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CURVATURE MEASURES AND GENERALIZED MORSE THEORY

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1. Introduction

1.1. In studying the differential geometry of a hypersurface M^n in euclidean space $\mathbf{E}^{n+1}$ it has often proved fruitful to view the integral of the Gauss-Kronecker curvature (or "Gauss-Bonnet integrand") as an integral instead over the unit sphere S^n: that is, as the *area of the Gauss map* $\nu : M^n \to S^n$. A notable success of this device is the work of Chern-Lashof relating the total absolute curvature of (compact) M to the sum of its Betti numbers. In this and other works (e.g. [1]) the further step has been taken to identify the value of the integrand on the sphere, at a point $v \in S^n$, with the sum of some topological indices associated to the "height function" $h_v(x) := x \cdot v$, $x \in M$, and to the points of $\nu^{-1}(v)$. In fact these latter points are exactly the *critical points* of this height function, and the topological index at each point is $(-1)^\lambda$, where λ is the Morse index of h_v there.

The resulting expression for the curvature, in terms of these height functions, possesses at least one other theoretical advantage: namely, that the indices above may exist in a generalized sense even when the surface M^n is highly singular. Thus it is natural to define the Gauss-Kronecker curvature (as well as certain other invariants) by this approach. This has been previously suggested by [10].

In this paper we will carry out a new treatment of an existing theory of "generalized curvature" in this way, namely Federer's theory [4] of the curvature measures of sets of positive reach. A set of positive reach is a closed subset A of a euclidean space $\mathbf{E}^{n+1}$ such that if a point x lies sufficiently close to A, then there is a unique point $\xi(x) \in A$ minimizing the distance to x. (Any compact $C^{1,1}$ manifold or closed convex set has this property.) Federer showed that, for small $r > 0$, the volume of the tubular neighborhood $A_r := \{x : \mathrm{dist}(x, A) \leq r\}$ is a polynomial of degree equal to the ambient dimension ("Steiner's formula"). He then defined

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the *curvature measures* of A in terms of the coefficients of this polynomial; these are Radon measures $\Phi_i(A, \cdot)$, $i = 0, \cdots, n+1$.

Our main theorem expresses the measure $\Phi_0(A, \cdot)$ as follows. Let f be a smooth function defined on a neighborhood of A, and suppose for convenience that A is compact. One may identify certain points of A as *nondegenerate critical points* of the restriction $f|A$, and also define the *index* $\lambda = \lambda(A, f, p)$ of $f|A$ at such a critical point p. If $f|A$ is nice (a "Morse function"), then just as in smooth Morse theory (cf. [12]), the set $f^{-1}(-\infty, f(p) + \varepsilon] \cap A$ has for small $\varepsilon > 0$ the homotopy type of $f^{-1}(-\infty, f(p) - \varepsilon] \cap A$ with a cell of dimension λ attached. Now let f be the "height function" $x \mapsto x \cdot v$, for a vector v of the unit sphere S^n, and put $\imath(A, v, p) := (-1)^\lambda$ with λ as above. Our theorem then states that for any Borel set K

$$(n+1)\alpha(n+1)\Phi_0(A, K) = \int_{S^n} \sum_{p \in K \cap A} \imath(A, v, p)\, d\mathscr{H}^n v,$$

where $\alpha(n+1)$ is the volume of the unit ball in $\mathbf{E}^{n+1}$, and $\mathscr{H}^n$ is the n-dimensional Hausdorff measure. Actually we will also give similar expressions for all of the curvature measures Φ_i (Corollary 6.3).

As an application we will prove a generalization of a theorem of Zähle, which extends the curvature measures in a geometrically satisfying way to certain locally finite unions of sets of positive reach.

As a final remark let us point out that this subject is less isolated than it may appear. For example it is a fact that under remarkably unrestrictive hypotheses on a compact set $S \subset \mathbf{E}^{n+1}$, the closure of the complements of the tubular neighborhoods S_r have positive reach–for example, for *any* such $S \subset \mathbf{E}^3$ this conclusion holds for all $r \in \mathbf{R}^+$ outside a compact set of measure zero (cf. [6]). Thus it is possible to analyze the curvature of the set S by means of these tubular approximations. The sequel to this paper will carry out this project in case S is a *subanalytic* subset of $\mathbf{E}^{n+1}$.

2. Basic facts and definitions

Let $\mathbf{E}^{n+1}$ be the euclidean space of $n+1$ dimensions, with the usual metric $d(x, y) = |x - y|$ and "dot product" $x \cdot y$. The interior of a set $A \subset \mathbf{E}^{n+1}$ will be denoted by A°, its closure by $\overline{A}$.

2.1. Definitions: sets of positive reach. Let $A \subset \mathbf{E}^{n+1}$. We define $d_A : \mathbf{E}^{n+1} \to \mathbf{R}$ by

$$d_A(x) := \inf\{|x - p| : p \in A\}.$$

Put

$$\pi_A(x) := \{p \in A : |x - p| = d_A(x)\},$$

and set for $r \geq 0$

$$A_r := d_A^{-1}([0, r]).$$

Then (cf. [4]) we define

$$\operatorname{Unp}(A) := \{x \in \mathbf{E}^{n+1} : \pi_A(x) \text{ is a singleton}\},$$

and distinguish the function $\xi_A : \operatorname{Unp}(A) \to A$ by $\{\xi_A(x)\} = \pi_A(x)$. Finally, if $p \in A$, then

$$\operatorname{reach}(A, p) := \sup\{r : B(p, r) \subset \operatorname{Unp}(A)\},$$

where $B(p, r) := \{x : |x - p| < r\}$, and

$$\operatorname{reach}(A) := \inf\{\operatorname{reach}(A, p) :\in A\}.$$

We will also use the notation $\overline{B}(p, r)$ for the closed ball $\{x : |x - p| \leq r\}$.

Note, for example, that any compact hypersurface of class $C^{1,1}$, or any body bounded by such a hypersurface has positive reach (cf. §3 and [4, 4.19]).

2.2. Definitions: weights and measures. We put $\mathbf{L}^k$ to be the Lebesgue measure in $\mathbf{E}^k$. For each $\alpha \geq 0$ we let $\mathscr{H}^\alpha$ be the Hausdorff measure of dimension α (cf. [5, 2.10.2]). These last-named measures are of course defined on any metric space; on $\mathbf{E}^k$, $\mathscr{H}^k$ coincides with $\mathbf{L}^k$. We also identify the constants

$$\alpha(k) := \mathbf{L}^k(\{x \in \mathbf{E}^k : |x| \leq 1\}), \qquad \beta(n, k) := \frac{\alpha(k)\alpha(n-k)}{\alpha(n)\binom{n}{k}},$$

whenever n, k are nonnegative integers, and $k \leq n$.

2.3. Definitions: tangents and normals. Let $A \subset \mathbf{E}^m$. We define the *tangent cone* to A at a point $p \in A$ by

$$\operatorname{Tan}(A, p) := \left\{v \in \mathbf{E}^m : \liminf_{t \downarrow 0} d_A(p + tv)/t = 0\right\},$$

and the *normal cone* to A at p as the *dual* to the tangent cone, namely

$$\operatorname{Nor}(A, p) := \{w \in \mathbf{E}^m : w \cdot v \leq 0 \text{ for all } v \in \operatorname{Tan}(A, p)\},$$

(cf. [5, 3.1.12]). We identify also the unit normals $\operatorname{nor}(A, p) := \operatorname{Nor}(A, p) \cap S^{m-1} = \{w \in \operatorname{Nor}(A, p) : |w| = 1\}$.

The *k-density* of A at p is given by

$$\mathbf{\Theta}^k(A, p) := \lim_{r \downarrow 0} \mathscr{H}^k(A \cap B(p, r))/\alpha(k) r^k,$$

(compare [5, 2.10.19]).

If $p \in M \subset \mathbf{E}^{n+1}$ is a C^1 submanifold, then we identify the tangent space T_pM with the parallel plane $\operatorname{Tan}(M,p)$ through the origin. In particular, if $M^n \subset \mathbf{E}^{n+1}$ is a hypersurface and $\nu : M^n \to S^n$ is the Gauss map, then $T_pM = T_{\nu(p)}S^n$.

2.4. Definition and remarks: Lipschitz manifolds. An m-dimensional *lipschitz manifold* is a paracompact metric space M such that there is a system of open sets U_α covering M and, for each α, a bilipschitzian homeomorphism φ_α of U_α onto an open subset of $\mathbf{E}^m$. A *lipschitz submanifold* of a metric space X is a subspace of X which is a lipschitz manifold under the induced (*not* the intrinsic) metric. Note, for example, that with this definition the singular plane curve $x^2 = y^3$ is *not* a lipschitz submanifold of $\mathbf{E}^2$.

If M is a k-dimensional lipschitz submanifold of a euclidean space $\mathbf{E}^m$, then for $\mathscr{H}^k$-a.e. $p \in M$ the cone $\operatorname{Tan}(M,p)$ is in fact a k-dimensional plane. For by Rademacher's theorem [5, 3.1.6] the inverse of each coordinate map $\varphi_\alpha : U_\alpha \to \mathbf{E}^k$ is $\mathbf{L}^k$-a.e. differentiable as a map into $\mathbf{E}^m$. Since φ_α is lipschitzian, it follows that any existing derivative $D[\varphi_\alpha^{-1}](x)$ is nonsingular; and if $p = \varphi_\alpha^{-1}(x)$, then the two tangent spaces above coincide with the image of $\mathbf{E}^k$ under $D[\varphi_\alpha^{-1}](x)$.

2.5. Definitions and remarks: normals to sets of positive reach. Suppose that $A \subset \mathbf{E}^{n+1}$ is a compact set with $\operatorname{reach}(A) > 0$. Then for each $p \in A$:

(a) $\operatorname{nor}(A,p) = \{(x-p)/|x-p| : x \in \xi_A^{-1}(p)\}$ (cf. [4, Theorem 4.8(12)];

(b) the tangent cone $\operatorname{Tan}(A,p)$ is the cone dual to $\operatorname{Nor}(A,p)$, i.e.,

$$\operatorname{Tan}(A,p) := \{v : v \cdot w \leq 0 \text{ for all } w \in \operatorname{Nor}(A,p)\} \quad \text{(loc. cit.).}$$

Furthermore if we put *nor*(A) to be the "generalized bundle" of unit normals

$$\operatorname{nor}(A) := \{(p,v) : v \in \operatorname{nor}(A,p)\} \subset \mathbf{E}^{n+1} \times S^n \subset \mathbf{E}^{n+1} \times \mathbf{E}^{n+1},$$

then this "bundle" is an n-dimensional lipschitz submanifold of $\mathbf{E}^{2n+2}$ (cf. [15, 1.1.7]). This is seen easily if we note that the sets $d_A^{-1}(r)$ are n-dimensional submanifolds of class C^1 (or even $C^{1,1}$) for $0 < r < \operatorname{reach}(A)$ (cf. [4, Theorem 4.8(3),(4),(5),(8)]), and that the map $\psi_r : d_A^{-1}(r) \to \operatorname{nor}(A)$, given by $\psi_r(x) := (\xi_A(x), [x - \xi_A(x)]/d_A(x))$, is a bilipschitzian homeomorphism of such $d_A^{-1}(r)$ onto $\operatorname{nor}(A)$. (The inverse map is given explicitly as $(x,v) \mapsto x + rv$.) In particular $\operatorname{Tan}[\operatorname{nor}(A),(p,v)]$ is an n-dimensional plane for $\mathscr{H}^n$-a.e. $(p,v) \in \operatorname{nor}(A)$.

2.6. Repeated use will be made of the following fundamental property of sets of positive reach.

Lemma (Federer). *If A is a set of positive reach, $p, q \in A$ and $v \in \operatorname{nor}(A, p)$, then*

$$v \cdot (q - p) \leq \tfrac{1}{2}|q - p|^2 \operatorname{reach}(A).$$

Proof. This is conclusion (7) of [4, Theorem 4.8].

2.7. We will often think of the euclidean space $\mathbf{E}^{2n+2}$ as the product space $\mathbf{E}^{n+1} \times \mathbf{E}^{n+1}$. With this representation we will denote the projections onto the first and second factors by π_1 and π_2.

3. $C^{1,1}$ Morse theory

3.1. In this section we indicate briefly how it is possible to extend the elementary notions of differential geometry and Morse theory to a $C^{1,1}$ hypersurface in $\mathbf{E}^{n+1}$.

Definition. Let $M^n \subset \mathbf{E}^{n+1}$ be an oriented C^1 hypersurface, and let $\nu : M \to S^n$ be its Gauss map. Then M is of *class* $C^{1,1}$ iff ν is lipschitzian.

3.2. Proposition. *The hypersurface M is of class $C^{1,1}$ iff, given any $p \in M$, there is a neighborhood $U \subset \mathbf{E}^{n+1}$ of p such that under some system of isometric coordinates on U the set $M \cap U$ appears as the graph of a C^1 function with lipschitzian gradient.*

Proof. The proof is trivial.

3.3. Remarks. If $B \subset \mathbf{E}^{n+1}$ has $\operatorname{reach}(B) =: R > 0$, then for $0 < r < R$ the naturally oriented hypersurface $d_B^{-1}(r)$ is of class $C^{1,1}$. This is a consequence of [4, Theorem 4.8(3) and (8)].

The Morse theory of a set of positive reach developed in the next section will be based on the approximation by these smoother sets. The reader will note that the $C^{1,1}$ Morse theory differs hardly at all from the C^2 or C^∞ theory, with the difference that certain points of a $C^{1,1}$ hypersurface cannot by their very nature occur as nondegenerate critical points. This distinction will have greater significance after passing to the general "positive reach" setting.

3.4. Let us now consider the Gauss map $\nu : M \to S^n$ as a lipschitzian map between C^1 manifolds. By Rademacher's theorem this map is differentiable at $\mathscr{H}^n$-a.e. point of M; we denote the set of points where the derivative exists by $\operatorname{Sm}(M)$, the set of *smooth points.* Given a map f between C^1 manifolds, we denote by $df(p)$ its derivative at those points p where this is defined. For $p \in \operatorname{Sm}(M)$, the derivative $d\nu(p) : T_pM \to T_{\nu(p)}S^n$ exists. (Recall however that with our conventions these two tangent spaces are identical to one another.)

At each smooth point p of M we define the *second fundamental form* $\mathrm{II}(p)$ to be the bilinear form on T_pM given by

$$\mathrm{II}(p)(\xi, \eta) = \xi \cdot d\nu(\eta).$$

As in the smooth case,

3.5. Proposition. $\mathrm{II}(p)$ *is symmetric.*

Proof. Given $\xi, \eta \in T_pM$, by 3.2 there are continuous vector fields $\bar{\xi}, \bar{\eta} : M \to \mathbf{E}^{n+1}$, defined in a neighborhood of p in M and tangent to M with $\bar{\xi}(p) = \xi$, $\bar{\eta}(p) = \eta$, which moreover are *differentiable* at the point p. Writing these out in coordinates, since the second derivatives are symmetric whenever they are defined (cf. [5, 3.1.11]) it follows in the usual way that the vector $[\bar{\xi}, \bar{\eta}](p) = d\bar{\eta}(p)(\xi) - d\bar{\xi}(p)(\eta)$ is well defined and *tangent* to M at p. Applying the Leibniz rule to the functions $\bar{\xi} \cdot \nu$ and $\bar{\eta} \cdot \nu$ we find that

$$\mathrm{II}(p)(\xi, \eta) - \mathrm{II}(p)(\eta, \xi) = \nu(p) \cdot [\bar{\xi}, \bar{\eta}](p) = 0.$$

3.6. Now let $f : \mathbf{E}^{n+1} \to \mathbf{R}$ be a C^∞ function, and for each $x \in \mathbf{E}^{n+1}$ let $Hf(x)$ denote the bilinear form on $\mathbf{E}^{n+1}$ defined by the Hessian matrix of second derivatives at x.

Definition. A point $p \in M$ is a *critical point* of the restriction $f|M$ iff $\operatorname{grad} f(p)$ is a multiple of $\nu(p)$. If such a point additionally belongs to $\mathrm{Sm}(M)$, then the *Hessian* of $f|M$ at p is defined to be the bilinear form on T_pM given by

$$H_M f(p) := Hf(p)|T_pM - (\operatorname{grad} f(p) \cdot \nu(p))\mathrm{II}(p).$$

3.7. Remarks. If p is as above and $\xi \in T_pN$, then for any C^1 curve α with $\alpha(0) = p$ and $\alpha'(0) = \xi$ we have

$$f \circ \alpha(t) - f(p) = H_M f(p)(\xi, \xi)t^2/2 + o(t^2)$$

as $t \downarrow 0$. Furthermore, if $\varphi : \mathbf{E}^n \supset U \to M$ is a C^1 diffeomorphism onto a neighborhood of p in M such that $\varphi^{-1}(p) = 0$ and φ (as a map into $\mathbf{E}^{n+1}$) is twice differentiable at 0, then $f \circ \varphi$ is twice differentiable at 0 with Hessian form given by

$$H(f \circ \varphi)(0)(\xi, \eta) = H_M f(p)(d\varphi(0)(\xi), d\varphi(0)(\eta)).$$

3.8. Definition. A critical point p of the restriction $f|M$ will be called *nondegenerate* iff $\operatorname{grad} f(p) \neq 0$ and the bilinear form $H_M f(p)$ is nondegenerate. The *index* of p is the number of negative eigenvalues of $H_M f(p)$.

3.9. Given $a \in \mathbf{R}$ let us put $M^a := M \cap f^{-1}(-\infty, a]$.

Theorem. *Let the $C^{1,1}$ hypersurface M be compact. Suppose that p is a nondegenerate critical point of $f|M$ of index λ, with $f(p) = c$, and that*

$f^{-1}(c)$ contains no other critical points of $f|M$. Then for $\varepsilon > 0$ sufficiently small, $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.

Proof. For $p \in M$ let us denote the projection of $\operatorname{grad} f(p)$ onto T_pM by

$$\operatorname{grad}_M f(p) := \operatorname{grad} f(p) - [\nu(p) \cdot \operatorname{grad} f(p)]\nu(p).$$

Let $U \subset \mathbf{E}^{n+1}$ be a neighborhood of p such that $U \cap M$ is in suitable coordinates the graph of a C^1 function $g : \mathbf{E}^n \supset U' \to \mathbf{R}$ with lipschitzian derivative. Since $p \in \operatorname{Sm}(M)$ by hypothesis, g is twice differentiable at the point x_0 corresponding to p. Let ρ be a smooth function on $\mathbf{E}^{n+1}$, equal to 1 outside of U and vanishing in a neighborhood of p. Then $V := -\rho \operatorname{grad}_M f$ is a lipschitzian vector field on M such that $V \cdot \operatorname{grad}_M f \leq 0$, with *strict* inequality in $f^{-1}(c)\backslash U$. The basic theory of ordinary differential equations (cf. e.g. [8]) asserts that V integrates to give a continuous flow on U, which for some $\varepsilon > 0$ deforms $M^{c+\varepsilon}$ into $M^{c-\varepsilon} \cup [M^{c+\varepsilon} \cap U]$. But in the coordinates above on U the function $x \mapsto f(x, g(x))$ mapping $U' \to \mathbf{R}$ is differentiable everywhere and by 3.7 has a nondegenerate critical point of index λ at the point x_0. The theorem now follows in the usual way from the result of [11].

3.10. Actually our interest lies not so much in the Morse theory of a $C^{1,1}$ manifold M but rather in that of a body in $\mathbf{E}^{n+1}$ *bounded* by such a manifold. If M is smooth then this remains classical (cf. [Morse and Cairns, Chapter 8]). Such a body, if compact, has positive reach. The next result is most naturally given in this more general context.

3.11. Definition. Let $A \subset \mathbf{E}^{n+1}$ be compact with $\operatorname{reach}(A) > 0$, and let as before $f : \mathbf{E}^{n+1} \to \mathbf{E}$ be a C^∞ function. A point $p \in A$ is a *regular point* of the restriction $f|A$ iff: either $p \in A^\circ$ and $\operatorname{grad} f(p) \notin 0$, or $p \in \operatorname{bdry} A$ and $-\operatorname{grad} f(p) \notin \operatorname{Nor}(A, p)$. A value $c \in \mathbf{R}$ is a *regular value* of $f|A$ iff every point of $A \cap f^{-1}(c)$ is regular. A point or value is *critical* for $f|A$ iff it is not regular.

Thus in the particular case when A is a body bounded by a $C^{1,1}$ hypersurface M, oriented so that the normal $\nu : M \to S^n$ points out of A, a boundary point p is critical for $f|A$ iff the vectors $\operatorname{grad} f(p)$ and $\nu(p)$ point in opposite directions (or else $\operatorname{grad} f(p) = 0$).

3.12. Let A and f be as above; then for $c \in \mathbf{R}$ we put $A^c := A \cap f^{-1}(-\infty, c]$.

Proposition. *If c is a regular value of $f|A$, then for all $\varepsilon > 0$ small enough the spaces $A^{c+\varepsilon}$ and $A^{c-\varepsilon}$ are homotopy equivalent.*

Proof. Let $F(x, t)$ be the flow of the vector field $-\operatorname{grad} f/|\operatorname{grad} f|^2$, by hypothesis defined for small t and x in some neighborhood of $A \cap f^{-1}(c)$.

Then $\frac{d}{dt}(f \circ F(x,t)) \equiv -1$. Let a neighborhood U of $f^{-1}(c) \cap A$ in $\mathbf{E}^{n+1}$, and a constant $K \in (0,1)$ be given such that, if $x \in U$, $p = \xi_A(x)$ and $w \in \operatorname{nor}(A,p)$, then

$$-w \cdot \operatorname{grad} f(x) < K|\operatorname{grad} f(x)| < |\operatorname{grad} f(x)|.$$

These exist by the regularity hypothesis and the fact that $\operatorname{nor}(A)$ is compact.

The flow $\xi_A \circ F$ is defined whenever $F(x,t) \in \operatorname{Unp}(A)$. Since F is continuous there are $\delta' > 0$ and a neighborhood V' of $f^{-1}(c) \cap A$ such that in fact $F(x,t) \in U \cap \operatorname{Unp}(A)$ whenever $|t| < \delta'$ and $x \in V$. Then we have for such points

$$\begin{aligned} f(\xi_A \circ F(x,t)) - f(x) &= f(\xi_A(\circ F(x,t)) - f(F(x,t)) + f(F(x,t)) - f(x) \\ &= d_A(F(x,t)) \int_0^1 (-w) \cdot \operatorname{grad} f(s\xi_A \circ F(x,t) \\ &\quad + (1-s)F(x,t))\, ds, \end{aligned}$$

where $w = (\xi_A \circ F(x,t) - F(x,t))\, d_A(F(x,t))^{-1} \in \operatorname{nor}(A, \xi_A \circ F(x,t))$. Now there are a neighborhood $V'' \subset V'$ and a constant $\delta'' < \delta'$, such that if $x \in V''$ and $|t| < \delta''$, then the segment σ joining $\xi_A \circ F(x,t)$ to $F(x,t)$ lies within U, whence

$$f(\xi_A \circ F(x,t)) - f(x) \leq K\, d_A(F(x,t)) \sup_\sigma |\operatorname{grad} f| - t.$$

Now if $y := F(x,t) \notin A$, then we have for $F(x,t) \in U$

$$\begin{aligned} \frac{d}{dt}[d_A \circ F(x,t)] &= \frac{(y - \xi_A(y))}{d_A(y)} \cdot \frac{(-\operatorname{grad} f(y))}{|\operatorname{grad} f(y)|^2} \\ &= \frac{-w \cdot \operatorname{grad} f(y)}{|\operatorname{grad} f(y)|^2}, \qquad w \in \operatorname{nor}(A, \xi_A(y)), \\ &< K|\operatorname{grad} f(y)|^{-1}. \end{aligned}$$

Thus

$$d_A \circ F(x,t) < Kt \sup\{|\operatorname{grad} f(y)|^{-1} : y \in F(x,[0,1])\},$$

$$\begin{aligned} &f(\xi_A \circ F(x,t)) - f(x) \\ &\qquad < K^2 t \sup\left\{ \frac{|\operatorname{grad} f(x)|}{|\operatorname{grad} f(y)|} : x \in \sigma, y \in F(x,[0,t]) \right\} - t. \end{aligned}$$

Let $\delta''' > 0$ be so small that

$$\sup\{|\operatorname{grad} f(x)|/|\operatorname{grad} f(y)| : |x-y| < 2\delta'''\} < K^{-1}.$$

If now $0 < t < \delta'''/\inf_U |\operatorname{grad} f|$, then for $x \in A$ we have

$$d_A \circ F(x,t) \leq |F(x,t) - x| < \delta''',$$

whence the segment σ above lies within δ''' of $F(x,t)$. Thus the right-hand side above $< Kt - t < 0$.

Thus there are $\varepsilon > 0$ and $\delta \in (0, \delta''')$ such that if $x \in V \cap A^{c+\varepsilon}$, then $\xi_A \circ F(x,\delta) \in A^{c-\varepsilon}$. If additionally ε is so small that $f^{-1}[c-2\varepsilon, c+\varepsilon] \subset V$, then using a mollifier the deformation $\xi_A \circ F$ extends to a homotopy of $A^{c+\varepsilon}$ into $A^{c-\varepsilon}$.

3.13. Remark. Using the same method together with a mollifier one may prove the somewhat more general result:

Let $c \in \mathbf{R}$, and $K \subset A$ be a compact subset such that $f^{-1}(c) \cap K$ consists only of regular points of $f|A$. Then for all small $\varepsilon > 0$ there is a homotopy $\rho : A^{c+\varepsilon} \times [0,1] \to A^{c+\varepsilon}$ such that for each x, $f(\rho(x,1)) \leq f(\rho(x,0))$ and $\rho(K,1) \subset A^{c-\varepsilon}$.

3.14. The next result is the main goal of this section and will be an important tool in the next.

Theorem. *Let $A \subset \mathbf{E}^{n+1}$ be a compact set bounded by a $C^{1,1}$ hypersurface M, and let $f : \mathbf{E}^{n+1} \to \mathbf{R}$ be a C^∞ function. Suppose that $p \in M$, $f(p) =: c$, is a critical point of the restriction $f|A$ and the only such point within $f^{-1}(c) \cap A$. Suppose further that, considered as a critical point of $f|M$, p is nondegenerate of index λ. Then for all $\varepsilon > 0$ small enough the set $A^{c+\varepsilon}$ has the homotopy type of $A^{c-\varepsilon}$ with a λ-cell attached.*

Proof. The case $A = M$ having been dealt with in 3.9, we may assume that $A = \overline{A^\circ}$. By hypothesis, $\operatorname{grad} f(p) = -|\operatorname{grad} f(p)|\nu(p)$, where $\nu(p)$ is the outward unit normal to A at p. For convenience of expression let us choose coordinates so that the vector $\nu(p)$ points downward. Then there is a convex neighborhood $U \subset \mathbf{E}^{n+1}$ of p such that $A^\circ \cap U$ lies above $M \cap U$. It is clear from Proposition 3.2 that U may be taken so small that pushing directly downward gives a deformation retraction of $A \cap U$ onto $M \cap U$. Let V be a second convex neighborhood of p with $\overline{V} \subset U$; then we can mollify the last deformation so that its restriction to $A \cap V$ is a retraction onto $M \cap V$, and so that it leaves $(\operatorname{bdry} U) \cap A$ fixed. Thus the mollified deformation may be extended to a deformation of all of A, leaving $A \backslash U$ fixed. Consider now, for small $\varepsilon > 0$, a deformation as in 3.13 with $K = A \backslash V$. Concatenating these two deformations we get a retraction of $A^{c+\varepsilon}$ into $A^{c-\varepsilon} \cup [M \cap V]$. The result now follows from the proof of 3.9.

3.15. Definition. Let $A \subset \mathbf{E}^{n+1}$ and $f : \mathbf{E}^{n+1} \to \mathbf{R}$ be as in 3.14. The restriction $f|A$ is *Morse* iff each set $A \cap f^{-1}(c)$ contains at most one critical point of $f|A$, and each such point is nondegenerate.

3.16. Corollary. *Let A, f be as above, with $f|A$ Morse. Then A has the homotopy type of a CW complex, with one cell of dimension λ for each critical point of index λ.*

Proof. The proof is just as in [12, Chapter 3]. (Of course we still have not shown that such Morse functions exist.)

4. Morse theory on a set of positive reach

4.1. Throughout this section A will denote a compact subset of $\mathbf{E}^{n+1}$ with $R := \operatorname{reach}(A) > 0$. Our goal is to develop the Morse theory of the restriction to A of a C^∞ function $f : \mathbf{E}^{n+1} \to \mathbf{R}$. Our strategy is to compare the behavior of f on A to the behavior of certain functions f_r associated to the tubular neighborhoods A_r, $0 < r < R$; the latter spaces are subject to the theory of §3. The topological changes of A through the changes in the levels of f are equivalent to those of A_r in the levels of f_r; meanwhile the algebraically defined index λ of f_r on $M_r := \operatorname{bdry} A_r$ agrees with a similar invariant for f on $\operatorname{nor}(A)$. Thus the Morse theory of $f|A$ can be stated without reference to the approximations f_r and A_r.

4.2. Letting $M_r := d_A^{-1}(r)$, $0 < r < R$, be the $C^{1,1}$ hypersurfaces bounding the tubes A_r, we put $\nu = \nu_r : M_r \to S^n$ for their Gauss maps. Recalling the natural bilipschitzian homeomorphisms

$$\psi_r : \operatorname{nor}(A) \to M_r, \qquad \psi_r(p,v) := p + rv,$$
$$\varphi_r : M_r \to \operatorname{nor}(A), \quad \varphi_r = \psi_r^{-1}, \quad \varphi_r(x) = (\xi_A(x)\nu(x)),$$

we have the

Proposition. *Let $(p,v) \in \operatorname{nor}(A)$ and $0 < r < R$. The following three conditions are equivalent.*

(i) $t := \operatorname{Tan}[nor(A),(p,v)]$ *is an n-dimensional plane in* $\mathbf{E}^{2n+2}$.

(ii) φ_r *is differentiable at* $\psi_r(p,v)$.

(iii) ν_r *is differentiable at* $\psi_r(p,v)$.

Proof. (i)$\Leftrightarrow$(ii). As the weighted addition map $(q,w) \mapsto q + rw$ is differentiable everywhere in $\mathbf{E}^{n+1} \times \mathbf{E}^{n+1}$, it follows that if (i) holds, then at (p,v) the restriction of its derivative to the space T is well defined and linear. Since ψ_r is bilipschitzian, this restriction is nonsingular, and it follows that the derivative of φ_r at $\psi_r(p,v)$ is the inverse of this restriction.

Conversely, if φ_r is differentiable there, then the derivative is nonsingular, and T is its image.

(ii)$\Leftrightarrow$(iii). $\Rightarrow$ is immediate from the expression above for φ_r. To get $\Leftarrow$, we notice also that if $x \in M_r$, then $\xi_A(x) = x - r\nu(x)$. q.e.d.

Such a point (p,v) will be called a *smooth point* of $\operatorname{nor}(A)$.

4.3. Proposition. *With the hypotheses above, if we put $x_0 := \psi_r(p,v)$, then the following hold*:

(a) *Any* $\xi \in T_{x_0}M_r$ *has the form* $\xi = \tau + r\sigma$ *for some* $(\tau, \sigma) \in T$.

(b) *If* ξ *has this form, then* $d\nu(x_0)(\xi) = \sigma$.

Proof. The proof is immediate.

4.4. Proposition. *Suppose that* $(p, v) \in \operatorname{nor}(A)$ *and that conditions* 4.2 (i)–(iii) *hold. If* (τ, σ) *and* $(\tau', \sigma') \in T \subset \mathbf{E}^{n+1} \times \mathbf{E}^{n+1}$, *then* $\tau \cdot \sigma' = \tau' \cdot \sigma$.

Proof. Let $0 < r < R$, and put $x_0 := \psi_r(p, v)$. Consider the symmetric bilinear form $\mathrm{II}_r(x_0) := \mathrm{II}_{M_r}(x_0)$. Then expressing ξ, $\xi' \in T_{x_o}M_r$ as in 4.3(a),

$$\begin{aligned}\mathrm{II}_r(x_0)(\xi, \xi') &= \mathrm{II}_r(x_0)(\tau + r\sigma, \tau' + r\sigma') \\ &= (\tau + r\sigma) \cdot d\nu(x_0)(\tau' + r\sigma') = (\tau + r\sigma) \cdot \sigma'.\end{aligned}$$

By symmetry this quantity can also be expressed as $(\tau' + r\sigma') \cdot \sigma$.

4.5. Definitions. Suppose that conditions 4.2(i)–(iii) hold. Define a vector subspace of $\mathbf{E}^{n+1}$ by

$$T_1 = T_1(p, v) := \pi_1(\operatorname{Tan}[\operatorname{nor}(A), (p, v)]).$$

Now define the *second fundamental form* $\mathrm{II}_A(p, v)$ as the symmetric bilinear form on T_1 given as

$$\mathrm{II}_A(p, v)(\tau, \tau') = \tau \cdot \sigma',$$

where $(\tau', \sigma') \in \operatorname{Tan}[\operatorname{nor}(A), (p, v)]$.

Note that if A is a $C^{1,1}$ hypersurface M, then $T_1 = T_pM$, and the definition of II_M just given agrees with the previous one. Now if f is a C^∞ function, and p is a critical point of $f|A$, with $|\operatorname{grad} f(p)| \cdot v = -\operatorname{grad} f(p)$, $v \in \operatorname{nor}(A, p)$, and such that conditions (4.2)(i)–(iii) hold at (p, v), then we define a symmetric bilinear form on $T_p(p, v)$ by

$$H_A(f(p) := Hf(p) \mid T_p(p, v) + |\operatorname{grad} f(p)| \mathrm{II}_A(p, v).$$

Under these circumstances we will say that p is a *nondegenerate* critical point of $f|A$ iff $H_Af(p)$ is nondegenerate, and we define the *index* $\lambda(f, A, p)$ to be the number of negative eigenvalues of this form.

4.6. Let $p \in \operatorname{bdry} A$ be a nondegenerate critical point of $f|A$, with $v = -\operatorname{grad} f(p)/|\operatorname{grad} f(p)| \in \operatorname{nor}(A, p)$, and for $0 < r < R$ let f_r be the C^∞ function $f_r(x) := f(x - rv)$.

Proposition. *If* $r > 0$ *is small enough, then* $\psi_r(p, v)$ *is a nongenerate critical point of* $f_r|A_r$ *with index*

$$\lambda(f_r, A_r, \psi_r(p, v)) = \lambda(f, A, p).$$

Proof. That $p_r := \psi_r(p, v)$ is a critical point of $f_r|A_r$ follows at once from the definition and the fact that $\nu(p_r) = v$. By 4.2, $p_r \in \operatorname{Sm}(M_r)$.

Thus for ξ, $\xi' \in T_{p_r}M_r$ we may compute the Hessian

$$\begin{aligned} H_r f_r(p_r)(\xi,\xi') &= Hf_r(p_r)(\xi,\xi') + |\operatorname{grad} f_r(p_r)| \mathrm{II}_r(p_r)(\xi,\xi') \\ &= Hf(p)(\xi,\xi') + |\operatorname{grad} f(p)| \mathrm{II}_r(p_r)(\xi,\xi') \\ &= Hf(p)(\tau + r\sigma, \tau' + r\sigma') + |\operatorname{grad} f(p)|(\tau + r\sigma)\cdot\sigma' \end{aligned}$$

by the proof of 4.4, where $\xi = \tau + r\sigma$, $\xi' = \tau' + r\sigma'$, (τ,σ), $(\tau',\sigma') \in T_{p,v}\operatorname{nor}(A)$. Finally we may write this last expression as

$$\begin{aligned} H_A f(p)(\tau,\tau') + r\{Hf(p)(\sigma,\tau') + Hf(p)(\tau,\sigma') + |\operatorname{grad} f(p)|\sigma\cdot\sigma'\} \\ + r^2 Hf(p)(\sigma,\sigma'), \end{aligned}$$

which defines a bilinear form F_r on (τ,σ), $(\tau',\sigma') \in T := T_{p,v}\operatorname{nor}(A)$, the pullback via ψ_r of $H_r f_r(p_)$.

Now put $T_2 := \{\sigma \in \mathbf{E}^{n+1} : (0,\sigma) \in T\}$, and let T_1^* be a subspace of T complementary to T_2 (thus $T_1^* \simeq T_1$). Writing F_r in matrix form, with a basis adapted to the decomposition $T = T_1^* \oplus T_2$ (independent of r),

$$F_r = \begin{vmatrix} A_1 + rA_2 + r^2 A_3 & rC \\ rC^t & r|\operatorname{grad} f(p)|I + r^2 B \end{vmatrix}$$

where the A_i, B, and C are fixed matrices (i.e., independent of r), and A_1 is the pullback of $H_A f(p)$ under the isomorphism $T_1^* \simeq T_1$. Thus,

$$\det F_r = (\det A_1)|\operatorname{grad} f(p)|^{n-\lambda} r^{n-\lambda} + O(r^{n-\lambda+1})$$

as $r \downarrow 0$, $\lambda := \lambda(f, A, p)$; as the leading term is by hypothesis $\neq 0$, this determinant $\neq 0$ when r is small enough. That is, the critical point p_r is nondegenerate.

To evaluate the index of F_r, let N be a maximal subspace of T_1^* on which A_1 is negative definite, and let $P \subset T_1^*$ be a complementary subspace on which A_1 is positive definite. Certainly $\dim N = \lambda$. Furthermore, for r small enough the form $F_r|T_1^* = A_1 + rA_2 + r^2 A_3$ remains negative definite on N and positive definite on P. At the same time $F_r|T_2 = r|\operatorname{grad} f(p)|I + r^2 B$ is certainly positive definite for all small r. Now if $\xi \in P$ and $\eta \in T_2$, then

$$\begin{gathered} F_r(\xi,\xi) \to A_1(\xi,\xi) > a|\xi|^2 > 0, \\ F_r(\xi,\eta) < br|\xi|\,|\eta|, \\ F_r(\eta,\eta) = r|\operatorname{grad} f(p)|\,|\eta|^2 + O(r^2)|\eta|^2 > rc|\eta|^2 > 0 \end{gathered}$$

as $r \downarrow 0$, for some positive constants a, b, c. Thus the Cauchy-Schwartz inequality

$$F_r(\xi,\eta)^2 < b^2 r^2 |\xi|^2 |\eta|^2 < acr|\xi|^2|\eta|^2 < F_r(\xi,\xi)F_r(\eta,\eta)$$

holds for all such ξ, η provided $r < ac/b^2$, which implies that F_r is positive definite on $P \oplus T_2$. Thus for such r the subspace N is a maximal subspace on which F_r is negative definite, $\dim N = \lambda$. But the dimension of such a subspace is the index of F_r.

4.7. Proposition. *Suppose that c is a regular value of $f|A$. Then for $r > 0$ small enough the spaces $A^c := A \cap f^{-1}(-\infty, c]$ and $A_r^c := A_r \cap f_r^{-1}(-\infty, c]$ are homotopy equivalent.*

Proof. Modifying f if necessary away from the compact set A we may assume that f is proper and that $\operatorname{grad} f$ never vanishes on $f^{-1}(c)$, and therefore that $\operatorname{grad} f_r$ never vanishes on $f_r^{-1}(c)$. Thus by [4, 5.19 and 4.13],

$$R' := \operatorname{reach} f^{-1}(-\infty, c] = \operatorname{reach} f_r^{-1}(-\infty, c] > 0.$$

Furthermore [4, 4.10] together with the hypothesis that c is a regular value of $f|A$ (cf. Definition 3.11) implies that there is $\eta > 0$ such that

$$\operatorname{reach}(A^c) \geq \eta \cdot \min\{R, R'\},$$
$$\operatorname{reach}(A_r^c) \geq \eta \cdot \min\{R - r, R'\}.$$

Now it is clear that, in the Hausdorff metric h, $\lim_{r \to 0} A_r^c = A^c$. Therefore we may take $r > 0$ so small that

$$h(A^c, A_r^c) < \min\{\operatorname{reach}(A^c), \operatorname{reach}(A_r^c)\}/2.$$

For such r we have

$$A^c \subset \operatorname{Unp}(A_r^c), \qquad A_r^c \subset \operatorname{Unp}(A^c).$$

Letting ξ^c and ξ_r^c be the projection maps onto these sets, we have furthermore for all $x \in A^c$ and $y \in A_r^c$

$$|\xi^c \circ \xi_r^c(x) - x| < \operatorname{reach}(A^c), \qquad |\xi_r^c \circ \xi^c(y) - y| < \operatorname{reach}(A_r^c).$$

It follows that the map $\xi^c \circ \xi_r^c : A^c \to A^c$ is homotopic to the identity map of A^c via the homotopy $H : A^c \times [0,1] \to A^c$ given by

$$H(x,t) := \xi^c(t\xi^c \circ \xi_r^c(x) + (1-t)x).$$

Similarly, the map $\xi_r^c \circ \xi^c : A_r^c \to A_r^c$ is homotopic to the identity of A_r^c.

4.8. Theorem. *Suppose that $p \in A$ is a nondegenerate critical point of $f|A$ of index λ, $f(p) = c$, and that $f|A$ has no other critical points within $A \cap f^{-1}(c)$. Then for all $\varepsilon > 0$ small enough the set $A^{c+\varepsilon}$ has the homotopy type of $A^{c-\varepsilon}$ with a λ-cell attached.*

Proof. Put $v := -\operatorname{grad} f(p)/|\operatorname{grad} f(p)| \in \operatorname{nor}(A, p)$. If $r > 0$ and $\varepsilon > 0$ are small enough, then all values within $[c - \varepsilon, c + \varepsilon] \setminus \{c\}$ are regular values both of $f|A$ and of $f_r|A_r$. By 4.6 we may take r so small that

$\psi_r(p,v) = p + rv$ is the unique critical point of $f_r|A_r$ within $A_r \cap f_r^{-1}(c)$, and so that this critical point is nondegenerate of index λ. By 3.12 and 3.14, for such r and ε each set $A_r^{c+\varepsilon}$ has the homotopy type of $A_r^{c-\varepsilon}$ with a λ-cell attached. But by 4.7 $A^{c\pm\varepsilon}$ is homotopy equivalent to $A_r^{c+\varepsilon}$.

4.9. The definition of a *Morse function* f on a compact set A of positive reach is the same as 3.15.

Corollary. *If $f|A$ is Morse, then A has the homotopy type of a CW complex, with one cell of dimension λ for each critical point of index λ.*

In the next section we will see that such Morse functions are plentiful.

5. Almost every height function is Morse

5.1. Let a compact set $A \subset \mathbf{E}^{n+1}$ be given with $\operatorname{reach}(A) > 0$. For each unit vector $v \in S^n$ we denote by h_v the height function $h_v(x) := x \cdot v$.

In this section we will prove:

Theorem. *For $\mathscr{H}^n$-a.e. $v \in S^n$, the restriction $h_v|A$ is Morse.*

5.2. Lemma. *Let $(p,v) \in \operatorname{nor}(A)$, with $(\tau,\sigma) \in \operatorname{Tan}[\operatorname{nor}(A),(p,v)]$. Then*

(i) $\lim_{\substack{q\to p\\ w\in\operatorname{nor}(A,q)}} w \cdot (q-p)|q-p|^{-1} = 0$, *and*

(ii) $\tau \cdot v = 0$.

Proof. (i) By compactness it is enough to consider a subsequence $(q_k, w_k) \in \operatorname{nor}(A)$ with $q_k \to p$, and such that

$$\lim_{k\to\infty} (q_k - p)|q_k - p|^{-1} = u \in \operatorname{Tan}(A,p),$$
$$\lim_{k\to\infty} w_k = w_0 \in \operatorname{nor}(A,p).$$

By continuity and the definition of the normal cone,

$$\lim_{k\to\infty} w_k \cdot (q_k - p)|q_k - p|^{-1} = w_0 \cdot u \le 0.$$

On the other hand Lemma 2.6 gives for each k

$$w_k \cdot (q_k - p)|q_k - p|^{-1} \ge -|q_k - p|/2\operatorname{reach}(A),$$

whence the limit above ≥ 0.

(ii) By the definition of the tangent cone, if $\tau \ne 0$, then a sequence $(q_k, w_k) \in \operatorname{nor}(A)$ may be found such that $(q_k - p)|q_k - p|^{-1} \to \tau/|\tau|$ and $w_k - v = O(|q_k - p|)$. By (i) we have

$$\begin{aligned} O &= \lim_{k\to\infty} w_k \cdot (q_k - p)|q_k - p|^{-1} \\ &= \lim_{k\to\infty} [v + O(|q_k - p|)] \cdot [\tau/|\tau| + o(1)] \\ &= v \cdot \tau/|\tau|. \end{aligned}$$

5.3. Corollary. *Suppose that*

$$\beta : (a,b) \to \operatorname{bdry} A, \qquad \gamma : (a,b) \to S^n$$

are continuous, with $\gamma(s) \in \operatorname{nor}(A, \beta(s))$ for all $s \in (a,b)$. If $\beta'(s_0)$ exists, then $\gamma(s_0) \cdot \beta'(s_0) = 0$.

Proof. If $\beta'(s_0) = 0$, the result is trivial. Otherwise, by 5.2(i),

$$\begin{aligned} 0 &= \lim_{s \to s_0} \gamma(s) \cdot (\beta(s) - \beta(s_0))|\beta(s) - \beta(s_0)|^{-1} \\ &= \gamma(s_0) \cdot \beta'(s_0)|\beta'(s_0)|^{-1}. \end{aligned}$$

5.4. Theorem. *Let $N \subset \mathbf{E}^{n+1} \times S^n$ be compact, and $(\mathscr{H}^n, n)$-rectifiable (cf. [5, 3.2.14]). Suppose that*

$$\lim_{\substack{q \to p \\ (q,w) \in N}} w \cdot (p-q)|p-q|^{-1} = 0$$

for all $p \in \mathbf{E}^{n+1}$. Then for $\mathscr{H}^n$-a.e. $v \in S^n$ the following hold:

(i) *$N \cap \pi_2^{-1}(v)$ is finite;*

(ii) *If $p, q \in \pi_1[N \cap \pi_2^{-1}(v)]$ and $p \neq q$, then $p \cdot v \neq q \cdot v$.*

Proof. (i) The approximate Jacobian $\operatorname{ap} J_n(\pi_2|N)$ (cf. [5, 3.2.1]) of the restriction of π_2 to N is clearly ≤ 1 wherever it is defined. Thus by the coarea formula [5, 3.2.20]

$$\infty > \mathscr{H}^n(N) \geq \int_N \operatorname{ap} J_n(\pi_2|N)\, d\mathscr{H}^n = \int_{S^n} \operatorname{card}[\pi_2^{-1}(v) \cap N]\, d\mathscr{H}^n v,$$

so the last integrand is a.e. finite.

(ii) By [5, 3.3.39] or [13, 11.1], there is a countable collection M_i' of C^1 submanifolds of $\mathbf{E}^{n+1} \times S^n$ such that $f' := N \setminus \bigcup M_1'$ has measure $\mathscr{H}^n(F') = 0$. Applying the coarea formula to the maps $\pi_2|M_i'$ we find that for a.e. $v \in S^n$

$$J_n(\pi_2|M_1')(p,v) \neq 0$$

for all $(p,v) \in \pi_2^{-1}(v) \cap M_i'$. Thus we may define new submanifolds (open subsets of the M_i') by

$$M_i := M_i' \cap \{(p,v) : J_n(\pi_2|M_i')(p,v) \neq 0\},$$

so that if $G := N \setminus \bigcup_i M_i$, then $\mathscr{H}^n(\pi_2(G)) = 0$. Refining the cover $\{M_i\}$ as necessary we may assume that the closure $\overline{M}_i$ of each submanifold is compact, and that the restriction of π_2 to $\overline{M}_i$ is one-to-one.

We want to show that

$$\begin{aligned} C := \{v \in S^n : \ & \text{there are } p, q \in \pi_1[\pi_2^{-1}(v) \cap N \cap \cup M_i] \\ & \qquad \text{with } p \neq q \text{ and } p \cdot v = q \cdot v\} \end{aligned}$$

has $\mathscr{H}^n(C) = 0$. For each pair of indices (i, j), $i \neq j$, let us put

$$C_{ij} := \{v \in S^n : \text{ there are } p_k \in \pi_1[\pi_2^{-1}(v) \cap M_k \cap N], \\ k = i, j,\ p_i \neq p_j \text{ and } p_i \cdot v = p_j \cdot v\}.$$

Since each $\pi_2|M_i$ is one-to-one, we have $C = \bigcup_{i \neq j} C_{ij}$. We will show that for every (i, j), $i \neq j$, the density $\Theta^n(C_{ij}, v) = 0$ at each $v \in C_{ij}$; this will imply that $\mathscr{H}^n(C_{ij}) = 0$ (cf. [15, 2.10.19]), and therefore that $\mathscr{H}^n(C) = 0$.

Choose any one of the C_{ij} and call it D; we may assume that $i = 0$, $j = 1$. Let $v \in D$ and let p, q be distinct points with $p \cdot v = q \cdot v$ and

$$(p, v) \in N \cap M_0, \qquad (q, v) \in N \cap M_1.$$

Let $v_k \in D$, $k = 1, 2, \ldots$, with $v_k \to v$; let p_k, q_k, $k = 1, 2, \ldots$, be sequences of points of $\mathbf{E}^{n+1}$ with

$$(p_k, v_k) \in N \cap M_0, \qquad (q_k, v_k) \in N \cap M_1,$$

and $p_k \cdot v_k \equiv q_k \cdot v_k$. Since π_2 is one-to-one on the compact sets $\overline{M}_0$ and $\overline{M}_1$, it follows that $p_k \to p$ and $q_k \to q$. The condition $J_n(\pi_2|M_i) \neq 0$ implies that

$$\limsup_{k \to \infty} |v_k - v|^{-1} |p_k - p| < \infty, \quad \limsup_{k \to \infty} |v_k - v|^{-1} |q_k - q| < \infty.$$

Combining these relations with the hypothesis we find that

$$\lim_{k \to \infty} v_k \cdot (p_k - p)|v_k - v|^{-1} = \lim_{k \to \infty} v_k \cdot (q_k - q)|v_k - v|^{-1} = 0$$

(see Figure 1).

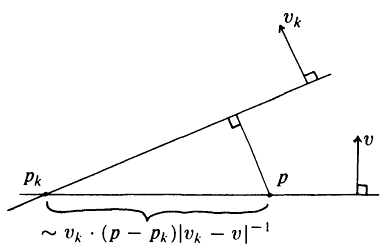

FIGURE 1

Since $p_k \cdot v_k = q_k \cdot v_k$ the difference of the left and middle expressions is

$$\lim_{k \to \infty} v_k \cdot (p - q)|v_k - v|^{-1} = 0.$$

Geometrically this means that the sequence of the v_k is asymptotic to the great hypersphere of S^n perpendicular to the vector $p - q$. As the sequence was arbitrarily chosen subject to $v_k \in D$, $v_k \to v$, it follows that

$$\mathscr{H}^n(D \cap B(v, r)) = o(r^n)$$

as $r \downarrow 0$. That is, $\Theta^n(D, v) = 0$ as claimed.

Proof of 5.1. By 5.2, the set $N = \operatorname{nor}(A)$ satisfies the hypothesis of 5.4. It follows that for a.e. $v \in S^n$ the height function $h_v|A$ has a finite number of critical points, and that no two such points correspond to the same critical value. It remains to be seen that, for a.e. $v \in S^n$, if $(p, v) \in \operatorname{nor}(A)$, then the conditions of 4.2 hold at (p, v) and that the second fundamental form $\mathrm{II}_A(p, v)$ is nondegenerate on the vector space $T_1(p, v)$. Recalling the coarea formula

$$\mathscr{H}^n(U) = \int_{\pi_2^{-1}(U)\cap\operatorname{nor}(A)} J_n(\pi_2|\operatorname{nor}(A))(p, v)\, d\mathscr{H}^n(p, v),$$

$U \subset S^n$, the first assertion holds since the conditions of 4.2 hold $\mathscr{H}^n$-a.e. in $\operatorname{nor}(A)$. To prove the second assertion the coarea formula implies also that for $\mathscr{H}^n$-a.e. $v \in S^n$, $J_n(\pi_2|\operatorname{nor}(A))(p, v) \neq 0$ at all points $(p, v) \in \pi_2^{-1}(v) \cap \operatorname{nor}(A)$. That is, if $(\tau, \sigma) \in T_{p,v} \operatorname{nor}(A)$ and $\tau \neq 0$, then $\sigma \neq 0$. It follows that $T_2(p, v) := \{\sigma : (\tau, \sigma) \in T_{p,v} \operatorname{nor}(A)\} = T_v S^n$; and by 5.2(ii), $T_1(p, v) \subset T_v S^n$. Thus there is $\tau' \in T_1(p, v)$ with $(\tau', \tau) \in T_{p,v} \operatorname{nor}(A)$, whence

$$\mathrm{II}_A(p, v)(\tau, \tau') = |\tau|^2 \neq 0.$$

6. Curvature measures

6.1. Here at last is the main theorem of this paper. Given $v \in S^n$ we will abbreviate the notation for the index of the height function h_v by $\lambda(v, A, p) := \lambda(h_v, A, p)$.

Theorem. *Let $A \subset \mathbf{E}^{n+1}$ be a compact set of positive reach, and let $\Phi_0(A, \cdot)$ be the Gauss-Kronecker curvature measure of A (cf.* [4, 5.7]*). Then for any Borel set $U \subset \mathbf{E}^{n+1}$*

$$\Phi_0(A, U) = [(n+1)\alpha(n+1)]^{-1} \int_{S^n} \sum_{\substack{p\in U \\ -v\in\operatorname{nor}(A,p)}} (-1)^{\lambda(v,A,p)}\, d\mathscr{H}^n v.$$

Proof. For $0 < r < \operatorname{reach}(A)$ let us put $U_r := \xi_A^{-1}(U) \cap M_r$. By [4, 5.8] we have

$$\Phi_0(A_r, U_r) = \int_{U_r} \det \mathrm{II}_r(p)\, d\mathscr{H}^n p. \tag{1}$$

Let $F_r : S^n \to \mathbf{Z}$ be the function

$$F_r(v) := \sum_{p\in\nu^{-1}(v)\cap U_r} (-1)^{\lambda(-v,A_r,p)}.$$

If we notice that $\det \mathrm{II}_r(p) = J_n\nu(p)$ (the Jacobian determinant of the Gauss map $\nu : M_r \to S^n$), we may apply the coarea formula to (1) and obtain

$$\begin{aligned}\Phi_0(A_r, U_r) &= [(n+1)\alpha(n+1)]^{-1} \int_{S^n} \sum_{p\in\nu^{-1}(v)\cap U_r} \operatorname{sign} J_n\nu(p)\, d\mathscr{H}^n v \\ &= [(n+1)\alpha(n+1)]^{-1} \int_{S^n} F_r(v)\, d\mathscr{H}^n v.\end{aligned}$$

But for each $v \in S^n$ the critical points p of $h_v|A$ are in one-to-one correspondence with the critical points $p + rv$ of $h_r|A_r$ (where $r \in (0, \operatorname{reach}(A))$ is fixed). Thus for each $v \in S^n$

$$\begin{aligned}F_r(v) &\le \operatorname{card}[\nu^{-1}(v) \cap U_r] \\ &= \operatorname{card}[\pi_1^{-1}(U) \cap \pi_2^{-1}(v) \cap \operatorname{nor}(A)].\end{aligned}$$

Applying the coarea formula to the function $\pi_2|[\pi_1^{-1}(U) \cap \operatorname{nor}(A)]$ we find that the last expression defines an integrable function of $v \in S^n$. Meanwhile by 4.6 and 5.1 $F_r(v)$ tends to

$$F_0(v) := \sum_{\substack{p\in U\\ v\in\operatorname{nor}(A,p)}} (-1)^{\lambda(-v,A,p)} \quad \text{for a.e. } v \in S,$$

so the dominated convergence theorem gives

$$\begin{aligned}(n+1)\alpha(n+1) \lim_{r\to 0} \Phi_0(A_r, U_r) &= \lim_{r\to\infty} \int_{S^n} F_r(v)\, d\mathscr{H}^n v \\ &= \int_{S^n} F_0(v)\, d\mathscr{H}^n v.\end{aligned}$$

But by [4, 5.6–5.8], $\Phi_0(A_r, U_r) = \Phi_0(A, U)$ for all such r.

6.2. Definitions. For $i = 1, \cdots, n$, let $G(n+1, i)$ denote the space of all i-dimensional affine planes in $\mathbf{E}^{n+1}$. Given $P \in G(n+1, i)$ we will write $\overline{P}$ for the i-plane parallel to P and passing through the origin.

The group G of euclidean motions of $\mathbf{E}^{n+1}$ acts naturally on $G(n+1, i)$, which space also carries a natural G-invariant Radon measure. Normalizing this measure so that the set of all i-planes intersecting the unit ball has measure $\alpha(n+1-i)$, we denote it by $\gamma(n+1, i)$.

6.3. Corollary. *With the hypotheses of* 6.1 *the other curvature measures* $\Phi_i(A, \cdot)$, $i = 1, \cdots, n+1$, *are given by*

$$\begin{aligned}&\Phi_{n+1-i}(A, U) \\ &\quad= \beta(n+1, i)^{-1} \int_{G(n+1-i)} \int_{\overline{P}\cap S^n} \sum_{-v\in\operatorname{nor}(A\cap P,p)} (-1)^{\lambda(v,A\cap P,p)}\, d\mathscr{H}^{i-1} v \\ &\qquad\qquad \cdot\, d\gamma(n+1, i)P.\end{aligned}$$

Proof. By [4, 6.11], $\operatorname{reach}(A \cap P) > 0$ for $\gamma(n+1, i)$ a.e. $P \in G(n+1, i)$; thus the formula makes sense. The formula follows from 6.1 together with [4, 6.13].

6.4. In order to apply these formulas more generally it will be necessary to express the indices $(-1)^\lambda$ in terms of the local topology of the height functions and without direct reference to λ. For this purpose we now give the following "Taylor theorem" for a set A of positive reach.

6.5. Theorem. *Let $(p, v) \in \operatorname{nor}(A)$ be a smooth point and let f be a C^∞ function $\mathbf{E}^{n+1} \to \mathbf{R}$ such that $-\operatorname{grad} f(p)/|\operatorname{grad} f(p)| = v$. Suppose that $(q_k, w_k) \in \operatorname{nor}(A)$, $k = 1, 2, \cdots$, with*

$$\lim_{k\to\infty} q_k = p, \qquad \lim_{k\to\infty} (q_k - p)|q_k - p|^{-1} = \tau \in S^n,$$

and $w_k - v = O(|q_k - p|)$ as $k \to \infty$. Then $\tau \in T_1(p, v)$ and, as $k \to \infty$,

$$f(q_k) - f(p) = \tfrac{1}{2} H_A f(p)(\tau, \tau)|q_k - p|^2 + o(|q_k - p|^2).$$

Proof. Taking a subsequence if necessary, we may assume that $(w_k - v)/|q_k - p| \to \sigma$ as $k \to \infty$, so that $|q_k - p|^{-1}(q_k - p, w_k - p) \to (\tau, \sigma) \in T_{p,v} \operatorname{nor}(A)$, which proves the first assertion.

To prove the second assertion let us leave aside the sequence $\{q_k\}$ for the moment. Let $0 < r < \operatorname{reach}(A)$ and let $\alpha : (-a, a) \to M_r$ be a C^1 function such that

$$\alpha(0) = p + rv, \quad \alpha'(0) = \tau + r\sigma, \quad |\alpha'| \equiv |\tau + r\sigma|,$$

where σ has been chosen so that $(\tau, \sigma) \in T_{p,v} \operatorname{nor}(A)$. Then $\beta := \xi_A \circ \alpha$ is a lipschitzian function into A with

$$\beta'(0) = \tau, \qquad \operatorname{lip}(\beta) \le (R/R - r)|\tau + r\sigma|$$

by [4, 4.8(8)]. Putting

$$\gamma := r^{-1}(\beta - \alpha) = \nu \circ \alpha : (-a, a) \to S^n$$

we also have

$$(\beta, \gamma) : (-a, a) \to \operatorname{nor}(A), \qquad \gamma'(0) = \sigma.$$

Let $\varepsilon > 0$ be given. Let $a > \delta > 0$ be so small that if $|s| < \delta$ then, putting $B(s) := s^{-1}(\beta(s) - p)$, we have

$$|B(s) - \tau| < \varepsilon \quad \text{and} \quad |B(s)|^2 > 1 - \varepsilon^3.$$

For $0 < s_0 < \delta$ we have

$$\begin{aligned} f(\beta(s_0)) - f(p) &= \int_0^{s_0} \operatorname{grad} f(\beta(s)) \cdot \beta'(s)\, ds \\ &= -|\operatorname{grad} f(p)| \int_0^{s_0} \gamma(s) \cdot \beta'(s)\, ds \\ &\quad + \int_0^{s_0} [\operatorname{grad} f(\beta(s)) + |\operatorname{grad} f(p)|\gamma(s)] \cdot \beta'(s)\, ds. \end{aligned}$$

The first integral vanishes by 5.3, and we may write the second integral as

$$\begin{aligned} &\int_0^{s_0} [\operatorname{grad} f(\beta(s)) - \operatorname{grad} f(p)] \cdot \beta'(s)\, ds \\ &\qquad + |\operatorname{grad} f(p)| \int_0^{s_0} [\gamma(s) - v] \cdot \beta'(s)\, ds \\ &= \int_0^{s_0} sHf(p)(\tau, \beta'(s)) + o(s)\, ds \\ &\qquad + |\operatorname{grad} f(p)| \int_0^{s_0} (s\sigma + o(s)) \cdot \beta'(s)\, ds \\ &= \frac{s_0^2}{2} H_A f(p)(\tau, \tau) + \int_0^{s_0} sHf(p)(\tau, \beta'(s) - \tau)\, ds \\ &\qquad + |\operatorname{grad} f(p)| \int_0^{s_0} s\sigma \cdot (\beta'(s) - \tau)\, ds + o(s_0^2) \end{aligned}$$

as $s_0 \downarrow 0$. The second and third terms are in magnitude

$$\leq s_0(\|Hf(p)\| + |\operatorname{grad} f(p)|\,|\sigma|) \int_0^{s_0} |\beta'(s) - \tau|\, ds.$$

We estimate this last integral over the parts of $[0, s_0]$, where $|\beta'(s) - \tau| < 2\varepsilon$ and $|\beta'(s) - \tau| \geq 2\varepsilon$. The former part has magnitude $< 2\varepsilon s_0$. For the latter part, if $0 < s \leq s_0 < \delta$, then $|\beta'(s) - \tau| \geq 2\varepsilon$ implies that $|\beta'(s) - B(s_0)| > \varepsilon$. Thus by Chebyshev's inequality

$$\begin{aligned} &\mathbf{L}^1([0, s_0] \cap \{s : |\beta'(s) - \tau| \geq 2\varepsilon\}) \\ &\quad \leq \mathbf{L}^1([0, s_0] \cap \{s : |\beta'(s) - B(s_0)| > \varepsilon\}) \\ &\quad < \varepsilon^{-2} \left(\int_0^{s_0} |\beta'(s)|^2\, ds - s_0 |B(s_0)|^2 \right) \\ &\quad < s_0 \varepsilon^{-2} (\operatorname{lip}(\beta)^2 + \varepsilon^3 - 1), \end{aligned}$$

whence

$$\begin{aligned} \int_0^{s_0} |\beta'(s) - \tau|\, ds &< 2\varepsilon s_0 + s_0 \varepsilon^{-2} (\operatorname{lip}(\beta)^2 + \varepsilon^3 - 1) \sup |\beta'(s) - \tau| \\ &\leq s_0(2\varepsilon + \varepsilon^{-2}(\operatorname{lip}(\beta)^2 + \varepsilon^3 - 1)(1 + \operatorname{lip}(\beta))). \end{aligned}$$

Thus

$$|f(\beta(s_0)) - f(p) - H_A f(p)(\tau,\tau)s_0^2/2| < s_0^2(\|Hf(p)\| + |\operatorname{grad} f(p)|\,|\sigma\|) \\ \times [2\varepsilon + \varepsilon^{-2}(\operatorname{lip}(\beta)^2 + \varepsilon^3 - 1)(1 + \operatorname{lip}(\beta))] + o(s_0^2).$$

Now let $0 < \varepsilon < 3^{1/3}$, and take $r \in (0, R)$ to be so small that

$$|\tau + r\sigma|^2(R/R-r)^2 - 1 < \varepsilon^3.$$

Let us use this value of r in the discussion above. In particular,

$$\operatorname{lip}(\beta) \le |\tau + r\sigma|(R/R-r) < (1+\varepsilon^3)^{1/2} < 2.$$

Thus

$$s_0^2(\|Hf(p)\| + |\operatorname{grad} f(p)|\,|\sigma|)[2\varepsilon + \varepsilon^{-2}(\operatorname{lip}(\beta)^2 + \varepsilon^3 - 1)(1 + \operatorname{lip}(\beta))] \\ < s_0^2(\|Hf(p)\| + |\operatorname{grad} f(p)|\,|\sigma|)(8\varepsilon) = C\varepsilon s_0^2,$$

where C is a constant depending only on f, p and σ, and we have

$$(*)\qquad |f(\beta(s_0)) - f(p) - H_A f(p)(\tau,\tau)s_0^2/2| < C\varepsilon s_0^2 + o(s_0^2).$$

Consider now the sequences $\{q_k\}$ and $\{w_k\}$, and put $a_k := |q_k - p|$. We have then $q_k - \beta(s_k) = o(s_k)$ and $w_k - v = O(s_k)$ as $k \to \infty$. Thus by 2.2

$$\gamma(s_k)\cdot(q_k - \beta(s_k)) \ge -o(s_k^2), \qquad w_k\cdot(\beta(s_k) - q_k) \ge -o(s_k^2)$$

as $k \to \infty$. But in the mean time

$$(v - \gamma(s_k))\cdot(q_k - \beta(s_k)) = o(s_k^2), \qquad (v - w_k)\cdot(\beta(s_k) - q_k) = o(s_k^2)$$

as $k \to \infty$, and if we add these relations to those above we find that

$$v\cdot(q_k - \beta(s_k)) = o(s_k^2)$$

as $k \to \infty$.

Thus if we put for $0 < t < 1$

$$\phi_k(t) := tq_k + (1-t)\beta(s_k),$$

then we have $\phi_k(t) - p = O(s_k)$ and

$$\begin{aligned} f(q_k) - f(\beta(s_k)) &= \int_0^1 \operatorname{grad} f(\phi_k(t))\cdot(q_k - \beta(s_k))\,dt \\ &= \int_0^1 [\operatorname{grad} f(\phi_k(t)) - \operatorname{grad} f(p)]\cdot(q_k - \beta(s_k))\,dt \\ &\quad - |\operatorname{grad} f(p)|v\cdot(q_k - \beta(s_k)) \\ &= o(s_k^2) \quad \text{as } k \to \infty. \end{aligned}$$

Putting $s_0 = s_1, s_2, \ldots$ in $(*)$ we find that

$$|f(q_k) - f(p) - H_A f(p)(\tau, \tau)s_k^2/2| < o(s_k^2) + C\varepsilon s_k^2.$$

But $\varepsilon > 0$ was chosen quite arbitrarily.

6.6. As a consequence of this last result we have

Proposition. *Let $p \in A$. Then for all small $r > 0$,*

$$\operatorname{reach}[A \cap \overline{B}(p, r)] > 0.$$

Furthermore, if p is a nondegenerate critical point of $h_v|A$, then for small $r > 0$, p is the only critical point of $h_v|A \cap \overline{B}(p, r)$ within $h_v^{-1}(h_v(p))$.

Proof. By [4, 4.10],

$$\operatorname{Nor}[A \cap \overline{B}(p, r), q] = \{u + t(q - p) : u \in \operatorname{Nor}(A, q), t \geq 0\}$$

at any point q with $|q - p| = r$, and by the same result $\operatorname{reach}[A \cap \overline{B}(p, r)] > 0$ provided $p - q \notin \operatorname{Nor}(A, q)$ whenever $|q - p| = r$. That this is the case for all small $r > 0$ follows from 5.2(i).

Now suppose that the second assertion of the proposition is false. Since p is a nondegenerate critical point of $h_v|A$, the result of Federer quoted above implies that there are sequences $A \ni q_k \to p$ and $w_k \in \operatorname{nor}(A, q_k)$ with $v \cdot (q_k - p) \equiv 0$ and w_k of the form

$$w_k = \frac{v + a_k(p - q_k)}{|v + a_k(p - q_k)|} = (1 + a_k^2|p - q_k|^2)^{-1/2}(v + a_k(p - q_k)), \qquad a_k > 0.$$

By 2.6 we have

$$\begin{aligned} \frac{a_k|q_k - p|^2}{(1 + a_k^2|q_k - p|^2)^{1/2}} &= w_k \cdot (p - q_k) \\ &\leq \frac{1}{2}|q_k - p|^2 \operatorname{reach}(A), \end{aligned}$$

whence (putting $R := \operatorname{reach}(A)$)

$$a_k^2\left(1 - \frac{|q_k - p|^2}{R^2}\right) \leq \frac{1}{4R^2},$$

$$\limsup_{k \to \infty} a_k \leq \frac{1}{2R};$$

it follows that $|w_k - v| = o(|q_k - p|)$. Taking a subsequence we may assume that

$$(q_k - p)/|q_k - p| \to \tau \quad \text{and} \quad (w_k - v)/|q_k - p| \to \sigma.$$

The hypothesis of Theorem 6.5 is now fulfilled, and we obtain

$$v \cdot (q_k - p) = \mathrm{II}_A(p, v)(\tau, \tau)|q_k - p|^2/2 + o(|q_k - p|^2).$$

By construction we have $\sigma = a\tau$ for some $a \leq 0$; thus $(\tau, a\tau) \in T_{p,v}\operatorname{nor}(A)$. Since $\mathrm{II}_A(p,v)$ is by hypothesis nondegenerate, it follows that $\mathrm{II}_A(p,v)(\tau,\tau) = a \neq 0$. Therefore the expression above is not 0 for large values of k, and hence we have a contradiction.

6.7. Corollary. *If $v \in S^n$ and p is a nondegenerate critical point of $h_v|A$ of index λ, $v \cdot p = c$, then for all sufficiently small $r > 0$*

$$\chi[A \cap \overline{B}(p,r) \cap h_v^{-1}(-\infty, c+x]]|_{x=-\varepsilon}^{\varepsilon} = (-1)^\lambda$$

for all sufficiently small $\varepsilon > 0$.

Proof. By 4.8, for such r and ε the space $A \cap \overline{B}(p,r) \cap h_v^{-1}(-\infty, c+\varepsilon]$ has the homotopy type of $X := A \cap \overline{B}(p,r) \cap h_v^{-1}(-\infty, c-\varepsilon]$ with a λ-cell attached. Since X has positive reach, by 4.8 and 5.1 X has the homotopy type of a finite CW complex. Therefore the result follows from a cellular approximation (cf. [12, p. 23]) and the long exact sequence associated to the attachment of the λ-cell to X.

6.8. Definition. If $A \subset \mathbf{E}^{n+1}$ is closed, $p \in A$ and $v \in S^n$, then we put

$$\imath(v, A, p) := \lim_{r\to 0}\lim_{\varepsilon \to 0} \chi[A \cap \overline{B}(p,r) \cap h_v^{-1}(-\infty, c+x]]|_{x=-\varepsilon}^{\varepsilon}.$$

6.9. A set of positive reach is obviously a euclidean neighborhood retract. It follows (cf. [3, VIII.6]) that if A, $B \subset \mathbf{E}^{n+1}$, and reach$(A)$, reach$(B)$ and reach$(A \cap B)$ are all positive, then

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

In view of 6.6 we have also for such A, B that for a.e. $v \in S^n$

$$\imath(v, A \cup B, p) = \imath(v, A, p) + \imath(v, B, p) - \imath(v, A \cap B), p)$$

for all $p \in A \cup B$.

Now notice that we may rewrite 6.1 as

$$(\dagger) \qquad \Phi_0(A, U) = [(n+1)\alpha(n+1)]^{-1} \int_{S^n} \sum_{p \in U} \imath(v, A, p)\, dv,$$

with similar formulas for the Φ_i, $i > 1$. From this expression, together with the additivity of $\imath$, we obtain at once

6.10. Theorem (*Zähle*). *Let U_{PR}^* denote the class of all locally finite unions $\bigcup_{i\in J} A_i$ of sets A_i of positive reach in $\mathbf{E}^{n+1}$, such that the intersection $\bigcap_{i \in I} A_i$ has positive reach for every choice of a subset $I \subset J$. Then there is a unique extension of the curvature measures Φ_i from the class of sets of positive reach to the class U_{PR}^* subject to the additivity condition*

$$\Phi_i(A \cup B, \cdot) = \Phi_i(A, \cdot) + \Phi_i(B, \cdot) - \Phi_i(A \cap B, \cdot),$$

and the measures are given by the formulas

$$\begin{aligned}&\beta(n+1,i)\cdot\Phi_{n+1-i}(A,U)\\&\quad=\int_{G(n+1,i)}\int_{v\in\overline{P}\cap S^n}\sum_{q\in P\cap A\cap U}\imath(v,P\cap A,q)\,d\mathscr{H}^{i-1}v\,d\gamma(n+1,i)P,\end{aligned}$$

$i = 1,\cdots,n+1$. *In particular the Gauss-Kronecker curvature measure* $\Phi_0(A,\cdot)$ *admits the expression* (†).

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