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GRIFFITHS' INFINITESIMAL INVARIANT AND THE ABEL-JACOBI MAP

MARK L. GREEN

0. Introduction

In recent years, a body of theorems has accumulated about the locus of smooth surfaces of degree d in $\mathbf{P}^3$ possessing curves which are not complete intersections of the given surface with another surface. Let us call this the *Noether-Lefschetz locus.* The classical theorem of Noether and Lefschetz states that this locus has positive codimension when $d \geq 4$. A simple infinitesimal proof of this theorem is now known [2], every component is known to have codimension at least $d-3$ [5], and for $d \geq 5$, the only component having this codimension is the variety of surfaces containing a line ([4], [9]). There are still some fascinating open problems about this locus (see [8]).

For smooth hypersurfaces X in $\mathbf{P}^n$ in higher dimensions the situation is very different. If we look at codimension-one subvarieties on X, then the Lefschetz theorems show immediately that the Noether-Lefschetz locus is empty in all degrees. If we look at higher codimension subvarieties of X, then the question becomes quite interesting. C. Voisin has recently shown that a general 3-fold in $\mathbf{P}^4$ always possesses curves which cannot be obtained by intersecting X with a surface. There is, however, a beautiful conjecture of Griffiths and Harris [8] which is not contradicted by Voisin's example: *On a general 3-fold X of degree $d \geq 6$, the Abel-Jacobi map from algebraic 1-cycles on X homologically equivalent to zero to the intermediate Jacobian $J^2(X)$ is zero.*

This conjecture is still open. In this paper, we will lay out a three step program for proving it, and do the first two steps. This yields the following partial result:

Theorem 0.1. *For a general 3-fold of degree ≥ 6, the image of the Abel-Jacobi map on algebraic 1-cycles homologically equivalent to zero has image contained in the torsion points of the intermediate Jacobian.*

Here is a sketch of the argument: Let $\mathscr{X} \xrightarrow{\pi} B$ be a family of smooth $(2m+1)$-folds with $\mathscr{X}$ smooth and π everywhere of maximal rank. Let

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$X_t = \pi^{-1}(t)$. Let

$$F^{2m+1} \subseteq F^{2m} \subseteq \cdots \subseteq F^0 \cong H^{2m+1}(X_0, \mathbf{C})$$

be the *Hodge bundles* on B. The bundle $\mathcal{J}^{m+1}$ of intermediate Jacobians is given by

$$J^{m+1}(X_t) = \Lambda \backslash F^0/F^{m+1},$$

where $\Lambda \cong H^{2m+1}(X_t, \mathbf{Z})$ is the integral lattice. For our purposes, a *normal function* for $\mathscr{X}$ will be analytic section ν of $\mathcal{J}^{m+1}$ which satisfies the *infinitesimal condition for normal functions*—let $\tilde{\nu}$ be any local lifting of ν to an analytic section of F^0. We then require that

$$\partial\tilde{\nu}/\partial t_i \in F^m$$

for all i, where $t_1, t_2, \cdots, t_N$ are local coordinates on B. It will not be necessary to include in our definition of normal function the behavior of ν at points of the (compactified) base space where the $(2m+1)$-folds become singular (see [13]). If $\mathscr{Z}$ is a codimension $m+1$ analytic cycle of $\mathscr{X}$ such that $Z_t = \mathscr{Z} \cap X_t$ is homologically equivalent to zero in X_t, then by the Abel-Jacobi map we obtain an analytically varying element $\nu_{\mathscr{Z}}(t) \in J^{m+1}(X_t)$ which satisfies the infinitesimal condition for normal functions (see [13]).

The family that will interest us the most is the *universal family of smooth* $(2m+1)$*-folds of degree* d. For this, let $B_d \subseteq \mathbf{P}^{\binom{2m+2+d}{d}-1}$ be the parameter space of smooth $(2m+1)$-folds of degree d in $\mathbf{P}^{2m+2}$, and $\mathscr{X}_d \to B_d$ the canonical family of $(2m+1)$-folds defined by

$$\mathscr{X}_d = \{(F, z) \in B_d \times \mathbf{P}^{2m+2} | F(z) = 0\}.$$

We will say that a normal function ν is *locally constant* if near every point of B there is a local lifting $\tilde{\nu}$ of ν to F^0 which is constant. Following [8], we will say that a normal function ν is *algebraically defined* over B if there is a generically finite surjective analytic map $\widetilde{B} \to B$ and ν is a normal function for the pullback of the family $\mathscr{X}$ to $\widetilde{B}$. With these linguistic conventions, the first step of the proof is:

Theorem 0.2. *Let* $\mathscr{X}_d \to B_d$ *be the universal family of* $(2m+1)$*-folds of degree* d*, where* $m \geq 1$*. If* Σ *is a subvariety of* B_d *and* ν *is an algebraically defined normal function for the induced family on* Σ*, then* ν *is locally constant provided* $\operatorname{codim} \Sigma \leq m(d-2) - 4$.

The proof of this theorem, which occupies §1, is based on an improvement of a beautiful *infinitesimal invariant of normal functions* introduced by Griffiths [6], together with a vanishing theorem for Koszul cohomology from [5].

The next step, carried out in §2, is a fairly simple monodromy argument, which shows that if we let $\mathscr{X}_d \to B_d$ be the universal family of $(2m+1)$-folds of degree d, then a locally constant algebraically defined normal function on B_d must be the projection of a torsion element of $\Lambda\backslash H^{2m+1}(X_0, \mathbf{C})$. This yields the following result, which was conjectured by Griffiths and Harris [8]:

Theorem 0.3. *Let $\mathscr{X}_d \to B_d$ be the universal family of $(2m+1)$-folds of degree d, where $m \geq 1$. Then any algebraically defined normal function on B_d is torsion provided $d \geq 2 + 4/m$.*

Let us now assume that the general $(2m+1)$-fold X of degree $d \geq 2+4/m$ has nonzero Abel-Jacobi map

$$\left\{\begin{array}{l}\textit{codimension } m+1 \textit{ algebraic cycles} \\ \textit{on } X \textit{ homologous to zero}\end{array}\right\} \to J^{m+1}(X).$$

There is then a smooth variety $\widetilde{B}_d$ and a generically finite dominant map $\widetilde{B}_d \xrightarrow{p} B_d$ and a codimension $(m+1)$ algebraic cycle $\mathscr{Z}$ of the pullback family $\widetilde{\mathscr{X}_d}$ of $\mathscr{X}_d$ to $\widetilde{B}_d$ with $Z_t = \mathscr{Z} \cap X_t$ homologous to zero, having nonzero associated normal function $\nu_{\mathscr{Z}}$. Indeed, the values $\nu_{\mathscr{Z}}(\tilde{t})$ taken over all $\tilde{t} \in p^{-1}(t)$ and all $\widetilde{B}_d$ and all $\mathscr{Z}$ constructed in this way give the image of the Abel-Jacobi map for X_t for t general. However, by the previous theorem these algebraically determined normal functions are all torsion. We thus have:

Theorem 0.4. *For a general $(2m+1)$-fold X of degree $d \geq 2+4/m$ in $\mathbf{P}^{2m+2}$, the image of the Abel-Jacobi map*

$$\left\{\begin{array}{l}\textit{codimension } m+1 \;\; \textit{algebraic cycles} \\ \textit{on } X \;\; \textit{homologous to zero}\end{array}\right\} \to J^{m+1}(X)$$

is contained in the torsion points of $J^{m+1}(X)$.

In §3, we will discuss some open problems and conjectures, as well as other loose ends left by the ideas in this paper.

Many of these results have been obtained independently by Claire Voisin, whose work will appear separately [10]. Because of my familiarity with the results of [5], I was able to go further in some directions, while because of her greater geometric insight into many parts of the problem, she was able to go further in others.

I am grateful to David Cox for pointing out a glitch in my original definition of the invariants $\delta_p\nu$, to Chad Shoen and Steve Zucker for educating me about algebraic cycles, to Joe Harris and Janos Kollar for some useful discussions on monodromy, and to Phillip Griffiths for his help and encouragement.

1. Generalization of Griffiths' infinitesimal invariant of normal functions

Let $\mathscr{X} \xrightarrow{\pi} B$ be an analytic family of smooth $(2m+1)$-folds, $m \geq 1$, with $\mathscr{X}$ and B smooth and π a submersion. Since all our considerations in this section will be local, we may take B to be the unit ball in $\mathbf{C}^N$. Let $T = T_0(B)$ and let $t_1, t_2, \cdots, t_N$ be local coordinates for B at 0. Let $\mathcal{F}^{2m+1}, \mathcal{F}^{2m}, \cdots, \mathcal{F}^0$ be the sheaves of analytic sections of the Hodge bundles $F^{2m+1}, F^{2m}, \cdots, F^0$. Differentiation

$$T \otimes \mathcal{F}^0 \to \mathcal{F}^0$$

is given by

$$\frac{\partial}{\partial t_k} \otimes f \mapsto \frac{\partial f}{\partial t_k}.$$

The *Griffiths infinitesimal period relations* state that differentiation takes $\mathcal{F}^p$ to $\mathcal{F}^{p-1}$ for every p, so that we have a map

$$T \otimes \mathcal{F}^p \to \mathcal{F}^{p-1}.$$

By standard multilinear algebra, this gives rise to a Koszul complex

$$\mathcal{F}^{p+1} \xrightarrow{\alpha} T^* \otimes \mathcal{F}^p \xrightarrow{\beta} \bigwedge^2 T^* \otimes \mathcal{F}^{p-1},$$

where

$$\alpha(f) = \sum_{k=1}^{N} dt_k \otimes \frac{\partial f}{\partial t_k},$$

$$\beta(dt_k \otimes f) = \sum_{l=1}^{N} dt_k \wedge dt_l \otimes \frac{\partial f}{\partial t_l}.$$

Let ν be a normal function for the family $\mathcal{X} \to B$, and let $\tilde{\nu}$ be a lifting of ν to F^0. Consider

$$d\tilde{\nu} = \sum_{k=1}^{N} \partial\tilde{\nu}/\partial t_k \otimes dt_k \in T^* \otimes F^m.$$

The fact that we land in F^m is due to the infinitesimal relation for normal functions. Let $\hat{\nu}$ be another lifting of ν. Then $\hat{\nu} = \tilde{\nu} + f$, where f is an analytic section of F^{m+1}. Now $d\hat{\nu} = d\tilde{\nu} + df$ and we may interpret df as the image of f under the differentiation map $\mathcal{F}^{m+1} \to T^* \otimes \mathcal{F}^m$. So we obtain from ν a well-defined element

$$\delta\nu \in (T^* \otimes \mathcal{F}^m)/\operatorname{im} \mathcal{F}^{m+1}.$$

This sits in one of the Koszul complexes just constructed:

$$\mathcal{F}^{m+1} \xrightarrow{\alpha} T^* \otimes \mathcal{F}^m \xrightarrow{\beta} \bigwedge^2 T^* \otimes \mathcal{F}^{m-1}.$$

Equality of mixed partials $d^2\tilde{\nu} = 0$ translates into saying that $\beta(\delta\nu) = 0$. We have therefore an invariant $\delta\nu \in \ker\beta/\operatorname{im}\alpha$.

The invariant $\delta\nu$ vanishes if and only if there is some lifting $\tilde{\nu}$ of ν for which $d\tilde{\nu} = 0$, i.e. for which $\tilde{\nu}$ is constant. We thus have:

$$\delta\nu = 0 \quad \textit{if and only if} \quad \nu \textit{ is locally constant.}$$

Using the identifications $H^{k,2m+1-k} \cong F^k/F^{k+1}$, differentiation induces maps of bundles

$$T \otimes H^{k,2m+1-k} \to H^{k-1,2m+2-k}$$

for every $k \geq 1$. Using these maps, it is possible to construct a series of complexes $A_p^{\cdot}$ given by

$$H^{m+p,m-p+1} \to T^* \otimes H^{m+p-1,m-p+2} \to \bigwedge^2 T^* \otimes H^{m+p-2,m-p+3},$$

where A_p^0 is the term on the left, A_p^1 is the term in the middle, and A_p^2 is the term on the right. The convention here is that a Hodge group is taken to be zero if one of its bidegrees is negative. Let $\mathscr{A}^{\cdot}$ be the complex (cf. [14])

$$\mathscr{F}^{m+1} \to T^* \otimes \mathscr{F}^m \to \bigwedge^2 T^* \otimes \mathscr{F}^{m-1}$$

indexed the same way. Thus $\delta\nu \in H^1(\mathscr{A}^{\cdot})$. We can filter $\mathscr{A}^{\cdot}$ so that the pth step of the filtration is the complex

$$\mathscr{F}^{m+p} \to T^* \otimes \mathscr{F}^{m+p-1} \to \bigwedge^2 T^* \otimes \mathscr{F}^{m+p-2}.$$

There is then a spectral sequence $E_r^{p,q}$ such that

$$E_1^{p,q} \cong H^{p+q}(A_p^{\cdot}) \quad \text{and} \quad E_\infty^{p,q} \cong \mathrm{Gr}^p(H^{p+q}(\mathscr{A}^{\cdot})).$$

The grading on $H^1(\mathscr{A}^{\cdot})$ may be interpreted as follows: we obtain from $\delta\nu$ a sequence of inductively defined invariants $\delta_1\nu, \delta_2\nu, \cdots, \delta_{m+2}\nu$, where $\delta_p\nu \in E_\infty^{p,1-p}$ and $\delta_p\nu$ can be defined only if the preceding invariants $\delta_1\nu, \delta_2\nu, \cdots, \delta_{p-1}\nu$ are all zero. Furthermore, $\delta\nu = 0$ if and only if $\delta_1\nu = 0$, $\delta_2\nu = 0, \cdots, \delta_{m+2}\nu = 0$. Thus we have:

Theorem 1.1. *A normal function ν is locally constant if and only if the infinitesimal invariants $\delta_1\nu, \delta_2\nu, \cdots, \delta_{m+2}\nu$ are all zero.*

Notice that

$$E_\infty^{1,0} \hookrightarrow E_1^{1,0} \cong H^1(A_1^{\cdot}),$$

and therefore the first of our infinitesimal invariants $\delta_1\nu$ lies in the cohomology at the middle term of the Koszul complex

$$H^{m+1,m} \xrightarrow{\bar{\alpha}} T^* \otimes H^{m,m+1} \xrightarrow{\bar{\beta}} \bigwedge^2 T^* \otimes H^{m-1,m+2}.$$

This latter space is the dual of the cohomology at the middle term of the complex

$$\bigwedge^2 T \otimes H^{m+2,m-1} \to T \otimes H^{m+1,m} \to H^{m,m+1}.$$

In this form, one may identify $\delta_1 \nu$ as the original infinitesimal invariant of Griffiths [6], the one change being that he did not set things up in a Koszul-theoretic formulation.

The remaining $\delta_p \nu$ are new invariants, and realize Griffiths' beautiful idea that there ought to exist a nice infinitesimal criterion for a normal function to be locally constant. At least in the local case, I think that this theorem is the only possible answer. To my knowledge, these are the first differential-geometric invariants that live in a Koszul group; I suspect that this does not reflect a lack of such invariants, but rather the fact that Koszul groups are unfamiliar to differential geometers.

The groups $E_\infty^{p,1-p}$ where the infinitesimal invariants live are admittedly somewhat daunting. However, we note immediately that:

A normal function ν is locally constant provided that $H^1(A_p^\bullet) = 0$ for $1 \leq p \leq m+2$.

We now consider the case of the universal family $\mathscr{X}_d \to B_d$ of $(2m+1)$-folds of degree d defined in the introduction. Let Σ be a subvariety of B_d and $\widetilde{\Sigma} \to \Sigma$ a generically finite dominant map. Consider the induced family on $\widetilde{\Sigma}$. Let $0 \in \widetilde{\Sigma}$ correspond to a smooth $(2m+1)$-fold X corresponding to a homogeneous polynomial F of degree d. Let $T = T_0(\widetilde{\Sigma})$. Then T is a linear subspace of $H^0(\mathscr{O}_{\mathbf{P}^{2m+2}}(d))$ containing the degree d part $J_d(F)$ of the Jacobi ideal of F. In particular, T is base-point free. The dual of $H^1(A_p^\bullet)$ is the cohomology at the middle term of the Koszul complex

$$\bigwedge^2 T \otimes H^{m-p+3,m+p-2} \to T \otimes H^{m-p+2,m+p-1} \to H^{m-p+1,m+p}.$$

Using the standard identifications via Poincaré residue or the methods of [3], that is equivalent to the cohomology at the middle term of the Koszul complex

$$\bigwedge^2 T \otimes R^{(m+p-2)d+d-2m-3} \to T \otimes R^{(m+p-1)d+d-2m-3} \to R^{(m+p)d+d-2m-3},$$

where

$$R^k = H^0(\mathscr{O}_{\mathbf{P}^{2m+2}}(k))/J_k(F).$$

An easy diagram chase shows that this will vanish if two conditions hold:

(1) The cohomology at the middle term of the Koszul complex

$$\bigwedge^2 T \otimes V^{(m+p-2)d+d-2m-3} \to T \otimes V^{(m+p-1)d+d-2m-3} \\ \to V^{(m+p)d+d-2m-3}$$

is zero. Here $V^k = H^0(\mathscr{O}_{\mathbf{P}^{2m+2}}(k))$.

(2) The multiplication map

$$T \otimes J_{(m+p-1)d+d-2m-3}(F) \to J_{(m+p)d+d-2m-3}(F)$$

is surjective.

Of course, (2) would be implied by the potentially stronger statement:

(2′) The multiplication map

$$T \otimes V^{(m+p-1)d-2m-2} \to V^{(m+p)d-2m-2}$$

is surjective.

Fortunately, the vanishing theorem of [5] states that (1) holds provided

$$\operatorname{codim} T \leq (m+p-1)d - 2m - 4$$

and (2′) holds provided

$$\operatorname{codim} T \leq (m+p-1)d - 2m - 2.$$

We will therefore have (1) and (2) for all $p \geq 1$ provided

$$\operatorname{codim} T \leq m(d-2) - 4.$$

We thus see that any normal function on $\widetilde{\Sigma}$ has vanishing infinitesimal invariants and hence is locally constant. We therefore have proved:

Theorem 1.2. *Let $\mathscr{X}_d \to B_d$ be the universal family of $(2m+1)$-folds of degree d, and let Σ be a subvariety of B_d. Then for $m \geq 1$, any algebraically determined normal function for the restriction of $\mathscr{X}_d$ to Σ must be locally constant, provided* $\operatorname{codim} \Sigma \leq m(d-2) - 4$.

2. Locally constant algebraically determined normal functions for the universal family of smooth $(2m+1)$-folds of degree d

Let $\mathscr{X}_d \to B_d$ be the universal family of smooth $(2m+1)$-folds of degree d. Pick a base point $0 \in B_d$ corresponding to a smooth $(2m+1)$-fold X. As we move toward a point of the boundary of B_d representing a $(2m+1)$-fold acquiring a node, there is a *vanishing cycle* $\gamma \in H^{2m+1}(X, \mathbf{Z})$ so that the monodromy as we loop around this point is given by the *Picard-Lefschetz transformation*

$$v \mapsto v + (v \cdot \gamma)\gamma$$

for all $v \in H^{2m+1}(X,\mathbf{C})$. Let $\mathscr{VAN} \subseteq H^{2m+1}(X,\mathbf{Z})$ be the subgroup spanned by the vanishing cycles. It is known [1] that $\mathscr{VAN}$ has finite index in $H^{2m+1}(X,\mathbf{Z})$.

Let $\widehat{B} \to B_d$ be a generically finite dominant map of varieties, and let $\widehat{\mathscr{X}} \to \widehat{B}$ be the pullback family from $\mathscr{X}_d \to B_d$. Assume that ν is a locally constant normal function for $\widehat{\mathscr{X}} \to \widehat{B}$. Finally, let $\widetilde{B}$ be the universal cover of $\widehat{B}$, and $\widetilde{\mathscr{X}} \to \widetilde{B}$ the pullback family. On this family, we may lift ν to a locally constant element $\tilde{\nu}$ of the pullback of F^0 which is now unambiguously identified with $H^{2m+1}(X,\mathbf{C})$. Then there is an open cover $\{U_\alpha\}$ for $\widetilde{B}$ so that $\tilde{\nu}$ is the constant vector v_α on U_α. On a nonempty overlap $U_\alpha \cap U_\beta$, the constant vector $v_\alpha - v_\beta$ belongs to F^{m+1}. Unless $v_\alpha - v_\beta$ vanishes, it gives a nonzero fixed element $v_{\alpha\beta}$ of F^{m+1}. Such a vector would be in the kernel of the derivative map

$$\mathscr{F}^{m+1} \to T^* \otimes \mathscr{F}^m.$$

These maps are filtered by the maps

$$H^{p,2m+1-p} \to T^* \otimes H^{p-1,2m+2-p},$$

where p is in the range $m+1 \leq p \leq 2m+1$. These maps, in the notation of §1, are just the maps

$$R^{(2m+1-p)d+d-2m-3} \to (V^d)^* \otimes R^{(2m+2-p)d+d-2m-3},$$

which are known to be injective by Macaulay's Theorem. Thus $v_\alpha = v_\beta$.

Descending to the family $\widehat{\mathscr{X}} \to \widehat{B}$, we obtain a vector $v \in H^{2m+1}(X,\mathbf{C})$ satisfying the property

$$\rho(v) - v \in \Lambda = H^{2m+1}(X,\mathbf{Z})$$

for all ρ belonging to the monodromy group Γ of the family. There is a positive integer N depending on the generic degree of the map $\widehat{B} \to B_d$ so that for every vanishing cycle $\gamma \in \mathscr{VAN}$, the transformation

$$v \mapsto v + N(v\cdot\gamma)\gamma$$

belongs to Γ. For the v just constructed, we have that $N(v\cdot\gamma)\gamma \in \Lambda$ for all $\gamma \in \mathscr{VAN}$. Since $\mathscr{VAN}$ has finite index in Λ, we see that for some positive integer M depending on this index, $NM(v\cdot\lambda)\lambda \in \Lambda$ for all $\lambda \in \Lambda$. By Poincaré duality, this implies $v \in \Lambda/(NM)$. Equivalently, $NM \cdot \nu = 0$ and thus ν is torsion. This proves:

Theorem 2.1. *Every algebraically determined normal function on the universal family of $(2m+1)$-folds of degree d is torsion when $d \geq 2 + 4/m$ and $m \geq 1$.*

For $m = 1$, this result was conjectured by Griffiths and Harris [8]. It has as an immediate consequence the following theorem.

Theorem 2.2. *The image of the Abel-Jacobi map for a general* $(2m+1)$*-fold of degree* d *in* $\mathbf{P}^{2m+2}$ *is contained in the torsion points of the intermediate Jacobian when* $d \geq 2 + 4/m$ *and* $m \geq 1$.

3. Open problems and further remarks

The most critical open problem suggested by this paper is:

Problem 3.1. *Show that for a general* $(2m+1)$*-fold* X *of degree* $d \geq 2 + 4/m$ *in* $\mathbf{P}^{2m+2}$, $m \geq 1$, *the image of the Abel-Jacobi map of* X *is torsion free.*

The local differential of §1 can be rephrased in a more general context than what we needed here. The results really are about maps

$$t \mapsto S_1(t) \subseteq S_2(t) \subseteq \cdots \subseteq S_N(t)$$

to flag manifolds which satisfy the infinitesimal relation $dS_i \subseteq S_{i+1}$ together with sections of one of the quotient bundles Q_i satisfying another infinitesimal relation. One natural situation to consider is the *osculating flag* of a subvariety of affine or projective space. Here, the Koszul groups where the invariants $\delta_p \nu$ lie involve the linear systems II, III,$\cdots$ introduced in this context by Griffiths and Harris [7], the *higher fundamental forms.* There should be some interesting local differential geometry in this situation.

There are also some interesting *differential systems* lurking in §1. The condition $d\tilde{\nu} \in T^* \otimes \mathscr{F}^p$ becomes an interesting looking differential system when written out.

Although we were able to show that any algebraically determined normal function on a subvariety Σ of low codimension of B_d must be locally constant, we were unable to show in this case that it must be torsion. Our monodromy argument breaks down—as Joe Harris pointed out to us, it is a frequently encountered problem in doing such arguments that the closure of Σ may meet the locus of singular hypersurfaces in an unfortunate way, e.g. always acquiring multiple nodes or worse singularities. Still, it is reasonable to pose:

Problem 3.2. *On a subvariety of sufficiently low codimension of* B_d, *must every algebraically determined locally constant normal function be torsion?*

A quite interesting phenomenon is to look at normal functions ν on a $2p$-fold Y for hypersurfaces X_t belonging to a linear system $|L|$, where L is taken to be a sufficiently ample line bundle, i.e., the first Chern class of L is taken extremely large. In this case, one can show that the Koszul group to which $\delta_1\nu$ belongs is isomorphic to $H^{p,p}_{pr}(Y)$. It is natural to conjecture that:

Problem 3.3. *For an abstract normal function in the situation above, i.e., an element of the 2p-th Deligne cohomology group of Y, the invariant* $\delta_1\nu$ *is obtained from the natural map* $H^{p,p}_{pr}(Y, \mathbf{Z}) \to H^{p,p}_{pr}(Y)$.

This has been shown by C. Voisin [11] in the case of a normal function arising from a codimension p analytic cycle. A solution to this problem would nicely complement the work of X. Wu [12] on nondegeneracy of normal functions.

Finally, a few comments about sharpness of our bounds on degree. For the case of 3-folds ($m = 1$), the locus of 3-folds of degree d containing a line has codimension $d-5$. These give rise to an algebraically defined normal function on this locus which is not locally constant. This shows that our main results are sharp for $m = 1$. The bound $d \geq 2 + 4/m$ may be rewritten $d \geq 6$ for $m = 1$, $d \geq 4$ for $m = 2, 3$; $d \geq 3$ for $m \geq 4$. For d smaller than these bounds, we readily check that every $(2m + 1)$-fold of degree d contains a $\mathbf{P}^m$. For $m = 2, 3$ one would thus expect this bound to be sharp. For $m \geq 4$, since the intermediate Jacobian is zero for $d \leq 2$, we see that the result extends to these lower degrees for trivial reasons. The bound $\operatorname{codim} \Sigma \leq (d-2)m - 4$ is unlikely to be sharp for $m > 1$, although as we just saw, it is sharp for $m = 1$.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES