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IRREDUCIBILITY OF THE EQUICLASSICAL LOCUS

STEVEN DIAZ

Let D be a reduced plane curve and p a singular point of D. We know there exists an étale miniversal deformation space for the pair (D, p) (see [2], [3]):

$$\begin{array}{ccc} p \in D & \rightarrow & X \\ \downarrow & & \downarrow \\ 0 & \in & B \end{array} \tag{1}$$

In B there are various loci that parametrize deformations of (D, p) that preserve certain properties of the singularity. These have been extensively studied (see [3], [7], [8]). In this article we will be interested in the equigeneric locus $EG \subset B$ which parametrizes deformations of (D, p) in which the singularity is allowed to possibly break up into several singularities but it is required that the total local contribution of all these singularities to the geometric genus of a plane curve must remain constant, and the equiclassical locus $EC \subset EG \subset B$ in which in addition we require that the total local contribution of the singularities to the class of a plane curve (that is, the degree of its dual) remains constant. (See [3] for some descriptive material on EC and EG.) We work over the complex numbers.

As indicated by the title of this article our main result will be that EC is irreducible near zero. We mean this in the strong sense that—even in the standard metric topology or in the complete local ring of 0 in B—EC has a single branch at 0. In fact we have the following theorem.

(2) Theorem. *Let $n\colon EC' \rightarrow EC$ be the normalization map. Then $n^{-1}(0)$ is a single point, and EC' is nonsingular at $n^{-1}(0)$.*

We also determine for which plane curve singularities EC is nonsingular.

First let us show how to use what is already known about plane curve singularities to reduce the proof of (2) to a tangent space computation. References for the next few paragraphs are [1], [3], [5], and [6].

We may assume that D is irreducible, p is the only singularity of D, and the degree d of D is as large as desired. Let $\mathbf{P}^N$ be the projective space which parametrizes plane curves of degree d. The point of $\mathbf{P}^N$ corresponding to D

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will also be denoted by D. From the versal property of the family (1) we get neighborhoods U of D in $\mathbf{P}^N$ and V of 0 in B and a morphism $f\colon U \to V$. Let g be the geometric genus of D and c the class of D. In $\mathbf{P}^N$ let $V^{d,g}$ be the set of points corresponding to reduced irreducible curves of degree d and geometric genus g, and let $V^{d,g,c} \subset V^{d,g}$ be the set of those that also have class c. Then for small enough U

$$\begin{aligned} f^{-1}(EC) &= V^{d,g,c} \cap U, \\ f^{-1}(EG) &= V^{d,g} \cap U. \end{aligned} \tag{3}$$

Since the degree of D is very large we have that, after possibly shrinking U and V, f is surjective with smooth fibers. U is locally in the standard metric topology the product of V and a nonsingular variety (see [3, proof of 4.15]). In view of these facts we see that (2) is equivalent to the following lemma.

(4) Lemma. *Let $n\colon \tilde{V}^{d,g,c} \to V^{d,g,c}$ be the normalization map. Then $n^{-1}(D)$ is a single point, and $\tilde{V}^{d,g,c}$ is nonsingular at $n^{-1}(D)$.*

To see how to prove (4) let us first recall one way of obtaining the normalization of $V^{d,g}$ near D. Let C be the normalization of D, and $\phi\colon C \to D \subset \mathbf{P}^2$ be the normalization map. There exists a deformation space for the map $\phi\colon C \to \mathbf{P}^2$:

$$\begin{array}{ccc} Y & \to & \mathbf{P}^2 \\ \downarrow & & \\ S & & \end{array} \tag{5}$$

We will denote the point of S corresponding to ϕ by ϕ. We have a natural morphism $h\colon S \to V^{d,g}$ with $h(\phi) = D$. Using the fact that the degree of D is very large it can be proven that a small neighborhood of ϕ in S (which we may as well assume is all of S) surjects onto a neighborhood of D in $V^{d,g}$; restricted to these neighborhoods h is one-to-one, and S is nonsingular at ϕ (see [1, pp. 486–488]). Thus, near D, S is the normalization of $V^{d,g}$, and h is the normalization map. In view of the fact that h is one-to-one we see that (4) will follow if we can show that $h^{-1}(V^{d,g,c})$ is nonsingular at ϕ. Since it is also known (again using that the degree of D is very large) that the dimension of every component of $V^{d,g,c}$ is $d+c-g+1$ [3, 5.2, 5.17] and that the dimension of $V^{d,g}$ is $g + 3d - 1$ [4, 2.3], we see that (4) will follow from the following claim.

(6) Claim. The Zariski tangent space to $h^{-1}(V^{d,g,c})$ at ϕ has codimension $2d + 2g - c - 2$ in the Zariski tangent space to S at ϕ.

As a first step toward identifying the tangent space to $h^{-1}(V^{d,g,c})$ at ϕ let us recall how one identifies the tangent space to S at ϕ. Let $\theta_{\mathbf{P}^2}$ and θ_C be the tangent bundles of $\mathbf{P}^2$ and C. There is a natural injective sheaf morphism from θ_C into $\phi^*\theta_{\mathbf{P}^2}$. Call the quotient the normal sheaf of ϕ and denote it by N_ϕ. We have an exact sequence

$$0 \to \theta_C \to \phi^*\theta_{\mathbf{P}^2} \to N_\phi \to 0. \tag{7}$$

We then have that the Zariski tangent space to S at ϕ is given by

$$T_{S,\phi} = H^0(C, N_\phi). \tag{8}$$

The standard way to prove (8) is via the Kodaira-Spencer theory of first order deformations. The Zariski tangent space to S at ϕ is isomorphic to the space of morphisms of $\operatorname{Spec}\mathbf{C}[\varepsilon]/(\varepsilon^2)$ ($\operatorname{Spec}\mathbf{C}[\varepsilon]$ for short) to S which take the point of $\operatorname{Spec}\mathbf{C}[\varepsilon]$ to ϕ. Such morphisms of $\operatorname{Spec}\mathbf{C}[\varepsilon]$ to S correspond to families

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathbf{P}^2 \\ \downarrow & & \\ \operatorname{Spec}\mathbf{C}[\varepsilon] & & \end{array} \tag{9}$$

which over the point of $\operatorname{Spec}\mathbf{C}[\varepsilon]$ give the morphism ϕ. Families like (9) can be described in terms of coordinate patches and transition data as follows.

Let $\mathscr{U} = \{U_\alpha\}$ be a finite open cover of C with z_α a local coordinate on U_α. The curve C is then described by transition data

$$z_\alpha = f_{\alpha\beta}(z_\beta) \quad \text{on } U_\alpha \cap U_\beta,$$

which satisfies the cocycle rule

$$f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) = f_{\alpha\gamma}(z_\gamma) \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \tag{10}$$

To give the map ϕ we may assume we have an open cover $\mathscr{V} = \{V_\alpha\}$ of $\mathbf{P}^2$ such that $\phi(U_\alpha) \subset V_\alpha$ for all α, and on each V_α we have local coordinates x_α, y_α. Again we have transition data:

$$x_\alpha = g^x_{\alpha\beta}(x_\beta, y_\beta), \quad y_\alpha = g^y_{\alpha\beta}(x_\beta, y_\beta) \quad \text{on } V_\alpha \cap V_\beta.$$

The map ϕ is given locally by

$$x_\alpha = \psi^x_\alpha(z_\alpha), \qquad y_\alpha = \psi^y_\alpha(z_\alpha).$$

These must satisfy the compatibility conditions

$$\begin{aligned} g^x_{\alpha\beta}(\psi^x_\beta(z_\beta), \psi^y_\beta(z_\beta)) &= \psi^x_\alpha(f_{\alpha\beta}(z_\beta)), \\ g^y_{\alpha\beta}(\psi^x_\beta(z_\beta), \psi^y_\beta(z_\beta)) &= \psi^y_\alpha(f_{\alpha\beta}(z_\beta)). \end{aligned} \tag{11}$$

To give a family as in (9) we must let this data vary with ε:

$$\begin{aligned} z_\alpha &= \tilde{f}_{\alpha\beta}(z_\beta, \varepsilon) = f_{\alpha\beta}(z_\beta) + \varepsilon b_{\alpha\beta}(z_\beta), \\ x_\alpha &= \tilde{\psi}^x_\alpha(z_\alpha, \varepsilon) = \psi^x_\alpha(z_\alpha) + \varepsilon a^x_\alpha(z_\alpha), \\ y_\alpha &= \tilde{\psi}^y_\alpha(z_\alpha, \varepsilon) = \psi^y_\alpha(z_\alpha) + \varepsilon a^y_\alpha(z_\alpha). \end{aligned} \tag{12}$$

One then checks that the obvious compatibility conditions which the equations of (12) must satisfy are equivalent to the former compatibility conditions (10) and (11), a condition on the $b_{\alpha\beta}$'s saying that we have constructed a first order deformation of the abstract curve C, and the fact that the cochain

$$\left\{a^x_\alpha \frac{\partial}{\partial x_\alpha} + a^y_\alpha \frac{\partial}{\partial y_\alpha}\right\} \in C^0(\mathscr{U}, \phi^* \theta_{\mathbf{P}^2}) \tag{13}$$

maps to an element of $H^0(C, N_\phi)$. This element is called the Horikawa class of the given first order deformation of ϕ. One checks that this class is independent of the choices we have made and that in this way we get the identification of (8). Now we must determine when the cochain (13) represents a tangent vector to $h^{-1}(V^{d,g,c})$.

The ramification divisor R of the morphism ϕ is defined to be the divisor on C defined by the first Fitting ideal of the sheaf of relative one-forms of C over D. Letting κ be the degree of R, it is well known that

$$c = 2d + 2g - 2 - \kappa. \tag{14}$$

In the family (5) degree and genus are constant, so a constant class is equivalent to constant κ.

Let us now write down what it means for a family given by the data from (9)-(12) to have constant κ. Let $p_1, \cdots, p_n$ be the distinct points of R. We may assume that $\mathscr{U}$ is chosen so that each U_α has at most one p_i (call it p_α) in it, that p_α occurs at $z_\alpha = 0$, and $p_\alpha \notin U_\beta$ for $\alpha \neq \beta$. We may further choose z_α, x_α, and y_α so that

$$\begin{aligned} \psi^x_\alpha(z_\alpha) &= z_\alpha^{k_\alpha+1} + \beta_1 z_\alpha^{k_\alpha+2} + \beta_2 z_\alpha^{k_\alpha+3} + \cdots, \\ \psi^y_\alpha(z_\alpha) &= z_\alpha^{l_\alpha+1} + \gamma_1 z_\alpha^{l_\alpha+2} + \gamma_2 z_\alpha^{l_\alpha+3} + \cdots \end{aligned} \tag{15}$$

with $l_\alpha > k_\alpha \geq 0$. Because near p_α the first Fitting ideal of the sheaf of relative one-forms of C over D is locally generated by $\partial\psi^x_\alpha/\partial z_\alpha$ and $\partial\psi^y_\alpha/\partial z_\alpha$, and the greatest common divisor of these two functions is $z_\alpha^{k_\alpha}$, we see that p_α occurs in R with multiplicity k_α.

By the same reasoning we see that our family (9) represents a tangent vector to $h^{-1}(V^{d,g,c})$ if and only if for each α with $k_\alpha > 0$ the greatest common divisor of $\partial\tilde{\psi}^x_\alpha/\partial z_\alpha$ and $\partial\tilde{\psi}^y_\alpha/\partial z_\alpha$ is of the form $z_\alpha^{k_\alpha} + \varepsilon(c_0^\alpha + c_1^\alpha z_\alpha + \cdots + c_{k_\alpha}^\alpha z_\alpha^{k_\alpha})$ where the c^α's are constants.

Next we must see what restrictions this puts on the cochain (13). These restrictions will clearly be local near each p_α, so in what follows we drop the α's to make the notation less cumbersome.

Set $a^x(z) = a_0^x + a_1^x z + a_2^x z^2 + \cdots$ and $a^y(z) = a_0^y + a_1^y z + a_2^y z^2 + \cdots$. We then have

$$\frac{\partial\tilde{\psi}^x}{\partial z} = (k+1)z^k + (k+2)\beta_1 z^{k+1} + \cdots + \varepsilon(a_1^x + 2a_2^x z + \cdots),$$
$$\frac{\partial\tilde{\psi}^y}{\partial z} = (l+1)z^l + (l+2)\gamma_1 z^{l+2} + \cdots + \varepsilon(a_1^y + 2a_2^y z + \cdots).$$

To say that the greatest common divisor of these two is $z^k + \varepsilon(c_0 + c_1 z + \cdots + c_k z^k)$ is to say that there exist constants d_i, e_i, f_i, and g_i, $i = 0, 1, 2, \cdots$, such that

$$\begin{aligned}(k+1)z^k &+ (k+2)\beta_1 z^{k+1} + (k+3)\beta_2 z^{k+2} + \cdots \\ &\qquad\qquad + \varepsilon(a_1^x + 2a_2^x z + 3a_3^x z^2 + \cdots) \\ &= (z^k + \varepsilon(c_0 + c_1 z + \cdots + c_k z^k))(d_0 + d_1 z + \cdots + \varepsilon(e_0 + e_1 z + \cdots)) \\ &= d_0 z^k + d_1 z^{k+1} + \cdots \\ &\qquad + \varepsilon(e_0 z^k + e_1 z^{k+1} + \cdots + (c_0 + c_1 z + \cdots + c_k z^k)(d_0 + d_1 z + \cdots)),\end{aligned}$$

(16)

$$\begin{aligned}(l+1)z^l &+ (l+2)\gamma_1 z^{l+1} + \cdots + \varepsilon(a_1^y + 2a_2^y z + \cdots) \\ &= (z^k + \varepsilon(c_0 + c_1 z + \cdots + c_k z^k))(f_0 + f_1 z + \cdots + \varepsilon(g_0 + g_1 z + \cdots)) \\ &= f_0 z^k + f_1 z^{k+1} + \cdots \\ &\qquad + \varepsilon(g_0 z^k + g_1 z^{k+1} + \cdots + (c_0 + c_1 z + \cdots + c_k z^k)(f_0 + f_1 z + \cdots)).\end{aligned}$$

Equating like terms we get the following relations:

$$d_0 = k+1, \qquad d_j = (k+j+1)\beta_j, \quad \text{for } j \geq 1, \tag{17}$$

$$j a_j^x = \sum_{i=0}^{j-1} c_i d_{j-i-1}, \quad \text{for } 1 \leq j \leq k, \tag{18}$$

$$\begin{gathered} f_j = 0 \quad \text{for } 0 \leq j \leq l-k-1, \qquad f_{l-k} = l+1, \\ f_{l-k+j} = (l+j+1)\gamma_j \quad \text{for } j \geq 0, \end{gathered} \tag{19}$$

$$j a_j^y = \sum_{i=0}^{j-1} c_i f_{j-i-1} \quad \text{for } 1 \leq j \leq k. \tag{20}$$

We have not written down the relations that come from terms of the form εz^j for $j \geq k$. It is easy to see that these relations will not lead to any relations among the a's.

Together (19) and (20) tell us that

$$a_j^y = 0 \quad \text{for } 1 \leq j \leq \min(k, l-k). \tag{21}$$

If $l \geq 2k$ these are all the relations; otherwise, we obtain more relations as follows. Notice that $d_0 = k+1 \neq 0$. Thus (17) and (18) tell us that:

$$c_0 = \frac{a_1^x}{k+1},$$
$$c_1 = \frac{2a_2^x - (k+2)\beta_1 c_0}{k+1} = \left(\frac{2}{k+1}\right) a_2^x - \left(\frac{(k+2)\beta_1}{(k+1)^2}\right) a_1^x,$$
$$\vdots$$

Continuing in this way we express c_i for $0 \leq i \leq k-1$ as a linear combination of $a_1^x, a_2^x, \cdots, a_{i+1}^x$ with the coefficient of a_{i+1}^x being nonzero. Similarly $f_{l-k} = l+1 \neq 0$. Thus (19) and (20) tell us how to express c_i for $0 \leq i \leq 2k-l-1$ as a linear combination of a_j^y's with $l-k < j \leq i+l-k+1$ and with the coefficient of $a_{i+l-k+1}^y$ nonzero. Equating these two expressions for c_i we get $2k-l$ linear relations among the a^x's and a^y's. These can be seen to be independent of each other because the equation obtained from c_{i+1} involves a_{i+2}^x whereas the equation obtained from c_i does not. These are independent of the relations in (21) because they do not involve a_j^y for $1 \leq j \leq l-k$.

This gives a total of k independent linear relations among the a's. We shall call them the equiclassical relations. It is easy to see that these are all the relations (16) gives; however, we must be careful about how we have shown these relations to be independent. To see whether these relations are independent on global sections of N_ϕ there are two further things we must take into account.

(a) The cochain (13) is an element of $C^0(\mathscr{U}, \phi^*\theta_{\mathbf{P}^2})$ and we are interested in its image in $H^0(C, N_\phi)$. Many different cochains of the form in (13) will represent the same element in $C^0(\mathscr{U}, N_\phi)$ and thus $H^0(C, N_\phi)$.

(b) Not every element of $C^0(\mathscr{U}, N_\phi)$ actually gives an element of $H^0(C, N_\phi)$. Even if the equiclassical relations are independent on elements of $C^0(\mathscr{U}, N_\phi)$ they need not be independent on elements of $H^0(C, N_\phi)$.

For (a) consider the exact sequence (7). Near p, $\theta_C \cong \mathscr{O}_C(\partial/\partial z)$ and $\phi^*\theta_{\mathbf{P}^2} \cong \mathscr{O}_C(\partial/\partial x, \partial/\partial y)$. The morphism $\theta_C \to \phi^*\theta_{\mathbf{P}^2}$ is locally given by

$$i\colon f(z)\frac{\partial}{\partial z} \to f(z)\frac{d\psi^x}{dz}(z)\frac{\partial}{\partial x} + f(z)\frac{d\psi^y}{dz}(z)\frac{\partial}{\partial y}. \tag{22}$$

The equiclassical relations will be independent on elements of $C^0(\mathscr{U}, N_\phi)$ when every cochain in the image of i satisfies the equiclassical relations. Clearly the image of i is contained in the set of cochains satisfying

$$a_1^x = a_2^x = \cdots = a_{k-1}^x = a_1^y = a_2^y = \cdots = a_{l-1}^y = 0. \tag{23}$$

One can check directly that any cochain satisfying (23) satisfies the equiclassical relations.

We now take care of (b). Equation (22) tells us that as soon as ϕ has a nontrivial ramification divisor R, N_ϕ will have a torsion subsheaf isomorphic to $\mathscr{O}_R$. Denote this torsion subsheaf by $\mathscr{K}_\phi$ and define N'_ϕ to be the quotient of N_ϕ by $\mathscr{K}_\phi$ so that we have an exact sequence

$$0 \to \mathscr{K}_\phi \to N_\phi \to N'_\phi \to 0. \tag{24}$$

Near $p \in R$, $N'_\phi \cong \mathscr{O}_C(\partial/\partial y)$. The morphism $N_\phi \to N'_\phi$ is given by

$$a^x(z)\frac{\partial}{\partial x} + a^y(z)\frac{\partial}{\partial y} \to \left[-a^x(z)\left(\frac{d\psi^y}{dz}\right)\left(\frac{d\psi^x}{dz}\right)^{-1} + a^y(z)\right]\frac{\partial}{\partial y}. \tag{25}$$

Note that $(d\psi^y/dz)(d\psi^x/dz)^{-1}$ is regular at p because we have assumed $l > k$. If we write down a section of N'_ϕ as

$$s = (q_0 + q_1 z + q_2 z^2 + \cdots)\frac{\partial}{\partial y},$$

we see from (25) that s is the image of a cochain satisfying the equiclassical relations if and only if certain linear relations among the q_i's are satisfied. Let us call a maximal locally independent set of these the pseudo-equiclassical relations. Observe that the equiclassical relations only involve a_i^x and a_j^y for $1 \le i \le k-1$, $1 \le j \le k$. Together with (25) this says that the pseudo-equiclassical relations will not involve q_i for $i > k$. N'_ϕ is a line bundle. A standard argument using the Riemann-Roch theorem shows that local conditions of the type just described when imposed at finitely many points are independent on global sections of a line bundle provided that the degree of the line bundle is sufficiently large. From the exact sequences (7) and (24) and the fact that $\mathscr{K}_\phi \cong \mathscr{O}_R$ we see that the degree of N'_ϕ is $2g-2-\kappa+3d$. Since d is very large—independent of the k_α's—we see that the pseudo-equiclassical relations are independent on elements of $H^0(C, N'_\phi)$.

This will imply that the equiclassical relations are independent on elements of $H^0(C, N_\phi)$ as follows. We have chosen our open cover $\mathscr{U}$ of C so that each $p_\alpha \in R$ is in only one U_α; therefore, we have $C^0(\mathscr{U}, \mathscr{K}_\phi) \cong H^0(C, \mathscr{K}_\phi)$ and $C^1(\mathscr{U}, \mathscr{K}_\phi) \cong H^1(C, \mathscr{K}_\phi) = 0$. From (24) comes the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & C^0(\mathscr{U}, \mathscr{K}_\phi) & \to & C^0(\mathscr{U}, N_\phi) & \to & C^0(\mathscr{U}, N'_\phi) & \to & 0 \\ & & \wr\Vert & & \uparrow & & \uparrow & & \\ 0 & \to & H^0(C, \mathscr{K}_\phi) & \to & H^0(C, N_\phi) & \to & H^0(C, N'_\phi) & \to & 0 \end{array} \tag{26}$$

A diagram chase shows that a cochain s in $C^0(\mathscr{U}, N_\phi)$ comes from a global section of N_ϕ if and only if the image of s in $C^0(\mathscr{U}, N'_\phi)$ comes from a global section of N'_ϕ. This fact together with another diagram chase shows that the equiclassical relations are independent on elements of $H^0(C, N_\phi)$ if and only if the pseudo-equiclassical relations are independent on elements of $H^0(C, N'_\phi)$.

Knowing the equiclassical relations to be independent on elements of $H^0(C, N_\phi)$ we see that the tangent space to $h^{-1}(V^{d,g,c})$ at ϕ has codimension $\kappa = \sum k_\alpha$ in $H^0(C, N_\phi)$, the tangent space to S at ϕ. (14) now finishes the proof of (6) and thus (2).

We now are in a position to say exactly when EC itself is nonsingular. We already know that when all the analytic branches of D at p are nonsingular, $EC = EG$ is nonsingular [3].

(27) Theorem. *EC is nonsingular at 0 if and only if each analytic branch of D at p is either nonsingular or, when given parametrically as in (15), $l_\alpha = k_\alpha + 1$.*

Proof. Using the same ideas which we used to reduce (2) to (4), we may (with the same assumptions on D) reduce (27) to (28).

(28) Lemma. *$V^{d,g,c}$ is nonsingular at D if and only if each analytic branch of D at p is either nonsingular or, when given parametrically as in (15), $l_\alpha = k_\alpha + 1$.*

Proof (of (28)). Let us consider the morphism $h\colon S \to V^{d,g}$. In [1, p. 487] it is shown that the differential of h at ϕ,

$$dh\colon T_{S,\phi} \to T_{V^{d,g},D},$$

has kernel equal to $H^0(C, \mathscr{K}_\phi)$ with its natural inclusion in $H^0(C, N_\phi)$. Knowing that $h^{-1}(V^{d,g,c})$ is nonsingular at ϕ and that h is one-to-one near ϕ we see that $V^{d,g,c}$ is nonsingular exactly when the intersection of the tangent space to $h^{-1}(V^{c,g,c})$ at ϕ and $H^0(C, \mathscr{K}_\phi)$ is $\{0\}$. Knowing that the equiclassical relations are independent on elements of $H^0(C, N_\phi)$ we see that this question is local near each point of R. From (22) we see that near a point of R any nonzero element of $H^0(C, \mathscr{K}_\phi)$ may be written in the form

$$\begin{aligned}(29)\qquad &(h_0 + h_1 z + \cdots + h_{k-1} z^{k-1}) \\ &\quad\times \left[((k+1) + (k+2)\beta_1 z + \cdots)\frac{\partial}{\partial x}\right. \\ &\qquad\left. + ((l+1)z^{l-k} + (l+2)\gamma_1 z^{l-k+1} + \cdots)\frac{\partial}{\partial y}\right].\end{aligned}$$

If D has a singularity for which we claim $V^{d,g,c}$ is singular, then one may check directly that

$$\begin{aligned}&[(k+1)z^{k-1} + (k+2)\beta_1 z^k + (k+3)\beta_2 z^{k+1} + \cdots]\frac{\partial}{\partial x} \\ &\quad + [(l+1)z^{l-1} + (l+2)\gamma_1 z^l + (l+3)\gamma_2 z^{l+1} + \cdots]\frac{\partial}{\partial y}\end{aligned}$$

is a nonzero element in both $H^0(C, \mathscr{K}_\phi)$ and the tangent space to $h^{-1}(V^{d,g,c})$ at ϕ. $V^{d,g,c}$ is singular as claimed.

Now suppose D has a singularity for which we claim $V^{d,g,c}$ is nonsingular. If all the branches of D are nonsingular, then $H^0(C, \mathscr{K}_\phi) = 0$ and we are done. Now suppose some branch is singular, and the element (29) of $H^0(C, \mathscr{K}_\phi)$ satisfies the equiclassical relations. We have by assumption $l = k+1$. (21) and (29) imply $h_0 = 0$. If $k = 1$, we are done. If not, (17)–(20) give

$$(30)\qquad a_1^x = c_0(k+1), \quad 2a_2^y = c_0(l+1); \quad \frac{1}{k+1}a_1^x = \frac{2}{l+1}a_2^y.$$

Then (29) together with $h_0 = 0$ yields

$$(31)\qquad a_1^x = h_1(k+1), \quad a_2^y = h_1(l+1); \quad \frac{1}{k+1}a_1^x = \frac{1}{l+1}a_2^y.$$

Together (30) and (31) imply $a_1^x = a_2^y = h_1 = c_0 = 0$. If $k = 2$, we are done. If not, having shown $a_s^x = a_t^y = h_v = c_w = 0$ for $1 \le s \le i$, $1 \le t \le i+1$, $0 \le v \le i$, $0 \le w \le i-1$, and $i < k-1$, one shows that $a_{i+1}^x = a_{i+2}^y = h_{i+1} = c_i = 0$ as follows. (17)–(20), together with what we already know is zero, gives

$$(32)\qquad \begin{gathered}(i+1)a_{i+1}^x = c_i(k+1), \quad (i+2)a_{i+2}^y = c_i(l+1);\\ \frac{i+1}{k+1}a_{i+1}^x = \frac{i+2}{l+1}a_{i+2}^y.\end{gathered}$$

Then (29), together with what we already know is zero, yields

$$(33)\quad a_{i+1}^x = h_{i+1}(k+1), \quad a_{i+2}^y = h_{i+1}(l+1); \quad \frac{1}{k+1}a_{k+1}^x = \frac{1}{l+1}a_{i+2}^y.$$

Together (32) and (33) give $a_{i+1}^x = a_{i+2}^y = h_{i+1} = c_i = 0$. Having shown all the h_j in (29) are zero we are done. q.e.d.

As a final remark note that while we have shown that the "local" variety EC is irreducible, Zariski [9, p. 223] has shown that the corresponding "global" variety $V^{d,g,c}$ is sometimes reducible. This is in contrast to the fact that both the "local" variety EG and the corresponding "global" variety $V^{d,g}$ are always irreducible ([1], [3], [4]).

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