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MAGNETIC MONOPOLES ON THREE-MANIFOLDS

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Introduction

In this paper we will investigate magnetic monopoles on an oriented, complete Riemannian 3-manifold M. Basically the result is that we associate to M a collection of moduli spaces of solutions of the (magnetic) monopole equations, provided that M is not compact and that the Riemannian metric on M is 'good near infinity'. Topologically M may be the interior of any compact 3-manifold with boundary, but the metric on an end $\mathbf{R}_{>0} \times S$, S a boundary surface of M, should be approximately of the form $dl^2 + e^{2l}\, ds_S^2$.

This situation is much the same as that for an oriented, Riemannian 4-manifold, which has a collection of instanton moduli spaces associated to it. More specifically, we shall prove that monopoles exist under reasonable conditions, we compute the dimensions of the moduli spaces and study smoothness, orientability and asymptotic behavior. Having obtained these moduli spaces together with their basic properties, the next step would be to exploit the topology of the moduli spaces to define topological invariants for 3-manifolds, just as instanton moduli spaces give invariants for smooth 4-manifolds. This will be discussed in a forthcoming paper.

To carry nontrivial monopoles M should not be compact. This gives rise to hard analytical problems on the 3-manifold, such as those considered in the work of Taubes and Floer. To avoid this we shall exploit the fact that a monopole is a 'time'-invariant instanton on $M \times S^1$. Using the conformal invariance of the instanton equations, we find ourselves working on a conformal compactification X of $M \times S^1$, and studying S^1-invariant instantons on X. The fixed point set of the S^1-action on X now plays the role which the boundary of M played in a direct, 3-dimensional approach. In order to exploit as much as possible the available knowledge about instantons we carefully compare various definitions of S^1-invariance. This enables us to realize the monopole moduli spaces as submanifolds of instanton moduli spaces. This will be carried out in §1.

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To compute dimensions of the monopole moduli spaces we have to compute the index of a certain Dirac operator. In order to do so, one has to come to grips with topological invariants of S^1-equivariant bundles on X. After this we can apply the Atiyah-Singer index formula as for instanton moduli spaces, only we now use the equivariant index formula. This is the subject matter of §2.

In §3 we describe the configuration spaces for the monopole problem and carry out deformation theory. This carries over almost word for word from the instanton case, inserting 'S^1-equivariant' at appropriate places.

Instanton moduli spaces are smooth for a generic metric, are orientable and have a compactification. This can be reinterpreted to show that, for a generic perturbation of the monopole equations (arising from perturbing the invariant metric), the monopole moduli spaces are smooth, and that they are orientable and can be compactified. This is the contents of §4.

In §5 we finally tackle the existence problem for monopoles by adapting Donaldson's alternating procedure. This gives the asymptotic structure of the moduli spaces and also provides information about monopoles on boundary connected sums of 3-manifolds. We have included some examples of these asymptotic models. Also we have made some further comparison with the direct approaches of Taubes and Floer.

Heuristically the alternating procedure constructs a monopole as follows. It places an approximate monopole far away in the manifold; if more than one monopole is glued in, then they are attached by a parameter in a circle. When there is no second homology in M an iterative procedure now shows that a nearby real monopole exists. For example we have

Proposition. *Let M be a 3-manifold with $H^2(M;\mathbf{R}) = 0$, such that a conformal compactification X_0 of $M \times S^1$ can be found. For any mass $m > 0$ a $(4k - 1 + b^1(M))$ parameter family of monopoles of charge k exists.*

Proof. The Proposition follows directly from Corollary 5.3 and Theorem 5.4.

This Proposition applies for example to knot complements, punctured homology spheres and handlebodies, provided the metric is as described in the beginning of the introduction. For manifolds with $H^2(M;\mathbf{R}) \neq 0$, gluing in 'lumps' is possible but the existence of a true monopole is thwarted by $b^2(M)$-obstructions, relating the centers of the lumps to the harmonic two-forms describing the homology. For example on $M = \mathbf{R}\times S^2$ with metric $dl^2 + \cosh^2 l\, ds^2_{S^2}$, monopoles with charge 1 at $-\infty$ and charge 0 at $+\infty$ do not exist; this would have been a lump near $-\infty \times S^2$. Monopoles with both charges equal to one do exist, but their centers are constrained to have equal distance from $\{0\} \times S^2$. Such phenomena seem to be the general laws governing existence, and they are carefully formulated in §5.

As the reader presumably realizes, all our main results rely entirely on fundamental work concerning instantons, which has been carried out by S. K. Donaldson, C. H. Taubes, K. K. Uhlenbeck and others.

The relation of our 4-dimensional approach with the conventional 3-dimensional approach can be seen most clearly by noting that the fixed surfaces of the S^1-action on the 4-manifold are in one-to-one correspondence with the boundary surfaces of the 3-manifold. For example, S^1-equivariant index computations become computations on the fixed surfaces through the Lefschetz theorems, much in the same way as Callias [12] developed index theorems on $\mathbf{R}^n$, playing back computations to the boundary sphere at infinity. Another example arises in the study of limits: any sequence of instantons on a compact 4-manifold has a subsequence which converges away from a finite set of points. For S^1-invariant instantons these points obviously have to lie in the fixed point set of the circle-action. The interpretation is easy: a sequence of monopoles on the 3-manifold can only fail to converge because some lumps move off to the boundary surfaces at infinity.

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1. Compactifications, monopoles and instantons

Let $\overline{M}$ be a compact, oriented 3-manifold with boundary $\delta M = \bigcup_{j=1,\cdots,N} S_j$, each S_j a compact surface without boundary. We shall assume that the interior $M = \overline{M} \backslash \delta M$ of $\overline{M}$ carries a complete Riemannian metric. The boundary surfaces S_j then lie at infinity. General references for the discussion of the Yang-Mills equations which follows are Freed-Uhlenbeck [23] and Jaffe-Taubes [25]. More specifically our approach is a generalization of that in Atiyah [2].

Let $Q \to M$ be a principal fiber bundle with fiber a compact Lie group G. In the sequel we shall take G to be SU(2) or SO(3) for simplicity. Let A_3 be a connection on Q, and Φ a *Higgs field*, i.e., a section of $\mathfrak{g}_Q = Q \times_{\mathrm{Ad}} \mathfrak{g}$, with

$\mathfrak{g}$ denoting the Lie algebra of G. The *Bogomol'nyi equations* for (A_3, Φ), the solutions of which are the (*magnetic*) *monopoles* on M, read

$$d_{A_3}\Phi = - *_3 F^{A_3} \qquad (*_3 \text{ the Hodge star on } M) \tag{1.1}$$

with d_{A_3} denoting covariant derivative and F^{A_3} the curvature of A_3. The standard boundary conditions for (1.1) are

$$\begin{aligned} &E(A_3, \Phi) = (8\pi)^{-1} \int_M |F^{A_3}|^2 + |d_{A_3}\Phi|^2 \, dV_M < \infty, \\ &|\Phi(x)| \to m_j \quad \text{if } x \to S_j \ (m_j \in \mathbf{R}_{\geq 0},\ j = 1, \cdots, N). \end{aligned} \tag{1.2}$$

The inner product on the Lie algebra su(2) has been chosen as $\langle \alpha, \beta \rangle = -2 \cdot \operatorname{tr}(\alpha \cdot \beta)$ and so(3) is isometric to su(2) through the adjoint representation. The m_j are called the *masses* of the monopole. As we indicated in the introduction, we shall not analyze these equations directly but revert to 4-dimensional geometry to study them.

Let $P' = Q \times S^1 \to M \times S^1$ be the pullback of Q under $\pi \colon M \times S^1 \to M$. Note that P' automatically has an S^1-action. Denote by $\partial/\partial\theta$ the vector field on P' induced by the action, and by $d\theta$ the dual one-form. Then $A = \pi^* A_3 + (\pi^*\Phi)\, d\theta$ is an S^1-invariant connection on P', and every S^1-invariant connection on P' can uniquely be written in this way. It is easy to check that (A_3, Φ) satisfies (1.1) iff A satisfies the *anti-self-duality equation*:

$$F^A = - *_4 F^A. \tag{1.3}$$

Here $*_4$ is the Hodge star on $M \times S^1$ considered as an oriented Riemannian product. Solutions of the anti-self-duality equation are called *instantons*.

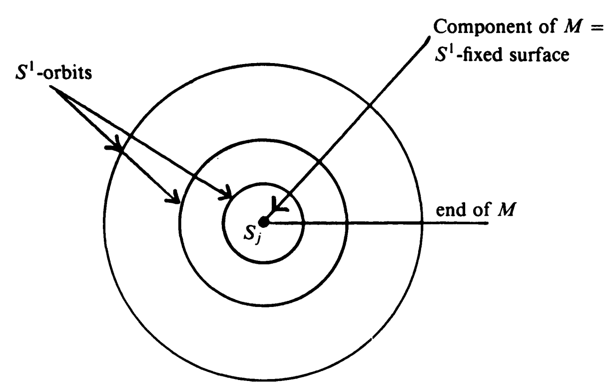

FIGURE 1.1

The next step is to exploit the conformal invariance of (1.3). Suppose that $M \times S^1$ has an oriented, S^1-equivariant conformal compactification

X with $X \backslash (M \times S^1) = \bigcup_j S_j$. Topologically we assume that X is M spun around its boundary surfaces (compare Figure 1.1). The manifold X has an S^1-action which is free away from the *fixed surfaces* S_j, which correspond exactly to the components of the boundary of $\overline{M}$. The normal bundles of the S_j are topologically trivial and of S^1-weight 1. Geometrically we assume that X carries a conformal structure which coincides with the conformal structure underlying the product metric on $M \times S^1$. In Braam [10] we showed that for a large class of *hyperbolic* 3-*manifolds* M such compactifications exist naturally. These are the quotients of H^3 by a geometrically finite group without cusps. It is always possible to give M a metric which is hyperbolic near the boundary and such that the compactification exists. Hyperbolicity is not necessary for the existence of the compactification: by deforming the conformal structure on X we see that there is an infinite dimensional space of metrics on M for which a conformal compactification of $M \times S^1$ exists. Nevertheless, for 'most' metrics on M the compactification cannot be given a *smooth* conformal structure. However, C^k conformal structures ($k \geq 0$) exist under fairly mild conditions on the metric. Most of the results of this paper should hold for such conformal structures (for $k \geq 1$ this will be obvious). In any case the metrics we are dealing with are approximately of the form $dl^2 + e^{2l}\, ds^2_{S_j}$ on the ends $\mathbf{R}_{>0} \times S_j$ of M.

Given X with the S^1-action consider an S^1-equivariant principal SO(3)-bundle P, i.e., a principal SO(3)-bundle with an S^1-action by bundle automorphisms covering the S^1-action on X. We shall be interested in S^1-invariant instantons modulo the group of S^1-invariant *gauge transformations* $GA(P) \cong \mathscr{G}(P)^{S^1}$, where $\mathscr{G}(P) = \Gamma(X, P \times_{\mathrm{Ad}} G)$ is the group of all gauge transformations, i.e., bundle automorphisms fixing the base.

Definition 1.1. An SO(3)-*monopole* on P is an S^1-invariant instanton on P. The *monopole moduli space* equals $\{S^1\text{-invariant instantons}\}/GA(P)$ and will be denoted by $\mathscr{M}(P)$.

Upon restriction to $M \times S^1 \subset X$ our monopoles certainly give rise to 'ordinary' monopoles on M. To check the boundary conditions (1.2), first remark that $E(A_3, \Phi)$ is equal to the *Yang-Mills functional* of A:

$$YM(A) = (16\pi^2)^{-1} \int_X |F^A|^2 \, dV_X, \tag{1.4}$$

so this is certainly finite. The limiting behavior of the Higgs field can be seen as follows. The S^1-action on $P_{|S_j}$ is by gauge transformations. This gives rise to a homomorphism $S^1 \to \mathscr{G}(P_{|S_j}) \cong \Gamma(S_j, P_{|S_j} \times_{\mathrm{Ad}} \mathrm{SO}(3))$, which is essentially a family of homomorphisms $S^1 \to \mathrm{SO}(3)$. Such a family is constant up to conjugation, and it follows that the vectorfield $\partial/\partial\theta$ on P, which is induced by the action, is vertical and of constant integral length on $P_{|S_j}$. Thus $|A(\partial/\partial\theta)|(x) = m_j \in \mathbf{Z}_{\geq 0}$ if $x \in S_j$ and therefore the boundary conditions (1.2) are satisfied.

Remark 1.2. (1) Atiyah conjectured in [2] that any solution of the Bogomol′nyi equation (1.1) on hyperbolic 3-space which satisfies (1.2) with integral m_j arises from a monopole on some S^1-equivariant bundle over S^4. The integrality condition is necessary as there are solutions of (1.1) satisfying (1.2) for nonintegral m_j. One may conjecture the same in our more general situation, and these conjectures have now been proved by L. M. Sibner and R. J. Sibner [27].

(2) The integrality of the m_j's is forced upon the monopole by our approach. However, the theory of connections with monodromy around a surface, developed by the Sibners, will almost certainly allow an easy extension of our results to $m_j \in \mathbf{R}_{\geq 0}$. A. Floer has exploited the fact that in a direct 3-dimensional approach the *masses* m_j can be varied continuously (see [19], [21]).

In our setup the definition of an SU(2)-monopole is slightly more complicated. To explain where the complications come from we make a short digression to discuss various definitions of invariant connections. Let X be a manifold with an S^1-action and suppose P is any principal SU(2)-bundle over X. Let $\mathscr{A}^*(P)$ be the space of irreducible connections on P and $\mathscr{G}(P)$ the group of gauge transformations. Since S^1 is connected, it is always possible to find a bundle automorphism of P covering the diffeomorphism of X induced by the action of an element of S^1 on X. Moreover such bundle automorphisms are unique up to gauge transformations, so S^1 acts on $\mathscr{B}^* = \mathscr{A}^*(P)/\mathscr{G}(P)$ in a natural way. If $[A] \in \mathscr{B}^*$ is a fixed point and if S^1 also acts on P (and therefore on $\mathscr{A}^*(P)$), then A is invariant up to gauge transformations. We shall relate these fixed points to invariant connections under some action of S^1 on P.

Let $\mathscr{H}$ be the group of all bundle automorphisms of P, which cover the action of some element of S^1 on X. Then an S^1-action on P is a homomorphism $S^1 \to \mathscr{H}$ such that under composition with the natural map $\pi\colon \mathscr{H} \to \mathrm{Diff}(X)$ we end up with the S^1-action on X. Remark that there is an exact sequence

$$e \to \mathscr{G}(P) \to \mathscr{H} \to S^1 \to e.$$

A class of connections $[A] \in \mathscr{B}^*$ is a fixed point iff the stabilizers $\mathscr{G}_A \subset \mathscr{G}(P)$ and $\mathscr{H}_A \subset \mathscr{H}$ satisfy that

$$e \to \mathscr{G}_A \to \mathscr{H}_A \to S^1 \to e \tag{1.5}$$

is exact. Note that for irreducible SU(2)-connections A the stabilizer $\mathscr{G}_A = \{\pm 1\}$. (For irreducible SO(3)-connections the stabilizer is trivial.) Thus the only two possibilities for $\mathscr{H}_A$ in (1.5) are $\mathscr{H}_A \cong \widetilde{S}^1$, the double cover of S^1, and $\mathscr{H}_A \cong (\mathbf{Z}/2\mathbf{Z}) \times S^1$. It follows that there exists a unique $\widetilde{S}^1$-action $\phi: \widetilde{S}^1 \to \mathscr{H}$ on P stabilizing A if $[A]$ is a fixed point in $\mathscr{B}^*$. A gauge transform $g \cdot A$ of A is stabilized by the action $g \circ \phi \circ g^{-1}$. (For irreducible SO(3)-connections the situation is simpler: $\mathscr{H}_A = S^1$, and there is a unique S^1-action on P stabilizing A.)

In the next section we shall see that the $\widetilde{S}^1$-actions up to conjugation by gauge transformations can easily be classified. Denote these actions by $\phi_j: \widetilde{S}^1 \to \mathscr{H}$, with j in an indexing set J, and let $\mathscr{A}_j^* \subset \mathscr{A}^*$ be the set of irreducible connections which are invariant under ϕ_j. We have just seen that $\bigcup_{j \in J} \mathscr{A}_j^* \to (\mathscr{B}^*)^{S^1}$ is surjective. Denoting by $\mathscr{G}_j$ the ϕ_j-invariant gauge transformations we see that also

$$\bigcup_{j \in J} (\mathscr{A}_j^* / \mathscr{G}_j) \to (\mathscr{B}^*)^{S^1} \tag{1.6}$$

is surjective. The final claim is

Theorem 1.3. *The map in* (1.6) *is a bijection.*

Proof. We need only show injectivity in view of the above. Now an irreducible connection is stabilized by a unique $\widetilde{S}^1$-action. Suppose that $A_0 = g \cdot A_1$ with $[A_i] \in (\mathscr{B}^*)^{S^1}$, $g \in \mathscr{G}(P)$ and $A_i \in \mathscr{A}_{j_i}$, $i = 0, 1$, for some $j_i \in J$. We have to prove $g \in \mathscr{G}_{j_0}$. From the uniqueness we get that $\phi_{j_1} = g \circ \phi_{j_0} \circ g^{-1}$. But then $j_0 = j_1$ and $g \in \mathscr{G}_{j_0}$, hence (1.6) is injective. q.e.d.

Different $\mathscr{A}_j$ may intersect at reducible $A \in \mathscr{A}$.

Remark. Obviously the above discussion generalizes to connections invariant under the action of a compact Lie group K which acts on X, such that $k^*P = P$ for all $k \in K$. Forgacs and Manton [22] gave a proof of Theorem 1.3 for invariant connections which superficially looks very different. However, solutions of their crucial partial differential equations are sections of the exact sequence (1.5), so the two approaches essentially agree. Other work on invariant gauge fields can be found in Harnad et al. [24].

It is now easy to see what the definition of an SU(2)-monopole should be. Let X be as above, and let P be a principal SU(2)-bundle on which the double cover $\widetilde{S}^1$ of S^1 acts by bundle automorphisms covering the action of S^1 on X.

Definition 1.4. An SU(2)-*monopole* on P is an $\widetilde{S}^1$-invariant instanton on P. The *moduli space of monopoles* is the space $\mathscr{M}(P) = \{\widetilde{S}^1$-invariant instantons on $P\}/\{\widetilde{S}^1$-invariant gauge transformations$\}$.

Example 1.5. (1) The four-sphere S^4 is the S^1-equivariant conformal compactification of $H^3 \times S^1$, with H^3 hyperbolic 3-space. The basic 't Hooft instanton is an instanton on the spin bundle S_- of S^4 which is an SU(2)-bundle. This instanton is S^1-invariant up to gauge transformations if the center of the instanton lies in the fixed point set $S^2 \subset S^4$, but there is no S^1-action on S_- which stabilizes this instanton. There is however an $\widetilde{S}^1$-action on S_- stabilizing the 't Hooft instanton (see Braam [10, §2]); thus the basic instanton is also a hyperbolic monopole with mass equal to 1.

(2) Next we discuss in some detail the hyperbolic version of the 't Hooft-Polyakov monopole of arbitrary mass m on H^3. In geodesic normal coordinates the metric of H^3 reads

$$ds^2 = dl^2 + \sinh^2 l \cdot (d\psi^2 + \sin^2 \psi \cdot d\phi^2),$$

with $l \in \mathbf{R}_{\geq 0}$ and (ψ, ϕ) the standard coordinates on S^2. If $\tau_j \in \mathrm{su}(2)$ is a basis satisfying $[\tau_i, \tau_j] = -\varepsilon_{ijk} \cdot \tau_k$, then the connection and Higgs field for the 1-monopole of mass $m = \alpha - 1$ with center at $l = 0$ are equal to

$$A_3 = \left(\frac{\alpha \cdot \sinh l}{\sinh(\alpha l)}\right) \cdot \tau_2 \cdot d\psi + \left(\frac{\alpha \cdot \sinh l}{\sinh(\alpha l)} \cdot \sin(\psi) \cdot \tau_1 + \cos(\psi) \cdot \tau_3\right) \cdot d\phi,$$

$$\Phi = [\coth l - \alpha \cdot \coth(\alpha l)] \cdot \tau_3,$$

(see Chakrabarti [13]). We see that Φ vanishes at the point $l = 0$, and this defines the *center of the monopole* to be the point $l = 0$. All other monopoles of charge 1 (see §2) and mass m can be obtained from the given one by applying an isometry of H^3. Having written down the formulas, it is worth making some further remarks.

First of all, observe that on the two-sphere at ∞ the curvature F^{A_3} equals

$$F^{A_3}_{|S^2_\infty} = \sin(\psi) \cdot \tau_3 \cdot d\phi \wedge d\psi,$$

which is the ordinary volume element times τ_3. We see that near infinity the connection approximates a U(1)-connection on the two-sphere and Φ approaches the unique covariantly constant section of $P \times_{\mathrm{Ad}} \mathrm{su}(2)$ of length m for this connection. Also we see that the bigger m is the more localized the non-Abelian part of the connection is.

If the mass m is an integer, then the monopole can also be interpreted as an S^1-invariant instanton on S^4, but for nonintegral mass there is monodromy around $S^2 \subset S^4$.

(3) Monopoles on $S^2 \times \mathbf{R}$. Let $M = S^2 \times \mathbf{R}$, with metric $ds^2 = \cosh^2 l \cdot ds^2_{S^2} + dl^2$, where $ds^2_{S^2}$ is the SO(3)-invariant metric of curvature 1 on S^2. An easy computation shows that the conformal compactification X of $M \times S^1$ is $S^2 \times S^2$, where the S^1-action is earth rotation in the second S^2, and with conformal structure induced by the product metric. Recall that any complex line bundle on S^2 is isomorphic to $\mathrm{O}(n)$, the nth power of the positive Hopf bundle over S^2. The bundle $\mathrm{O}(n)$ admits an $\widetilde{S}^1$-action for any weights $w_0, w_\infty \in \mathbf{Z}$ satisfying $2n = w_0 - w_\infty$. Let $\pi_j \colon X \to S^2$ be the projection on the jth factor, and denote by $\mathrm{O}(n, m)$ the bundle $\pi_1^* \mathrm{O}(n) \otimes \pi_2^* \mathrm{O}(m)$ over X.

The line bundles $\mathrm{O}(k, -k)$ $(k \in \mathbf{Z})$ carry a unique (up to gauge transformations) anti-self-dual connection. For any weights $w_0, w_\infty \in \mathbf{Z}$ satisfying $-2k = w_0 - w_\infty$, there is an $\widetilde{S}^1$-action with these weights, which leaves this connection invariant. Hence one finds reducible monopoles on the vector bundle $E = \mathrm{O}(k, -k) \oplus \mathrm{O}(-k, k)$. The masses of these monopoles satisfy $m_j = |w_j|$.

We shall come back to these examples throughout the paper to illustrate the theory. Other examples can be found in §5 of Braam [10] and in §6 below. We proceed to study the topology of the equivariant bundles occurring in the definitions above.

2. Topology of bundles and index computations

Our aim here is twofold. We start by discussing the topological invariants of the equivariant bundles occurring in the definitions of monopoles, and, secondly, we compute the equivariant index of the omnipresent Dirac operator, which governs the deformation theory of monopoles.

We shall start discussing $\widetilde{S}^1$-equivariant principal SU(2)-bundles. Afterwards we will indicate the changes which have to be made for SO(3)-bundles. As we explained in §1, for every fixed surface $S_j \subset X$ we get a *mass* or *weight* $m_j \in \mathbf{Z}_{\geq 0}$. Let $E = P \times_{\mathrm{SU}(2)} \mathbf{C}^2$. If $m_j \neq 0$ then $E_{|S_j}$ splits as

$$L_j \oplus L_j^*, \tag{2.1}$$

where $\widetilde{S}^1$ acts on L_j (L_j^*) by scalar multiplication with weight m_j $(-m_j)$. Now a further invariant is the *charge* k_j, which is defined by

$$k_j \cdot x_j = c_1(L_j^*) \in H^2(S_j; \mathbf{Z}), \tag{2.2}$$

with $k_j \in \mathbf{Z}$ and x_j a positive generator of $H^2(S_j;\mathbf{Z})$; this requires an orientation of S_j: X gets an orientation from the orientation of the 3-manifold, the normal bundle N_j of S_j in X is oriented by the S^1-action, thus S_j inherits an orientation. If $m_j = 0$ then $E_{|S_j}$ is trivial as an $\widetilde{S}^1$-equivariant bundle, and we shall leave k_j undefined.

There is one important constraint on the m_j. Recall that $-1 \in \widetilde{S}^1$ acts trivially on X, so it acts on E by a gauge transformation of order two, that is, it acts as -1 or $+1$ on E. It follows that all m_j are either even or odd. That this is the only constraint follows from

Proposition 2.1. *Isomorphism classes of $\widetilde{S}^1$-equivariant* SU(2)*-bundles are in one-to-one correspondence with tuples of integers $(m_j, k_j)_{j=1,\cdots,N}$ where $k_j \in \mathbf{Z}$ is undefined for $m_j = 0$, and the $m_j \in \mathbf{Z}_{\geq 0}$ all are either even or odd.*

Proof. Clearly the m_j and k_j depend only on the isomorphism class.

To understand the structure of these $\widetilde{S}^1$-equivariant SU(2)-bundles first consider the restriction to the open set $M \times S^1 \subset X$. Use the double cover $p\colon M \times \widetilde{S}^1 \to M \times S^1$ to form the pullback $P' = p^* P_{|M\times S^1}$, which carries a free $\widetilde{S}^1$-action. Because the action is free, P' is a pullback of an SU(2)-bundle P'' on M. As all SU(2)-bundles on M are trivial, it follows that the $\widetilde{S}^1$-equivariant isomorphism class of $P_{|M\times S^1}$ is determined by the sign of the action of $-1 \in \widetilde{S}^1$.

Next we concentrate on a neighborhood of the $S_j \subset X$: such a neighborhood is always S^1-equivariantly diffeomorphic to $S_j \times \mathbf{C}$, because the normal bundles N_j of S_j are trivial. It is easy to see that $E_{|S_j\times\mathbf{C}}$ is $\widetilde{S}^1$-equivariantly isomorphic to $\pi^* E_{|S_j}$ with $\pi\colon S_j \times \mathbf{C} \to S_j$ the projection. Clearly $\widetilde{S}^1$-equivariant isomorphism classes of $\widetilde{S}^1$-equivariant SU(2)-bundles on S_j are determined by a pair (m_j, k_j).

Finally we need an $\widetilde{S}^1$-equivariant transition function $(S_j \times \mathbf{C}) \cap (M \times S^1) \cong S_j \times (\mathbf{C}\backslash\{0\}) \to \mathrm{SU}(2)$. This is the same as a $\mathbf{Z}/2\mathbf{Z}$-equivariant transition function on a slice $S_j \times \mathbf{R}_{>0}$. Such a function exists if and only if the parity of m_j agrees with the sign of the action on $P_{|M\times S^1}$. The transition functions in question are trivial up to equivalence, as Maps(S_j, SU(2)) is connected. This proves the proposition.

For an S^1-equivariant SO(3)-bundle Q let $\mathfrak{g}_Q = Q \times_{\mathrm{Ad}} \mathrm{so}(3)$ be the bundle of Lie algebras. If the action on the restriction of Q to S_j is nontrivial, then the S^1-action splits $\mathfrak{g}_{Q|S_j}$ as

$$\mathfrak{g}_{Q|S_j} = K_j \oplus \mathbf{R}, \tag{2.3}$$

where S^1 acts on K_j by scalar multiplication of weight $m_j \in \mathbf{Z}_{>0}$ thereby turning K_j into a complex line bundle with

$$c_1(K_j) = -q_j \cdot x_j \qquad (q_j \in \mathbf{Z}).$$

If the action on $Q_{|S_j}$ is trivial, we shall put $m_j = 0$ and leave q_j undefined. If Q is associated to an $\widetilde{S}^1$-equivariant SU(2)-bundle by the adjoint representation, then the two definitions of m_j agree, and $q_j = 2 \cdot k_j$, because $K_j = L_j \otimes L_j$. Using the map $j\colon M \to X\colon m \to (m, 1)$ another invariant of SO(3)-bundles is the second Stiefel-Whitney class $w_2(j^*Q) \in H^2(M; \mathbf{Z}/2\mathbf{Z})$. The following proposition shows that these invariants determine the bundle, and tells which relations exist among the invariants.

Proposition 2.2. *Isomorphism classes of S^1-equivariant* SU(3)-*bundles on X are in one-to-one correspondence with sets of invariants $((m_j, q_j)_{j=1,\cdots,N}, w_2)$ with $m_j \in \mathbf{Z}_{\geq 0}$, $q_j \in \mathbf{Z}$, and $w_2 \in H^2(M; \mathbf{Z}/2\mathbf{Z})$. The only constraints on these variables are that $w_2(Q)_{|S_j} = q_j \cdot x_j \pmod 2$ for all j for which q_j is defined.*

Proof. The proof is very similar to that of Proposition 2.1. Now $Q_{|M\times S^1}$ is the pullback of an SO(3)-bundle Q' on M. Such Q' are determined by $w_2 = w_2(Q') \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ (see Freed-Uhlenbeck [23, Appendix E]). Any $w_2 \in H^2(M; \mathbf{Z}/2\mathbf{Z})$ occurs as the class of a bundle. To see this, choose a map $M \to K(\mathbf{Z}/2\mathbf{Z}, 2)$; to get a lift $M \to \mathrm{BSO}(3)$, apply obstruction theory to the pullback of the fibration $\mathrm{BSU}(2) \to \mathrm{BSO}(3) \to K(\mathbf{Z}/2\mathbf{Z}, 2)$ under the given map (for the fibration see Freed a.o., loc. cit.). Existence of the transition functions requires that $Q_{|S_j}$ is isomorphic to Q restricted to the end of M going out towards S_j. This implies the relation between q_j and w_2, because $w_2(Q_{|S_j}) = q_j \cdot x_j \pmod 2$ if q_j is defined. q.e.d.

There is a difference in the nature of the constraints occurring in the two propositions above. The equality of the parity in Proposition 2.1 is an artifact of our construction, whereas the constraint on the w_2 would also appear in a direct approach on M. Let us finally remark that if P and Q are as above, then there are various ways of proving that

$$c_2(P) = \sum_j m_j \cdot k_j, \qquad p_1(Q) = \sum_j 2 \cdot m_j \cdot q_j, \tag{2.4}$$

(see Braam [10, §5] and Atiyah [2]).

Next we turn to the index calculation. We only do the case of an $\widetilde{S}^1$-equivariant SU(2)-bundle P. Let $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathrm{su}(2)$ be the bundle of Lie algebras associated to P. Denote by Λ^j the vector bundle of j-forms on X, by Λ^2_+ the bundle of self-dual 2-forms on X and by P_+ the projection

$\Lambda^2 \to \Lambda^2_+$. The basic elliptic complex $\{P\}$ occurring in the analysis of instantons reads ($A \in \mathscr{A}(P)^{\widetilde{S}^1}$ a monopole on X)

$$\Gamma(X\mathfrak{g}_P) \xrightarrow{d_A} \Gamma(X\Lambda^1 \otimes \mathfrak{g}_P) \xrightarrow{P_+d_A} \Gamma(X\Lambda^2_+ \otimes \mathfrak{g}_P). \tag{2.5}$$

Here the complex, and its cohomology vector spaces H^j are acted upon by $\widetilde{S}^1$. In what follows we shall consider only this circle action and therefore denote it by S^1; thus S^1-equivariant, S^1-invariant etc. is to be understood with respect to the $\widetilde{S}^1$-action. For our study of monopoles we shall be interested in the weight zero subspace H^1_0 of H^1: under the assumption $H^0_0 = H^2_0 = 0$, H^1_0 is the tangent space at A to the moduli space of monopoles on X (see §3).

The Atiyah-Segal-Singer-Lefschetz formula expresses the S^1-character

$$\mathrm{ind}_{S^1}(\{P\} \otimes \mathbf{C}) \in R(S^1) = \mathbf{Z}[t, t^{-1}] \tag{2.6}$$

of the virtual representation $HT = (H^0 - H^1 + H^2) \otimes \mathbf{C}$, in terms of topological data. We find

$$\begin{aligned} I &\stackrel{\text{def}}{=} \dim_{\mathbf{R}} H^1_0 - \dim_{\mathbf{R}} H^0_0 - \dim_{\mathbf{R}} H^2_0 \\ &= -\int_{S^1} \mathrm{ind}(\{P\} \otimes \mathbf{C})(t)\, dt \\ &= -\text{constant term in } \mathrm{ind}_{S^1}(\{P\} \otimes \mathbf{C})(t) \in \mathbf{Z}[t, t^{-1}]. \end{aligned} \tag{2.7}$$

To compute this index first recall from Atiyah-Hitchin-Singer [6] that the operator

$$d_A \oplus (P_+d_A)^* \colon \Gamma((\mathfrak{g}_P \oplus (\mathfrak{g}_P \otimes \Lambda^2_+)) \otimes \mathbf{C}) \to \Gamma(\mathfrak{g}_P \otimes \Lambda^1 \otimes \mathbf{C})$$

is nothing but the Dirac operator on $S_+ \otimes \mathfrak{g}_P$:

$$D_A \colon \Gamma(S_+ \otimes S_+ \otimes \mathfrak{g}_P) \to \Gamma(S_- \otimes S_+ \otimes \mathfrak{g}_P), \tag{2.8}$$

where S_+, S_- are spin bundles of positive and negative chirality on X. Denote the complex (2.8) by $\{P\} \otimes \mathbf{C}$ too.

A computationally pleasant way to find the index is to invoke equivariant cohomology (see Atiyah-Bott [3] for background). Let ES^1 and BS^1 be the universal bundle and classifying space for S^1. For any S^1-manifold Y, define the homotopy quotient Y_{S^1} to be the associated bundle $Y_{S^1} = ES^1 \times_{S^1} Y$ over BS^1. Then the equivariant cohomology $H^*_{S^1}(Y; \mathbf{Z})$ is by definition $H^*(Y_{S^1}; \mathbf{Z})$, which is a module over $H^*(BS^1; \mathbf{Z})$. For an S^1-equivariant vector bundle $V \to Y$ define $\mathrm{ch}_{S^1}(V)$ to be $\mathrm{ch}(V_{S^1}) \in H^*_{S^1}(Y; \mathbf{Z})$ and similarly for other characteristic classes. For an S^1-representation V, the map

$$\begin{gathered} R(S^1) = \mathbf{Z}[t, t^{-1}] \to H^{**}(BS^1) = \text{completion of } H^*_{S^1}(\text{point}) = \mathbf{Z}[[u]], \\ V \to \mathrm{ch}_{S^1}(V) = \mathrm{ch}(ES^1 \times_{S^1} V) \end{gathered}$$

is given by $\sum a_j t^j \to \sum a_j (e^u)^j$ (see Atiyah-Hirzebruch [4, 4.3]). Therefore (2.7) equals minus the constant term in an expansion of $\mathrm{ch}(HT_{S^1})$ in e^u.

Now the index formula (Atiyah-Segal [7], Atiyah-Singer [8]) gives

$$\mathrm{ch}_{S^1}(HT) = \pi_*^{S^1}\{\mathrm{ch}_{S^1}(\mathfrak{g}_P \otimes S_+) \cdot \widehat{A}_{S^1}(X)\},$$

with $\pi_*^{S^1}$ the push-forward to a point in equivariant cohomology. By the localization formula in equivariant cohomology (Atiyah-Bott [3]) this equals

$$\sum_j \int_{S_j} e_{S^1}(N_j)^{-1} \cdot \mathrm{ch}_{S^1}(\mathfrak{g}_P \otimes S_+)_{|S_j} \cdot \widehat{A}_{S^1}(X)_{|S_j},$$

with $e_{S^1}(N_j)$ the equivariant Euler class of the normal bundle N_j of S_j. We proceed to compute the relevant characteristic classes restricted to S_j.

The equivariant Euler class of N_j is equal to

$$2 \cdot u \in H_{S^1}^*(S_j) \cong H^*(S_j) \otimes H^*(BS^1) \cong H^*(S_j) \otimes \mathbf{Z}[u].$$

Using the decomposition (2.1) we obtain (with x_j a positive generator of $H^2(S_j)$):

$$\begin{aligned}\mathrm{ch}_{S^1}(\mathfrak{g}_P)_{|S_j} &= \mathrm{ch}_{S^1}(L_j^2) + \mathrm{ch}_{S^1}(L_j^{-2}) + \mathrm{ch}_{S^1}(\mathbf{C})\\ &= e^{2(-k_j x_j + m_j u)} + e^{2(k_j x_j - m_j u)} + 1\\ &= (1 + 2 \cdot \cosh(2 \cdot m_j \cdot u)) - 4 \cdot k_j \cdot x_j \cdot \sinh(2 \cdot m_j \cdot u).\end{aligned}$$

Furthermore, $S_+ \otimes S_+ = (\Lambda_+^2 \otimes \mathbf{C}) \oplus \mathbf{C}$ and $\Lambda_+^2 \otimes \mathbf{C}_{|S_j} \cong \mathbf{C} \oplus (\Lambda^{1,0}(S_j) \otimes_{\mathbf{C}} N_j^*) \oplus (\Lambda^{0,1}(S_j) \otimes_{\mathbf{C}} N_j)$, where N_j has been given a $\mathbf{C}$-structure, using the S^1-action. It follows that

$$\begin{aligned}\mathrm{ch}_{S^1}(S_+) &= e^{(-u + \frac{1}{2} \cdot c_j \cdot x_j)} + e^{(u - \frac{1}{2} \cdot c_j \cdot x_j)}\\ &= 2 \cdot \cosh(u) - c_j \cdot x_j \cdot \sinh(u),\end{aligned}$$

where $c_j \cdot x_j = c_1(\Lambda^{1,0} S_j) = 2 \cdot (g_j - 1) \cdot x_j$, with g_j the genus of S_j.

Finally we need the equivariant $\widehat{A}$ genus of X restricted to S_j. It follows from $T^*X_{|S_j} \cong T^*S_j \oplus N_j^*$ that

$$(p_1)_{S^1}(T^*X)_{|S_j} = (c_1)_{S^1}^2 - 2(c_2)_{S^1} = (c_j \cdot x_j - 2 \cdot u)^2 + 4 \cdot u \cdot c_j \cdot x_j = (2u)^2,$$

so

$$\widehat{A}_{S^1}(X)_{|S_j} = \frac{2 \cdot u}{e^u - e^{-u}} = u \cdot \sinh^{-1}(u).$$

Combining the formulas above we obtain:

$$\begin{aligned}
\mathrm{ch}_{S^1}(HT) &= \pi_*^{S^1}\{\mathrm{ch}_{S^1}(\mathfrak{g}_P \otimes S_+) \cdot \widehat{A}_{S^1}(X)\} \\
&= \sum_j \left(\int_{S_j} e_{S^1}(N_j)^{-1} \cdot \mathrm{ch}_{S^1}(\mathfrak{g}_P \otimes S_+)_{|S_j} \cdot \widehat{A}_{S^1}(X)_{|S_j} \right) \\
&= \sum_j \left[\int_{S_j} \tfrac{1}{2} \cdot \sinh^{-1}(u) \cdot (-2 \cdot \cosh(u) \cdot 4 \cdot k_j \cdot x_j \cdot \sinh(2 \cdot m_j \cdot u) \right. \\
&\qquad \left. - c_j \cdot x_j \cdot (1 + 2 \cdot \cosh(2 \cdot m_j \cdot u) \cdot \sinh(u))) \right] \\
&= \sum_{m_j \neq 0} (-4k_j - \tfrac{1}{2} \cdot c_j) - \sum_{m_j = 0} \tfrac{3}{2} \cdot c_j + \sum_{n \neq 0} (\cdots) e^{nu}.
\end{aligned}$$

This results in

Theorem 2.3. *The constant term in* $-\mathrm{ind}(\{P\} \otimes \mathbf{C})(t)$ *is equal to*

$$I(m_j, k_j) = \sum_{m_j \neq 0} (4 \cdot k_j + (g_j - 1)) + \sum_{m_j = 0} 3 \cdot (g_j - 1), \tag{2.9}$$

with g_j *the genus of* S_j. *Moreover if all* m_j *are nonzero, then this simplifies to*

$$I(m_j, k_j) = \sum_j 4 \cdot k_j - (1 - b^1(M) + b^2(M)), \tag{2.10}$$

$$I(0, 0) = 3 \cdot (b^1(M) - b^2(M) - 1), \tag{2.11}$$

where $b^i(M) = \dim H^i(M; \mathbf{R})$.

Proof. Only (2.10) and (2.11) remain to be proved and follow directly from Braam [10, Proposition 2.2].

Remarks 2.4. (1) For S^1-equivariant SO(3)-bundles Q with invariants $((m_j, q_j), w_2)$ minus the constant term in the S^1-equivariant index of the Dirac operator on $\mathfrak{g}_Q \otimes S_+$ equals

$$I((m_j, q_j), w_2) = \sum_{m_j \neq 0} (2 \cdot q_j + (g_j - 1)) + \sum_{m_j = 0} 3 \cdot (g_j - 1). \tag{2.12}$$

The proof is the same as that of Theorem 2.3.

(2) If X is a Kähler manifold we could have used the equivariant Riemann-Roch formula for the Dolbeault complex on $\mathfrak{g}_P \otimes \mathbf{C}$. One sees easily that $a \to a^{0,1}$ induces an isomorphism $H^1 \to H^1(\mathfrak{g}_P \otimes \mathbf{C})$ (see Donaldson [17]).

The restrictions to S_j of the characteristic classes occurring in this procedure show u^{-1} terms. These terms cancel upon summation. Conversely, one deduces in this way that the X built from 3-manifolds are not complex manifolds if the poles do not cancel.

Example 2.5. (1) The basic monopole on H^3 (see Example 1.5(2)) has $F^{A_3}_{|S^2_\infty} = \tau_3 \cdot \sin(\psi) \cdot d\phi \wedge d\psi$. As an endomorphism of $L_1 \oplus L_1^*$, the matrix τ_3 equals

$$\begin{bmatrix} \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i \end{bmatrix},$$

so

$$k_1 = c_1(L_1^*) = \frac{-1}{2\pi i} \cdot \int_{S^2} (-\tfrac{1}{2}i) \cdot \sin(\psi) \cdot d\phi \wedge d\psi = 1.$$

Hence the basic monopole of mass m has charge 1.

The dimension of H_0^1 equals 3 (we shall see in §§3 and 4 that H^0 and H^2 vanish), and this agrees with the fact that one-monopoles of mass m are determined by their center in H^3.

(2) Monopoles on $S^2 \times \mathbf{R}$ (compare Example 1.5(3)). There is a reducible monopole A_0 on the vector bundle $E = \mathrm{O}(k, -k) \oplus \mathrm{O}(-k, k)$. The $\widetilde{S}^1$-action on $\mathrm{O}(k, -k)$ has weights w_0, w_∞ satisfying $-2k = w_0 - w_\infty$, and the masses m_j equal $|w_j|$. The charges satisfy $k_0 = -\mathrm{sign}(w_0)k$, $k_\infty = \mathrm{sign}(w_\infty)k$ (observe that the orientation we assign to $S^2 \times \{\infty\}$ is opposite to the orientation it carries as a complex submanifold of $S^2 \times S^2$). So nontrivial monopoles exist with two charges of opposite sign, provided the masses m_0 and m_∞ are not equal. Physically this may indicate that some gravitational effect is balancing the monopole charge attraction.

Next we pay attention to the cohomology groups of the deformation complex for this monopole. There are several cases to be considered. To simplify things we assume $k > 0$ and $w_0 > 0$, i.e., we discard flat monopoles and possibly reverse the roles of 0 and ∞. The three cases to be considered are (compare (2.9)):

(1) $m_\infty = 0$, $I = 4k - 4$,

(2) $m_\infty > 0$, $k_\infty > 0$, $I = 8k - 2$,

(3) $m_\infty > 0$, $k_\infty < 0$, $I = -2$.

In order to compare this with sheaf cohomology recall that:

$$H^0(S^2, \mathrm{O}(l)) = \begin{cases} 0, & l < 0, \\ \mathbf{C}^{l+1}, & l \geq 0, \end{cases}$$
$$H^1(S^2, \mathrm{O}(l)) = H^0(S^2, \mathrm{O}(-l-2)).$$

If $\widetilde{S}^1$ acts on $O(l)$ with weights p_0, p_∞, then

$$H^0(S^2, O(l))^{\widetilde{S}^1} = \begin{cases} \mathbf{C} & \text{if } p_i \text{ is even, } p_0 \geq 0 \text{ and } p_\infty \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$H^1(S^2, O(l))^{\widetilde{S}^1} = \begin{cases} \mathbf{C} & \text{if } p_i \text{ is even, } p_0 + 2 \leq 0 \text{ and } -p_\infty + 2 \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using Remark 2.4(2) and the Künneth formula we obtain

$$\begin{aligned} H_0^1 &\cong H^1(X, \mathbf{C} \oplus O(2k, -2k) \oplus O(-2k, 2k))^{\widetilde{S}^1} \\ &\cong \{H^0(S^2, O(2k)) \otimes H^1(S^2, O(-2k))^{\widetilde{S}^1}\} \\ &\quad \oplus \{H^1(S^2, O(-2k)) \otimes H^0(S^2, O(2k))^{\widetilde{S}^1}\}. \end{aligned}$$

So H_0^1 is isomorphic to:

$$\begin{array}{ll} H^1(S^2, O(-2k)) \cong \mathbf{C}^{2k-1} & \text{in case (1),} \\ H^0(S^2, O(+2k)) \oplus H^1(S^2, O(-2k)) \cong \mathbf{C}^{2k+1} \oplus \mathbf{C}^{2k-1} & \text{in case (2),} \\ 0 & \text{in case (3).} \end{array}$$

It is not hard to show that in all three cases

$$H_0^0 = \mathbf{R} \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cong \mathbf{R}, \qquad H_0^2 = \mathbf{R} \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \omega \cong \mathbf{R}$$

with ω the self-dual Kähler form of $S^2 \times S^2$, in agreement with the index formulas.

3. Deformation theory of monopoles

Let $A_0 \in \mathscr{A}(P)^{S^1}$ be a monopole, i.e.,

$$P_+ F^{A_0} = \tfrac{1}{2}(F^{A_0} + *_4 F^{A_0}) = 0.$$

We shall look for nearby solutions $A = A_0 + p$ of these equations which are not gauge-equivalent to A_0. This deformation theory will turn out to be very similar to the deformation theory for instantons (see Donaldson [16, §4] for the most complete treatment thus far). Before we start, it is necessary to establish some basic facts concerning the configuration space.

Choose an S^1-invariant metric representing the conformal structure on X, and let P be an $\widetilde{S}^1$-equivariant SU(2)-bundle; we leave the SO(3) case to the reader. Recall that $\mathscr{A}(P)$, the space of connections on P, has completions

$$\mathscr{A}_{p,k}(P) = A + L_k^p[\Lambda^1(\mathfrak{g}_P)] \qquad (A \in \mathscr{A}(P),\ p \geq 1,\ k \geq 0),$$

turning it into an affine Banach manifold; here $L_k^p(E)$ means sections of the Hermitian vector bundle E whose derivatives up to order k are in L^p. $\widetilde{S}^1$ acts continuously by affine transformations, but not smoothly on $\mathscr{A}_{p,k}(P)$. Define the configuration space $C_{p,k}(P)$ of the monopole problem to be

$$C_{p,k}(P) = \mathscr{A}_{p,k}(P)^{\widetilde{S}^1}. \tag{3.1}$$

Then $C_{p,k}(P)$ is a closed affine submanifold of $\mathscr{A}_{p,k}(P)$, equal to the L_k^p-closure of $\mathscr{A}(P)^{\widetilde{S}^1}$ in $\mathscr{A}(P)$. The underlying vector space is $\mathscr{W}_{p,k}^1$ where $\mathscr{W}_{p,k}^i$ and $\mathscr{W}^i$ are defined as

$$\mathscr{W}_{p,k}^i = L_k^p(\Lambda^i(\mathfrak{g}_P))^{\widetilde{S}^1}, \qquad \mathscr{W}^i = \Gamma\Lambda^i(\mathfrak{g}_P)^{\widetilde{S}^1}. \tag{3.2}$$

In the same way

$$GA_{p,k+1}(P) = \mathscr{G}_{p,k+1}(P)^{\widetilde{S}^1} \tag{3.3}$$

is a closed Banach Lie subgroup of the group of gauge transformations $\mathscr{G}_{p,k+1}(P)$, provided $k + 1 - p/4 > 0$ (see Freed-Uhlenbeck [23]). The Lie algebra of $GA_{p,k+1}$ is $\mathscr{W}_{p,k+1}^0$. The action of $GA_{p,k+1}(P)$ on $C_{p,k}(P)$ is smooth, being the restriction of the $\mathscr{G}_{p,k+1}(P)$ action on $\mathscr{A}_{p,k}(P)$. We shall assume that $k \geq 1$, $p \geq 2$ and $k - p/4 > 0$, then we are in the stable range for multiplication with L_k^p continuous and this will ensure continuity of the operators which we use below; the condition $p \geq 2$ gives our L_k^p-spaces a positive definite inner product.

Remark. The various Sobolev spaces can be chosen in essentially different ways if one works on the 3-dimensional manifold itself. However, a direct approach on the 3-manifold seems hard at the moment because the Fredholm theory for the elliptic operators involved has yet to be developed. A. Floer has developed such a theory for *asymptotically Euclidean* 3-manifolds (see [20]).

We proceed to construct a slice at $A \in C_{p,k}(P)$ for the $GA_{p,k+1}(P)$ action on $C_{p,k}(P)$. Identifying the tangent space $T_A C_{p,k}(P)$ with the space of $\widetilde{S}^1$-invariant forms $\mathscr{W}_{p,k}^1$ we see that

$$T_A(GA_{p,k+1}(P) \cdot A) = d_A(\mathscr{W}_{p,k+1}^0) \subset T_A C_{p,k}(P). \tag{3.4}$$

Next remark that the Green operator $G_A\colon \mathscr{W}_{p,k+1}^0 \to \mathscr{W}_{p,k-1}^0$ of $d_A^* \circ d_A$, which we define to be equal to $(d_A^* \circ d_A)^{-1}$ on $(\ker d_A)^{\perp_{L^2}}$ and zero on $\ker d_A$, is automatically $\widetilde{S}^1$-invariant. Writing $p \in T_A C_{p,k}(P)$ as

$$p = d_A G_A d_A^* \cdot p + (1 - d_A G_A d_A^*)p, \tag{3.5}$$

we see that $(T_A(GA_{p,k+1}(P)\cdot A))^{\perp_{L^2}} \subset T_A C_{p,k}(P)$ is given by

$$H_A = \mathscr{W}^1_{p,k} \cap \ker d_A^*, \tag{3.6}$$

and that $H_A \oplus T_A(GA_{p,k+1}(P)\cdot A) = T_A C_{p,k}(P)$. This is exactly the same as in the instanton case, apart from the fact that we only consider $\widetilde{S}^1$-invariant objects here.

In precisely the same way as in Atiyah-Hitchin-Singer [6] or Freed-Uhlenbeck [23], it follows that $S_A = A + W \subset C_{p,k}(P)$ is a slice for the $GA_{p,k+1}(P)$-action, with $W \subset H_A$ a small neighborhood of 0. The stabilizer $\Gamma_A \subset GA_{p,k+1}(P)$ acts on H_A, and W can always be chosen in such a way that S_A is Γ_A-invariant. This stabilizer Γ_A consists of $\widetilde{S}^1$-invariant, covariantly constant sections of $\mathscr{G}(P) \cong P \times_{\mathrm{Ad}} \mathrm{SU}(2)$, and its Lie algebra equals $\ker(d_A) \cap \mathscr{W}^0$. If one of the integral invariants m_j of P is nonzero, then Γ_A is isomorphic to $S^1 \subset \mathrm{SU}(2)$ for reducible A and to $\{\pm 1\}$ for irreducible A. If all $m_j = 0$, and A is a connection with holonomy contained in $\{\pm 1\}$, then $\Gamma_A \cong \mathrm{SU}(2)$.

Just as for instantons, $B_{p,k}(P) = C_{p,k}(P)/GA_{p,k+1}(P)$ is a smooth Banach manifold away from the reducible connections, and there are singularities at the reducible connections (cones in $\mathbf{C}P^\infty$), which are caused by a jump of stabilizers. In any case $B_{p,k}(P)$ is a Hausdorff topological space, and this property is inherited by the moduli spaces of monopoles. Putting the results together we have proved

Theorem 3.1. *For $k - 4/p > 0$, $p \geq 2$, the configuration space $C_{p,k}(P)$ (see (3.1)) is a smooth affine Banach manifold with a smooth action of the Banach Lie group $GA_{p,k+1}(P)$ (see (3.3)) on it. Slices for the action at any $A \in C_{p,k}(P)$ exist, and are equal to $A+W$, where W is a small neighborhood of 0 in H_A (see (3.6)). The quotient space $B_{p,k}(P) = C_{p,k}(P)/GA_{p,k+1}(P)$ is a Hausdorff topological space, and a smooth Banach manifold away from the reducible connections.*

We now proceed with a standard Lyapunov-Schmidt procedure to treat the deformation theory of a monopole; we stick to the assumptions made about p, k in Theorem 3.1. Suppose that $A_0 \in C_{p,k}(P)$ is a monopole. Of interest are the zeros of

$$K\colon H_{A_0} \to \mathscr{W}^2_{+,p,k-1}\colon p \to P_+F^{A_0+p} = P_+(d_{A_0}p + \tfrac{1}{2}\cdot[p,p]), \tag{3.7}$$

where $\mathscr{W}^2_+$ is the space of $\widetilde{S}^1$-invariant, self-dual 2-forms on X. Recall that associated to A_0 there is the elliptic complex (2.5). Just as in (3.5) we obtain direct sum decompositions by using the Greens function for this complex:

$$
\begin{aligned}
H_{A_0} &= H_0^1 \oplus (\mathrm{im}(P_+ d_{A_0})^* \cap \mathscr{W}^1_{p,k}) = H_0^1 \oplus C, \\
\mathscr{W}^2_{+,p,k-1} &= V_{A_0} \oplus (\mathrm{im}(P_+ d_{A_0}) \cap \mathscr{W}^2_{+,p,k-1}) = V_{A_0} \oplus I
\end{aligned}
\tag{3.8}
$$

with V_{A_0} a finite dimensional subspace of $\mathscr{W}^2_+$, such that the L^2-projection to the $\tilde{S}^1$-invariant part H_0^2 of the 2nd cohomology of (2.5) is an isomorphism. By Aronszajn's Theorem [1] the forms in V_{A_0} can be assumed to have support in the complement of small open sets in X. The spaces C and I are defined to be the second summands of the sums occurring in the middle in (3.8).

By definition the derivative

$$DK_{A_0} = P_+ d_{A_0} \colon C \to I$$

is an isomorphism. The implicit function theorem therefore supplies us with smooth maps, defined in a neighborhood of $0 \in H_0^1$:

$$
\begin{aligned}
p &\to \tilde{p} \colon H_0^1 \to H_{A_0}, \\
p &\to \phi(p) \colon H_0^1 \to V_{A_0},
\end{aligned}
\tag{3.9}
$$

which are Γ_{A_0}-equivariant and solve the equation

$$K(q) = P_+ F^{A_0+q} = \phi(q) \tag{3.10}$$

for $q \in \mathscr{W}^1$. Conversely every small solution $q \in \mathscr{W}^1$ of

$$d^*_{A_0} q = 0, \qquad K(q) \in V_{A_0}$$

is of the form $q = \tilde{p}$. In the sequel we shall need the following estimates:

$$\|p - \tilde{p}\| = \mathrm{O}(p^2), \qquad \|d\tilde{p}/dp - \mathrm{id}\| = \mathrm{O}(p). \tag{3.11}$$

Both of these readily follow from the implicit function theorem.

From the above we extract a local model of the moduli space as the quotient by Γ_{A_0} of the zeros of the Γ_{A_0}-equivariant map $\phi_{|W} \colon W \to V_{A_0}$, with W a neighborhood of $0 \in H_0^1$. Under the assumptions $p \geq 2$, $k - 4/p > 0$ all L^p_k-solutions of the Bogomol'nyi equation are automatically smooth, and the topology induced on the moduli spaces is independent of the precise choice of p, k.

The leading term of ϕ is generically quadratic and can be expressed very explicitly. The equation $\phi(p) = 0$ is equivalent with $\langle \omega, \phi(p) \rangle_{L^2} = 0$ for all harmonic forms $\omega \in \ker(P_+ d_{A_0})^* \cong H_0^2$. Now

$$
\begin{aligned}
\langle \omega, \phi(p) \rangle_{L^2} &= \langle \omega, P_+(d_{A_0}\tilde{p} + \tfrac{1}{2}[\tilde{p}, \tilde{p}]) \rangle_{L^2} \\
&= \langle \omega, \tfrac{1}{2}[\tilde{p}, \tilde{p}] \rangle_{L^2} = \langle \omega, \tfrac{1}{2}[p, p] \rangle_{L^2} + \mathrm{O}(p^3).
\end{aligned}
\tag{3.12}
$$

Hence ϕ is a small C^1-perturbation of the quadratic form

$$(3.13) \qquad q\colon H_0^1 \to H_0^{2*} \cong (\ker(P_+ d_{A_0}^*))^*\colon p \to \tfrac{1}{2}\langle ?, P_+[p,p]\rangle_{L^2}.$$

The stability theorem for submersions teaches that if 0 is a regular value of the map

$$q\colon H_0^1\backslash\{0\} \to (H_0^2)^*,$$

then there exists a Γ_{A_0}-equivariant local homeomorphism around $0 \in U_{A_0}$, carrying the zero set of ϕ into the null-cone of q. This local homeomorphism is smooth away from the origin. Summarizing one has

Theorem 3.2. *Let A_0 be a monopole on P, Γ_{A_0} its stabilizer in $GA(P)$, and H_0^1, H_0^2 the $\widetilde{S}^1$-invariant parts of the cohomology groups of* (2.5). *There exist a Γ_{A_0}-invariant neighborhood W of $0 \in H_0^1$, an embedding $\sim\colon W \to H_{A_0}$, with derivative* 1 *at $0 \in W \subset H_0^1$, and a smooth map $\phi\colon W \to H_0^2$ (as in* (3.9)) *such that:*

(1) *$\sim$ is a Γ_{A_0}-equivariant homeomorphism of the zero set of ϕ in W to all $\widetilde{S}^1$-invariant solutions of the anti-self-duality equation near A_0 in a slice,*

(2) *the moduli space of monopoles is in a neighborhood of A_0, the quotient of the zero set of the smooth function ϕ in W by Γ_{A_0}.*

If $\Gamma_{A_0} = \{\pm 1\}$ and $H_0^2 = 0$, then the moduli space near A_0 is a smooth manifold of dimension given by Theorem 2.3.

Example 3.3. (1) The curvature properties of the 4-sphere ensure that $H_0^2 = 0$ for a monopole on H^3; also no reducible SU(2)-monopoles exist. We shall see that $\mathscr{M}(m,k)$ is not empty, so that it is a smooth manifold of dimension $4k-1$. This agrees with the results in Atiyah [2].

(2) Deformations of the reducible monopole on the bundle $E = \mathrm{O}(k,-k) \oplus \mathrm{O}(-k,k)$ over $S^2 \times S^2$ are more interesting. For notation compare Example 2.5. We used the identification

$$H^1_{\text{sheaf}}(X, \mathrm{O}(2k,-2k))^{\widetilde{S}^1} \oplus H^1_{\text{sheaf}}(X, \mathrm{O}(-2k,2k))^{\widetilde{S}^1} \to H_0^1,$$
$$(\alpha,\beta) \to \begin{bmatrix} 0 & \alpha-\beta^* \\ \beta-\alpha^* & 0 \end{bmatrix}.$$

A short computation shows that the quadratic form (ω the Kähler form of $S^2 \times S^2$)

$$q\colon H_0^1 \to \mathbf{R}\colon p \to \left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \omega, \tfrac{1}{2}P_+[p,p] \right\rangle_{L^2}$$

is equal to $q(\alpha,\beta) = \|\alpha\|_{L^2}^2 - \|\beta\|_{L^2}^2$, which is a nondegenerate quadratic form on H_0^1. In case (1) the form is definite, so the reducible monopole is isolated, despite the fact that there are many linearly independent infinitesimal deformations. In case (2) the form is indefinite and a neighborhood of the reducible monopole in the moduli space is the cone $q(\alpha, \beta) = 0$ modulo $\widetilde{S}^1$.

A very remarkable property of deformations of monopoles on $\mathbf{R}^3$ is that deformations $(a_3, \phi) \in \Omega^1(\mathfrak{g}_P) \oplus \Omega^0(\mathfrak{g}_P)$ can after gauge transformation be assumed to have a finite L^2-norm on $\mathbf{R}^3$; this is rather nontrivial (see Taubes [29] and Atiyah-Hitchin [5]). A consequence of this fact is that monopoles on $\mathbf{R}^3$ can be moved slowly with finite energy. In hyperbolic spaces, which are a special case of our situation, this is very different. For example moving a $k = 1$ monopole on H^3 by an isometry induces a transformation on the curvature on $\delta H^3 = S^2$ which is the pullback under a fractional linear transformation (unless the isometry leaves the monopole invariant). A moment's reflection shows that such a deformation cannot be removed by a gauge transformation and, indeed, the resulting infinitesimal deformations are not L^2. It will be interesting to see if motion of hyperbolic monopoles can be defined in a way which reflects geometrical properties of the moduli spaces.

Related to this is the following. If (A_3, Φ) is any monopole on $\mathbf{R}^3$ then A restricted to the 2-sphere S^2_∞ exists, and is always the unique (up to gauge transformations) homogeneous connection on the kth power of the Hopf bundle on S^2. In contrast with this, the restriction of A to S^2_∞ for hyperbolic monopoles depends on the monopole, and one can in fact show that it determines the monopole (see Braam-Austin [11]).

4. Smoothness, orientability and compactifications

In Freed-Uhlenbeck [23] it is proved that for a generic metric on X, the moduli spaces of irreducible instantons are smooth manifolds. It follows almost directly from this that for a generic S^1-invariant metric on X, the moduli spaces of irreducible monopoles are smooth. However, such metrics do not necessarily come from metrics on the three-manifold M as in §2. To see this observe that an S^1-invariant metric on X induces the circle bundle $X - \bigcup S_j$ over M equipped with a connection which is not necessarily trivial. Triviality of this connection is a necessary condition for an S^1-invariant metric to arise from a metric on M.

Perturbing the metric to a nearby invariant metric on X affects the 3-dimensional Bogomol'nyi equations in the following way. First of all, the metric on M is being perturbed. This is not the only thing which happens, also the Hodge star map from 1-forms to 2-forms on M is replaced by a nearby map from 1-forms to 2-forms, which no longer is the Hodge star for the metric. This is the effect of not keeping the circle orbits orthogonal to M. Thus we show that for such perturbations of the equations the moduli spaces are generically smooth. It would be interesting to know if the moduli spaces can be smoothed by just perturbing the metric on M.

After this we show that it follows from Donaldson's orientability theorem [18], that the monopole moduli spaces can be oriented in a canonical way, starting from an orientation of the real homology of M. Another important fact concerning instanton moduli spaces is that they have a compactification. Again, monopole moduli spaces can be compactified similarly.

Let P be an $\widetilde{S}^1$-equivariant SU(2)-bundle with invariants (m_j, k_j). Recall that $p_1(P) = \sum m_j k_j$, and that $\|F^A\|_{L^2} \geq 8\pi^2 \cdot p_1(P)$.

Proposition 4.1. *If $\sum m_j \cdot k_j \neq 0$, then for an open dense set of S^1-invariant C^q-metrics ($q \geq 1$) on X the space $\mathscr{M}^*(m_j, k_j)$ of irreducible monopoles in $\mathscr{M}(m_j, k_j)$ (with respect to these metrics) is a smooth, possibly empty, manifold of dimension $I(m_j, k_j)$ (see (2.9)).*

Proof. We shall outline the approach of Freed-Uhlenbeck [23, pp. 60–73] and indicate what changes one has to make to obtain our result. Fix an S^1-invariant C^q-metric g on X. The anti-self-duality equations read $P_+ F^A = 0$, where P_+ is projection onto self-dual forms. Let $\mathscr{D} = C^q(\mathrm{GL}(TM))^{S^1}$ be the Banach manifold of S^1-invariant C^q-automorphisms of TX. Note that $\phi \in \mathscr{D}$ acts on $\mathscr{W}^2 = \Omega^2(\mathfrak{g}_P)$ and that the anti-self-duality equations for $A \in C(P)$ with respect to the metric $\phi^* g$ read $P_+(\phi^* F^A) = 0$. Also observe that $\mathscr{D}$ acts transitively on the space of C^q-metrics. However, elements of $\mathscr{D}$ may change the orthogonality of S^1-orbits and M, and give rise to metrics on X which are not compactified metrics on M.

Following Freed-Uhlenbeck study the map

$$\mathscr{P}\colon C^*(P) \times \mathscr{D} \to \mathscr{W}_+^2\colon (A, \phi) \to P_+(\phi^* F^A),$$

with $C^*(P)$ denoting the space of irreducible S^1-invariant connections.

Lemma 4.2. *$\mathscr{P}$ is smooth and has 0 as a regular value.*

Proof. We follow Cho [14]. Let $B\colon V \to W$ be a linear S^1-equivariant surjection. Then B restricted to the zero weight space in V maps surjectively to the zero weight space in W. We shall now apply this.

It should be proved that if $\mathscr{P}(A, \mathrm{id}) = 0$, then $d\mathscr{P}_{(A,\mathrm{id})}$ is surjective. Freed-Uhlenbeck prove that $\mathscr{P}$ extended to the space of all connections cross all bundle automorphisms has 0 as a regular value, because the assumption $\sum m_j k_j \neq 0$ implies that $F^A \neq 0$. Now just observe that our $\mathscr{P}$ is restriction to fix point sets.

From the lemma it follows that $\mathscr{P}^{-1}(0) \subset C^*(P)\times\mathscr{D}$ is a manifold. One finishes the proof by showing that the projection $\mathscr{P}^{-1}(0)/GA(P) \to \mathscr{D}$ is a Fredholm map of index $I(m_j, k_j)$; the infinite dimensional Sard theorem then gives the result.

Notice that Proposition 4.1 proves that the cohomology group H_0^2 vanishes for generic elements of $\mathscr{D}$ and irreducible monopoles. Next we indicate when no reducible monopoles will be present.

Proposition 4.3. *If $b^2(M) > b^1(M)$ then for an open dense set of invariant metrics on X there are no reducible monopoles in $\mathscr{M}(m_j, k_j)$ if $\sum m_j k_j \neq 0$, i.e., $\mathscr{M}^*(m_j, k_j) = \mathscr{M}(m_j, k_j)$.*

Proof. Proceeding exactly as in Freed-Uhlenbeck [23, Corollary 3.21] the result follows.

An open dense subset of $\mathscr{D}$ contains smooth metrics. Combining Propositions 4.1 and 4.3 we get

Theorem 4.4. *For an open dense set of smooth S^1-invariant metrics on X the moduli space $\mathscr{M}^*(m_j, k_j)$ is a smooth manifold of dimension $I(m_j, k_j)$ if $\sum m_j \cdot k_j \neq 0$. If additionally $b^2(M) > b^1(M)$ holds, then one may assume that for a generic metric also $\mathscr{M}(m_j, k_j) = \mathscr{M}^*(M_j, k_j)$.*

Another condition which ensures that $\mathscr{M}^*(m_j, k_j)$ is smooth is that on X there is an anti-self-dual metric in the given conformal class of positive scalar curvature (see Atiyah-Hitchin-Singer [6]). In Braam [10] we analyzed which X's arising from hyperbolic manifolds satisfy this condition.

Remark 4.5. (1) If $Q \to X$ is an S^1-equivariant SO(3)-bundle with $w_2(Q) \neq 0$, then $\mathscr{M}((m_j, q_j), w_2) = \mathscr{M}^*((m_j, q_j), w_2)$. If additionally $\sum m_j \cdot q_j \neq 0$, then one can prove as above that $\mathscr{M}((m_j, q_j), w_2)$ is smooth for a generic S^1-invariant metric.

(2) A. Floer [19] investigated monopole moduli spaces on asymptotically Euclidean 3-manifolds. One of his results is that these are smooth if the 3-manifold has positive definite Ricci curvature.

Next we shall show that the monopole moduli spaces are orientable.

Theorem 4.6. *Assume that $\mathscr{M}^*(m_j, k_j)$ is smooth and cut out transversely. An orientation of $H^1(M; \mathbf{R}) \oplus H^2(M; \mathbf{R})$ gives a canonical orientation of $\mathscr{M}^*(m_j, k_j)$.*

Proof. The operators (compare (2.8))

$$D_A : (\Lambda_+^2 \oplus \Lambda^0) \otimes \mathfrak{g}_P \to \Lambda^1 \otimes \mathfrak{g}_P$$

form a family of Dirac operators on $\mathscr{B}^* \times X \to \mathscr{B}^*$, with $\mathscr{B}^* = \mathscr{A}^*(P)/\mathscr{G}(P)$. Associated to this is a real determinant line bundle $\lambda \to \mathscr{B}^*$ with fiber at $[A]$ equal to

$$\lambda_A = \Lambda^{\max}(\ker D_A)^* \otimes \Lambda^{\max}(\operatorname{coker} D_A),$$

where $\Lambda^{\max} V = \Lambda^{\dim V} V$ for any vector space V.

From Donaldson [18] we know that an orientation of $H^2_+(X; \mathbf{R}) \oplus H^1(X; \mathbf{R})$ canonically gives rise to a nonvanishing section of λ (up to pointwise multiplication by a positive function). Now $H^2_+(X; \mathbf{R}) \cong H^2(M; \mathbf{R})$ using the Hodge star (compare Braam [10, Proposition 2.2]). It follows that $H^2_+(X; \mathbf{R})$ is oriented canonically. As $H^1(X; \mathbf{R}) \cong H^1(M; \mathbf{R})$ our data also give a nonvanishing section of λ.

We may assume that $\mathscr{M}^*(m_j, k_j) \subset \mathscr{B}^*$, by Theorem 1.3. If $A \in \mathscr{M}^*(m_j, k_j)$, then $\ker D_A \cong T_A \mathscr{M}(m_j, k_j) \oplus N_A$ where N_A is an $\widetilde{S}^1$-representation, in which weight 0 does not occur. This also holds for $\operatorname{coker} D_A$, because we assumed $\mathscr{M}(m_j, k_j)$ to be cut out transversely. Hence N_A and $\operatorname{coker} D_A$ are canonically oriented. Now

$$\lambda_A \cong \Lambda^{\max}(T_A \mathscr{M}^*(m_j, k_j))^* \otimes \Lambda^{\max}(N_A)^* \otimes \Lambda^{\max}(\operatorname{coker} D_A),$$

so our data give a canonical nonvanishing section of $\Lambda^{\max}(T\mathscr{M}^*(m_j, k_j))$, i.e., an orientation of $\mathscr{M}^*(m_j, k_j)$.

Remark 4.7. For monopole moduli spaces $\mathscr{M}(Q)$ with Q an S^1-equivariant SO(3)-bundle, we get a canonical orientation from an orientation of $H^1(M; \mathbf{R}) \oplus H^2(M; \mathbf{R})$ if $w_2(Q) \in H^2(X; \mathbf{Z}/2\mathbf{Z})$ lies in the image of the natural map $H^2(X; \mathbf{Z}) \to H^2(X; \mathbf{Z}/2\mathbf{Z})$. This follows from Donaldson [18] in the same way as Theorem 4.6, using the fact that X is always a Spin manifold (see Braam [10, Proposition 2.2]).

The final topic to be discussed in this section is that of compactifying $\mathscr{M}(m_j, k_j)$. We first recall briefly the situation for instantons. Let $[A_k] \in \mathscr{I}_p$ be a sequence of instantons on a bundle P with $c_2(P) = p$, where $\mathscr{I}_p$ denotes the *instanton moduli space*. From Uhlenbeck's weak compactness theorem it follows (see Donaldson [16, §3.2] and Sedlacek [26]) that, after going over to a subsequence $[A_k]$, there are a sequence of gauge transformations $g_k \in \mathscr{G}(P)$ and a finite set of points $\{x_1, \cdots, x_n\} \in X$, counted with multiplicities, such that

(4.1)
(1) On $X \backslash \{x_1, \cdots, x_n\}$, $g_k \cdot A_k \to A_\infty$ in the C^∞-topology, with A_∞ an instanton on a bundle $P' \to X$ with $c_2(P') = c_2(P) - n$.
(2) The functions $|F^{A_k}|^2$ converge to $16\pi^2 \delta_{x_j} + |F^{A_\infty}|^2$ in the sense of measures.

Using this control, it is possible to define a compactification $\overline{\mathscr{I}}_p$ of $\mathscr{I}_p$, equal to the closure of $\mathscr{I}_p$ in the space

$$\mathscr{I} = \mathscr{I}_p \cup (\mathscr{I}_{p-1} \times X) \cup (\mathscr{I}_{p-2} \times \mathscr{S}^2(X)) \cup \cdots \cup (\mathscr{I}_0 \times \mathscr{S}^p(X)),$$

which one gives a topology using (4.1); here $\mathscr{S}^j(X)$ is the space of unordered j-tuples of points in X. Observe that $\mathscr{I}$ is compact, so $\overline{\mathscr{I}}_p$ is compact. We shall show that a similar compactification of the monopole moduli spaces exists, adding lower strata which consist of pairs of a set of points in boundary surfaces of M and a monopole of lower charge. As we shall see, masses are preserved under limits.

Let $[A_i]$ be a sequence in $\mathscr{M}(m_j, k_j)$, and again consider the convergence (4.1). Clearly by the $\widetilde{S}^1$-invariance the x_i lie in some S_j, so we change the indices to j_i, meaning x_{j_i} is the i-th point in S_j. Denote by μ_{j_i} its multiplicity.

Proposition 4.8. (1) $\mu_{j_i} = \lambda_{j_i} \cdot m_j$ *for some* $\lambda_{j_i} \in \mathbf{Z}_{>0}$.

(2) $[A_\infty] \in \mathscr{M}(m_j, l_j)$ *with* $l_j = k_j - \sum_i \lambda_{j_i}$ *and* $\sum_j m_j \cdot l_j \geq 0$.

Proof. There is a sequence of gauge transformations $g_i \in \mathscr{G}(P)$ (i.e., not necessarily commuting with the $\widetilde{S}^1$-action) such that $g_i \cdot A_i \to A_\infty$ in the C^∞-topology on X with the points removed. Denote our original $\widetilde{S}^1$-action on P by a homomorphism $\theta\colon \widetilde{S}^1 \to \mathrm{Aut}(P)$.

Lemma 4.9. *Suppose A_i, A_∞ and $B(u)$ are connections on some principal* SU(2)*-bundle over a smooth manifold with $B(u)$ depending smoothly on u (u an element of a compact space). Let $h_i(u)$ be gauge transformations also depending smoothly on u. If $A_i \to A_\infty$ and $h_i(u) \cdot A_i \to B(u)$ both converge uniformly on u in the C^∞-topology (convergence of all derivatives on precompact subsets of X), then a subsequence $h_j(u)$ converges uniformly in u in the C^∞-topology to a family of gauge transformations $h(u)$, where $h(u)$ has the property that $h(u) \cdot A_\infty = B(u)$.*

Proof. The proof is standard and can be modelled on that of Proposition A.5 in Freed-Uhlenbeck [23].

Now apply this to

$$g_i \cdot A_i \to A_\infty,$$
$$h_i(u) \circ g_i \cdot A_i \to \theta(u) \cdot A_\infty, \quad h_i(u) = \theta(u) \circ g_i \circ \theta(-u) \circ g_i^{-1}.$$

Then we may assume $h_i(u) \to h(u)$ and $u \to h(u)^{-1} \circ \theta(u)$ define an action on $P_{|X\setminus\{x_{j_i}\}}$, stabilizing A_∞. This action is the limit of the actions $u \to g_i \circ \theta(u) \circ g_i^{-1}$, so its weights are equal to those of θ.

Considering A_∞ as a connection on some SU(2)-bundle P' has two disadvantages: A_∞ is $\widetilde{S}^1$-invariant on $P'_{|X\setminus\{x_{j_i}\}}$, but not yet invariant on P' and, furthermore, it is not clear that the gauge transformation removing the singularity respects the $\widetilde{S}^1$-action. Using formula 2.4 (see also Braam [10, §5]) the latter implies that the multiplicities are multiples of m_j.

For this reason consider A_∞ as a singular connection on P. We shall study the $\widetilde{S}^1$-equivariance properties of the gauge transformation which removes the singularities. Start by taking a small S^1-invariant ball B, centered at one of the singular points x_{j_i}, and a section $s: B \to P$ such that the action of $\widetilde{S}_1$ on P is described by

$$u \cdot s(b) = s(u \cdot b) \cdot \lambda(u) \qquad (u \in S^1, b \in B)$$

for a homomorphism $\lambda: \widetilde{S}^1 \to \mathrm{SU}(2)$. Identifying $s^* A_\infty$ with A_∞, the $\widetilde{S}^1$-invariance of A_∞ reads

$$u^* A_\infty = \lambda(u) \cdot A_\infty. \tag{4.2}$$

Here the action of a gauge transformation $g: B \to \mathrm{SU}(2)$ on A_∞ is given by $g \cdot A_\infty = g dg^{-1} + g A_\infty g^{-1}$.

The removability of singularities theorem (see Freed-Uhlenbeck [23]), which depends on $\int_B |F^{A_\infty}|^2 \, dV < \infty$, asserts that (possibly after shrinking B) there exists a gauge transformation g such that:

(1) $d^*(g \cdot A_\infty) = 0$.

(2) $g \cdot A_\infty(\partial/\partial r) = 0$ on δB.

(3) $\|g \cdot A_\infty\|_{L_1^2(B)} \leq \mathrm{const} \cdot \|F^{A_\infty}\|_{L^2(B)}$.

(4) g is smooth on $B\backslash\{0\}$.

With such a g, elliptic regularity implies that $g \cdot A_\infty$ is smooth. Furthermore, there are no infinitesimal L_2^2-deformations of g preserving the properties (1)–(4), apart from those arising from composition on the left with constant gauge transformations.

Next remark that $d^*(u^* g \cdot A_\infty) = 0$ and that

$$u^* g \cdot A_\infty = h(u) \cdot g \cdot A_\infty \tag{4.3}$$

for the gauge transformation $h(u) = (u^* g) \circ \lambda(u) \circ g^{-1}$ which is possibly singular at 0. It remains to show that $h(u): B \to \mathrm{SU}(2)$ is in L_2^2, because then it is constant (by uniqueness) and equal to $\lambda(u)$, up to conjugation. This implies that $g \cdot A_\infty$ is invariant as in (4.2), and g is an $\widetilde{S}^1$-equivariant gauge transformation which can only change $c_2(P)$ by multiples of m_j.

Now (4.3) gives that the function h satisfies the ordinary differential equation

$$dh(u) + (g \cdot A_\infty) h(u) = h(u)(u^* g \cdot A_\infty),$$

so h is certainly L_2^2. This finishes the proof of Proposition 4.8.

It follows that we can define a compactification $\overline{\mathscr{M}}(m_j, k_j)$ of $\mathscr{M}(m_j, k_j)$ which is the closure of $\mathscr{M}(m_j, k_j)$ in the compact space of ideal monopoles $\mathscr{I}$:

$$\mathscr{J} = \bigcup_{\left\{ (l_j);\, \substack{j \text{ is s.t. } m_j \neq 0 \\ l_j \leq k_j, \sum m_j l_j \geq 0} \right\}} \left[\mathscr{M}(m_j, l_j) \times \prod_{\substack{j \text{s.t.} \\ m_j \neq 0}} \mathscr{S}^{k_j - l_j}(S_j) \right],$$

where $\mathscr{S}^l(S_j)$ is the lth symmetric product of S_j, normalized such that $\mathscr{S}^0(S_j)$ is a set with one element. The topology on $\mathscr{J}$ is defined using (4.1); thus a point $(A_\infty, \{x_{j_i}\}) \in \mathscr{J}$ lies in $\overline{\mathscr{M}}(m_j, k_j)$ precisely if there is a sequence $A_i \in \mathscr{M}(m_j, k_j)$ such that (4.1) holds. This expresses the fact that a monopole can loose $k_j - l_j$ lumps which move to definite points in surfaces S_j for which $m_j \neq 0$. It should be noted that some of the charges l_j could become < 0; an example of this will be discussed in the next section.

In §1 we indicated that $\mathscr{M}^*(m_j, k_j)$ could be considered as a subset of an instanton moduli space. The same now holds for the compactified moduli spaces, but attention should be drawn to the fact that the multiplicities of the tuples of points in $\mathscr{M}(m_j, k_j)$ differ by factors m_j from the multiplicities in the compactifications of instanton moduli spaces.

Taubes [31] has studied the convergence properties of sequences of monopoles on $\mathbf{R}^3$, and he shows that essentially the same compactification exists for monopoles on $\mathbf{R}^3$ as the one we constructed for monopoles on H^3.

5. Existence theorems and asymptotic models of moduli spaces

For the construction of solutions of the Bogomol'nyi equation near the lower strata in the compactification we shall use Donaldson's [16] alternating procedure, which describes instantons on connected sums of two 4-manifolds. As we shall see, this needs only minor modifications in order to deal with S^1-invariant instantons. Just as for instantons, it can be shown that all monopoles near a 'good' point in a lower stratum of the compactified moduli space are constructed by the alternating method. This will be explained in the second part of this section.

Let X_0, X_1 be two Riemannian 4-manifolds with S^1-actions which are constructed from two 3-manifolds M_0 and M_1 as in §1. Let $x_0 \in X_0$ and $x_1 \in X_1$ be points lying in fixed surfaces, and first assume that the Riemannian metrics on X_0 and X_1 are conformally flat in neighborhoods of x_0 and x_1. This holds automatically for those X's arising from Kleinian groups (see Braam [10]). In this situation we can form an S^1-equivariant conformal connected sum $X_0 \# X_1$. Choose coordinates ξ, η on X_0, X_1, such that the metric in these coordinates is Euclidean and such that $\xi(x_0) = 0, \eta(x_1) = 0$. Now define the Riemannian manifold $X = X_0 \# X_1$ by the identification

$$\eta = \lambda \cdot |\xi|^{-2} \cdot \sigma(\xi),$$

where $\lambda > 0$ is sufficiently small, and $\xi \to \sigma(\xi)$ is an orientation reversing S^1-equivariant isometry $T_{x_0}X_0 \to T_{x_1}X_1$. In Braam [10, §2], it has been explained that X is the 4-manifold which arises from the 3-manifold $M = M_0 \#_b M_1$, where $\#_b$ denotes a boundary connected sum. If we take $X_1 = S^4$, then $X_0 \# X_1 = X_0$ as conformal manifolds, and in this case we shall see that the construction below gives monopoles on X_0 'supported' near a boundary surface of X_0, by 'gluing in' monopoles from H^3. In the boundary connected sum procedure, a half-ball in M, with its bounding disc in δM, without monopoles is replaced by a half ball from H^3, which does carry monopoles. The alternating procedure will give a description of when exactly these approximate grafted monopoles are close to a real monopole. Gluing in means that by using a partition of unity the part of the monopole on X_0 in the identification area is made flat, and the same is done to the monopole on X_1. In the overlap there is now an S^1-worth of bundle morphisms $P_0 \to P_1$ to give a family of connections on the connected sum; if the connected sum parameters have been suitably chosen these are almost monopoles. Other parameters in the construction arise from variation in the attaching points, etc.

For the following discussion compare Figure 5.1, in which arrows indicate the boundaries of the open sets which will be defined. Define spheres in X_0 of radii $N^{-1}\sqrt{\lambda}$ and $N\sqrt{\lambda}$, and shells

$$R_{-1} = \{x \in X_0; |x - x_0| \in [kN^{-1}\sqrt{\lambda}, N^{-1}\sqrt{\lambda}]\},$$
$$R_1 = \{x \in X_0; |x - x_0| \in [N\sqrt{\lambda}, k^{-1}N\sqrt{\lambda}]\}.$$

Like Donaldson, we shall fix k, say at 0.9, while N will have to be chosen large enough for the proof to work. More precisely, one needs

$$(k^{-4} - 1) \cdot (1 - k) \cdot N \cdot (N - N^1)^{-3} < 8,$$

which is a relation independent of λ, but for our spheres and shells to be well defined, λ should be sufficiently small.

The identification maps R_{-1} and R_1 to shells in X_1 whose sizes are reversed. The image of R_j in X_1 will be denoted by R_j too. Let $U_0 \subset X_0$ be the complement of the ball $|\xi| \leq kN^{-1}\sqrt{\lambda}$ and $\widehat{U}_0 \subset X_0$ the complement

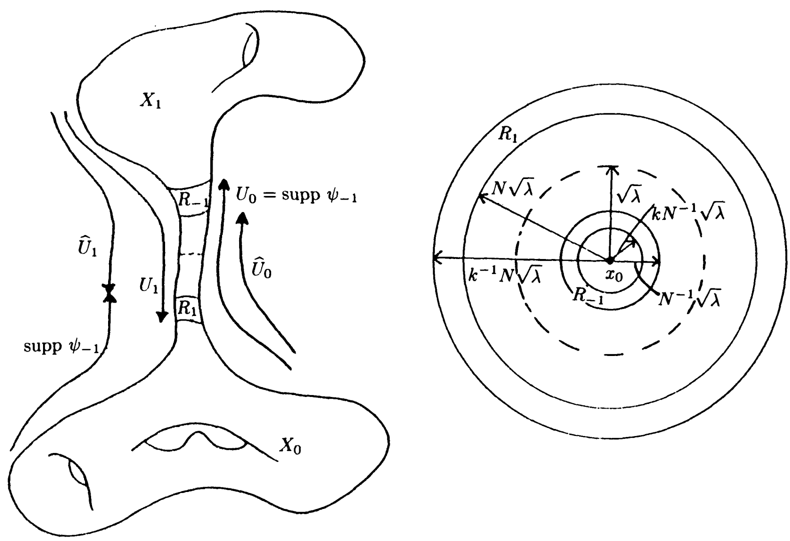

FIGURE 5.1

of the ball $|\xi| \leq N^{-1}\sqrt{\lambda}$. Define $U_1, \widehat{U}_1 \subset X_1$ symmetrically, so that $X = X_0 \# X_1$ is covered by the open sets $\widehat{U}_0, \widehat{U}_1$, which intersect in an annulus bounded by the inner boundary spheres of the shells $R_1 \subset X_0$ and $R_{-1} \subset X_1$.

Suppose that A_0, A_1 are monopoles on $\widetilde{S}^1$-equivariant SU(2)-bundles $P_j \to X_j$, satisfying the following acyclicity condition on the $\widetilde{S}^1$-invariant part of the cohomology groups of (2.5):

$$H^0_{0,A_j} = H^1_{0,A_j} = H^2_{0,A_j} = 0, \qquad j = 0, 1. \tag{5.1}$$

Consider for $\eta_0 > 0$ the set of $\widetilde{S}^1$-invariant connections on $X = X_0 \# X_1$ which can be represented in the form

$$A = (A_0 + a, A_1 + a', \rho), \tag{5.2}$$

where

$$a \text{ is defined on } \widehat{U}_0, \quad d^*_{A_0} a = 0, \quad \|a\|_{L^{2p}(\widehat{U}_0)} < \eta_0,$$

$$a' \text{ is defined on } \widehat{U}_1, \quad d^*_{A_1} a' = 0, \quad \|a'\|_{L^{2p}(\widehat{U}_1)} < \eta_0,$$

$$\rho \in C^\infty((\widehat{U}_0 \cap \widehat{U}_1), \operatorname{Hom}^{\widetilde{S}^1}(P_{0,x_0}, P_{1,x_1})) \quad \text{s.t. } A_1 + a' = \rho \cdot (A_0 + a).$$

Following Donaldson, we take $p > 6$ to ensure that all nonlinear maps involved will be smooth on the relevant Banach spaces. The relation between $A_0 + a$ and $A_1 + a'$ should be satisfied in $\widehat{U}_0 \cap \widehat{U}_1$, in exponential trivializations emanating from x_0 and x_1. Recall that the points x_j lie in fixed surfaces in X_j, so the $\widetilde{S}^1$-actions on P_j assign a mass m_j to x_j. For an identification ρ to exist, the masses m_j must be equal.

Theorem 5.1. *Under the conditions* (5.1) *and for sufficiently small* λ, η_0, *the monopoles* A *on* $X_0 \# X_1$, *which can be represented in the form* (5.2), *are smoothly parametrized by* $\mathrm{Hom}^{\widetilde{S}^1}(P_{0,x_0}, P_{1,x_1})/\{\pm 1\}$.

Proof. This is an $\widetilde{S}^1$-equivariant version of Theorem 4.17 in Donaldson [16]. His proof is natural with respect to $\widetilde{S}^1$-actions and carries over to our situation. q.e.d.

Notice that the set of gluing data

$$I = \mathrm{Hom}^{\widetilde{S}^1}(P_{0,x_0}, P_{1,x_1})$$

is diffeomorphic to SU(2) if the masses m_j at x_j vanish, and to S^1 if $m_0 = m_1 \neq 0$.

Before proceeding, let us recall briefly how Theorem 5.1 constructs monopoles on $X_0 \# X_1$. In trying to understand the construction it may be helpful to compare it with the Mayer-Vietoris argument in de Rham cohomology. In fact, Donaldson's alternating procedure can be interpreted as a nonlinear version of a proof of the statement $H^1(X;\mathbf{R}) \cong H^1(X_0;\mathbf{R}) \oplus H^1(X_1;\mathbf{R})$ on the level of harmonic differential 1-forms.

First, following Donaldson, glue A_0 to A_1 by using an exponential gauge around x_0, x_1: let ψ_1, ψ_{-1} be S^1-invariant cutoff functions $X_0 \# X_1 \to \mathbf{R}$ such that

$$\psi_{1|X \setminus U_1} = 1, \quad \mathrm{supp}\,\psi_1 \subset X \setminus \widehat{U}_1, \quad \mathrm{supp}\, d\psi_1 \subset R_1,$$
$$\psi_{-1|\widehat{U}_0} = 1, \quad \mathrm{supp}\,\psi_{-1} \subset U_0, \quad \mathrm{supp}\, d\psi_{-1} \subset R_{-1}.$$

For $\rho \in I = \mathrm{Hom}^{\widetilde{S}^1}(P_{0,x_0}, P_{1,x_1})$ define the glued connection to be

$$A^0(\rho) = (A_0 + a_0, A_1 + a_0', \rho) \tag{5.3}$$

with

$$\begin{aligned} a_0 &= (\psi_1 - 1) \cdot A_0 + (1 - \psi_1) \cdot \rho^{-1} A_1 \rho \text{ on } U_0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{extended to } X_0 \text{ by } 0, \\ a_0' &= -\psi_1 \cdot A_1 + \psi_1 \cdot \rho A_0 \rho^{-1} \text{ on } U_1, \quad \text{extended to } X_1 \text{ by } 0. \end{aligned}$$

Clearly $\mathrm{supp}\, P_+ F^{A^0(\rho)} \subset R_1 \subset X$.

Next suppose (induction) that a_j, a_j', with j even, are given, and assume that $\sigma = P_+ F^{A_0 + a_j}_{|U_0}$ satisfies $\mathrm{supp}\,\sigma \subset R_1$. We shall solve an equation on X_0. Extend a_j over $X_0 \backslash U_0$ by 0 and solve the following equation for $b \in \mathscr{W}^1_{p,1}(X_0)$:

$$P_+(d_{A_0}b + [a_j, b] + \tfrac{1}{2} \cdot [b, b]) = -\sigma, \qquad d^*_{A_0}b = 0. \tag{5.4}$$

To see that there are solutions and that they are well behaved, interpret the left-hand side of (5.4) as an operator:

$$\begin{gathered} Q(\cdot,\cdot)\colon \mathscr{W}^1_{p,1}(X_0) \times \mathscr{W}^1_{2p,0}(X_0) \to \mathscr{W}^2_{+,p,0}(X_0) \times \mathscr{W}^0_{p,0}(X_0), \\ (b, a) \to (P_+\{d_{A_0}b + [a, b] + \tfrac{1}{2} \cdot [b, b]\}, d^*_{A_0}b). \end{gathered}$$

The derivative with respect to the first variable is precisely the deformation operator studied in §3 and is assumed to be invertible. Therefore the implicit function theorem supplies solutions of (5.4) for small a_j and σ. Next define

$$a_{j+1} = a_j + \psi_{-1} \cdot b, \qquad a'_{j+1} = a'_j + \psi_{-1} \cdot \rho b \rho^{-1}.$$

The effect of this is that the self-dual part of $F^{(A_0+a_{j+1}, A_1+a'_{j+1}, \rho)}$ now has its support in $R_{-1} \subset X$. Now we 'alternate', i.e., we solve a similar equation on X_1 by reversing the roles of X_0 and X_1. This completes one cycle of the alternating procedure.

Using estimates on the linearization of (5.4) and the initial data, Donaldson proves that the a_j, a'_j converge to some a_∞, a'_∞, and that $A^\infty(\rho) = (A_0 + a_\infty, A_1 + a'_\infty, \rho)$ has anti-self-dual curvature. Of course it is here that conditions on λ and η_0 appear. Next, a lemma gives that for $\rho \neq \rho'$ the connections $A^\infty(\rho)$ and $A^\infty(\rho')$ are not gauge equivalent. The construction is completed by putting the a_∞, a'_∞ in the right gauge giving new $a_\infty(\rho), a'_\infty(\rho)$ satisfying $d^*_{A_0}a_\infty(\rho) = d^*_{A_1}a'_\infty(\rho) = 0$.

We now have the right preliminaries to explain what must be modified if we relax the $\widetilde{S}^1$-acyclicity condition on the deformation complexes of A_0, A_1 and the condition that the metrics be conformally flat. Again, this is Donaldson's original result in an equivariant setup. First choose liftings V_{A_0}, V_{A_1} of H^2_{0,A_0}, H^2_{0,A_1}, given by forms which are supported away from the identification area, as in §3. Let

$$p_0 \to \tilde{p}_0\colon H^1_{0,A_0} \to \mathscr{W}^1(X_0), \qquad p_1 \to \tilde{p}_1\colon H^1_{0,A_1} \to \mathscr{W}^1(X_1)$$

be the deformations described in formula (3.9). (5.4) is now replaced by

$$P_+(d_{A_0}b + [a_j + \tilde{p}_0, b] + \tfrac{1}{2} \cdot [b, b]) - \phi_0 = -\tilde{\sigma}, \quad d^*_{A_0}b = 0, \tag{5.5}$$

with $\phi_0 \in V_{A_0}$ and $\tilde{\sigma} = P_+F^{A_0+\tilde{p}_0+a_j}$. This equation has a unique solution $b \in (H^1_{0,A_0})^{\perp_{L^2}} \subset \mathscr{W}^1_{p,1}(X_0)$, and solves the infinite dimensional part of the equations just as the implicit function theorem took care of the infinite dimensional part in formulas (3.7)–(3.10). As above, we start the iteration by cutting off the deformed connections $A_0 + \tilde{p}_0, A_1 + \tilde{p}_1$ and solving for a_i, a_i'. For small p_i we get a limiting connection

$$A^{\infty}(\rho, p_0, p_1) = (A_0 + \tilde{p}_0 + a_{\infty}(\rho), A_1 + \tilde{p}_1 + a'_{\infty}(\rho), \rho)$$

satisfying

$$P_+ F(A^{\infty}(\rho, p_0, p_1)) = \phi_0(\rho, p_0, p_1) + \phi_1(\rho, p_0, p_1)$$

for smooth functions

$$\phi_i \colon I \times H^1_{0,A_0} \times H^1_{0,A_1} \to V_{A_i}. \tag{5.6}$$

Therefore the $\widetilde{S}^1$-equivariant version of Donaldson's Theorem 4.53 reads

Theorem 5.2. *Let g_0, g_1 be S^1-invariant metrics on X_0, X_1, which are conformally flat near x_0, x_1, and A_0, A_1 monopoles with respect to these metrics. If λ and η_0 are sufficiently small, and g is the S^1-invariant metric on $X = X_0 \# X_1$, then:*

(1) *there is a $(\Gamma_{A_0} \times \Gamma_{A_1})$-invariant open neighborhood N of $I \times \{0\} \times \{0\}$ in $I \times H^1_{0,A_0} \times H^1_{0,A_1}$,*

(2) *there is a $(\Gamma_{A_0} \times \Gamma_{A_1})$-equivariant map*

$$\Phi = (\varphi_0, \varphi_1) \colon N \to H^2_{0,A_0} \times H^2_{0,A_1}$$

such that the monopoles with respect to the metric g on $X_0 \# X_1$ representable in the standard form (5.1) are parametrized (up to gauge equivalence) by $\Phi^{-1}(0)/(\Gamma_{A_0} \times \Gamma_{A_1})$.

Before proceeding, we remark that away from the identification region $U_0 \cap U_1$ in $X_0 \# X_1$ solutions $(a_{\infty}, a'_{\infty})$ of the anti-self-duality equation are small in the C^{∞}-topology if η_0 is small. This follows from the standard elliptic estimates ($p > 6$!) which hold on small open sets in $(X_0 \# X_1) \backslash (U_0 \cap U_1)$.

Theorem 5.2 is quite powerful. First of all it allows us to construct a $4k-1$ parameter family of monopoles on H^3 (recall that the corresponding X is S^4) of any mass $m \in \mathbf{Z}_{>0}$ and charge $k \in \mathbf{Z}_{>0}$: using induction, start with an $(m, k-1)$-monopole A_0 on H^3, and let A_1 be the $(m, 1)$-monopole on H^3. Now apply Theorem 5.2. The H^2_0's vanish by the remark after Proposition 4.3, $I \cong S^1$ and $\Gamma_{A_i} = \{\pm 1\}$, so by induction it follows that we construct a $4k-1$ dimensional family of monopoles. Starting from the fact that the 1-monopole has 3 degrees of freedom, we do not even need the index formulas of §2 for this; in fact Theorem 5.2 can be used to give an alternative computation of the indices, based on excision.

It is worth discussing another direct corollary of Theorem 5.2. Recall (see Braam [10, 2.5]) that a 3-manifold M with $H^2(M; \mathbf{R}) = 0$ can have at most one boundary surface. Thus for monopoles on such an M there is just one charge and one mass.

Corollary 5.3. *Let M be a 3-manifold with $H^2(M;\mathbf{R}) = 0$, such that a compactification X_0 of $M \times S^1$ as in §1 can be found, which is conformally flat in a neighborhood of the fixed surface. For any mass $m \in \mathbf{Z}_{>0}$ a $(4k - 1 + b^1(M))$-parameter family of monopoles of charge k exists.*

Proof. Start with a flat monopole A_0 (i.e., $F^{A_0} = 0$) of mass $m \in \mathbf{Z}_{>0}$ on M. These are parametrized up to gauge transformations by

$$(H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})) \times H_1(M;\mathbf{Z})_{\mathrm{tor}}$$

(see Braam [10, Example 5.3.2]). The stabilizer of such a monopole is $\Gamma_{A_0} = S^1$ and $H^2_{0,A_0} \cong H^2_+(X;\mathbf{R}) \cong H^2(M;\mathbf{R}) = 0$. Now apply Theorem 5.2 with A_1 an (m,k)-monopole on H^3. q.e.d.

In particular this applies to hyperbolic handlebodies $M \cong H^3/\text{Schottky}$ group, to punctured homology spheres and to complements of S^1's, i.e., knots, in a homology 3-sphere, provided the metric near the knot is chosen such that the compactification X exists. Observe that the monopoles constructed in this way are close to a lower stratum in the compactified moduli space.

Example 5.4. Let M be a manifold as in Corollary 5.3. Choose a mass $m \in \mathbf{Z}_{>0}$ and consider monopoles of charge 1. Using Corollary 5.3 one constructs a map

$$S \times (0,\varepsilon) \times \{(H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})) \times H_1(M;\mathbf{Z})_{\mathrm{tor}}\} \to \mathscr{M}(m,1),$$

where the parameter in S is the attaching point of $\infty \in S^4$ to X, and the second one is the scale λ of the identification. Conjecturally for large m this map extends to a diffeomorphism

$$M \times \{(H_1(M;\mathbf{R})/H_1(M;\mathbf{Z})) \times H_1(M;\mathbf{Z})_{\mathrm{tor}}\} \to \mathscr{M}(m,1).$$

The exact identification of points in M with monopoles should go through the zero of the Higgs field. Floer [21] proved a theorem of this kind for asymptotically Euclidean manifolds with $H_1(M;\mathbf{Z}) = 0$.

We shall now relax the condition that the metrics on X_0, X_1 be flat in the identification region. If g_0, g_1 are any S^1-invariant metrics on X_0, X_1 then the connected sum, as a manifold, can be defined as before, using a geodesic coordinate system centered on x_0, x_1. The gluing data ($\lambda > 0$, $\sigma\colon T_{x_0}X_0 \to T_{x_1}X_1$) remain the same, just as the shells R_i and open sets U_i. We shall use Donaldson's notion of conformal structures being close. Let g be a metric on X. We shall say that g is conformally ε-close to g_0, g_1 if there are functions f_i on U_i such that

$$\|(g_i - f_i \cdot g)_{|U_i}\|_{L^\infty(U_i)} < \varepsilon.$$

A short computation shows that metrics g on X exist, which are $C \cdot \lambda$ conformally close to g_0, g_1, with C a constant depending on the Riemannian curvature of g_0 and g_1. As in Donaldson [16, 4.6] one concludes

Theorem 5.5. *Let g_i be metrics on X_i and let A_i be monopoles with respect to these metrics. Choose a constant $K > 0$. There are $\tilde{\eta}_0 > 0$ and $\tilde{\lambda} > 0$ such that for any $\lambda \leq \tilde{\lambda}$ and $\eta_0 \leq \tilde{\eta}_0$ and any metric g on X which is $K \cdot \lambda^{1/2}$ close to g_0, g_1, statements* (1) *and* (2) *of Theorem* 5.2 *hold.*

The formula of Donaldson [16, 4.57], giving a highest order approximation of $\langle \varphi_0, \omega \rangle_{L^2}$ ($\omega \in H^2_{0,A_0}$) in the case of metrics conformally flat near x_0, x_1, remains unaltered; only ω is an $\widetilde{S}^1$-invariant form here. The formula reads

$$\langle \varphi_0(\rho, p_0, p_1), \omega \rangle_{L^2} = q_0^\omega(\rho, p_0, p_1) + O(\lambda^3 + |p_0|^3 + (|p_0| + |p_1|)\lambda^2), \tag{5.7}$$

with q_0^ω the quadratic form we encountered in the deformation theory for monopoles, modified with a term involving the bundle clutching parameter ρ:

$$q_0^\omega(\rho, p_0, p_1) = \langle \tfrac{1}{2}[p_0, p_0], \omega \rangle_{L^2} - \tfrac{1}{2} \cdot v_3 \cdot \lambda^2 \cdot \mathrm{Tr}(F^{(A_1+p_1)}(x_1) \cdot \rho^{-1} \omega^\alpha \rho). \tag{5.8}$$

(In Donaldson's formula, 4 should be 4^{-1}, and our inner product is two times his inner product.) Here v_3 is the 3-volume of the standard 3-sphere, σ is the orientation reversing isometry $T_{x_0}X_0 \to T_{x_1}X_1$, and $\omega^\sigma = (\sigma^{-1*}\omega)(x_1)$. Observe that q_0^ω is linear in ω, therefore $q_0^\cdot$ and $q_1^\cdot$ define a map

$$q = q_0^\cdot \oplus q_1^\cdot : I \times H^1_{0,A_0} \times H^1_{0,A_1} \to H^2_{0,A_0} \oplus H^2_{0,A_1}.$$

If q is C^1-close to Φ, then stability theorems imply that the zero set of Φ is modelled on that of q.

The procedure above extends in the obvious way to construct $\widetilde{S}^1$-invariant connections over $X_0 \# X_1 \# \ldots \# X_l$, where the X_i, $i = 1, \cdots, l$ are attached to X_0 at different points. Specifically we shall take X_0 to be our original manifold X, all X_i to be S^4, and attach $\infty \in S^4$ to different points $x_{j_i} \in S_j \subset X$, where $m_j \neq 0$. This is essentially the same situation as considered in Taubes [31], [30] and in Donaldson [16, §V]. The index j_i relates to the ith point in S_j as in §4. Let $\overline{\lambda} = \max \lambda_{j_i}$, where λ_{j_i} are the scales in the identifications. The metric on X can be seen to be conformally (const $\cdot \overline{\lambda}$)-close to the metrics on X and the S^4's.

To start the alternating procedure take for A_0 a monopole in $\mathscr{M}(m_j, k_j - l_j)$, where l_j is the number of j_i's; so $l = \sum_j l_j$. For A_{j_i} we take the standard one-monopoles I_{m_j} on S^4 with mass m_j, i.e., $I_{m_j} \in \mathscr{M}(m_j, 1)$, which we discussed in §1.

We shall now work out formula (5.8) for this case. There are no obstructions coming from the 4-spheres. Take $\omega \in H^2_{0,A_0} \subset \mathscr{W}^2_+(X)$; then our interest lies in the second term of (5.8). Let $E = P \times_{\mathrm{SU}(2)} \mathbf{C}^2$, and recall that $E_{|S_j} = L_j \oplus L_j^*$, where $\widetilde{S}^1$ acts on L_j with weight $+m_j$. If $x \in S_j$ and $Z_x \in \Lambda^2 T_x X$ is the two-vector dual to $dV_{S_j} + *_X dV_{S_j} \in \Lambda^2 T^*_x X$ (with dV_{S_j} denoting the volume 2-form using the orientation of S_j), then $\omega(Z_x)$ is a skew adjoint, S^1-invariant endomorphism of E_x, which restricts to a purely imaginary scalar on L_j. Define

$$e(\omega(x)) = -i \cdot \omega(Z_x)_{|L_{j,x}} \in \mathbf{R}.$$

Proposition 5.6. *If we glue l standard one-monopoles A_{j_i} of masses m_j to A_0, then*

$$(5.9) \qquad \langle q_0(p_0, 0, \rho_{j_i}), \omega\rangle = \langle \tfrac{1}{2}[p_0, p_0], \omega\rangle_{L^2(X_0)} - \sum_{\text{all } j_i} v_3 \cdot \lambda^2_{j_i} \cdot e(\omega(x_{j_i})).$$

Proof. Denote by P_I, E_I etc. the bundles on S^4 associated to the standard one-monopoles. Then, as a 2-form with values in endomorphisms of $L_I \oplus L_I^*$, the curvature of A_I restricted to $S^2 \subset S^4$ equals

$$F_I = \frac{1}{2} \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot (dV_{S^2} - *_4 dV_{S^2}),$$

(see Example 1.5(2)). So F_I takes values in the trivial summand $\mathbf{R}$ of the endomorphism bundle $\mathfrak{g}_I$. The S^1-equivariant clutching parameter $\rho\colon \mathfrak{g}_{P_0} \to \mathfrak{g}_I$ restricts to the identity map $\mathbf{R} \to \mathbf{R}$. Furthermore, the orientation reversing isometry $\sigma\colon T_{x_{j_i}} X \to T_\infty S^4$ maps dV_{S_j} to $-dV_{S^2}$, also because it is S^1-equivariant.

Now put

$$\omega(x_{j_i}) = \begin{bmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{bmatrix} \cdot (dV_{S_j} + *_4 dV_{S_j}) + R,$$

with R perpendicular in $(\Lambda^2_+ X \otimes \mathfrak{g}_P)_{x_{j_i}}$ to the first term. Then $e(\omega(x_{j_i})) = \alpha$. But

$$\begin{aligned} &-\frac{1}{2} \cdot v_3 \cdot \lambda^2_{j_i} \cdot \mathrm{tr}(F_I(\infty) \cdot \rho^{-1} \omega^\sigma \rho) \\ &= v_3 \cdot \lambda^2_{j_i} \cdot \mathrm{tr}\left(\frac{1}{2} \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{bmatrix}\right) \\ &= -v_3 \cdot \alpha \cdot \lambda^2_{j_i} = -v_3 \cdot \lambda^2_{j_i} \cdot e(\omega(x_{j_i})), \end{aligned}$$

so the proposition follows.

Remark 5.7. Floer [21] expressed the 'constraint formula' (5.9) a bit differently. By expanding ω in normal coordinates around S_j one recovers Floer's formula.

There is an important difference between formula (5.8) for instantons and (5.9) for monopoles. In (5.8) the sign of the second term can be changed by a different choice of ρ, provided ω and F^{A_1} do not vanish at the point in question. To get an existence theorem one has to take care of more than one ω, and therefore the condition for existence is, roughly, that one attaches sufficiently many 4-spheres at points which are in general position with respect to the ω's. For monopoles the signs of these terms are determined by the value distribution of the $e(\omega(x))$, where x ranges through the fixed surfaces in X, and ω through H_0^2. This is a more global affair: indeed, examples exist (see Example 5.8 and Braam [9]) which show that for existence it may be necessary to have k_j nonzero for more than just one of the j. Also this property of (5.9) makes it rather more cumbersome to formulate a general existence theorem for monopoles than for instantons, because one would first have to create an overview of the value distribution of the $e(\omega(x))$. This can certainly be carried out if all m_j are equal, because then we can start with the trivial monopole and H_0^2 is just the de Rham cohomology of M. If there are no obvious, e.g. reducible, solutions with unequal masses, then it seems very hard to prove a general existence theorem for monopoles with such unequal masses, because it is not clear how to 'start' the gluing procedures.

Next we continue the general discussion. The parameters in H_0^1 for the standard one-monopoles can, after grafting, be considered as small variations in the attaching points and scales of the identification. Notice that in coordinates in M near $x_{j_i} \in \delta M$, the zero of the Higgs field of a grafted monopole is approximately the point (x_{j_i}, λ), with the second coordinate a normal coordinate to δM. Our construction gives a family of connections parametrized by

$$N_r = (B_r(0) \subset H^1_{0,A_0}) \times \prod_{\text{all } j_i} ((B_r(x_{j_i}) \subset S_j) \times S^1 \times ((1-r)\lambda_{j_i}, (1+r)\lambda_{j_i})),$$

with $r > 0$ sufficiently small, and $B_\varepsilon(y)$ denoting a ball of radius ε around y. The monopoles in N_r are the zero set of $\Phi\colon N_r \to H^2_{0,A_0}$. There are no problems in defining the bigger family of connections described by

$$N = (B_r(0) \subset H^1_{0,A_0}) \times \prod_{\text{all } j_i} (B_r(x_{j_i}) \times S^1 \times (0, \varepsilon))$$

(for some $\varepsilon > 0$) such that the monopoles are still zeros of a $\Gamma_{A_0} \times \{\pm 1\}^l$-equivariant smooth map $\Phi\colon N \to H^2_{0,A_0}$. To prove this it is necessary to check that letting $\lambda_{j_i} \to 0$ and varying the attaching points x_{j_i} does not force one to shrink the ball in H^1_{0,A_0} drastically. For the scales this follows from the fact that all estimates in the alternating procedure improve upon letting

$\lambda \to 0$ (see Donaldson [16, 4.3–4.6]). The convergence of the alternating procedure is also easily seen to be locally uniform under small variations in the attaching points.

We now insert an example to illustrate the material discussed so far.

Example 5.8. Monopoles on $S^2 \times \mathbf{R}$.

(1) Take $E = \mathrm{O}(k, -k) \oplus \mathrm{O}(-k, k)$ with the reducible A_0 connection as a monopole with $m_0 = 2k, k_0 = k, m_\infty = 0$. The quadratic form (3.13) is given by

$$H_0^1 \to \mathbf{R}\colon p_0 \to \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \omega, P_+[p_0, p_0] \right\rangle_{L^2} = \|p_0\|_{L^2}^2,$$

(see Example 3.3(2)). This is positive definite, so the given reducible solution is isolated. In fact, using algebraic geometry, one can prove that $\mathscr{M}((2k, 0), (k, 0))$ is a point (see Braam [9]).

Next we try to glue in a monopole of mass $2k$ at a point in $S^2 \times \{0\}$ in order to obtain elements of $\mathscr{M}((2k, 0), (k+1, 0))$. For the function q we find (compare (5.9))

$$q\colon N \cong \mathbf{C}^{2k-1} \times S^2 \times S^1 \times (0, \varepsilon) \to \mathbf{R},$$
$$(p_0, x, \rho, \lambda) \to |p_0|^2 - v_3 \cdot \lambda^2.$$

This is a family of nondegenerate quadratic forms parametrized by (x, ρ), so one expects the moduli space of $((2k, 0), (k+1, 0))$-monopoles to look asymptotically like $(S^2 \times S^1 \times \{(p_0, \lambda); |p_0|^2 - \lambda^2 = 0, \lambda > 0\})/S^1$, a bundle of cones over S^2. (Observe that this has real dimension $4(k+1)-4$, as predicted by the index theorem, whereas the dimension of $\mathscr{M}((2k, 0), (k, 0))$ is 'wrong'.) To make this rigorous, we need to know how Φ is approximated by q for λ and p small, and we shall make some remarks concerning this below.

It can also be proved that $\mathscr{M}((2k, 0), (l, 0))$ is empty for $l < k$ (see Braam [9]). Therefore the compactification of this moduli space is much smaller than one might naively be led to expect, because many of the possible lower strata do not occur.

(2) Next consider the trivial bundle E with the trivial connection as an element of $\mathscr{M}((2m, 2m), (0, 0))$, $m \in \mathbf{Z}_{>0}$. For the trivial connection one has $H_0^1 = 0, H_0^0 = \mathbf{R}, H_0^2 = \mathbf{R}$, where the latter is generated by what is essentially the Kähler form of X, as in Example 2.5(2).

It can be proved that $\mathscr{M}((2m, 2m), (k, 0))$ is empty for any $k > 0$ (see Braam [9]), which exemplifies the statement that it may be necessary to have more than just one $k_j > 0$ for existence. This does not come totally unexpectedly, for the constraint formula (5.9) is easily seen to be definite if one applies it to attaching one-monopoles to one fixed surface. However, monopoles in $\mathscr{M}((2m,2m),(k_0,k_\infty))$ with $k_0, k_\infty > 0$ exist; the signs of $e(\omega(x))$ are different for the two fixed S^2's ($\omega \in H_0^2$).

(3) Finally consider again the reducible solution on $E = \mathrm{O}(k,-k) \oplus \mathrm{O}(-k,k)$, but now as a monopole with

$$m_0, m_\infty > 0, \quad k_0 = k, \quad k_\infty = -k, \quad m_0 - m_\infty = k,$$

i.e., as an element of $\mathscr{M}((m_0,m_\infty),(k,-k))$. Our existence theorem supplies monopoles in $\mathscr{M}((m_0,m_\infty),(k+l,-k+m))$ for any $l, m > 0$. Thus there is a sequence of monopoles with positive charges which converges to a monopole with a negative charge.

If $[A_n]$ is a sequence of monopoles in N such that the coordinates $(p_0, x_{j_i}, \rho_{j_i}, \lambda_{j_i})_n$ converge to $(0, x_{j_i}, \rho_{j_i}, 0)$, then the sequence $[A_n]$ converges to $([A_0],(x_{j_i}))$ in the compactified moduli space; this is easy to check.

Our final aim is to show that the methods of this section describe a complete neighborhood of $([A_0],(x_{j_i}))$ in the compactified moduli space. Interpret $S^1 \times (0,\varepsilon)$ as the punctured disc $B_\varepsilon(0)\backslash\{0\} \subset \mathbf{C}$ and define

$$\overline{N} = (B_r(0) \subset H^1_{0,A_0}) \times \prod_{\text{all } j_i} \{B_r(x_{j_i}) \times (B_\varepsilon(0) \subset \mathbf{C})\}.$$

Putting one of the scales, say $\lambda_{j_{0_{i_0}}}$, equal to zero amounts to looking for a monopole in the lower stratum $S_{j_0} \times \mathscr{M}(m_j, k_j')$ of the compactified moduli space $\overline{\mathscr{M}}(m_j,k_j)$, with $k_j' = k_j$ for $j \neq j_0$ and $k_{j_0}' = k_{j_0} - 1$. The function q (see (5.9)) extends naturally to a map $\overline{N} \to H^2_{0,A_0}$, and from the formula preceding (5.8) it follows that also Φ extends continuously to $\overline{N}$. Appealing to Donaldson [16, 5.4], we can obtain more.

Proposition 5.9. *The function $\Phi\colon \overline{N} \to H^2_{0,A_0}$ is C^1, and is C^1-approximated by q for small λ_{j_i} and p_0. Furthermore, $\Phi^{-1}\{0\}/(\Gamma_{A_0} \times \{\pm 1\}^l)$ is a family of gauge inequivalent ideal monopoles.*

Thus we have an injection immersion $(\Phi^{-1}\{0\} \cap \overline{N})/(\Gamma_{A_0} \times \{\pm 1\}^l) \to \overline{\mathscr{M}}$. It remains to show that any $A \in \mathscr{M}(m_j,k_j)$ close to (x_{j_i}, A_0) lies in the image of this map. For monopoles with $m_j = 1$ the basic 1-monopoles are also the basic 1-instantons, and the result follows from Donaldson [16, Proposition 4.11], which asserts that such an A can be put in the standard form (5.2). For arbitrary m_j one once more has to adapt instanton proofs to monopoles. This results in:

Theorem 5.10. *A neighborhood of $([A_0],(x_{j_i}))$ in $\overline{\mathscr{M}}(m_j,k_j)$ is modelled on the quotient of the zero set of $\Phi\colon \overline{N} \to H^2_{0,A_0}$ by $\Gamma_{A_0} \times \{\pm 1\}^l$. The stratification is induced by taking intersections of $\Phi^{-1}\{0\}$ with hyperplanes*

$\lambda_{j_i} = 0$. *The map* Φ *is a* C^1*-perturbation of the function* $q\colon \overline{N} \to H^2_{0,A_0}$ *(see* (5.9)) *for small* p_0 *and* λ_{j_i}.

We end with a refinement of Example 5.4 (compare Floer [19]):

Proposition 5.11. *Let* M *be a* 3*-manifold as in Corollary* 5.3. *There are compact sets* $K \subset M$ *and* $\mathscr{K} \subset \mathscr{M}(m, 1)$ *such that the map* $\mathscr{M}(m, 1)\backslash\mathscr{K} \to M\backslash K$ *which assigns the zero of the Higgs field to a monopole is a smooth fibration with fiber* $\{H^1(M; \mathbf{R})/H^1(M; \mathbf{Z})\} \times H_1(M; \mathbf{Z})_{\text{tor}}$.

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