 0, 0, 0, 0)$ and $H\colon x_4 = 0$ and verify, in convenient affine coordinates, the corresponding statements (8.8)(0)–(5) for $\mathscr{C}_{x,H}$.

Now we justify fixing, for each possible rank, the quadratic part of the equation of the hyperplane section. Let $\mathscr{S}_x$ be the space of surfaces of degree d in $\mathbb{P}^3$ which are singular at x. In $\mathscr{S}_x$ we have the Zariski-open subset $\mathscr{S}_{x,2}$ of surfaces with a *double point* at x and a morphism $\tau\colon \mathscr{S}_{x,2} \to \mathscr{Q}$ = the space of conics in $\mathbb{P}T_x(\mathbb{P}^3) \cong \mathbb{P}^2$, $S \to Q_x$, associating to a surface with a double point at x the projectivized tangent cone at x. Now $\mathscr{Q}$ is stratified by rank; let $\mathscr{Q}^i \subset \mathscr{Q}$ denote the (Zariski locally closed subset of) *corank* i quadrics ($i = 0, 1, 2$). Let $\mathscr{S}^i \subset \mathscr{S}_{x,2}$ denote the preimage of $\mathscr{Q}^i$, i.e., the surfaces with a corank i double point at x. The subgroup G_x of $\mathbb{P}\mathrm{GL}(4)$ fixing x acts on $\mathscr{S}_x$ preserving each $\mathscr{S}^i$ and also induces an action on $\mathscr{Q}$ so that τ is equivariant.

(8.12) Lemma. *Let S be a G_x-invariant subset of $\mathscr{S}^i$. Then S is a Zariski "iso-trivial" bundle over $\mathscr{Q}^i$. Precisely, over an étale cover of $\mathscr{Q}^i$ there exists an algebraic trivialization of the map $\tau\colon \mathscr{S}^i \to \mathscr{Q}^i$ which induces a product structure on (the pullback of) S.*

Proof. We will take $i = 1$; the other cases are similar. Fix a point $[q] \in \mathscr{Q}^1$, i.e., a rank 2 conic in $\mathbb{P}^2$, which is a union of two distinct lines. Applying a *fixed* linear transformation of the x, y, z variables, we may assume that $q = x^2 + y^2$. Now we have to give a Zariski open neighborhood U of $x^2 + y^2$ in $\mathscr{Q}^1$, an étale surjection $\tilde{U} \to U$, and a trivialization of

$\mathscr{S}^1 \to \mathscr{Q}^1$ over $\tilde{U}$ which preserves S. Let U be the conics for which the $(1,1)$-entry and the upper left 2×2 minor are nonzero; on U we assume that the coefficient of x^2 is 1. Then we have an (algebraic) algorithm for transforming a quadratic form into $x^2 + \mu y^2$, $\mu \neq 0$; namely, if the coefficient of xy is α, we replace x by $x - \frac{1}{2}\alpha y$ and eliminate the xy term, preserving U. Similarly we eliminate the xz and yz terms and arrive at $x^2 + \mu y^2$, $\mu \neq 0$ (the coefficient of z^2 is zero since the rank is 2). Now we need the étale cover; letting $\tilde{\mu}$ be a square root of μ, we replace y by $\tilde{\mu}^{-1}y$ to obtain $x^2 + y^2$. Finally, the finite, étale surjection $\mathbf{C} - \{0\} \to \mathbf{C} - \{0\}$, $\tilde{\mu} \to \tilde{\mu}^2 = \mu$, defines by pullback the desired finite, étale surjection $\tilde{U} \to U$. q.e.d.

Therefore, for any G_x-invariant subset S of $\mathscr{S}^i$, to check that S is irreducible, Zariski locally closed (resp. Zariski closed) of codimension $\geq r$ in $\mathscr{S}_x$, it suffices to check, for some $q \in \mathscr{Q}^i$, that $S \cap \tau^{-1}(q)$ is irreducible, Zariski locally closed (resp. Zariski closed) of codimension $\geq r_i$ in $\mathscr{S}_x \cap \tau^{-1}(q) = \{$surfaces of degree d in $\mathbf{P}^3$ with a double point at x and tangent cone $[q]$ there$\}$. Here, $r_0 = r$ since $\operatorname{codim}_{\mathscr{Q}}(\mathscr{Q}^3) = 0$, $r_1 = r - 1$ since $\operatorname{codim}_{\mathscr{Q}}(\mathscr{Q}^1) = 1$, and $r_2 = r - 3$ since $\operatorname{codim}_{\mathscr{Q}}(\mathscr{Q}^2) = 3$.

The proof of the following result (using Bertini's theorem) is left to the reader.

(8.13) Lemma. *Let S be a surface of degree d in a hyperplane H of $\mathbf{P}^4$ with only isolated singularities. Then there exists a nonsingular hypersurface M of degree d in $\mathbf{P}^4$ with hyperplane section $M \cap H = S$. Therefore, if we let $\mathbf{P}(d,S) = \{$hypersurfaces M of degree d with prescribed hyperplane section $M \cap H = S\}$ and $\mathscr{M}_{d,S} = \{$nonsingular M in $\mathbf{P}(d,S)\}$, then $\mathscr{M}_{d,S}$ is a nonempty, Zariski-open subset of $\mathbf{P}(d,S)$.*

Consequently, if we prove the irreducibility of a Zariski locally closed subset of $\mathbf{P}(d,S)$, then the intersection with $\mathscr{M}_{d,S}$, being open in the irreducible subset, is irreducible. Similarly, if we prove a subset of $\{$surfaces in H singular at $x\}$ is Zariski locally closed and irreducible, then its preimage in $\mathscr{C}_{x,H}$ is Zariski locally closed and irreducible.

To prove (8.8) we need to be able to recognize singularities of hyperplane sections $f_0(x,y,z) = 0$ of M of types A_1, A_2, A_3, A_4, and D_4. We start with the characterizations of these singularities by their normal forms (cf. §7):

(i) A point p of a surface S is an A_n-singularity if and only if the germ (S,p) is analytically isomorphic to the germ of $x^2 + y^2 + z^{n+1} = 0$ at 0 in $\mathbf{C}^3$.

(ii) A point p of a surface S is a D_4-singularity if and only if the germ (S,p) is analytically isomorphic to the germ of $x^2+y(y^2+z^2)=0$ at 0 in $\mathbb{C}^3$. (In general, for D_n, $n \geq 4$, use $x^2+y(y^{n-2}+z^2)$.)

In addition to the quadratic tangent cone, there are two tools we will use to analyze these surface singularities: (a) their blow-up properties, and (b) the structure of the space T^1 of first order deformations and its intrinsic filtration.

The following characterization of A_n and D_4 by blowing up is a special case of D. Kirby's analysis [21] of isolated double points of surfaces in 3-space.

(8.14) Proposition. *Let (S,p) be the germ of a surface singularity in $\mathbb{C}^3$.*

(i) *For $n \geq 3$, (S,p) is an A_n-singularity $\Leftrightarrow$ (S,p) is a rank* 2 *double point and the blow-up $\tilde{S}$ has along the exceptional curve an A_{n-2}-singularity and no other singularities.* (*The blow-ups of A_1 and A_2 are nonsingular.*)

(ii) *(S,p) is a D_4-singularity $\Leftrightarrow$ (S,p) is a rank* 1 *double point and the blow-up $\tilde{S}$ has along the exceptional curve three A_1-singularities and no other singularities.*

Remark. Kirby characterized the "DuVal-Arnold singularities" $\{A_n, n=1,2,\cdots; D_n, n=4,5,\cdots; E_6, E_7, E_8\}$ as the absolutely isolated double points of surfaces in $\mathbb{C}^3$. (Absolutely isolated means that the singularity can be resolved by finitely many point blow-ups, i.e., each blowing-up is only allowed to have a single, reduced point as center.) It follows that a germ (S,p) of a surface in $\mathbb{C}^3$ at a double point is one of $\{A_n, D_n, E_6, E_7, E_8\}$ if and only if the blow-up $\tilde{S}$ has only such singularities along the exceptional set E.

(8.15) Proposition. *Let (S,p) be a germ of surface singularity in $\mathbb{C}^3$.*

(i) *(S,p) is an A_n-singularity, $n \geq 2$ $\Leftrightarrow$ it is a rank* 2 *double point and* $\dim(T^1_{(S,p)}) = n$.

(ii) *(S,p) is a D_4-singularity $\Leftrightarrow$ it is a rank* 1 *double point and* $\dim(T^1_{(S,p)}) = 4$.

Proof. (i) $\Rightarrow$: Assume A_n, $n \geq 2$. Then we can use the analytic normal form $x^2+y^2+z^{n+1}$. Now, since $n \geq 2$, the quadratic form is x^2+y^2, which has rank 2, and $T^1 \cong \mathbb{C}[[x,y,z]]/(x,y,z^n) \cong \mathbb{C}[[z]]/(z^n)$, which has dimension n.

$\Leftarrow$: Assume (S,p) is a rank 2 double point and $\dim T^1 = n$. Then (S,p) is analytically equivalent to $x^2+y^2+z^{k+1}$ for some integer $k \geq 2$ (cf. [24, 7.16, p. 127] or [21, Theorem 4i]). But now, $k = \dim T^1 = n$ so (S,p) is an A_n-singularity.

(ii) $\Rightarrow$: Trivial.

$\Leftarrow$: Assume (S,p) is a rank 1 double point. Then the equation for (S,p) is analytically equivalent to $x^2+g(y,z)$ for an analytic $g=\varphi+\cdots$, where φ is a homogeneous cubic polynomial. Now

$$T^1=\mathbb{C}[[x,y,z]]/(x^2+g,x,g_y,g_z)\cong\mathbb{C}[[y,z]]/(g,g_y,g_z),$$

and (S,p) has a D_4-singularity $\Leftrightarrow$ $\varphi(y,z)$ has three simple roots in $\mathbb{P}^1$ (cf. [21, §2.6]). Next, φ has three simple roots in $\mathbb{P}^1$ $\Leftrightarrow$ φ_y and φ_z have no common zeros in $\mathbb{P}^1$ $\Leftrightarrow$ $y\varphi_y$, $z\varphi_y$, $y\varphi_z$, $z\varphi_z$ span the cubics in y and z. Now we see that our second hypothesis $\dim T^1=4$ implies D_4. Indeed, if φ did not have three simple roots then $\dim T^1$ would be at least 5 because $1,y,z$, a quadratic, and then a cubic not in the span of $y\varphi_y$, $z\varphi_y$, $y\varphi_z$, $z\varphi_z$ would be linearly independent in T^1. To see this, note that $(g, g_y, g_z$, all quartics$)=(\varphi_y,\varphi_z$, all quartics$)$ since $g\equiv\varphi$ (mod the ideal generated by all quartics) and $\varphi=(1/3)(y\varphi_y+z\varphi_z)$.

(8.16) Corollary. *Let (S,p) be a germ of surface singularity in $\mathbb{C}^3$. Then (S,p) is one of the five types A_1, A_2, A_3, A_4, D_4 $\Leftrightarrow$ $\dim(T^1_{(S,p)})\le 4$.*

For an isolated singularity $(X,0)$, T^1 together with its structure as an $\mathcal{O}_{X,0}$-module is a complex analytic invariant; indeed, for a hypersurface this is just the invariance of the Jacobian ideal J generated by the defining equation and its partials. In particular, we have an intrinsic filtration of T^1 defined by the powers of the maximal ideal $m=m_{X,0}$ of $\mathcal{O}_{X,0}$:

$$T^1\supset m\cdot T^1\supset m^2\cdot T^1\supset\cdots.$$

Note that always (if $(X,0)$ is singular), $\dim(T^1/m\cdot T^1)=1$ and an element $f_1(\text{mod }J)\in T^1$ induces a basis for $T^1/m\cdot T^1$ $\Leftrightarrow$ $f_1(0,0,0)\neq 0$. Also, by Nakayama's lemma, $m^{k+1}\cdot T^1$ is properly contained in $m^k\cdot T^1$ unless $m^k\cdot T^1=0$.

(8.17) Lemma. *For any normal form x^2+y^2+ (higher) of an equation for A_n, $z^k(\text{mod }J)\in m^k\cdot T^1$ induces a basis for the (1-dimensional) vector space $m^k\cdot T^1/m^{k+1}\cdot T^1$ for $k=0,\cdots,n-1$. Hence $\{1,z,\cdots,z^{n-1}\}$ induces a $\mathbb{C}$-basis for T^1.*

Proof. Since $\dim T^1=n$, it suffices to show that, for all k, z^k spans $G_k=m^k\cdot T^1/m^{k+1}\cdot T^1$; it then follows, by using the filtration, that $\{1,z,\cdots,z^{n-1}\}$ spans T^1 as a $\mathbb{C}$-vector space, hence forms a basis. Assume, by induction, that z^k spans G_k (this is true for $k=0$). Then $\{xz^k,yz^k,z^{k+1}\}$ certainly spans $G_{k+1}=m^{k+1}\cdot T^1/m^{k+2}\cdot T^1$. But, by the form of the equation, x and y are in the ideal (J,m^2), so xz^k and yz^k are 0 in G_{k+1} and z^{k+1} spans. q.e.d.

We postpone a precise look at T^1 for a D_4-singularity until later. Here we simply note that for any normal form x^2+ (higher in x,y,z) of an equation for D_4, $1 \pmod J \in T^1$ induces a basis for $T^1/m \cdot T^1$, $y, z \pmod J \in m \cdot T^1$ induce a basis for $m \cdot T^1/m^2 \cdot T^1$, and last, $m^2 \cdot T^1/m^3 \cdot T^1$ is 1-dimensional.

Proof of (8.8). By (8.11), we fix the point x and the hyperplane H, and look at $\mathscr{C}_{x,H}$ (so the hyperplane H is tangent to M at x) in affine coordinates. To simplify notation, we write $\mathscr{C}$, $\mathscr{C}(A_1)$, etc. for $\mathscr{C}_{x,H}$, $\mathscr{C}(A_1)_{x,H}$, etc. We consider all the affine equations f_0 with a singularity at the origin, so

$$f_0(x,y,z) = q(x,y,z) + c(x,y,z) + d(x,y,z) + \cdots,$$

where q is quadratic, c is cubic, d is quartic, etc.

Now we will prove (8.8)(0)–(5) in turn.

(0) This is well known. Indeed, after our normalization, we have a projective space of polynomials $f = f_0 + uf_1 + \cdots$ of dimension $= \dim(\mathscr{M}_d) - 4$ (since the constant and linear terms of f_0 are zero), and $\dim(\mathscr{F}) = \dim(\mathbf{P}^4 \times \mathbf{P}^{4*}) - 1$.

(1) We have an A_1-singularity if and only if $q(x,y,z)$ is a nondegenerate quadratic form if and only if the discriminant of q is nonzero. Therefore $\mathscr{C}(A_1)$ is open in $\mathscr{C}$.

To see $\mathscr{C}(A_1) = V(A_1)$, we apply (8.2) and note that since $f_1(0,0,0) \neq 0$, $f_1 \pmod J \in T^1$ is nonzero and therefore spans T^1, which is a 1-dimensional vector space in this case.

(2) Now consider an A_2-singularity. We want to show $\mathscr{C}(A_2)$ is irreducible, locally closed, of codimension 1. We know by (8.15)(i) that A_2 is characterized by q's being a rank 2 quadratic form and $\dim T^1 = 2$, so by (8.6) we have that $\mathscr{C}(A_2)$ is locally closed in $\mathscr{C}$. Note that $\mathscr{C} - \mathscr{C}(A_1)$ has pure codimension 1 in $\mathscr{C}$ since it is defined by one equation (the discriminant). But $\mathscr{C}(A_2)$ is open in $\mathscr{C} - \mathscr{C}(A_1)$ as it is the complement of $\{\operatorname{rank}(q) \leq 1 \text{ or } \dim T^1 \geq 3\}$. This proves $\mathscr{C}(A_2)$ has codimension 1 in $\mathscr{C}$.

Now we prove the irreducibility of $\mathscr{C}(A_2)$. The set of rank 2 quadratic forms is Zariski locally closed in the vector space of all quadratic forms in x, y, z. The set is irreducible since it is an orbit under the action of the connected algebraic group GL(3). Hence {surfaces in H, singular with a rank 2 quadratic form at p} is Zariski locally closed and irreducible and {surfaces in H with an A_2-singularity at p} is an open subset; therefore it is also Zariski locally closed and irreducible, and so is $\mathscr{C}(A_2)$.

By (8.12) we may normalize the equation projectively to the form $f_0 = x^2 + y^2 +$ higher. To see $\mathscr{C}(A_2) = V(A_2)$, we only have to show that $zf_1 \pmod J$, which lies in $m \cdot T^1$, is not in $m^2 \cdot T^1 = 0$. But $f_1 = \lambda +$ (linear $+ \cdots$), where λ is a nonzero constant, so $zf_1 \equiv \lambda z \pmod J$ since

$z \cdot (\text{linear} + \cdots) \in m^2 \cdot T^1 = 0$. Thus we only need to check that $z \pmod J$ is not in $m^2 \cdot T^1$; this is given by Lemma (8.17).

(3) For A_3, the closed conditions are that the quadratic form have rank 2, hence 1-dimensional kernel, and that the cubic vanish on this kernel. This is one linear condition on the cubic. The A_3 locus is open in this Zariski locally closed, irreducible set.

Next we prove that $\operatorname{codim}_{\mathscr{C}(A_3)} B(A_3) \geq 2$. To show this we may normalize the equation f_0 for our A_3-singularity by *linear* transformation in $\mathbb{C}^3$ to the form $x^2 + y^2 + c(x, y, z) + \cdots$, where $c(0, 0, 1) = 0$. Then the Jacobian ideal J is $(x^2 + y^2 + c(x, y, z) + \cdots, 2x + c_x + \cdots, 2y + c_y + \cdots, c_z + \cdots)$.

(8.18) Lemma. *$\{f_1, xf_1, yf_1, zf_1\}$ fails to span T^1 if and only if the coefficients of xz^2 and yz^2 in c are both* 0.

Proof. First recall from (8.17) that f_1 spans $T^1/m \cdot T^1$, zf_1 spans $m \cdot T^1/m^2 \cdot T^1$, $\dim_{\mathbb{C}} T^1 = 3$, and $m^3 \cdot T^1 = 0$. Now note that modulo J, $x \equiv -\frac{1}{2}(c_x + \cdots) \in m^2$ and $y \equiv -\frac{1}{2}(c_y + \cdots) \in m^2$. Thus, in T^1, xf_1 and yf_1 lie in $m^2 \cdot T^1$ and equal λx and λy respectively where $f_1 = \lambda + (\text{linear} + \cdots)$. It follows that f_1, xf_1, yf_1, zf_1 do not span T^1 if and only if x and y are both 0 in T^1, i.e., if and only if $(x, y) \subset J$, or equivalently, $(x, y, z^3) = J$ in $\mathbb{C}[[x, y, z]]$. Next, $(x, y, z^3) = J$ if and only if the image of J is 0 in $\mathbb{C}[x, y, z]/(x, y, z^3) \cong \mathbb{C}[z]/(z^3)$. The image of J is $\overline{J} = (c_x, c_y, c_z) \subset (z^2)/(z^3)$; now the z^2 term in c_x comes from the xz^2 term in c, in c_y from the yz^2 term, and c_z has no z^2 term since c has no z^3 term. q.e.d.

Therefore, by (8.2), after our normalizations, $B(A_3)$ is defined in $\mathscr{C}(A_3)$ by two independent linear conditions.

(4) For the analysis of A_4 we use (8.14)(i). Suppose we blow up our surface singularity S: $f_0(x, y, z) = 0$ in $\mathbb{C}^3$ at the origin. The exceptional curve (as a scheme) in the blow-up $\tilde{S}$ is the projectivized tangent cone Q to S at 0. Since the exceptional curve is a Cartier divisor in the surface $\tilde{S}$, at points where Q is nonsingular $\tilde{S}$ must be nonsingular. Assume $(S, 0)$ is an A_n-singularity, $n \geq 2$, and that the quadratic part $q(x, y, z)$ of f_0 is $x^2 + y^2$ (or xy). Then to analyze the singularities of $\tilde{S}$ it is only necessary to look at a single point of $\tilde{S}$, namely the vertex $(0, 0, 1)$ of the degenerate plane conic Q: $q = 0$ in $\mathbb{P}^2$.

The blown-up surface $\tilde{S}$ can be realized as the proper transform of S in the blow-up $\tilde{\mathbb{C}}^3$ of $\mathbb{C}^3$ at 0. Now $\tilde{\mathbb{C}}^3$ is covered by U, V, W, three affine open subsets isomorphic to $\mathbb{C}^3$. On the coordinate patch W the blowing-up σ is given by $(u, v, w) \to (uw, vw, w) = (x, y, z)$ and the (possibly) singular point of $\tilde{S}$ is at the origin. Now pulling back f_0 under σ we get

$$\begin{aligned} f_0(uw, vw, w) &= q(uw, vw, w) + c(uw, vw, w) + d(uw, vw, w) + \cdots \\ &= w^2[q(u, v, 1) + wc(u, v, 1) + w^2 d(u, v, 1) + \cdots]. \end{aligned}$$

Let us write out the cubic part of f_0,

$$c(x, y, z) = (\alpha x + \beta y + \xi z)z^2 + (\text{other terms, not divisible by } z^2),$$

and the quartic part of f_0,

$$d(x, y, z) = \gamma z^4 + (\text{terms involving an } x \text{ or } y).$$

Then the equation in W for the proper transform is

$$\tilde{f}_0(u, v, w) = \xi w + [u^2 + v^2 + w(\alpha u + \beta v + \gamma w)] + (\text{cubic terms}) + \cdots .$$

The (u, v, w)-origin is nonsingular on $\tilde{S} \Leftrightarrow \xi \neq 0$; hence by (8.14)(i), this A_n-singularity, $n \geq 2$, is $A_2 \Leftrightarrow \xi \neq 0$. Now assume $\xi = 0$, i.e., that we have an A_n-singularity, $n \geq 3$. Then the singularity is $A_3 \Leftrightarrow$ the quadratic form $\tilde{q}(u, v, w) = u^2 + v^2 + w(\alpha u + \beta v + \gamma w)$ is nondegenerate. We find that $\tilde{q}$ is degenerate, i.e., has rank $= 2 \Leftrightarrow \gamma - [(\frac{1}{2}\alpha)^2 + (\frac{1}{2}\beta)^2] = 0 \Leftrightarrow \alpha^2 + \beta^2 = 4\gamma$. (If $q(x, y, z) = xy$ then $\tilde{q}$ is degenerate $\Leftrightarrow \alpha\beta = \gamma$.)

To summarize:

(8.19) The equation

$$f_0(x, y, z) = x^2 + y^2 + [(\alpha x + \beta y)z^2 + \cdots] + [\gamma z^4 + \cdots] + \cdots$$

defines an A_4-singularity $\Leftrightarrow \alpha^2 + \beta^2 = 4\gamma$ and an open condition on the 5-jet of f_0 holds. Namely, $\tilde{f}_0$ has an A_2-singularity; i.e., the cubic term $\tilde{c}$ of $\tilde{f}_0$ does not vanish on the kernel of the rank 2 quadratic term $\tilde{q}$.

Let us see now that $\mathscr{C}(A_4)$ is an irreducible, codimension 3, Zariski locally closed subset of $\mathscr{C}$. In $\mathscr{C}$ we have the irreducible, codimension 1, Zariski locally closed subset R defined by $\operatorname{rank}(q) = 2$. In R we have an irreducible, codimension 1, Zariski closed subset S defined by the condition that the cubic $c(x, y, z)$ vanish on the kernel of the quadratic $q(x, y, z)$; the complement of S in R is $\mathscr{C}(A_2)$. Now define a subset T of S by the condition (in normalized coordinates): $\alpha^2 + \beta^2 = 4\gamma$. When the degree $d \geq 4$, we can view α and β as free parameters and γ as a polynomial function of them; so we obtain irreducibility of T. If the degree $d = 3$, then $\gamma = 0$ so we have $\alpha^2 + \beta^2 = 0$. This locus (with Q fixed) has two irreducible components; however, the components $\beta = \pm i\alpha$ can be interchanged by monodromy.

Next we will give explicitly the Zariski open condition for $\mathscr{C}(A_4) \cap T$ in T. Write:

$$\begin{aligned} c(x,y,z) &= (\alpha x + \beta y)z^2 + (\lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2)z + \cdots, \\ d(x,y,z) &= \gamma z^4 + (\mu_1 x + \mu_2 y)z^3 + \cdots, \\ e(x,y,z) &= \nu z^5 + \cdots . \end{aligned}$$

Then $\tilde{c}(u,v,w) = w(\lambda_1 u^2 + \lambda_2 uv + \lambda_3 v^2) + w^2(\mu_1 u + \mu_2 v) + w^3 \nu$. The kernel of $\tilde{q}$ is spanned by the vector $(\alpha, \beta, -2)$ so the final condition for exactly an A_4-singularity is:

$$2(\mu_1 \alpha + \mu_2 \beta) - (\lambda_1 \alpha^2 + \lambda_2 \alpha\beta + \lambda_3 \beta^2) \neq 4\nu.$$

To see whether the A_4-singularity is versally deformed we compute the span of $\{f_1, xf_1, yf_1, zf_1\}$ in terms of the basis for T^1 induced by the powers of z. The question is whether xf_1 and yf_1 span $m^2 \cdot T^1$. For this we first express x and y (as operators on T^1) as linear combinations of z^2 and z^3. The results of the calculations are as follows:

(o) Modulo J, $x \equiv -\frac{1}{2}(c_x + d_x)$ and $y \equiv -\frac{1}{2}(c_y + d_y)$ since $m^4 \subset J$ for A_4.

(i) Modulo (J, m^3), $x \equiv -\frac{1}{2}\alpha z^2$ and $y \equiv -\frac{1}{2}\beta z^2$.

(ii) Note that x^2, xy, y^2 and all the cubic monomials except z^3 lie in J.

(iii) Modulo J,

$$\begin{aligned} x &\equiv -\tfrac{1}{2}\alpha z^2 - \tfrac{1}{2}(\mu_1 - \lambda_1 \alpha - \tfrac{1}{2}\lambda_2 \beta)z^3, \\ y &\equiv -\tfrac{1}{2}\beta z^2 - \tfrac{1}{2}(\mu_2 - \tfrac{1}{2}\lambda_2 \alpha - \lambda_3 \beta)z^3. \end{aligned}$$

(iv) If $f_1(x,y,z) = 1 + Ax + By + Cz + \cdots$, then modulo J,

$$\begin{aligned} xf_1 &\equiv x + Cxz \equiv -\tfrac{1}{2}\alpha z^2 - \tfrac{1}{2}(\mu_1 - (\lambda_1 - C)\alpha - \tfrac{1}{2}\lambda_2 \beta)z^3, \\ yf_1 &\equiv -\tfrac{1}{2}\beta z^2 - \tfrac{1}{2}(\mu_2 - \tfrac{1}{2}\lambda_2 \alpha - (\lambda_3 - C)\beta)z^3. \end{aligned}$$

(v) Thus, xf_1 and yf_1 are given by the vectors $-\frac{1}{2}(\alpha, \mu_1 - (\lambda_1 - C)\alpha - \frac{1}{2}\lambda_2\beta)$ and $-\frac{1}{2}(\beta, \mu_2 - \frac{1}{2}\lambda_2\alpha - (\lambda_3 - C)\beta)$, whose linear dependence is expressed by the 2×2 determinant $\mu_2\alpha - \mu_1\beta - \frac{1}{2}\lambda_2\alpha^2 + (\lambda_1 - \lambda_3)\alpha\beta + \frac{1}{2}\lambda_2\beta^2$. (It is interesting that the C-terms cancel out in the determinant; hence the versality of the deformation of the A_4-singularity depends only on the hyperplane section and not on the nonsingular hypersurface M of $\mathbb{P}^4$ from which the given section arises.) From this, the existence of A_4-singularities which are versally deformed on a nonsingular hypersurface of degree $d \geq 3$ in $\mathbb{P}^4$ certainly follows.

In any event, it is easy to check directly that $f_0 = xy+xz^2+y^2z+(*) = 0$, where $(*)$ is any sufficiently small choice of higher order terms such that the coefficient of z^4 is zero, is an A_4-singularity at $(0,0,0)$ and is versally deformed on the hypersurface $f_0 + u \cdot 1 + \cdots$.

Here is the analysis of D_4. Suppose $(S,0)$ has a rank 1 double point, with equation

$$f_0(x,y,z) = x^2 + c(x,y,z) + d(x,y,z) + \cdots .$$

To give the condition for the point to be exactly a D_4-singularity, we blow up. On an (r,s,t) coordinate patch V, $\sigma(r,s,t) = (rs,s,st)$, so the equation in V for the proper transform is

$$\tilde{f}_0(r,s,t) = r^2 + sc(r,1,t) + s^2 d(r,1,t) + \cdots .$$

The singularities along the exceptional curve $s = 0$ occur at $(0,0,t)$, where $c(0,1,t) = 0$. Next we make the same calculation on the (u,v,w) patch W, on which $\sigma(u,v,w) = (uw,vw,w)$; since the tangent cone to S at 0 is $x = 0$ (counted twice), the proper transform of S in $\tilde{\mathbb{C}}^3$ is already contained in the union of V and W. On W the proper transform is

$$\tilde{f}_0(u,v,w) = u^2 + wc(u,v,1) + w^2 d(u,v,1) + \cdots .$$

The singularities along $w = 0$ occur at $(0,v,0)$, where $c(0,v,1) = 0$.

Thus, using the two coordinate patches we have located all the singular points of the blow-up $\tilde{S}$ but we need to relate the two patches. On $V \cap W$ (given in V by $t \neq 0$, or in W by $v \neq 0$), from $(rs,s,st) = (uw,vw,w)$, we have $r = u/v$, $s = vw$, $t = 1/v$. If $E \cong \mathbb{P}^1$ denotes the reduced exceptional curve in $\tilde{S}$, we know $E \cap V$ is $\{(0,0,t) | t \in \mathbb{C}\}$ and $E \cap W$ is $\{(0,v,0) | v \in \mathbb{C}\}$ with the identification $t = 1/v$ in $V \cap W$. We conclude that the singular points of $\tilde{S}$ are the points of $E \cong \mathbb{P}^1$ at which the cubic form $\varphi = c(0,y,z)$ vanishes; there are three such points, counting multiplicity.

Let $\tilde{p} \in E$ be a point at which $\varphi = c(0,y,z)$ vanishes. A computation shows that $(\tilde{S},\tilde{p})$ is an ordinary double point $(A_1) \Leftrightarrow \tilde{p}$ is a simple root of φ. Therefore, by (8.14)(ii) we have:

(8.20) (S,p) is a D_4-singularity $\Leftrightarrow$ φ has three simple roots.

Now we look at the condition for a D_4-singularity of a hyperplane section $f_0 = 0$ to be versally deformed on the hypersurface $f_0 + uf_1 + \cdots$, i.e., by (8.2), the condition for f_1, xf_1, yf_1, zf_1 to span $T^1 = T^1_{f_0}$. First we note, since $f_1(0,0,0) \neq 0$, that f_1 induces a basis for $T^1/m \cdot T^1$ and yf_1, zf_1 induce a basis for $m \cdot T^1/m^2 \cdot T^1$. Since $x \equiv -\frac{1}{2}c_x \bmod J$ $(\supset m^3)$, $xf_1 \pmod J$ lies in $m^2 \cdot T^1$ so f_1, xf_1, yf_1, zf_1 span $T^1 \Leftrightarrow xf_1$ is not zero in $T^1 \Leftrightarrow x$ is not zero in $T^1 \Leftrightarrow c_x \notin J$. Now, since c_x is a homogeneous quadratic, we look at the "quadratic part" of J, $(J \cap m^2)/m^3$ (in general the "quadratic part" of J would be $([J+m^3]\cap m^2)/m^3$, but here $J \supset m^3$). Clearly, the quadratic part of J is the linear span of c_y, c_z, x^2, xy, xz so $c_x \in J \Leftrightarrow c_x$ is a linear combination of c_y, c_z, x^2, xy, $xz \Leftrightarrow \overline{c_x}$ is a linear combination of $\varphi_y = \overline{c_y}$ and $\varphi_z = \overline{c_z}$, where the bars mean reduction modulo x. To give the condition in terms of the coefficients of f_0, write out c as a linear combination of the cubic monomials in x, y, z and hence φ_y,φ_z, and $\overline{c_x}$ as linear combinations of the basis y^2, yz, z^2 for quadratic forms in y, z. Now φ_y and φ_z are certainly linearly independent (they have no common roots since $\varphi(y,z)$ has simple roots) so the condition that $\overline{c_x}$ be a linear combination of φ_y and φ_z is that the corresponding 3×3 determinant vanish. If

$$c(x,y,z) = \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3 + Axy^2 + Bxyz + Cxz^2 + (\text{lin. comb. of } x^3, x^2y, x^2z),$$

the determinant is

$$(6\beta\delta - 2\gamma^2)A - (9\alpha\delta - \beta\gamma)B + (6\alpha\gamma - 2\beta^2)C.$$

In particular, the D_4-singularity

$$f_0 = x^2 + (y^3 + z^3 + xyz) + (\text{any choice of higher order terms})$$

is versally deformed on any nonsingular hypersurface having it as a hyperplane section.

(5) Recall first that $W = \{\dim T_x^1 \geq 5\}$ by (8.16), and then that the latter set is Zariski closed by (8.6). It remains to show $\operatorname{codim}_{\mathscr{C}} W \geq 4$. To do this, it suffices to check that the codimension in $\mathscr{C}$ is ≥ 4 after intersecting W with each of the three Zariski locally closed sets: $\mathscr{C}^1 = \{\operatorname{rank}(q) = 2\}$, $\mathscr{C}^2 = \{\operatorname{rank}(q) = 1\}$, and $\mathscr{C}^3 = \{\operatorname{rank}(q) = 0\}$. We have seen that $T = \{\operatorname{rank}(q) = 2 \text{ and not } A_2 \text{ or } A_3\}$ is an irreducible, Zariski locally closed subset of codimension 3 in $\mathscr{C}$, and that $\mathscr{C}(A_4)$ is nonempty and Zariski open in this set. Therefore, $T - \mathscr{C}(A_4)$, which is $W \cap \mathscr{C}^1$, has codimension ≥ 4 in $\mathscr{C}$. Next, since we know that $\mathscr{C}^2$ is irreducible, of codimension 3 in $\mathscr{C}$ and $\mathscr{C}(D_4)$ is a nonempty, Zariski-open subset, $\mathscr{C}^2 - \mathscr{C}(D_4) = W \cap \mathscr{C}^2$ has codimension ≥ 4 in $\mathscr{C}$. Finally, $\mathscr{C}^3 = \mathscr{C}^3 \cap W$ has codimension 6 in $\mathscr{C}$.

This completes the proof of Theorem (8.8). q.e.d.

Notice the following consequence of the irreducibility of $\mathscr{C}(A_4)$ and $\mathscr{C}(D_4)$.

(8.21) Proposition. *Let $Z \subset V(A_4)$ (or $V(D_4)$) be a proper Zariski closed subset. Then $\{M|(x,\gamma(x),M) \notin Z$ for all $x \in M\}$ is a nonempty Zariski-open subset of $\mathcal{G}$, the set of Gauss-stable hypersurfaces of $\mathcal{M}_d$.*

Proof. Since $Z \neq V(A_4)$ and $V(A_4)$ is irreducible, $\dim Z < \dim V(A_4) = \dim \mathcal{G}$. The restriction of $p\colon V \to \mathcal{G}$ to $V(A_4)$ is proper. Hence the image of Z in $\mathcal{G}$ is a Zariski-closed subset of positive codimension. The latter case is obtained by substituting D_4 for A_4. q.e.d.

With regard to the actual occurrence of the five expected types of singularities on the hypersurfaces in $\mathcal{G}$, note that the various (good) loci we have in $\mathcal{C}$ have codimension < 4 so the images in $\mathbb{P}^{4*} \times \mathcal{G}$ have codimension < 5 and also have the expected fiber dimensions over $\mathcal{G}$. Hence the images in $\mathcal{G}$ have the expected dimensions, i.e., are all of $\mathcal{G}$.

9. Genericity and jet transversality

In this section we give a second proof of Theorem (8.1).

The smooth hypersurface $M \subset \mathbb{P}^4$ is Gauss-stable if and only if $J^4p\colon \Gamma \to J^4(\Gamma, \mathbb{P}^{4*})$ is transverse to $\Sigma(\Gamma, \mathbb{P}^{4*})$ for all orbits $\Sigma \subset J^4(6,4)$ (cf. (6.3)). Since the orbits of $J^4(6,4)$ of codimension ≤ 6 are Thom-Boardman loci (6.4), it suffices to prove that the set of hypersurfaces such that J^4p is transverse to the Thom-Boardman loci is a nonempty Zariski-open subset of $\mathcal{M}_d$.

We will use two properties of the Thom-Boardman sets Σ^I. First, each Σ^I is invariant under unfolding (i.e., it is "u-stable" in Ronga's terminology [31]). Second, the Thom-Boardman sets of codimension ≤ 6 in $J^4(6,4)$ form an algebraic Whitney stratification (cf. (6.4)).

Recall that an r-parameter *unfolding* of a map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ is a germ $F\colon (\mathbb{C}^n \times \mathbb{C}^r,0) \to (\mathbb{C}^p \times \mathbb{C}^r,0)$ such that $F(x,t) = (\tilde{f}(x,t),t)$ and $\tilde{f}(x,0) = f(x)$ for all x,t. (Thus $\tilde{f}$ is an r-parameter *deformation* of f.) Given integers n_0, c, k, with $n_0 > 0$ and $k > 0$, let $\Sigma = \{\Sigma(n,p)\}$, $n \geq n_0$, $p = n + c$, be a sequence of singularity types, with $\Sigma(n,p) \subset J^k(n,p)$ for all n,p. ($\Sigma(n,p)$ is a *singularity type* if $\Sigma(n,p)$ is invariant under analytic changes of coordinates in source and target.) We say that the sequence Σ has *order* k and *degree* c.

Definition. The sequence Σ is *unfolding invariant* if for every map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, $n \geq n_0$, $p = n + c$, for every $r \geq 1$ and every r-parameter unfolding F of f, the following two conditions hold:

(1) $J^kf(0) \in \Sigma(n,p) \Leftrightarrow J^kF(0) \in \Sigma(n+r,p+r)$,

(2) $J^kf \pitchfork \Sigma(n,p) \Leftrightarrow (J^kF)|(\mathbb{C}^n \times 0) \pitchfork \Sigma(n+r,p+r)$.

(9.1) Lemma [31]. *The Thom-Boardman sequence* $\Sigma_c^I = \{\Sigma^I(n,p) \mid p - n = c\}$ *is unfolding invariant.*

Proof. This follows from [6, Theorem 7.15, p. 56], since Boardman's intrinsic derivative descriptions of Σ^I and the transversality condition (ι_s) are unfolding invariant. q.e.d.

By an analytic *family* $F_t\colon X_t \to Y_t$ of maps, we mean an analytic map $F\colon X \to Y$ of manifolds, together with proper analytic submersions $p_X\colon X \to T$, $p_Y\colon Y \to T$ with $p_X = p_Y \circ F$:

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & T & \end{tikzcd}$$

For $t \in T$, $F_t\colon X_t \to Y_t$ is the restriction of F to the fibers over t.

The following is an immediate consequence of the definition of an unfolding invariant sequence.

(9.2) Lemma [31, 2.1]. *Let* $\Sigma = \{\Sigma(n,p)\}$, $n \geq n_0$ *be an unfolding invariant sequence of singularity types of order* k *and degree* c, *and let* $F_t\colon X_t \to Y_t$ *be an analytic family of maps, with* $\dim Y_t - \dim X_t = c$ *and* $\dim X_t \geq n_0$.

(1) *If* $J^kF_t \pitchfork \Sigma(X_t, Y_t)$ *at* $x \in X_t$, *then* $J^kF \pitchfork (X,Y)$ *at* x.

(2) *If* $J^kF \pitchfork \Sigma(X,Y)$ *at* $x \in X_t$, *then* $J^kF_t \pitchfork \Sigma(X_t,Y_t)$ *at* x *if and only if* $X_t \pitchfork (J^kF)^{-1}\Sigma(X,Y)$ *at* x.

An *algebraic stratification sequence* $\mathcal{S} = \{\mathcal{S}(n,p)\}$ of degree c and order k through codimension s consists of the following. For each $n \geq n_0$, $p = n + c$, $\mathcal{S}(n,p)$ is an algebraic Whitney stratification of $U(n,p) - \Omega(n,p)$, where $U(n,p)$ is a singularity type that is a Zariski-open subset of $J^k(n,p)$ and $\Omega(n,p)$ is a singularity type that is a closed algebraic subset of $U(n,p)$ of codimension $> s$. (An algebraic Whitney stratification is a decomposition for which each stratum is a Zariski locally closed set, and each pair of strata is Whitney regular and satisfies the frontier condition [13].) Furthermore, we require that $\{U(n,p)\}$ and $\{\Omega(n,p)\}$ be unfolding invariant sequences. Finally, we require that there be unfolding invariant sequences of singularity types $\Sigma^I = \{\Sigma^I(n,p) \mid n \geq n_0,\ p = n + c\}$, $I \in \mathcal{S}$, such that $\mathcal{S}(n,p) = \{\Sigma^I(n,p) \mid I \in \mathcal{S}\}$.

If $\mathcal{S}$ is an algebraic stratification sequence of order k and degree c, and X^n, Y^p, $n \geq n_0$, $p = n + c$ are smooth algebraic varieties, let $\mathcal{S}(X,Y)$ denote the stratification of $J^k(X,Y)$ corresponding to $\mathcal{S}(n,p)$, and let $U(X,Y)$, $\Omega(X,Y)$, $\Sigma^I(X,Y)$ correspond to $U(n,p)$, $\Omega(n,p)$, $\Sigma^I(n,p)$, respectively. For $F\colon X \to Y$, we say $JF \pitchfork \mathcal{S}$ if $J^kF(x) \in U(X,Y)$ for all $x \in X$, $(J^kF)^{-1}\Omega(n,p)$ has codimension $> s$ in X, and $J^kF \pitchfork \Sigma^I(X,Y)$ for each stratum $\Sigma^I(X,Y)$ of $\mathcal{S}(X,Y)$.

(9.3) Lemma. *Let $\mathscr{S} = \{\mathscr{S}(n,p)\}$, $n \geq n_0$, $p = n + c$, be an algebraic stratification sequence of degree c and order k, through codimension s. Let $F_t\colon X_t \to Y_t$ be an algebraic family of maps, with* $\dim X_t \geq n_0$, *and* $\dim Y_t = \dim X_t + c$.

(1) *If $JF_t \pitchfork \mathscr{S}$ for all $t \in T$, then $JF \pitchfork \mathscr{S}$.*

(2) *If $JF \pitchfork \mathscr{S}$, then $\{t | JF_t \pitchfork \mathscr{S}\}$ is a nonempty Zariski-open subset of T.*

Proof. Part (1) follows from (9.2)(1). Part (2) follows from (9.2)(2) together with the Whitney regularity condition. If $JF \pitchfork \mathscr{S}$, then $C(\Omega) = \{t | \dim(JF_t)^{-1}\Omega(X_t, Y_t) \geq \dim X_t - s\}$ is a closed algebraic subset of T of codimension at least one. If $JF \pitchfork \mathscr{S}$ and Σ is a stratum of $\mathscr{S}$, then (9.2)(2) implies that $C(\Sigma) = \{t | JF_t \text{ is not transverse to } \Sigma\}$ is a constructible subset of T of codimension at least one. Whitney regularity yields that the closure of $C(\Sigma)$ is contained in the union of $C(\Omega)$ and the sets $C(\Sigma')$ for $\Sigma' \subset \overline{\Sigma}$. Thus $\{t | JF_t \text{ is not transverse to } \Sigma\}$ is a proper closed algebraic subset of T. q.e.d.

Now to prove Theorem (8.1) we put the incidence projection p into the universal family $P\colon \mathscr{X} \to \mathscr{Y}$ and use the unfolding invariance of the Thom-Boardman sets. Recall $\mathscr{X} = \{(x, H, M) \in \mathbf{P}^4 \times \mathbf{P}^{4*} \times \mathscr{M}_d | x \in M \cap H\}$, $\mathscr{Y} = \mathbf{P}^{4*} \times \mathscr{M}_d$, $P\colon \mathscr{X} \to \mathscr{Y}$, $P(x, H, M) = (H, M)$. The map P can be viewed as a family of maps either over $\mathscr{M}_d$ or over $\mathbf{P}^{4*}$:

$$\begin{tikzcd} & \mathscr{X} \arrow[dl] \arrow[d, "P"] \arrow[dr] & \\ \mathscr{M}_d & \mathscr{Y} \arrow[l] \arrow[r] & \mathbf{P}^{4*} \end{tikzcd}$$

Let $\mathscr{S}(6,4)$ be the stratification of $J^4(6,4)$ through codimension 6 using the Thom-Boardman loci Σ^1, $\Sigma^{3,0}$, $\Sigma^{3,1,0}$, $\Sigma^{3,1,1,0}$, $\Sigma^{3,1,1,1}$, $\Sigma^{3,2,0}$, and let $\mathscr{S}$ be the corresponding algebraic stratification sequence (with $n_0 = 6$, $c = -2$), which exists by (6.4) and (9.1). We want to show that $\{M | JP_M \pitchfork \mathscr{S}\}$ is a nonempty Zariski-open subset of $\mathscr{M}_d$. By Lemma (9.3) it is enough to show $JP \pitchfork \mathscr{S}$, or that $JP_H \pitchfork \mathscr{S}$ for all $H \in \mathbf{P}^{4*}$. For $d \geq 4$ this is a consequence of the case $m = 3$ of the following basic proposition.

Let $\mathbf{M}_d^m$ be the space of all degree d hypersurfaces (singular or nonsingular) in $\mathbf{P}^m$. Thus $\mathbf{I}_d^m$ is the projectivization of the vector space of homogeneous degree d complex polynomials in $m+1$ variables, and $\dim \mathbf{M}_d^m = \binom{m+d}{d} - 1$. Let $\mathbf{I}_d^m = \{(x, f) \in \mathbf{P}^m \times \mathbf{M}_d^m | f(x) = 0\}$, and let $q\colon \mathbf{I}_d^m \to \mathbf{M}_d^m$ be the projection $q(x, f) = f$. The fiber of q over f is the hypersurface $\{x \in \mathbf{P}^m | f(x) = 0\}$; thus $\mathbf{I}_d^m$ is the *universal degree d hypersurface* in $\mathbf{P}^m$, and $\dim\ \mathbf{I}_d^m = \binom{m+d}{d} + m - 2$.

(9.4) Proposition. *$J^d q \pitchfork \Sigma(\mathbf{I}_d^m, \mathbf{M}_d^m)$ for all orbits Σ of*

$$J^d\left(\binom{m+d}{d} + m - 2,\ \binom{m+d}{d} - 1\right).$$

Let $\mathbf{H}_d^{m+1} = \{(x,f) \in \mathbf{I}_d^{m+1} | x \in \mathbf{P}^m,\ f(y) \neq 0 \text{ for some } y \in \mathbf{P}^m\}$ and $(\mathbf{H}_d^{m+1})' = \{(x,f) \in \mathbf{H}_d^{m+1} | f \text{ is smooth}\}$. Since $(\mathbf{H}_d^{m+1})'$ is a Zariski-open subset of $\mathbf{H}_d^{m+1}$, and $\mathbf{H}_d^{m+1}$ is a Zariski fiber bundle over $\mathbf{I}_d^m$, Proposition (9.4) implies the corresponding result for $(\mathbf{H}_d^{m+1})'$. Taking $m = 3$, it follows that, for $d \geq 4$, $JP_H \pitchfork \mathscr{S}$ for all $H \in \mathbf{P}^{4*}$, completing the proof of Theorem (8.1).

Proof of (9.4). We abbreviate $\mathbf{M}_d^m$ by $\mathbf{M}$ and $\mathbf{I}_d^m$ by $\mathbf{I}$. Let $(\mathbf{z}_0, \cdots, \mathbf{z}_m)$ be the standard homogeneous coordinates on $\mathbb{P}^m$. Let U be the affine subset $\mathbf{z}_0 \neq 0$, and let $(x_1, \cdots, x_m)$ be the affine coordinates $x_i = \mathbf{z}_i/\mathbf{z}_0$. By dehomogenization, $\mathbf{M} = \mathbb{P}(\mathbf{N})$, where $\mathbf{N}$ is the vector space of polynomials of degree $\leq d$ in $x_1, \cdots, x_m$. Let $\mathbf{N}_0 = \{f \in \mathbf{N} - \{0\} | f(0) = 0\}$ and $\mathbf{M}_0 = \mathbb{P}(\mathbf{N}_0)$. Define $h\colon \mathbb{C}^m \times \mathbf{M}_0 \to \mathbf{I}$ by $h(x,f) = (x, f - f(x))$, an isomorphism onto $\mathbf{I} \cap (U \times \mathbf{M})$. Consider the map $Q\colon \mathbb{C}^m \times \mathbf{N}_0 \to \mathbf{N}$, $Q(x,f) = f - f(x)$. Since Q is a trivial unfolding of $q \circ h$ (i.e., Q is locally isomorphic to the suspension of $q \circ h$), it is enough to prove (9.4) for Q. In other words, it suffices to check that $J^d Q \pitchfork \Sigma(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ for all orbits Σ of $J^d(\binom{m+d}{d} + m - 1, \binom{m+d}{d})$.

Let $J_0^d(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ be the space of jets that map 0 to 0. The standard coordinates identify $J_0^d(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ with $J^d(\binom{m+d}{d} + m - 1, \binom{m+d}{d})$. Let

$$J_0^d Q\colon \mathbb{C}^m \times \mathbf{N}_0 \to J_0^d(C^m \times \mathbf{N}_0, \mathbf{N})$$

assign to the vector $v \in \mathbb{C}^m \times \mathbf{N}_0$ the d-jet at 0 of the map $Q_v(w) = Q(v + w) - Q(v)$. Then $J^d Q \pitchfork \Sigma(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ if and only if $J_0^d Q \pitchfork \Sigma$.

For $g \in \mathbf{N}$ write $g(x) = \Sigma a_I x^I$, using the multiindex $I = (i_1, \cdots, i_m)$. The a_I are the standard coordinates on the vector space $\mathbf{N}$, and $g \in \mathbf{N}_0$ if and only if $a_0 = 0$. With respect to the coordinates $x = (x_i)$ and $a = (a_I)$, we have $Q(x,a) = b$, where $b_0 = -\Sigma a_I x^I$ and $b_I = a_I$ for $I \neq 0$.

An element F of $J_0^4(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ is a polynomial map (truncated to degree 4) in the variables $X = (X_i)$ and $A = (A_I)$ such that $F(0) = 0$. For each multiindex J, let F_J denote the Jth component of F. Then $J_0^4 Q\colon \mathbb{C}^m \times \mathbf{N}_0 \to J_0^4(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ is given by $J_0^4 Q(x,a) = F$, where

$$\begin{aligned} F_0 &= -\Sigma(A_I + a_I)(X + x)^I + \Sigma a_I x^I, \\ F_J &= A_J, \qquad J \neq 0. \end{aligned} \tag{9.5}$$

We will verify that the differential D of $J_0^4 Q$ at (x,a) is transverse to the orbit of $J_0^4 Q(x,a)$ in $J_0^4(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$.

We compute with the coordinates x_i, a_I on $\mathbb{C}^m \times \mathbf{N}_0$ and coordinates ψ_J^α on $J_0^4(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ defined as follows. Let $\{\varphi^\alpha\}$ be a basis of monomials in X_i, A_I, and let $\{\psi^\alpha\}$ be the dual basis. Let ξ_J be the dual basis to the standard basis on $\mathbf{N}$, and let $\psi_J^\alpha = \psi^\alpha \otimes \xi_J$. For $F \in J_0^4(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$, $\psi_J^\alpha(F)$ is the coefficient of φ^α in the polynomial F_J.

First note that by (9.5), we have

$$\left(D\left(\frac{\partial}{\partial a_I}\right)\right)_J = \frac{\partial F_J}{\partial a_I} = \begin{cases} -(X+x)^I + x^I, & J = 0, \\ 0, & J \neq 0, \end{cases}$$

so by induction on $|I|$ the image of D contains the subspace V spanned by the tangent vectors $\partial/\partial\psi_0^\alpha$ for $\varphi^\alpha = X^I$, $I \neq 0$. Therefore, letting T denote the tangent space to the orbit of $J_0^4 Q(x,a)$, it is enough to show that $\partial/\partial\psi_J^\alpha \in T \pmod V$ for all other coordinates ψ_J^α. We show

(1) $\partial/\partial\psi_0^\alpha \in T$ for all φ^α except $\varphi^\alpha = X^I$, $I \neq 0$,

(2) $\partial/\partial\psi_J^\alpha \in T \pmod V$ for all φ^α and all $J \neq 0$.

To show that these vectors are in $T \pmod V$, we construct curves through $J_0^4(x,a)$, in the orbit of $J_0^4 Q(x,a)$ in $J_0^4(\mathbb{C}^m \times \mathbf{N}_0, \mathbf{N})$ (under the action of the group of germs at 0 of analytic coordinate changes in the source $\mathbb{C}^m \times \mathbf{N}_0$ and the target $\mathbf{N}$). These curves are constructed as orbits of $J_0^4 Q(x,a)$ under germs of 1-parameter subgroups.

Let φ be a monomial in the variables A_I. The 1-parameter family of target coordinate changes $F \to F + tG$ in (9.5), where $G_0 = \varphi$ and $G_J = 0$ for $J \neq 0$, shows that (1) holds for all monomials not involving the X_i. Next let $K \neq 0$ be a fixed multiindex, and consider the simultaneous source-target coordinate changes in (9.5): $(X,A) \to (X, A + tB)$, $F \to F - tG$, $|t| < \varepsilon$, where $B_K = \varphi$, $B_J = 0$ for $J \neq K$, and $G_K = \varphi$, $G_J = 0$ for $J \neq K$. The tangent vector at $t = 0$ of the resulting curve has 0-component $-\varphi(X+x)^K$ and J-component zero for $J \neq 0$. Thus (1) follows by induction on $|K|$.

Finally, let φ be an arbitrary monomial in X_i and A_I, let $K \neq 0$ be a fixed multiindex, and consider the source coordinate changes $(X,A) \to (X, A+tB)$ in (9.5), where $B_K = \varphi$, $B_J = 0$ for $J \neq K$. The tangent vector at $t = 0$ of the resulting curve has 0-component $-\varphi(X+x)^K$, which lies in $T \pmod V$ by (1). The K-component of this vector is φ, and all other components are zero. This implies (2).

Remark. An obstruction to extending the technique of proof (b) of Theorem (8.1) to higher dimensions is that the Thom-Boardman strata do not, in general, satisfy the frontier condition. So one is forced to work with a finer decomposition of the jet space. For $M^n \subset \mathbb{P}^{n+1}$, the incidence correspondence $\Gamma \subset M^n \times \mathbb{P}^{n+1*}$ has dimension $2n$. A nice stratification of $J^{n+1}(2n, n+1)$ would give, via the technique of proof (b) of (8.1), a classification of singularities of the incidence projection (or the Gauss map) of a generic n-dimensional hypersurface of degree $d \geq n+1$.

Genericity for contact with lines. If M is a smooth hypersurface in $\mathbb{P}^4$, let $\Gamma_1 \subset M \times \mathbb{G}(1,4)$ be the incidence correspondence $\Gamma_1 = \{(x, L) | x \in L\}$, and let $p_1 : \Gamma_1 \to \mathbb{G}(1,4)$ be the incidence projection, $p_1(x, L) = L$. The singular locus $\Sigma(p_1)$ is $\{(x, L) | L$ tangent to M at $x\}$, which can be identified with the projectivized tangent bundle $\mathbb{P}TM$. In analogy with the Gauss map $M \to \mathbb{P}^{4*} = \mathbb{G}(3,4)$, the correspondence $M \leftarrow \Sigma(p_1) \to \mathbb{G}(1,4)$ is the *Gauss correspondence.*

Now $\dim \Gamma_1 = \dim \mathbb{G}(1,4) = 6$. The fibers of p_1 are the one-dimensional linear sections $M \cap L$. Of particular interest are the Morin singularities of p_1:

$$\mathscr{Z}_k = \Sigma^{1_k}(p_1) = \{(x, L) | L \text{ has at least } k\text{th order contact with } M \text{ at } x\},$$
$$\mathbb{P}TM = \mathscr{Z}_1 \supset \mathscr{Z}_2 \supset \mathscr{Z}_3 \supset \cdots \supset \mathscr{Z}_d = \mathscr{Z}_{d+1} = \cdots,$$

where d is the degree of M.

(9.6) Theorem. *The set of smooth hypersurfaces in $\mathbb{P}^4$ such that $\mathscr{Z}_k$ is a smooth $(6-k)$-manifold is a nonempty Zariski-open subset of $\mathscr{M}_d$ for all $k < d$.*

The proof is similar to proof (b) of Theorem (8.1), using Ronga's method. The same proof works for $M^n \subset \mathbb{P}^{n+1}$ for all n. Note that the case $k = d$, which corresponds to lines contained in M, does not follow from this proof, but is an immediate consequence of [5].

Proof. We put the incidence projection into the universal family parametrized by $\mathscr{M}_d$. Let $\mathscr{X} = \{(x, L, M) \in \mathbb{P}^4 \times \mathbb{G}(1,4) \times \mathscr{M}_d | x \in M \cap L\}$, $\mathscr{Y} = \mathbb{G}(1,4) \times \mathscr{M}_d$, $P : \mathscr{X} \to \mathscr{Y}$, $P(x, L, M) = (L, M)$. The map P can be viewed either as a family over $\mathscr{M}_d$ or over $\mathbb{G}(1,4)$:

$$\begin{tikzcd}
& \mathscr{X} \arrow[dl] \arrow[d, "P"] \arrow[dr] & \\
\mathscr{M}_d & \mathscr{Y} \arrow[l] \arrow[r] & \mathbb{G}(1,4)
\end{tikzcd}$$

Note that dP has corank ≤ 1 everywhere, since for each M, P_M is the restriction of the big incidence projection $\{(x, L) \in \mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{G}(1,4) | x \in L\} \to \mathbb{G}(1,4)$.

Let $J^d_{(1)}(6,6)$ be the subset of $J^d(6,6)$ consisting of jets with corank ≤ 1. Consider the Thom-Boardman loci $\Sigma^{1_k,0} \subset J^d_{(1)}(6,6)$, $0 \leq k \leq d-1$, with $\Sigma^{1_0,0} = \Sigma^0$. Then $\Sigma^0, \Sigma^{1,0}, \cdots, \Sigma^{1_{d-1},0}$ is an algebraic Whitney stratification of $J^d_{(1)}(6,6) - \Omega$, where Ω has codimension d. Let $\mathscr{S}$ be the corresponding algebraic stratification sequence. To show that $\{M | JP_M \pitchfork \mathscr{S}\}$ is generic, it suffices by Lemma (9.3) to show that $JP \pitchfork \mathscr{S}$, or that $JP_L \pitchfork \mathscr{S}$ for each $L \in \mathbb{G}(1,4)$. This follows from Proposition (9.4) with $m = 1$. (The case $k = 2$ is not covered by this argument, since then $\operatorname{codim}\Omega = 3$. But we saw in (3.4) that $\mathscr{Z}_2$ if smooth if M is Gauss-stable.)

Remark. A similar analysis can be made of contact of hypersurfaces with l-planes for any l, at least for small n and large d. Contact of 2-planes with hypersurfaces in $\mathbb{P}^4$ seems particularly interesting.

References

[1] V. I. Arnold, *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities*, Functional Anal. Appl. **6** (1972) 254–272.

[2] ——, *Singularities of systems of rays*, Russian Math. Surveys **38** (1983) 87–176.

[3] V I. Arnold, S. M. Gusein-Zade & A. N. Varchenko, *Singularities of differentiable maps*, Vol. 1, Birkhäuser, Boston, 1985.

[4] M. Atiyah & I. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969.

[5] W. Barth & A. Van deVen, *Fano varieties of lines on hypersurfaces*, Arch. Math. **31** (1978) 96–104.

[6] J. M. Boardman, *Singularities of differentiable maps*, Inst. Hautes Études Sci. Publ. Math. **33** (1967) 383–419.

[7] J. W. Bruce, *The duals of generic hypersurfaces*, Math. Scand. **49** (1981) 36–60.

[8] A. Cayley, *A memoir on cubic surfaces*, Philos. Trans. Roy. Soc. London Ser. A **159** (1869) 231–326.

[9] C. H. Clemens & P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972) 281–356.

[10] A. Durfee, *Fifteen characterizations of rational double points and simple critical points*, Enseignement Math. (2) **25** (1979) 131–163.

[11] G. Fano, *Ricerche sulla varietà cubica generale dello spazio a quattro dimensioni e sopra i suoi spazi pluritangenti*, Ann. Mat. Pura Appl. (3) **10** (1904) 251–285.

[12] W. Fulton, *Intersection theory*, Springer, Berlin, 1984.

[13] C. G. Gibson, K. Wirthmüller, A. A. du Plessis & E. J. N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Math., Vol. 552, Springer, Berlin, 1976.

[14] M. Golubitsky & V. Guillemin, *Stable mappings and their singularities*, Springer, New York, 1973.

[15] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[16] ——, *Algebraic geometry and local differential geometry*, Sci. École Norm. Sup. (4) **12** (1979) 355–452.

[17] J. Harris & L. Tu, *On symmetric and skew-symmetric determinantal varieties*, Topology **23** (1984) 71–84.

[18] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, New York, 1966.

[19] T. Josefiak, A. Lascoux & P. Pragacz, *Classes of determinantal varieties associated with symmetric and antisymmetric matrices*, Izvestija **45** (1981) 662–673.

[20] A. Kas & M. Schlessinger, *On the versal deformation of a complex space with an isolated singularity*, Math. Ann. **196** (1972) 23–29.

[21] D. Kirby, *The structure of an isolated multiple point of a surface*. II, Proc. London Math. Soc. (3) **7** (1957) 1–18.

[22] V. S. Kulikov, *Calculation of singularities of an imbedding of a generic algebraic surface in projective space* $\mathbf{P}^3$, Functional Anal. Appl. **17** (1983) 176–186.

[23] S. Lichtenbaum & M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Math. Soc. **128** (1967) 41–70.

[24] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. Lecture Note Series 77, Cambridge, 1984.

[25] J. Martinet, *Déploiments versels des applications différentiables et classification des applications stables*, Lecture Notes in Math., Vol. 535, Springer, Berlin, 1976, 1–44.

[26] ——, *Singularities of smooth functions and maps*, London Math. Soc. Lecture Note Series 58, Cambridge, 1982.

[27] J. Mather, *Stratifications and mappings*, Dynamical Systems (Peixoto, ed.), Academic Press, New York, 1973, 195–232.

[28] ——, *On Thom-Boardman singularities*, Dynamical Systems (Peixoto, ed.), Academic Press, New York, 1973, 233–248.

[29] C. McCrory & T. Shifrin, *Cusps of the projective Gauss map*, J. Differential Geometry **19** (1984) 257–276.

[30] F. Ronga, *On the number of planes triply tangent to a surface in projective space*, Canadian Math. Soc. Conf. Proc., Vol. 6, Amer. Math. Soc., Providence, RI, 1986, 389–395.

[31] ——, *On the singularities of the surface reciprocal to a generic surface in projective space*, J. Differential Geometry, **29** (1989), 105–112.

[32] G. Salmon, *A treatise on the analytic geometry of three dimensions*, Vol. II, fifth edition (1914), reprinted by Chelsea, New York, 1965.

[33] B. Teissier, *Déformations à type topologique constant*. I, II, Séminaire Douady-Verdier 1971-72, Astérisque **16** (1974) 215–249.

[34] L. Tu, *Degeneracy loci*, Proc. Internat. Conference on Algebraic Geometry, Berlin, 1985, Teubner, Leipzig, 1986, 296–305.

[35] I. Vainsencher, *On the degrees of certain strata of the dual variety*, Compositio Math. **38** (1979) 241–252.

UNIVERSITY OF GEORGIA