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ANALYTIC EXTENSIONS OF THE ZETA FUNCTIONS FOR SURFACES OF VARIABLE NEGATIVE CURVATURE

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0. Introduction

The purpose of this note is to prove that for a compact C^∞ Riemannian surface of (variable) negative curvature the associated zeta function ς satisfies: $\varsigma(s)$ is nonzero and analytic on a half-plane $\mathrm{Re}(s) > h - \delta$ $(h, \delta > 0)$ except for a simple pole at $s = h$.

The result is well known in the special case that the surface has constant negative curvature (cf. [3], for example). For constant curvature surfaces one can use the Selberg trace formula, whose very existence seems to depend strongly on the Lie group construction of the surface. More generally it appears different techniques are required.

We adopt a dynamical viewpoint and study the associated geodesic flow. By an earlier result of the author (on more general Axiom A flows) we know that $\varsigma(s)$ can be extended *meromorphically* to a domain of the above form [9]. The difficulty is to show that no poles (other than at $s = h$) actually occur. For variable curvature geodesic flows we give a simple necessary condition for the occurrence of poles in this region (Lemma 4). The result follows by showing this condition is void (Lemma 5 and Theorem).

The proof that we give works at a slightly more general level than for geodesic flows. Our main result remains valid for the case of any transitive weak-mixing three-dimensional Anosov flow for which the stable and unstable horocycle foliations are continuously differentiable.

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1. Definitions and basic constructions

We begin by introducing basic material we shall need for the proof.

Let $\phi_t \colon M \to M$ be the geodesic flow on M, the unit tangent bundle of a compact C^∞ Riemannian surface S of strictly negative curvature. This

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flow is of Anosov type, i.e., $TM = E^0 \oplus E^u \oplus E^s$ where E^0, E^u and E^s are continuous $D\phi_t$-invariant one-dimensional sub-bundles with E^0 tangent to the flow, and $C, \lambda > 0$ such that $\|D\phi_t v\| \le Ce^{-\lambda t}\|v\|$ for $v \in E^s$, $t \ge 0$, and $\|D\phi_{-t}v\| \le Ce^{-\lambda t}\|v\|$ for $v \in E^u$, $t \ge 0$ [1].

Let $h > 0$ be the topological entropy of the flow. The flow ϕ is (topologically) weak-mixing, i.e., there are no nontrivial solutions to $F\phi_t = e^{iat}F$, $a > 0$, $F \in C(M)$.

Given $\varepsilon > 0$ we denote the ε-local stable and unstable manifolds through $x \in M$ by

$$W^s(x) = \{y \in M \mid d(\phi_t, x, \phi_t y) \le \varepsilon \ \forall t \ge 0, d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to +\infty\},$$
$$W^u_\varepsilon(x) = \{y \in M \mid d(\phi_{-t} x, \phi_{-t} y) \le \varepsilon \ \forall t \ge 0, d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to +\infty\},$$

respectively. These are C^∞-embedded one-dimensional discs with $T_x W^s_\varepsilon(x) = E^s_x$ and $T_x W^u_\varepsilon(x) = E^u_x$. These are neighborhoods of x in the (global) stable and unstable manifolds through x denoted by

$$W^s(x) = \{y \in M \mid d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to +\infty\},$$
$$W^u(x) = \{y \in M \mid d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to +\infty\},$$

respectively. These form C^1-foliations of M which we denote by $\mathscr{F}^s, \mathscr{F}^u$.

Let S_i ($i = 1, \cdots, n$) be *parallelograms* (cf. [10] for full definitions). In particular, each S_i is a flow box which is the C^1-embedding of a set $\{(x, y, t) \in [0,1] \times [0,1] \times \mathbf{R} \mid 0 \le t \le r(y)\}$ where $r\colon [0,1] \to \mathbf{R}$ is a strictly positive C^1-function. We denote the embedding map by π_i. The sets T_i should satisfy

(i) $\pi_i(x, y, (0, r(y))) = S_i \cap \phi_{[-\varepsilon,\varepsilon]}\pi_i(x, y, 0)$ and $\pi_i(x, y, t) = \phi_t \pi_i(x, y, 0)$ $(0 \le t \le r(y))$.

(ii) $\pi_i([0,1], y, t) = S_i \cap W^s_\varepsilon(\pi_i(0, y, t))$.

(iii) $\phi_{g_i(x,y,t)}\pi_i(x, y, t) \in W^u_\varepsilon(\pi_i(x, 0, t))$, where g_i is C^1.

We assume throughout that $\operatorname{diam} S_i \ll \varepsilon \ll 1$.

The family of parallelograms $\{S_i\}$ forms a *partition* if the S_i have mutually disjoint interiors and $M = \bigcup_{i=1}^n S_i$. We call $\{S_i\}$ *Markovian* provided:

(a) if $\pi_i((0,1), y, r(y)) \cap \pi_j((0,1), y', 0) \ne \varnothing$ then $\pi_i([0,1], y, r(y)) \subset \pi_j([0,1], y', 0)$;

(b) if $\pi_i(x, (0,1), 0) \cap \pi_k(x', y, r(y)) \ne \varnothing$ for some $0 < y < 1$ then $\pi_i(x, [0,1], 0) \subset \bigcup_{y \in [0,1]} \pi_k(x', y, r(y))$.

Let $J = \coprod_{i=1}^n [0,1]$ the indexing corresponding to the parallelograms. We define $f\colon J \to J$ by $f(x) = x'$ (as in (b)). This map is C^1. We can always choose smaller parallelograms to make this true, if necessary. Let A be an $n \times n$ matrix with (i,j)th entry 1 if (a) is satisfied and 0 otherwise.

Lemma 1. *For the geodesic flow $\phi_t\colon M \to M$ there exists a Markovian partition of C^1 parallelograms.*

Proof. This is a standard result due to Ratner [10] and Bowen [2]. The only additional observation is the Hirsch-Pugh result that the foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ are C^1 implies that the embeddings π_i can be chosen C^1 [4].

Lemma 2. *For each $i = 1, \cdots, n$, $0 \le y \le 1$, $0 \le t \le r(y)$ there exists a measure $\mu^i_{(y,t)}$ supported on $\pi_i([0,1], y, t)$ such that $\phi^*_{t'}\mu^i_{(y,t)} = e^{ht'}\mu^i_{(y,t+t')}$ for $0 \le t, t+t' \le r(y)$. If $\phi_{t'}\pi_i([0,1], y, t) \subset \pi_j([0,1], y'', t'')$, then*

$$\phi^*_{t'}\mu^i_{(y,t)} = e^{ht'}\mu^j_{(y'',t'')}\big|_{\phi_{t'}\pi_i([0,1],y,t)}.$$

Furthermore, if $f\colon S_i \to \mathbf{C}$ is C^1, then $(y,t) \to \int f\, d\mu^i_{(y,t)}$ is also C^1.

Proof. We can construct $\mu^i_{(y,t)}$ from the Margulis measure μ_M. This is a transverse measure to the foliation $\mathscr{F}^u$ which transforms as $\phi^*_t\mu_M = e^{ht}\mu_M$ [5]. To define $\mu^i_{(y,t)}(B)$ for a Borel set $B \subset \pi_i([0,1], y, t)$ we consider $\phi_{[-\delta,\delta]}(B)$. This lies in the transverse section $\phi_{[-\delta,\delta]}\pi_i([0,1], y, t)$ to $\mathscr{F}^u$. We define

$$\mu^i_{(y,t)}(B) = \lim_{\delta \to 0} \mu_M(\phi_{[-\delta,\delta]}B)/\delta h.$$

This is well defined by the property $\phi^*_t\mu_M = e^{ht}\mu_M$. By construction $\mu^i_{(y,t)}$ transforms in the way described.

For the final part we observe that for fixed (y,t):

$$\int f(x, y, t+t')\, d\mu^i_{(y,t+t')} = \int [f(x, y, t+t')e^{ht'}]\, d\mu^i_{(y,t)}(x),$$

$$\int f(x, y+y', t)\, d\mu^i_{(y+y',t)} = \int [f(x, y+y', t)e^{hg_i(x, y+y', t)}]\, d\mu^i_{(y,t)}(x).$$

Hence $(y', t') \to \int f(x, y+y', t+t')\, d\mu^i_{(y+y',t+t')}$ is C^1.

2. Zeta-functions and their poles

We define the zeta function $\varsigma(s)$, $s \in \mathbf{C}$, by the Euler product $\varsigma(s) = \prod_\tau (1 - e^{-s\lambda(\tau)})^{-1}$, where τ denotes a closed geodesic of length $\lambda(\tau)$. This is well defined for $\mathrm{Re}(s)$ sufficiently large.

For general Anosov flows we can define $\varsigma(s)$ in a similar fashion where τ now denotes a closed orbit for the flow of least period $\lambda(\tau)$.

To characterize the poles we proceed as follows: Let $C^1(J)$ denote complex C^1-functions on J with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. We define a Perron-Frobenius-type operator $L_s\colon C^1(J) \to C^1(J)$, $s \in \mathbf{C}$, by

$$(L_s h)(x) = \sum_{z\colon fz = x} e^{-sr(z)} h(z),$$

where $f\colon J \to J$ is as defined in the previous section. The dual operator $L^*_s\colon C^1(J)^* \to C^1(J)^*$ is $(L^*\nu)(h) = \nu(L_s h)$.

Lemma 3. (i) $\varsigma(s)$ *is nonzero and analytic for* $\mathrm{Re}(s) > h$;

(ii) $\varsigma(s)$ *has a nonzero meromorphic extension to* $\mathrm{Re}(s) > h - \delta$ *for some* $\delta > 0$;

(iii) *If* $s = s_0$ *is a pole for* $\varsigma(s)$ *with* $\mathrm{Re}(s) > h - \delta$, *then* $L_s^* \nu = \nu$ *for some* $\nu \in C^1(J)^*$.

Proof. Parts (i) and (ii) follow directly from [9]. For part (iii) the original condition was that 1 should be an isolated eigenvalue for L_s acting on Hölder continuous functions on a subshift of finite type Σ. The space $C^1(J)$ corresponds to a strictly smaller space in the space of Hölder continuous functions H^α (the correspondence being the injection induced by the semi-conjugacy $\pi\colon \Sigma \to J$, i.e., $\pi^*\colon C^1(J) \hookrightarrow H^\alpha$), where $\alpha > 0$ is some Hölder exponent related to the foliation.

The (generalized) eigenspaces for the isolated eigenvalues of $L_{s_0}\colon H^\alpha \to H^\alpha$, which are disjoint from the essential spectrum, lie in the subspace $C^1(J)$. To see this consider the eigenvalues ordered by modulus. Assume that λ is a unique eigenvalue of maximum modulus, having eigenprojection P_λ. Then the eigenspace V_λ is contained in $C^1(J)$ since for any $h \in C^1(J)$ the iterates $L_{s_0}^n h/\lambda^n$ remain in $C^1(J)$ but converge to $P_\lambda h \in V$. If there are two, or more, eigenvalues of equal modulus we can modify this simple argument using Cesaro averages. We can then proceed inductively, replacing L_{s_0} by $L_{s_0} - \lambda P_\lambda$, dealing with successive isolated eigenvalues.

We can conclude that if 1 is an isolated eigenvalue for $L_{s_0}\colon H^\alpha \to H^\alpha$ then it is also an isolated eigenvalue (of finite multiplicity in both cases) for $L_{s_0}\colon C^1(J) \to C^1(J)$.

By duality the spectrum of L_{s_0} acting on $C^1(J)$ is the same as the spectrum of $L_{s_0}^*$ acting on $C^1(J)^*$. In particular, 1 is an eigenvalue for $L_{s_0}^*\colon C^1(J)^* \to C^1(J)^*$, of finite multiplicity. Thus we can choose $\nu \in C^1(J)$ such that $L_{s_0}^* \nu = \nu$.

For the remainder of this section we shall assume that s is a pole for the zeta function.

Let $C^1(M)$ denote the C^1-functions $f\colon M \to \mathbf{C}$. This is a Banach space with norm $\|f\| = \|f\|_\infty + \|f\|_1$, where $\|f\|_1 = \sup\{\|D_x f\|' \mid x \in M\}$; here $\|\ \|'$ is the norm of $D_x f \in L(T_x M, T_{fx} M)$. We can construct linear functionals on $C^1(M)$ by the following:

Lemma 4. *If* $L^* \nu = \nu$, $\nu \in C^1(J)^*$, *then there exists* $m \in C^1(M)^*$ *such that* $\phi_t^* m = e^{(s-h)t} m$.

Proof. We define m by

$$m(f) = \sum_{i=1}^{n} \nu \left(\int_0^{r(y)} e^{-st} \int f \pi_i \, d(\pi_i^* \mu_{(y,t)}^i(x)) \, dt \right).$$

To show $\phi_t^* m = e^{(s-h)t} m$ we shall assume (without loss of generality) that $0 < t = T < \inf r$. We can write:

$$\begin{aligned}
f_i &= \int_0^{r(y)} e^{st} \left[\int f\pi_i(x,y,t)\, d(\pi_i^* \mu^i_{(y,t)}(x)) \right] dt \\
&= \int_T^{r(y)} e^{st} \left[\int f\pi_i(x,y,t)\, d(\pi_i^* \mu^i_{(y,t)}(x)) \right] dt \\
&\quad + \sum_{A(j,i)=1} \int_0^T e^{st} \left[\int f|_{\phi_T S_j} \pi_i(x,y,t)\, d(\pi_i^* \mu^i_{(y,t)}(x)) \right] dt \\
&= e^{sT} \int_0^{r(y)-T} e^{st} \left[\int f\pi_i(x,y,t+T)(e^{-hT} d\pi_i^* \mu^*_{(y,t)}(x)) \right] dt \\
&\quad + \sum_{A(j,i)=1} e^{-sr(\rho_j,y)} e^{sT} \int_{r(\rho_j y)-T}^{r(\rho_j,y)} e^{-st} \\
&\qquad\qquad \times \left[\int f\pi_j(x,\rho_j y, t+T)(e^{-hT} d\pi_j^* \mu^j_{(\rho_j y,t)}(x)) \right] dt \\
&= e^{(s-h)T}(f_j^0 + L_s f_i^1),
\end{aligned}$$

where ρ_j is the local inverse to $f\colon J \to J$,

$$\begin{aligned}
f_i^0(y) &= \int_0^{r(y)-T} e^{st} \left[\int f\pi_i(x,y,t+T)\, d(\pi_i^* \mu^i_{(y,t)}(x)) \right] dt, \\
f_i^1(y) &= \int_{r(y)-T}^{r(y)} e^{st} \left[\int f\pi_i(x,y,t+T)\, d(\pi_i^* \mu^i_{(y,t)}(x)) \right] dt.
\end{aligned}$$

Thus

$$\begin{aligned}
m(f) = \nu\left(\sum_{i=1}^n f_i\right) &= \sum_{i=1}^n e^{(s-h)T} \nu(f_i^0 + L_s f_i^1) \\
&= \sum_{i=1}^n e^{(s-h)T} \nu(f_i^0 + f_i^1) = e^{(s-h)T} m(f\phi_T),
\end{aligned}$$

where $\nu(L_s f_i^1) = \nu(f_i)$ by assumption and by comparing definitions: $f_i^0 + f_i^1 = (f\phi_T)_i$. q.e.d.

The functional m is nonzero. This is a consequence of the functional ν being nonzero and having full support on J in a distributional sense, i.e., for any open interval $I \subseteq J$ there exists $\psi \in C^1(J)$ with $\operatorname{supp}(\psi) \subseteq I$ and $\nu(\psi) \neq 0$. This simple condition can be deduced from the identity $L_{s_0}^* \nu = \nu$.

We can derive estimates for the spectrum of the induced action of the flow on C^1-functions, i.e., for $\phi_t^*\colon C^1(M) \to C^1(M)$ by $(\phi_t^* f) = f \circ \phi_t$.

Lemma 5. *There exist $0 < \lambda_1 < \lambda_2 < 1 < \mu_1 < \mu_2$ such that* $\mathrm{sp}(\phi_t^*)$ *is contained in the union of:* (i) *the unit circle K*; (ii) *the annulus A_λ with inner and outer radii λ_2, λ_1, respectively, and* (iii) *the annulus A_μ with inner and outer radii μ_2, μ_1, respectively.*

Proof. Since constant functions are ϕ_t^*-invariant, consider the induced operator $\phi_t^*\colon C^1(M)/\mathbf{C} \to C^1(M)/\mathbf{C}$ on the quotient space $C^1(M)/\mathbf{C}$, where $\mathbf{C}$ denotes constant functions. $C^1(M)/\mathbf{C}$ is a Banach space with the quotient norm $\|f + \mathbf{C}\| = \|f\|_1$. We can identify this space with continuous sections $s\colon M \to TM$, which we denote by $C^0(M,TM)$. Since $TM = E^0 \oplus E^u \oplus E^s$ we can decompose $C^0(M,TM)$ into ϕ_t^*-invariant spaces: $C^0(M,TM) = C^0(M,E^0) \oplus C^0(M,E^u) \oplus C^0(M,E^s)$. We can complexify each of these spaces; we denote, for example, the complexification $C^0(M,TM) \oplus_{\mathbf{R}} \mathbf{C}$ by $\hat{C}^0(M,TM)$.

By definition, $D\phi_t\,|_{E^u}$ is uniformly expanding; thus by the spectral radius theorem applied to $D\phi_t\,|_{E^u}$ and $D\phi_{-t}\,|_{E^u}$ we have $\mathrm{sp}(\phi_t^* \,|\, \hat{C}^0(M,E^u)) \subset A_\mu$ for appropriate expansion bounds $1 < \mu_1 < \mu_2$. Similarly, $\mathrm{sp}(\phi_t^* \,|\, \hat{C}^0(M,E^s)) \subset A_\lambda$ for appropriate contraction bounds $0 < \lambda_1 < \lambda_2 < 1$ on $D\phi_t\,|_{E^s}$. For $\phi_t^* \,|\, \hat{C}^0(M,E^0)$ we observe that ϕ_t^* preserves the norm and so $\mathrm{sp}(\phi_t^* \,|\, \hat{C}^0(M,E^0) \subset K$. The spectra are related by:

$$\text{(i)} \qquad \mathrm{sp}(\phi_t^* \,|\, \hat{C}^1(M)) \subset \mathrm{sp}(\phi_t^* \,|\, \mathbf{C}) \cup \mathrm{sp}(\phi_t^* \,|\, \hat{C}^1(M)/\mathbf{C});$$

and

$$\text{(ii)} \qquad \begin{aligned} \mathrm{sp}(\phi_t^* \,|\, \hat{C}^1(M)/\mathbf{C}) \subseteq \mathrm{sp}(\phi_t^* \,|\, \hat{C}^0(M,E^u)) &\cup \mathrm{sp}(\phi_t^* \,|\, \hat{C}^0(M,E^s)) \\ &\cup \mathrm{sp}(\phi_t^* \,|\, \hat{C}^0(M,E^0)). \end{aligned}$$

Thus $\mathrm{sp}(\phi_t^*) \subset A_\lambda \cup A_\mu \cup K$, as claimed.

3. Main result

We are now in a position to prove the following:

Theorem. *Let S be a compact C^∞ Riemannian surface of* (*variable*) *negative curvature. Let $\varsigma(s)$ be the associated zeta function. There exists $\varepsilon > 0$ so that $\varsigma(s)$ is nonzero and analytic for* $\mathrm{Re}(s) > h - \varepsilon$, *except for a simple pole at $s = h$.*

Proof. For $\mathrm{Re}(s) = h$ we know that there is exactly one pole on $\mathrm{Re}(s) = h$ (which is a simple pole at $s = h$) since ϕ_t is weak-mixing (cf. [6]). Thus by Lemma 3, parts (i) and (ii), the theorem can only fail if there exists a sequence of poles $s_n = \sigma_n + it_n$ with $\sigma_n < h$ and $\sigma_n \to h$. If this were true then for each s_n there exists $m_n \in C^1(M)^*$ with $\phi_t^* m_n = e^{(s_n - h)t} m_n$, by Lemma 3, part (iii), and Lemma 4. In particular, $e^{(s_n-h)t} \in \mathrm{sp}(\phi_{t^*})$, where $\phi_{t*}: C^1(M)^* \to C^1(M)^*$ by $(\phi_{t*}m)(f) = m(f\phi_t)$, since $e^{(s_n-h)t}$ is an eigenvalue with eigenfunction m_n. However, since ϕ_{t*} is the dual operator to ϕ_t^* we have $\mathrm{sp}(\phi_t^*) = \mathrm{sp}(\phi_{t*})$, and in particular $e^{(s_n-h)t} \in \mathrm{sp}(\phi_t^*)$. However, for sufficiently large n, $e^{(s_n-h)t} \notin K \cup A_\lambda \cup A_\mu$ (since $|e^{(s_n-h)t}| = e^{(\sigma_n-h)t} \to 1$ but $|e^{(s_n-h)t}| \neq 1$). This contradicts Lemma 5.

Remark. Unfortunately, it is not possible to give effective estimates on the value ε in the above theorem. The major problem is that ε depends on the value δ occurring in Lemma 3(ii). The methods used in [9] to prove this result do not yield good estimates on the size of δ.

Corollary 1. *Let $Z(s) = \prod_{n=0}^{+\infty} \prod_\tau (1 - e^{-(s+n)\lambda(\tau)})$ be the Selberg zeta function for a compact C^∞ Riemannian surface of (variable) negative curvature. There exists $\varepsilon > 0$ so that $Z(s)$ is nonzero and analytic for* $\mathrm{Re}(s) > h-\varepsilon$, *except for a simple zero at $s = h$.*

Let m be the unique measure of maximal entropy for the flow. Let $F, G \in C^\infty(M,)$ and define $\rho(t) = \int F\phi_t G\, dm - \int F\, dm \int G\, dm$. We consider the Fourier transform $\hat{\rho}(s) = \int_{-\infty}^{+\infty} \rho(t) e^{ist}\, dt$ $(s \in \mathbf{C})$.

Corollary 2. *The Fourier transform $\hat{\rho}(s)$ has an analytic extension to a strip* $|\mathrm{Re}(s)| \le \varepsilon$ *for some* $\varepsilon > 0$.

Proof. There is a direct correspondence between the domains of $\varsigma(s)$ and $\hat{\rho}(s)$ (cf. [8] and [11] for details). This, together with the theorem, proves the corollary.

The above theorem and its corollaries should have important consequences for the asymptotic behavior of the geodesic flow. We shall postpone a consideration of these aspects until a later date.

The proof we have given of the above results is valid in a slightly more general setting. In particular all the above results remain valid for any transitive weak-mixing three-dimensional C^1 Anosov flow for which both the stable and unstable manifold foliations are of class C^1.

Remark. The C^1 condition on the foliations is assumed so that the (eventually fictitious) functional m lies in $C^1(M)^*$. The reason that we prefer to work with C^1 functions is that this is the most convenient space in which the effects of the hyperbolicity of the flow can be detected (cf. Lemma 5). One reason for dealing only with surfaces is that the C^1 condition on the foliations is automatic, whereas for manifolds of negative sectional curvature the C^1 condition generally requires additional pinching assumptions. A second advantage of surfaces is that the boundaries of Markov partitions are C^1. This condition is convenient in the definition of m. For manifolds of higher dimension the boundaries to the Markov partition are generally not C^1.

We believe the above results should be true at the level of generality of Axiom A attractors. A proof in that context should make use of Hölder continuous functions rather than C^1 functions.

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