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NECESSARY CONDITIONS FOR LOCAL SUBELLIPTICITY OF $\square_b$ ON CR MANIFOLDS

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1. Introduction

This paper presents a proof that a certain geometric condition on a pseudoconvex CR manifold is necessary for a subelliptic estimate to hold for $\square_b$ for tangential $(0, q)$-forms. This work extends previous results which applied only to pseudoconvex boundaries in $\mathbf{C}^n$ [3]. These results have as their archetype the conditions that Catlin proved are necessary for subellipticity of the $\bar{\partial}$-Neumann problem for pseudoconvex open subdomains of $\mathbf{C}^n$ [1, Theorem 3]. Sufficient conditions for subellipticity of $\square_b$ on pseudoconvex CR-manifolds are presented in [6]. Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex subdomains of $\mathbf{C}^n$ is discussed in [5] and [2]. Necessary conditions for subellipticity of the $\bar{\partial}$-Neumann problem on nonpseudoconvex subdomains of $\mathbf{C}^n$ appear in [4, Theorem 4.2].

1.1. Terminology and notation. Let M be a smooth compact manifold of real dimension $(2n-1)$. A codimension-1 CR structure for M is a vector subbundle $T^{0,1}$ of the complexified tangent bundle with the following properties:

(i) the complex rank of $T^{0,1}$ is $(n-1)$;

(ii) $T^{0,1} \cap \overline{T}^{0,1} = \{0\}$;

(iii) if the vectorfields $\overline{L}_1, \overline{L}_2$ are local sections of $T^{0,1}$, then the Lie bracket vectorfield $[\overline{L}_1, \overline{L}_2]$ is also a section of $T^{0,1}$.

There exists a nonvanishing imaginary 1-form τ on M uniquely determined up to multiplication by a nonvanishing real function by the condition that τ annihilates $T^{0,1} + \overline{T}^{0,1}$. The CR structure on M shall be assumed to be pseudoconvex, which means that $\langle \tau, [L, \overline{L}] \rangle$ is a semi-definite Hermitian form on $\overline{T}^{0,1} \times T^{0,1}$.

Given a point $p_0 \in M$, a germ at p_0 of a CR imbedding into $\mathbf{C}^n$ consists of an n-tuple $(z_1, \cdots, z_n)$ of functions defined on some neighborhood U of p_0 with the properties that

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(i) $\overline{L}z_j = 0$ for every section L of $T^{0,1}$ defined near p_0 and for every $j \in \{1, \cdots, n\}$;

(ii) $z\colon M \cap U \mapsto \mathbf{C}^n$ is an imbedding into $\mathbf{C}^n$.

The imbedding is said to be almost-CR if (i) is replaced by the condition that $\overline{L}z_j$ vanishes to infinite order at p_0. The bold-faced letter $\mathbf{M}$ denotes the image of M by a germ of an almost-CR imbedding.

The tangential Cauchy-Riemann complex associated to a CR manifold is defined in §5.4. The second-order, linear, nonelliptic differential operator $\Box_b$ associated to this complex is defined in §7.1. The L^2-Sobolev space of order ε, and subelliptic estimates of order ε for $\Box_b$ are discussed in §7.2.

The symbol $B_m(a, r)$ denotes the m-dimensional ball of radius r centered at a.

1.2. Statement of results. If M is a compact manifold of real dimension $2n - 1$ that carries a codimension 1 CR structure, then at every point $p_0 \in M$ there exists a germ of an almost-CR imbedding of M into $\mathbf{C}^n$ as a real hypersurface $\mathbf{M}$. Let $z(p_0)$ denote the imbedded image of p_0. Theorem (7.9.3) below asserts that that if $\mathbf{M}$ is holomorphically flat in dimension q of order at least η at $z(p_0)$, then a subelliptic Sobolev estimate of order ε does not hold for $\Box_b$ on tangential $(0, q)$ forms on any open set containing p_0 for any value of $\varepsilon > 1/\eta$. Here *holomorphic flatness of order η in dimension q* means that there exists a sequence of patches of q-dimensional holomorphic submanifolds of $\mathbf{C}^n$ shrinking to the imbedded point $z(p_0) \in \mathbf{M}$ such that the distance from each point on the submanifold to $\mathbf{M}$ is no more than $\mathcal{O}$ (diameter of patch $^\eta$). Catlin [1, Theorem 2] has proved that this condition is implied by the condition that some q-dimensional complex manifold makes contact of order at least η with $\mathbf{M}$ at $z(p_0)$. Our results are based on the assumption that the global L^2-theory for $\Box_b$ has been established for the pseudoconvex CR manifold M in a sense made precise in §7.2.

1.3. Organization of text. §2 summarizes properties of almost-CR imbeddings. The intrinsic nature of the criterion for holomorphic flatness, which ostensibly depends on the choice of the imbedding of the abstract CR manifold M as a hypersurface $\mathbf{M}$ is discussed in §3. In §3, it is also shown that $\mathbf{M}$ can be approximated by pseudoconvex hypersurfaces in $\mathbf{C}^n$, and that there exist holomorphic functions defined essentially on one side of $\mathbf{M}$ whose derivatives grow at a prescribed rate as one approaches $\mathbf{M}$. §4 presents a simple Green's formula for the positive currents represented by integration over complex q-dimensional submanifolds of an open subdomain in $\mathbf{C}^n$. We use the formula to express the integral of a holomorphic function over a q-dimensional complex submanifold in terms of an interior integral and a boundary integral on the hypersurface $\mathbf{M}$. In §5 the integral over the q-dimensional complex submanifold is minorized by a certain power of the distance of the submanifold to **M**. §6 shows that the interior integral is less important than the boundary integral, and that the main part of the boundary integrand can be approximated by a $\overline{\partial}_b$-closed q-form on M. Finally, in §7 we show that an ε-subelliptic estimate for $\Box_b$ yields an upper bound for the boundary integral. Comparison of the lower bound derived in §5 with the upper bound of §7 provides the conclusion that $\varepsilon \leq 1/\eta$.

2. Almost-CR imbeddings of an abstract CR manifold

2.1. The following properties of almost-CR imbeddings can be proved using standard arguments with formal power series.

(2.1.1) Proposition. *Let M be a manifold of real dimension $2n-1$ which carries a codimension one CR structure. Let $p_0 \in M$. There exist n smooth complex-valued functions $(z_1 \dots z_n)$ defined on some neighborhood of p_0 such that*

(2.1.2) *local coordinates are provided near p_0 by the real and imaginary parts of the first $n-1$ functions, and by the real part of z_n;*

(2.1.3) *each of the complex-valued functions is annihilated to infinite order at p_0 by the antiholomorphic vectorfields associated to the abstract CR structure of M;*

(2.1.4) *in terms of the function $\rho = \operatorname{Im} z_n$ which is functionally dependent on the local coordinates of* (2.1.2), *the $n-1$ vectorfields*

$$\overline{L}_1 = \frac{\partial}{\partial \overline{z}_1} + \left[\frac{i(\partial\rho/\partial\overline{z}_1)}{1 - i(\partial\rho/\partial x_n)}\right]\frac{\partial}{\partial x_n},$$

$$\vdots \tag{2.1.5}$$

$$\overline{L}_{n-1} = \frac{\partial}{\partial \overline{z}_{n-1}} + \left[\frac{i(\partial\rho/\partial\overline{z}_{n-1})}{1 - i(\partial\rho/\partial x_n)}\right]\frac{\partial}{\partial x_n}$$

agree to infinite order at p_0 with a set of local generators for the local sections of the abstract CR structure M.

(2.1.6) Proposition. *Let $z = (z_1, z_2, \cdots, z_n)$ and $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_n)$ be two n-tuples of complex-valued functions defined on a neighborhood in M of $p_0 \in M$ such that* (2.1.2) *and* (2.1.3) *hold true for both sets of functions. There exists a local diffeomorphism Φ of $\mathbf{C}^n$ such that Φ satisfies*

(2.1.7) *Φ is defined on an open subset of $\mathbf{C}^{n+1}$ that contains z_0;*

(2.1.8) $\bar{\partial}\Phi$ *vanishes to infinite order at* z_0;

(2.1.9) $\varsigma - \Phi^*(z)$ *vanishes to infinite order at* p_0.

3. Order of contact of asymptotically holomorphic submanifolds

3.1. Invariant descriptions of holomorphic flatness. In this section we introduce a test which detects holomorphic flatness of a hypersurface in $\mathbf{C}^n$. A hypersurface is holomorphically flat in dimension q at a given point on the hypersurface if there exists any sequence of essentially holomorphic, q-dimensional patches of submanifolds of $\mathbf{C}^n$ that shrink to the given point and that have large diameter in comparison to their distance to the hypersurface. Catlin formulated a precise criterion based on this notion in [1]. The purpose of this section is to modify his criterion to create a test which is manifestly invariant not merely under biholomorphic transformations, but also under the broader class of transformations which are $\bar{\partial}$-closed to infinite order at the given point on the hypersurface.

(3.1.1) Definition. Let $\{r_m\}_{m=1}^{\infty}$ be a sequence of positive real numbers. For each ball $B_{2q}(0, r_m) \subset \mathbf{C}^q$, let $G_m\colon B_{2q}(0, r_m) \to \mathbf{C}^n$ be a smooth map. The image $G_m(B_{2q}(0, r_m)) = V_m \subset \mathbf{C}^n$ is a subset of $\mathbf{C}^n$ parametrized by G_m. Call the sequence $\{V_m\}_{m=1}^{\infty}$ *admissible* if the sequences $\{r_m\}_{m=1}^{\infty}$ and $\{G_m\}_{m=1}^{\infty}$ satisfy the following conditions:

(3.1.2) (Shrinking of radii).

$$\lim_{m\to\infty} r_m = 0.$$

(3.1.3) (Uniform Lipschitz control). There exists a constant $C < \infty$ such that for every m and every $u_1, u_2 \in B_{2q}(0, r_m)$,

$$\frac{|G_m(u_1) - G_m(u_2)|}{|u_1 - u_2|} \le C.$$

(3.1.4) (Uniform nondegeneracy of immersions). There exists a constant $c > 0$ such that for every m

$$c \le \sum_{|I|=q} \|G_m^*(dz^I \wedge d\bar{z}^I)\|^2 \quad \text{at the center of the ball } B_{2q}(0, r_m).$$

(3.1.5) (Asymptotic holomorphicity). For every positive integer N there exists C_N such that for every m

$$\|\bar{\partial} G_m\|_{L^\infty(B_{2q}(0,r_m))} \le C_N r_m^N.$$

(3.1.6) (Controlled convergence to limit point). There exist a $z_0 \in \mathbf{C}^n$, $\alpha > 0$, and C such that for every m

$$|z_0 - G_m(0)| \le C r_m^{\alpha}.$$

(3.1.7) Definition. Let $\mathbf{r}$ be a smooth real-valued defining function for the germ at z_0 of a hypersurface $\mathbf{M} \subset \mathbf{C}^n$. Suppose that $d\mathbf{r} \neq 0$ at z_0. Suppose that $\{V_m\}_{m=1}^{\infty}$ is an admissible sequence whose limit point is z_0. The order of contact of $\{V_m\}_{m=1}^{\infty}$ with $\mathbf{M}$ is said to be at least η if there exists a constant C such that for all sufficiently large m

$$\sup_{z \in V_m} |\mathbf{r}(z)| \leq C r_m^{\eta}. \tag{3.1.8}$$

Of course, the preceding definition is independent of the particular nondegenerate defining function $\mathbf{r}$ for $\mathbf{M}$. The existence of a sequence $\{V_m\}_{m=1}^{\infty}$ whose order of contact with a hypersurface is at least η is also invariant under certain coordinate changes. One can prove the following invariance property.

(3.1.9) Proposition. *Let $\{V_m\}_{m=1}^{\infty}$ be an admissible sequence converging to z_0. Let Φ be the germ at z_0 of a diffeomorphism of $\mathbf{C}^n$ such that $\overline{\partial}\Phi$ vanishes to infinite order at z_0. Then $\{r_m\}_{m=1}^{\infty}$, $\{\Phi \circ G_m \colon B(0, r_m) \to \mathbf{C}^n\}_{m=1}^{\infty}$, determine an admissible sequence $\{\Phi(V_m)\}_{m=1}^{\infty}$ whose limit point is $\Phi(z_0)$. Furthermore, if there is a germ at z_0 of a hypersurface M with which $\{V_m\}_{m=1}^{\infty}$ makes contact of order at least η, then the image sequence $\{\Phi(V_m)\}_{m=1}^{\infty}$ makes contact of order at least η with $\Phi(M)$.*

3.2. Transition to holomorphic submanifolds. Many computations are simplified by the observation that it is possible to modify the sequence of asymptotically holomorphic submanifolds to obtain a similar sequence of submanifolds which are, however, actually holomorphic.

(3.2.1) Proposition. *Let $\{V_m\}_{m=1}^{\infty}$ be an admissible sequence whose order of contact with the hypersurface $\mathbf{M}$ at z_0 is at least η. There exists an admissible sequence of* complex *submanifolds $\{V_m'\}_{m=1}^{\infty}$ whose order of contact with $\mathbf{M}$ at z_0 is at least η.*

Proof. Define V_m' as the image of a holomorphic map $G_m' \colon B_{2q}(0, r_m/2) \mapsto \mathbf{C}^n$, where G_m' is the Bergman projection of G_m. Standard mapping properties of the Bergman projection for the unit ball in $\mathbf{C}^q$ yield the estimates (3.1.3)–(3.1.4) for G_m'.

One advantage of dealing with complex submanifolds such as those generated by Proposition (3.2.1) is that the analytic implicit function theorem provides a lower bound on the diameter of the patches of q-dimensional complex subspaces of $\mathbf{C}^n$ over which the submanifolds may be regarded as graphs. Passing to a subsequence of $\{V_m'\}_{m=1}^{\infty}$ if necessary, one can use the condition on uniform nondegeneracy of immersions (3.1.4) to show that there exist a multi-index I and a constant $c > 0$ such that

$$c \leq \|G_m'^{*} dz^I \wedge d\overline{z}^I\| \quad \text{at the center of } B_{2q}(0, r_m/2). \tag{3.2.2}$$

This condition implies that a portion of V'_m can be graphed over the q-dimensional subspace $\pi = \pi_I$ associated to I. To obtain a lower bound on the size of the patch of the coordinate subspace π over which each submanifold is graphed, we note that the power series coefficients for the graphing functions can be determined by implicit differentiation of the differential relations which arise upon solving for the differentials of the $n-q$ dependent coordinate variables on the submanifold. The radius of convergence of such a power series is minorized by a constant multiple of the radius of convergence of the power series for the map $G'\colon B_{2q}(0, r_m/2) \to \mathbf{C}^n$. The constant multiple in question can be taken to be the uniform lower bound in (3.2.2) multiplied by the $(2q)$th power of the Lipschitz constant of (3.1.3). Such an argument proves the following result originally stated by Catlin [1].

(3.2.3) Proposition (*Catlin*). *Let $\{V'_m\}_{m=1}^{\infty}$ be an admissible sequence of complex submanifolds, parametrized by maps $G'_m\colon B_{2q}(0, r_m/2) \to \mathbf{C}^n$. Replacing $\{V'_m\}_{m=1}^{\infty}$ be a subsequence if necessary, one can find a constant $c > 0$, a complex q-dimensional coordinate subspace π of $\mathbf{C}^n$, a sequence of points $\{u_m\}_{m=1}^{\infty}$ such that $u_m \in \pi$, and a sequence of holomorphic maps $\{H_m\colon \pi \cap B_{2q}(u_m, cr_m) \to \pi^{\perp}\}_{m=1}^{\infty}$ such that*

(3.2.4) *the graph of H_m in $\mathbf{C}^n = \pi \oplus \pi^{\perp}$ is contained in V'_m, and*

(3.2.5) *there is a Lipschitz bound for $H_m\colon \pi \cap B_{2q}(u_m, cr_m) \to \pi^{\perp}$ that is uniform in m.*

Here $\pi^{\perp}$ is the complex $(n-q)$-dimensional subspace of $\mathbf{C}^n$ orthogonal to π.

(3.2.6) Proposition. *The complex coordinate subspace π of* (3.2.3) *can be chosen perpendicular to the normal vector to $\mathbf{M}$ at z_0 if $\{V'_m\}_{m=1}^{\infty}$ makes contact with $\mathbf{M}$ at z_0 of order $\eta > 1$.*

Proof. Since $\eta > 1$, the tangent plane of V'_m cannot be close to the complex normal of $\mathbf{M}$.

3.3. Approximation of M by pseudoconvex hypersurfaces. After a holomorphic, affine coordinate transformation in $\mathbf{C}^n$, $\mathbf{M}$ can be represented locally as a graphed hypersurface through $z_0 = 0 \in \mathbf{C}^n$ of the form

$$\{0 = \mathbf{r}(z) = -\operatorname{Im} z_n + \rho(z_1, z_2, \cdots, z_{n-1}, \operatorname{Re} z_n)\}$$

with $0 = \rho(0) = d\rho(0)$. The almost-CR imbedding $z\colon M \mapsto \mathbf{C}^n$ will not necessarily map M to a pseudoconvex hypersurface, but the image $\mathbf{M}$ can be closely approximated by pseudoconvex hypersurfaces.

(3.3.1) Definition. An *admissible sequence* of graphed pseudoconvex hypersurfaces over $0 \in \mathbf{C}^n$ consists of

(i) a sequence of $(2n-1)$-dimensional balls $\{B_{2n-1}(0, s_m)\}_{m=1}^{\infty}$ in $\mathbf{C}^{n-1} \times \mathbf{R} \times \{0\}$ for which the radii $s_m \downarrow 0^+$;

(ii) a sequence of graphing functions $\rho_m\colon B_{2n-1}(0, s_m) \mapsto \mathbf{R}$ for which there is an m-independent Lipschitz bound valid throughout $B_{2n-1}(0, s_m)$;

(iii) a choice of sgn $\in \{-1, 1\}$ such that for all sufficiently large m, the Levi-form of the open subdomain $\Omega_m = \{(-y_m + \rho_m)\mathrm{sgn} < 0\}$ is positive semi-definite at every boundary point lying in the cylinder $C_m = B_{2n-1}(0, s_m) \times \mathbf{R}$.

(3.3.2) Definition. The *order of contact* of an admissible sequence of pseudoconvex hypersurfaces with $\mathbf{M}$ at 0 is at least N if $\sup\{|\mathbf{r}(z)|\colon z \in C_m \cap b\Omega_m\} \leq Cs_m^N$.

(3.3.3) Proposition. *For every $s_m \downarrow 0$ and every positive integer N, there exists an admissible sequence of pseudoconvex hypersurfaces in $\mathbf{C}^n$ whose order of contact with $\mathbf{M}$ at 0 is at least N.*

Proof. Set $\rho_m = \rho + s_m^N(|z_1|^2 + \cdots + |z_{n-1}|^2)$. The existence of an m-uniform Lipschitz bound for ρ_m is obvious, as is the fact that the order of contact with $\mathbf{M}$ is at least $N + 2$. To prove that the portion of the graph of ρ_m lying over $B_{2n-1}(0, s_m)$ is a pseudoconvex hypersurface, it is convenient to introduce the symbol t as a perturbation parameter, set $\rho(t) = \rho + t(|z_1|^2 + \cdots + |z_{n-1}|^2)$, and investigate the semi-definiteness of the matrix $(c_{j,k})(t) = \langle \tau(t), [L_j(t), \overline{L}_k(t)] \rangle$ in which $\tau(t)$ is an imaginary 1-form that annihilates a set of generators $\{\overline{L}_1(t), \cdots, \overline{L}_{n-1}(t)\}$ for the CR structure induced by $\mathbf{C}^n$ on the hypersurface which is the graph of $\rho(t)$. Formula (2.1.5) is an exact description of such a set of generators for the induced CR structure, except that in (2.1.5) ρ must be replaced by $\rho(t)$. Expanding $c_{j,k}(t)$ as a Taylor series in t gives

$$c_{j,k}(t) = c_{j,k}(0) + tc'_{j,k}(0) + \mathscr{O}(t^2),$$

where

$$c'_{j,k} = \langle \tau', [L_j, \overline{L}_k] \rangle + \langle \tau, [L'_j, \overline{L}_k] \rangle + \langle \tau, [L_j, \overline{L}'_k] \rangle,$$

in which primes denote derivatives with respect to t evaluated at $t = 0$. A simple calculation using (2.1.5) shows that $\tau(t)$ can be taken to be

$$\tau(t) = -i dx_n + t\mathscr{O}(|z_1| + \cdots + |z_{n-1}| + |x_n|) + \mathscr{O}(t^2); \tag{3.3.4}$$

and that at $t = 0$,

$$\begin{aligned} \tau' &= \mathscr{O}(|z_1| + \cdots + |z_{n-1}| + |x_n|); \\ L'_k &= (iz_k)(1 - i\partial\rho/\partial x_n)^{-1}\partial/\partial x_n; \\ [L'_j, \overline{L}_k] &= i\delta_{j,k}\partial/\partial x_n + \mathscr{O}(|z_1| + \cdots + |z_{n-1}| + |x_n|). \end{aligned}$$

Thus $c'_{j,k}(0) = 2\delta_{j,k} + \mathscr{O}(|z_1| + \cdots + |z_{n-1}| + |x_n|)$. Inserting this into the Taylor expansion gives

$$c_{j,k}(s_m^N) = c_{j,k}(0) + 2s_m^N\delta_{j,k} + o(s_m^N)$$

uniformly in m on $B_{2n-1}(0, s_m)$. Proposition (2.1.1) implies that in local coordinates for M the data $\tau(0), c_{j,k}(0)$ on M which are induced by the complex structure of $\mathbf{C}^n$ differ from corresponding data $(c_{j,k})_{\text{abstract}}, \tau_{\text{abstract}}$ by error terms which vanish to infinite order at 0. The Levi-matrix $(c_{j,k})_{\text{abstract}}$ is semi-definite. Suppose that it is positive semi-definite for the choice of τ_{abstract} which is approximated by (3.3.4). In this case, since the error terms are $o(s_m^N)$ on $B_{2n-1}(0, s_m)$,

$$c_{j,k}(s_m^N) \geq 2s_m^N \delta_{j,k} + o(s_m^N) > 0$$

for $s_m > 0$.

If the Levi-matrix of the abstract CR structure is negative semi-definite for the choice of τ_{abstract} approximated by (3.3.4), simply replace t by $-t$ in the previous discussion; that is, replace s_m^N by $-s_m^N$, to obtain a sequence of negative definite Levi-forms for hypersurfaces in $\mathbf{C}^n$. In this case choose $\operatorname{sgn} \in \{-1, 1\}$ to be -1 to fulfill (3.3.1)(iii).

The preceding theorem implies that $\mathbf{M}$ separates the cylinder $C_m = B_{2n-1}(0, s_m) \times \mathbf{R}$ into two open subdomains, at least one of which is well approximated by pseudoconvex domains in $\mathbf{C}^n$. Let Ω denote such a "pseudoconvex" side of $\mathbf{M}$.

(3.3.5) Corollary. *Let $s_m \downarrow 0^+$. For every $z \in \Omega \cap B_{2n}(0, s_m)$, and every positive integer N there exist C'_N and a function f analytic on $\Omega \cap B_{2n}(0, s_m)$ such that for every positive integer k there exists C_k so that*

$$(3.3.6) \qquad |(\partial/\partial y_n)^k f(z)| \geq C_k(|\mathbf{r}(z)| + C'_N(s_m^N))^{-(k-e)} \|f\|_{L^2(\Omega \cap B_{2n}(0, s_m))}$$

for an exponent e that depends only on the dimension n.

Proof. Proposition (3.3.3) applied to the sequence $\{2s_m\}_{m=1}^{\infty}$ implies that $\Omega \cap B_{2n}(0, s_m) \subset \Omega \cap C_m$ is contained within a pseudoconvex domain $D_m \subset \mathbf{C}^n$ which is obtainable by translating the graph of $\rho_m : B_{2n-1}(0, 2s_m) \mapsto \mathbf{R}$ along the y_n-axis a distance $d_m = C_N(s_m^N)$ away from Ω, and intersecting with $B_{2n}(0, s_m)$. Apply [1, Lemma 1] to D_m to deduce that for every $z \in D_m$ the norm of the continuous linear functional $L^2(D_m) \ni f \mapsto (\partial/\partial y_n)^k f(z)$ is minorized by $C_k|y - \rho_m \pm d_m|^{-(k-e)}$ for a constant e depending only on n. This last expression is minorized by $C'_k(|\mathbf{r}(z)| + C'_N(s_m^N))^{-(k-e)}$.

(3.3.7) Comment. It is clear from the proof of Lemma 1 in [1] that in (3.3.6) Ω can be replaced by a thickening of Ω by an amount $\approx |\mathbf{r}(z)| + d_m$. This implies that f can be taken to be smooth on the portion of $\mathbf{M}$ that lies over C_m. Furthermore, in (3.3.6) the interior L^2-norm can be replaced by the boundary norm $\|f\|_{L^2(\mathbf{M} \cap C_m)}$ for some other value of e.

(3.3.8) Proposition. *If $\{V_m\}_{m=1}^{\infty} = \{G_m(B_{2q}(0, r_m)\}_{m=1}^{\infty}$ is an admissible sequence of complex manifolds whose order of contact with $\mathbf{M}$ at 0 is at* *least $\eta > 1$, then there exists an admissible sequence of graphed pseudoconvex hypersurfaces $\{\rho_m : B_{2n-1}(0, s_m) \mapsto \mathbf{R}\}_{m=1}^{\infty}$ such that for every m the associated pseudoconvex domain D_m contains a complex manifold V'_m belonging to an admissible sequence $\{V'_m\}_{m=1}^{\infty}$ whose order of contact with $\mathbf{M}$ at 0 is still at least η.*

Proof. Translate V_m a distance $\approx r_m^{\eta}$ into Ω along the y_n-axis to obtain a sequence V'_m for which $V'_m \subset \Omega$ and $\inf\{|\mathbf{r}(z)| : z \in V_m\} \approx r_m^{\eta}$. From (3.1.3) and (3.1.6), $\sup\{\operatorname{dist}(z, 0) : z \in V_m\} \leq \mathscr{O}(r_m^{\alpha}) + \mathscr{O}(r_m^{\eta}) + \mathscr{O}(r_m)$ for some $\alpha > 0$. Define

$$s_m = r_m^p \quad \text{for } p < \min(\alpha, \eta, 1). \tag{3.3.9}$$

Then the ball $B_{2n}(0, s_m)$ contains V'_m.

(3.3.10) Corollary. *The admissible sequence $\{V'_m\}$ has the property that for every $z \in V'_m$ there is an f with the property that for every positive integer k, there exists $C_k > 0$ such that*

$$|(\partial/\partial y_n)^k f(z)| \geq C_k |r_m|^{-\eta(k-e)} \|f\|_{L^2(\Omega \cap B_{2n}(0, r_m^p))}.$$

Proof. Choose N so large in Proposition (3.3.3) that $Np > \eta$. With this N, invoke Corollary (3.3.5). Note that the estimate $|\mathbf{r}(z)| \approx r_m^{\eta}$ valid on V'_m, and the estimate $s_m^N = r_m^{pN} = o(r_m^{\eta})$, imply that $|\mathbf{r}(z)| + \mathscr{O}(s_m^N) \approx r_m^{\eta}$ for $z \in V'_m$.

4. Reproducing formulas via currents

4.1. Let Ω be an open subdomain of $\mathbf{C}^n$. If zero is a regular value of the holomorphic map $F : \Omega \to \mathbf{C}^{n-q}$ then $V = F^{-1}(0)$ is a q-dimensional complex submanifold of $\Omega \subset \mathbf{C}^n$. Integration over V of test forms of bidegree (q, q) defines a current $[V] \in \mathscr{D}'(\Omega, \wedge^{q,q})$ of bidimension (q, q). Explicitly $[V](\phi) = \int_V \phi$.

It is convenient to express $[V]$ in terms of the Bochner-Martinelli form of $\mathbf{C}^{n-q}$, which is

$$\begin{aligned}\mathscr{B}_{n-q} &= c_{n-q}|z|^{-2(n-q)} \left(\sum_{j=1}^{n-q} (-1)^j \bar{z}_j d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_{n-q} \right) \\ &\qquad \wedge dz_1 \wedge \cdots \wedge dz_{n-q} \\ &= \partial(S_{n-q}).\end{aligned}$$

Here

$$S_{n-q} = c'_{n-q}|z|^{-2(n-q-1)} \sum_{j=1}^{n-q} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_j \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge dz_{n-q} \wedge d\bar{z}_{n-q}$$

stands for deletion and c_{n-q}, c'_{n-q} are constants.

(4.1.1) Proposition. *For every test form ϕ compactly supported in Ω,*

$$\int_V \phi = \int_\Omega F^*(S_{n-q}) \wedge \overline{\partial}\partial\phi.$$

That is, the following is an equality in the distribution sense:

$$[V] = \overline{\partial}\partial F^*(S_{n-q}) = \overline{\partial} F^*(\mathscr{B}_{n-q}). \tag{4.1.2}$$

The following formula includes the correction term that must be incorporated when $\operatorname{supp}\phi$ extends to $b\Omega$.

(4.1.3) Corollary. *Let Ω be a bounded subdomain of $\mathbf{C}^n$. Let ϕ be a differential form which is smooth up to the boundary of Ω such that* $\operatorname{supp}\phi \cap V \Subset \Omega$. *Then,*

$$\int_V \phi = \int_{b\Omega} F^*(\mathscr{B}_{n-q}) \wedge \phi + \int_\Omega F^*(\mathscr{B}_{n-q}) \wedge \overline{\partial}\phi. \tag{4.1.4}$$

Proof of (4.1.1) *and* (4.1.3). See [3].

Note how simply (4.1.4) changes under the substitution $\phi \to f\phi$, where f is any function which is holomorphic on Ω and smooth on $b\Omega$. It is instructive to set

$$\phi_\varepsilon = f\psi_\varepsilon(z_1, \cdots, z_q) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_q \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q,$$

where $\{\psi_\varepsilon\}_{\varepsilon>0}$ is an approximate identity approaching the delta function supported on the submanifold $0 = z_1 = z_2 = \cdots = z_q$. For every $\varepsilon > 0$, $\overline{\partial}\phi_\varepsilon = 0$. Assume that the projection map $\pi\colon \mathbf{C}^n \to \mathbf{C}^q$, defined by $\pi(z_1, \cdots, z_n) = (z_1, \cdots, z_q)$, is injective when restricted to V; so that $\operatorname{supp}\phi_\varepsilon \cap V$ approaches a unique point $p = \pi^{-1}(0) \cap V$ as $\varepsilon \to 0$. With these assumptions, one can show that as $\varepsilon \to 0$, (4.1.4) stabilizes to yield a reproducing formula for holomorphic functions f:

$$f(p) = \int_{b\Omega \cap \{z_1 = \cdots = z_q = 0\}} f F^*(\mathscr{B}_{n-q}). \tag{4.1.5}$$

We shall use the heuristic principle that (4.1.4) generates a reproducing formula to link the growth properties of a holomorphic function on a q-dimensional submanifold inside a domain Ω to the boundary analysis of the differential (q,q) forms $\phi_\varepsilon = f\psi_\varepsilon dz_1 \wedge \cdots \wedge dz_q \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q$ which are $\overline{\partial}_b$ closed on $b\Omega$.

5. Construction of test forms

5.1. It is necessary to localize the holomorphic functions given by Corollary (3.3.10) in order to construct test forms φ_m suitable for pairing with $[V_m]$ as in (4.1.4). The dimensions of the region on which each localization occurs play a crucial role in the subsequent majorization of derivatives of the test forms.

Let $\chi\colon R \to [0,1]$ be a smooth cutoff function that satisfies $\chi(t) = 0$ if $|t| \geq \frac{1}{3}$ and $\chi(t) = 1$ if $|t| \leq \frac{1}{4}$.

(5.1.1) Definition. Let $\chi_{1,m}(z_1, z_2, \cdots, z_q) = \chi((z - u_m)/cr_m)$, where u_m and cr_m are the *center* and *radius* of the ball $B_m = B_{2q}(u_m, cr_m)$ over which V_m is graphed as in (3.2.3).

(5.1.2) Definition. Let $\chi_{2,m}(z_{q+1}, \cdots, z_n) = \chi(|(z_{q+1}, \cdots, z_n)|/s_m)$, where $s_m = r_m^p$ is as in (3.3.9).

Note that $\chi_{1,m}$ localizes in the first set of q complex variables, and $\chi_{2,m}$ localizes in the second set of $n - q$ variables. Also note that although $\chi_{1,m}$ localizes to a cylinder whose radius r_m is effectively the diameter of V_m, the cutoff function $\chi_{2,m}$ localizes to a much larger cylinder of radius $\mathscr{O}(r_m^p)$. Since p can be chosen to be any sufficiently small positive number satisfying the estimate in (3.3.9), it follows that derivatives of $\chi_{2,m}$ are much milder than derivatives of $\chi_{1,m}$.

(5.1.3) Definition. Let $F_m : B_m \times \mathbf{C}^{n-q} \to \mathbf{C}^n$ be given by $F_m(z_1, \cdots, z_n) = (z_{q+1}, \cdots, z_n) - H_m(z_1, \cdots, z_q)$, where $H_m\colon B_m \to \mathbf{C}^{n-q}$ represents V_m' as a graph over B_m as in (3.2.3).

(5.1.4) Definition. Given $w \in V_m'$, construct f as in Corollary (3.3.10). For every m and k define

$$\varphi_m^{(k)} = [(\partial/\partial z_n)^k f]\chi_{1,m} F_m^*(\chi_{2,m}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_q.$$

(5.1.5) Definition. The *basic pairing formula* is the identity

$$\int_{V_m'} \varphi_m^{(k)} = \int_{\mathbf{M}} F_m^*(\mathscr{B}_{n-q}) \wedge \varphi_m^{(k)} + \int_{\Omega} F_m^*(\mathscr{B}_{n-q}) \wedge \bar{\partial}\varphi_m^{(k)}.$$

This integral identity expresses the natural pairing between a test form $\varphi_m^{(k)}$ and a current $[V_m']$ in terms of integrals amenable to boundary analysis.

(5.1.6) Proposition. *The basic pairing formula is valid for all sufficiently large values of m.*

Proof. This follows from (4.1.4) under the substitution $F = F_m$, $\varphi = \varphi_m^{(k)}$, and $V = V_m'$. One must verify that f is smooth on the closure of $\Omega \cap \operatorname{supp} \chi_{1,m} \cap \operatorname{supp} F_m^*(\chi_{2,m})$. It suffices to show that $B_{2n}(0, s_m/2)$ contains $\operatorname{supp} \chi_{1,m} \cap \operatorname{supp} F_m^*(\chi_{2,m})$. If $z = (z_1, \cdots, z_n) \in \operatorname{supp} \chi_{1,m} \cap \operatorname{supp} F_m^*(\chi_{2,m})$ then $(z_1, z_2, \cdots, z_q) \in B_m$ and $F_m(z_1, \cdots, z_n) \in \operatorname{supp} \chi_{2,m}$. From (5.1.2),

$|F_m(z)| \le \frac{1}{3}s_m$. From (5.1.3),

$$\begin{aligned} |z|^2 &= |(z_1, \cdots, z_q)|^2 + |(z_{q+1}, \cdots, z_n)|^2 \\ &= |(z_1, \cdots, z_q)|^2 + |F_m(z) - H_m(z_1, \cdots, z_q)|^2 \\ &\le |(z_1, \cdots, z_q)|^2 + 2|F_m(z)|^2 + 2|H_m(z_1, \cdots, z_q)|^2 \\ &\le 2|(z_1, \cdots, z_q, H_m(z_1, \cdots, z_q))|^2 + 2(s_m/3)^2. \end{aligned}$$

But $w = (z_1, z_2, \cdots, z_q, H_m(z_1, \cdots, z_q))$ is a point on V'_m because $(z_1, \cdots, z_q) \in B_m$. It merely remains to show that $|w| \le o(s_m)$. From the proof of (3.3.8) it is clear that $|w| \le \mathscr{O}(r_m^\alpha) + \mathscr{O}(r_m) + \mathscr{O}(r_m^\eta) \le o(r_m^p = s_m)$ since $p < \min(\alpha, 1, \eta)$.

5.2. Minorization of the basic pairing. From (3.2.3), each V'_m contains a distinguished center point $w_m = (u_m, H_m(u_m)) \in \mathbf{C}^n \cap V'_m$.

(5.2.1) Proposition. *Let $f = f_m$ be the analytic function of* (3.3.10) *associated to the point $w = w_m$. Use this f to define $\varphi_m^{(k)}$ as in* (5.1.4). *With these choices for each k there exists C_k so that uniformly in m*

$$\left| \int_{V'_m} \varphi_m^{(k)} \right| \ge C_k r_m^{-\eta(k-e)+2q} \|f\|_{L^2(\Omega \cap B_{2n}(0, r_m^p))}$$

for an exponent e dependent only on n.

Proof. The submanifold V'_m is parametrized as the graph of the map $H_m \colon B_m \to \mathbf{C}^{n-q}$ defined by (3.2.3), at least on $\operatorname{supp} \chi_{1,m}$. Using the fact that $F_m^*(\chi_{2,m}) = 1$ on $V'_m = F_m^{-1}(0)$, pulling back the integral $\int_{V'_m} \varphi_m^{(k)}$ to B_m by means of the map $H_m \colon B_m \to V'_m$, and using the mean value theorem for the holomorphic function $[(\partial/\partial z_n)^{(k)} f_m] \circ (z_1, \cdots, z_q, H_m(z_1, \cdots, z_q))$ yields

$$\int_{V'_m} \varphi_m^{(k)} = (\partial/\partial z_n)^{(k)} f_m(w_m) \int_{B_m} \chi_{1,m}(z_1, \cdots, z_q) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_q \wedge d\bar{z}_q$$

since $\chi_{1,m}$ is a radial function. Thus

$$\begin{aligned} \left| \int_{V'_m} \varphi_m^{(k)} \right| &\ge |(\partial/\partial z_n)^k f_m(w_m)| r_m^{2q} \\ &\ge C_k r_m^{-\eta(k-e)+2q} \|f\|_{L^2(\Omega \cap B_{2n}(0, r_m^p))} \quad \text{by (3.3.10).} \end{aligned}$$

5.3. Reformulation of the basic pairing. The remainder of the paper is devoted to majorization of the basic pairing (5.1.5). The dominant term is the boundary integral on $\mathbf{M}$, which can be majorized when local subelliptic estimates are valid for $\Box_b$. In this section we shall cast this boundary integral into a form suited to analysis in terms of solutions to the inhomogeneous $\Box_b$ equation.

It is important to note that both integrands on the right side of (5.1.5) contain the maximal holomorphic form $(dz)^n = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ in $\mathbf{C}^n$.

(5.3.1) Definition. (i) Let b_{n-q} denote the form of type $(0, n-q-1)$ obtained from the Bochner-Martinelli form $\mathscr{B}_{n-q}$ in $\mathbf{C}^{n-q}$ by suppressing $(dz)^{n-q}$, so that $\mathscr{B}_{n-q} = b_{n-q} \wedge dz_1 \wedge \cdots \wedge dz_{n-q}$.

(ii) Let $\alpha_m^{(k)} = [(\partial/\partial z_n)^k f]\chi_{1,m} F_m^*(\chi_{2,m}) d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q$.

(iii) Let $\beta_m = F_m^*(b_{n-q})$.

(5.3.2) Comment. Note that $\alpha_m^{(k)}$ is of bidegree $(0,q)$ and β_m is of bidegree $(0, n-q-1)$.

With these definitions, it is easy to rewrite the two integrals on the right side of (5.1.5). We record the result below.

(5.3.3) Definition. Let I_0 and I_1 be the integrals

$$I_0 = (-1)^{n+1} \int_\Omega \overline{\partial}\alpha_m^{(k)} \wedge \beta_m \wedge (dz)^n,$$

$$I_1 = (-1)^{q^2} \int_{\mathbf{M}} \alpha_m^{(k)} \wedge \beta_m \wedge (dz)^n,$$

so that the right side of (5.1.5) equals $I_0 + I_1$.

It is useful to note that the kth order derivatives of f which occur in (5.3.1)(ii) can be replaced by derivatives taken tangentially along the boundary $\mathbf{M}$.

(5.3.4) Definition. Let T denote the vectorfield $T = \partial/\partial z_n - c\partial/\partial\overline{z}_n$, where c is chosen so that T annihilates the defining function for $\mathbf{M}$ in a neighborhood of the origin of $\mathbf{C}^n$.

(5.3.5) Proposition. *The vectorfield T has the following properties:*

(i) $0 = Tz_1 = Tz_2 = \cdots = Tz_{n-1} = T\overline{z}_1 = T\overline{z}_2 = \cdots = T\overline{z}_{n-1}$;

(ii) $Tz_n = 1$;

(iii) $(\partial/\partial z_n)^k f = T^k f$ *for every holomorphic function defined on some neighborhood of the origin in* $\mathbf{C}^n$; *and*

(iv) T *is tangential to* $\mathbf{M}$ *on some neighborhood of the origin.*

Proof. These are trivial consequences of the construction of T. Note that (i) and (ii) imply (iii).

We shall use properties of Lie differentiation to simplify I_0 and I_1.

(5.3.6) Notation. The Lie derivative with respect to T shall be written as $\mathscr{L}_T$. The derivation on the exterior algebra of differential forms given by contraction with T shall be written as $T\vee$.

(5.3.7) Proposition (*Integration by parts identities*). *For all forms ω, ω' of arbitrary degree*

(i) $\mathscr{L}_T\omega = (T \vee d\omega) + d(T \vee \omega)$,

(ii) $[\mathscr{L}_T, d] = 0$, *and*

(iii) $\mathscr{L}_T(\omega \wedge \omega') = \mathscr{L}_T\omega \wedge \omega' + \omega \wedge \mathscr{L}_T\omega'$.

If ω_{2n-1} *is a top-order form on* $\mathbf{M}$ *compactly supported where* T *is defined, then*

(iv) $\int_{\mathbf{M}} \mathscr{L}_T(\omega_{2n-1}) = 0$.

If ω_{2n} *is a top-order form on* $\mathbf{C}^n$ *compactly supported where* T *is defined, then*

(v) $\int_\Omega \mathscr{L}_T\omega_{2n} = 0$.

Proof of (i)–(v). The first three equations are well known elementary properties of $\mathscr{L}_T$, and the remaining identities are easy consequences of these equations. See [3] for details.

(5.3.8) Proposition. *The quantity* $\alpha_m^{(k)}$ *defined by* (5.3.1)(ii) *satisfies*

$$\alpha_m^{(k)} = [\mathscr{L}_T^{(k)} f]\chi_{1,m} F_m^*(\chi_{2,m}) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q.$$

Proof. Substitute (5.3.5)(iii) into (5.3.1)(ii).

Thus the coefficient of $d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$ in the expansion of $\alpha_m^{(k)}$ is intrinsic to $\mathbf{M}$. We now wish to approximate $d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$ by an expression which is intrinsic to the abstract CR structure of M.

5.4. Comparison of abstract and imbedded CR structures.

(5.4.1) Notation. Given any CR structure of anti-holomorphic vectorfields on M, one can define a tangential $(0,q)$ form canonically as an equivalence class of differential q-forms in which the null class consists of all annihilators of all q-fold exterior products of sections of the given CR structure. For each differential q-form φ on M, let $[\varphi]^{0,q}$ denote this equivalence class, called the tangential $(0,q)$ form associated to φ. The exterior derivative operator d induces the tangential Cauchy-Riemann operator $\bar{\partial}_{\tan}$ via the formula

$$\bar{\partial}_{\tan}([\varphi]^{0,q}) = [d\varphi]^{0,q+1},$$

which is well defined because of the closure property under d of the differential ideal generated by the null class of 1-forms.

Because our manifold M carries two possibly distinct CR structures, the preceding notation shall be modified to distinguish between the abstract tangential $(0,q)$ forms and the imbedded $(0,q)$ forms which arise from the concrete realization of M as a germ of hypersurface $\mathbf{M}$ in $\mathbf{C}^n$. Let $([\varphi]_b^{0,q}, \bar{\partial}_b)$ and $([\varphi]_{\mathbf{M}}^{0,q}, \bar{\partial}_{\mathbf{M}})$ denote these two systems.

Note that any Hermitian metric $\langle\ ,\ \rangle$ on the complexified cotangent bundle of M allows one to define distinguished representatives for $[\varphi]_b^{0,q}$, $[\varphi]_{\mathbf{M}}^{0,q}$, $\bar{\partial}_b[\varphi]_b^{0,q}$, and $\bar{\partial}_{\mathbf{M}}[\varphi]_{\mathbf{M}}^{0,q}$ which are orthogonal to their naturally associated null classes. These shall be denoted by $\langle\varphi\rangle_b^{0,q}$, $\langle\varphi\rangle_{\mathbf{M}}^{0,q}$, $\bar{\partial}_b\langle\varphi\rangle_b^{0,q}$, and $\bar{\partial}_{\mathbf{M}}\langle\varphi\rangle_{\mathbf{M}}^{0,q}$ respectively. The brackets and superscripts will be omitted occasionally. Finally, $\bar{\partial}$ denotes the standard operator acting in $\mathbf{C}^n$.

The following is an elementary property of almost-CR imbeddings.

(5.4.2) Lemma. *Let $(z_1, \cdots, z_n)$ be the coordinate functions in an almost CR imbedding of M into $\mathbf{C}^n$ at the point $p_0 \in M$. For any metric $\langle\ ,\ \rangle$ and every $q \in \{1, \cdots, n-1\}$ there exist forms φ and ψ such that the following decomposition holds on a neighborhood of p_0 in M:*

$$d\bar{z}_1 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_q = \langle \overline{\partial}_b \bar{z}_1 \wedge \overline{\partial}_b \bar{z}_2 \wedge \cdots \wedge \overline{\partial}_b \bar{z}_q \rangle_b^{0,q} + \varphi + \psi. \tag{5.4.3}$$

Here φ is an element of the exterior algebra generated by $dz_1, \cdots, dz_n$ over the ring of germs at p of smooth complex-valued functions, and ψ is a differential q-form that vanishes to infinite order at p.

6. Estimates for I_0 and I_1

The integrals $I_0 = I_0(m, k)$ and $I_1 = I_1(m, k)$ which are defined in (5.3.3) shall be majorized for each k by a sum of powers of $1/r_m$ uniformly in m as $m \to \infty$. Each exponent of $1/r_m$ depends linearly on k. Because k can be chosen arbitrarily large, the important exponents are those containing the largest coefficient of k. Any power of $1/r_m$ whose exponent has k-coefficient less than η, for example, will ultimately prove to be an insignificant, lower order term because it can be absorbed into the right side of (5.2.1).

6.1. Auxiliary estimates for components of the integrands in I_0, I_1. Here we record useful bounds for components of I_0 and I_1 that do not rely on subelliptic estimates for $\Box_b$.

(6.1.1) Notation. Let $\|\ \|_{C^\nu(K)}$ denote the semi-norm for functions (or forms) whose derivatives of all orders not exceeding ν are continuous on K given by

$$\|\varphi\|_{C^\nu(K)} = \sup_K \sum_{|\alpha| \leq \nu} \|D^\alpha \varphi\|.$$

In the following we use the notation of §5.1 and let f_m be as in (5.2.1) except normalized to satisfy $\|f\|_{L^2(\Omega \cap B_{2n}(0, r_m^p))} = 1$.

We set $R_m = \operatorname{supp} \chi_{1,m} \cap \operatorname{supp} F_m^*(\chi_{2,m})$ so that $\operatorname{supp} \langle \alpha_m^{(k)} \rangle_b^{0,q} \subset R_m$.

(6.1.2) Lemma. *For every $\nu \geq 0$ there exists C_ν such that the following estimates hold uniformly in m:*

$$\|\chi_{1,m}\|_{\mathbf{C}^\nu(\mathbf{M})} \leq C_\nu r_m^{-\nu}, \tag{6.1.3}$$

$$\|\chi_{2,m}\|_{\mathbf{C}^\nu(\mathbf{M})} \leq C_\nu r_m^{-\nu p}, \tag{6.1.4}$$

$$\|F_m\|_{\mathbf{C}^\nu(\mathbf{M}\cap \operatorname{supp}\chi_{1,m})} \leq C_\nu r_m^{-\nu}, \tag{6.1.5}$$

$$\|F_m^*(\chi_{2,m})\|_{\mathbf{C}^\nu(\mathbf{M}\cap R_m)} \leq C_\nu^{-\nu(1+p)}, \tag{6.1.6}$$

$$\|f_m\|_{\mathbf{C}^\nu(\overline{\Omega}\cap\overline{R}_m)} \leq C_\nu r_m^{-(\eta\nu)-e}, \tag{6.1.7}$$

$$\|F_m^*(b_{n-q})\|_{\mathbf{C}^\nu(\mathbf{M}\cap \operatorname{supp}\chi_{1,m})} \leq C_\nu r_m^{-\nu-\eta[\nu+2(n-q)-1]}, \tag{6.1.8}$$

$$\|F_m^*(b_{n-q})\|_{\mathbf{C}^\nu(R_m\cap \operatorname{supp} \operatorname{grad} F_m^*(\chi_{2,m}))} \leq C_\nu r_m^{-\nu-p[\nu+2(n-q)-1]}, \tag{6.1.9}$$

$$\|F_m^*(b_{n-q})\|_{\mathbf{C}^\nu(\operatorname{supp}\chi_{1,m}\setminus R_m)} \leq C_\nu r_m^{-\nu-p[\nu+2(n-q)-1]}, \tag{6.1.10}$$

$$\|f\|_{L^2(\mathbf{M}\cap C_m)} \leq C r_m^{-e}. \tag{6.1.11}$$

Comment. Except for (6.1.7) and (6.1.8), these terms are harmless for large ν, in the sense that the coefficient of ν in the exponent of $(1/r_m)$ is strictly less than η. The constant e depends only on the dimension n of $\mathbf{C}^n$, and varies from line to line.

Proof. Estimates (6.1.3)–(6.1.4) follow directly from Definitions (5.1.1) and (5.1.2).

Estimate (6.1.5) follows from Cauchy estimates on the components of F_m, defined by (5.1.3), using the uniform estimate (3.2.5) for the Lipschitz norm of each component.

Estimate (6.1.6) follows from (6.1.4), (6.1.5), and the chain rule for differentiation [cf. the proof of 6.1.8 below].

Estimate (6.1.7) follows from (3.3.6) and Cauchy estimates for f.

Estimate (6.1.8) can be obtained by first noting that for each multi-index α,

$$D^\alpha F_m^*(b_{n-q}) = \sum_{|\beta|\leq|\alpha|} C_{\alpha,\beta} F_m^*(D^\beta b_{n-q}),$$

where $C_{\alpha,\beta}$ is a polynomial in the derivatives of F_m. If one assigns weight l to each lth order derivative of F_m, then the polynomial $C_{\alpha,\beta}$ is homogeneous of weighted degree $|\alpha|$. From (6.1.5) it therefore follows that

$$\sup_{\mathbf{M}\cap\operatorname{supp}\chi_{1,m}} |C_{\alpha,\beta}| \leq C_{|\alpha|} r_m^{-|\alpha|} \quad \text{uniformly in } m.$$

On the other hand, the homogeneity of the Bochner-Martinelli kernel implies that $D^\beta(b_{n-q})$ has coefficients that are homogeneous of (unweighted) degree $-[2(n-q)+|\beta|-1]$. Pulling back each coordinate differential that occurs in

b_{n-q} yields an expression that is majorized by the uniform Lipschitz bound (3.2.5) for F_m. Thus

$$|D^\alpha F_m^*(b_{n-q})| \leq r_m^{-|\alpha|} \sum_{|\beta|\leq|\alpha|} |F_m|^{-[2(n-q)+|\beta|-1]}. \tag{6.1.12}$$

To minorize $|F_m(z)|$ for $z \in \mathbf{M} \cap \operatorname{supp} \chi_{1,m}$ note that

$$|F_m(z)| = |(z_{q+1}, \cdots, z_n) - H_m(z_1, \cdots, z_q)| = \operatorname{dist}(z, w),$$

where $z = (z_1, \cdots, z_n) \in \mathbf{M} \cap \operatorname{supp} \chi_{1,m}$ and $w = (z_1, \cdots, z_q, H_q(z_1, \cdots, z_q))$ $\in V_m$. Thus $|F_m(z)| \approx r_m^\eta$. Substituting this into (6.1.12) yields (6.1.8).

Estimates (6.1.9)–(6.1.10) are proved by combining (6.1.12) with the lower bounds

$$|F_m(z)| \geq \begin{cases} \mathscr{O}(r_m^p) & \text{for } z \in \operatorname{supp} \chi_{1,m} \cap \operatorname{supp} \operatorname{grad} F_m^*(\chi_{2,m}), \\ \mathscr{O}(r_m^p) & \text{for } z \in \operatorname{supp} \chi_{1,m} \backslash R_m, \end{cases}$$

which follow trivially from (5.1.2).

Estimate (6.1.11) follows from (3.3.7) and the normalization condition on f adopted at the beginning of this section.

6.2. Majorization of I_0.

(6.2.1) Proposition. *The integral $I_0 = I_{0,m,k}$ defined by* (5.3.3) *satisfies*

$$|I_{0,m,k}| \leq C_k r_m^{-[(1+p)k-e]} \quad \textit{uniformly in } m,$$

where the exponent e is independent of k and m.

Proof. Recall that

$$I_0 = (-1)^{n+1} \int_{\Omega \cap R_m} \overline{\partial} \alpha_m^{(k)} \wedge \beta_m \wedge (dz)^n,$$

where $\alpha_m^{(k)}$ is given by (5.3.8). From type considerations, it is clear that $\overline{\partial}\alpha_m^{(k)}$ can be replaced by $d\alpha_m^{(k)}$. Commute $\mathscr{L}_T$ past d by means of (5.3.7)(ii); integrate by parts using (5.3.7) and (5.3.5)(i)–(ii); and then use type considerations once again to replace $d\alpha_m^{(0)}$ by $\overline{\partial}\alpha_m^{(0)} = f\chi_{1,m} F_m^*(\overline{\partial}\chi_{2,m}) \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q$. The result is

$$\begin{aligned} |I_0| &= \left| \int_{\Omega \cap R_m} \overline{\partial}\alpha_m^{(0)} \wedge \mathscr{L}_T^{(k)} \beta_m \wedge (dz)^n) \right| \\ &\leq \|\overline{\partial}\alpha_m^{(0)}\|_{L^2(\Omega\cap R_m)} \|\beta_m\|_{C^k(\operatorname{supp} \overline{\partial}\alpha_m^{(0)})}. \end{aligned} \tag{6.2.2}$$

Clearly,

$$\begin{aligned} \|\overline{\partial}\alpha_m^{(0)}\|_{L^2(\Omega\cap R_m)} &\leq \|\chi_{1,m} f_m \overline{\partial}[F_m^*(\chi_{2,m})]\|_{L^2(\Omega\cap R_m)} \\ &\leq \|f_m\|_{L^2(\Omega\cap R_m \cap \operatorname{supp}\operatorname{grad} F_m^*(\chi_{2,m}))} \\ &\quad \cdot \|F_m^*(\chi_{2,m})\|_{C^1} \cdot \|\chi_{1,m}\|_{C^0}. \end{aligned} \tag{6.2.3}$$

The three factors on the right of (6.2.3) can be dominated using (6.1.7) with $\nu = 0$, (6.1.4) with $\nu = 1$, and (6.1.3) with $\nu = 0$. This gives

$$\|\overline{\partial}\alpha_m^{(0)}\|_{L^2(\Omega\cap R_m)} \leq C r_m^{-e}, \tag{6.2.4}$$

where e is independent of k and m.

The factor in (6.2.2) involving $\beta_m = F_m^*(b_{n-q})$ can be dominated using (6.1.9) with $\nu = k$, which is applicable because $\operatorname{supp}\overline{\partial}\alpha_m^{(0)} \subset \operatorname{supp}\chi_{1,m} \cap \operatorname{supp}\operatorname{grad} F_m^*(\chi_{2,m})$. The result is

$$\|\beta_m\|_{C^k(\operatorname{supp}\overline{\partial}\alpha_m^{(0)})} \leq C_k r_m^{-k-p[k+2(n-q)-1]} = C_k r_m^{-[(1+p)k+e']}. \tag{6.2.5}$$

Substituting (6.2.4) and (6.2.5) into (6.2.2) gives (6.2.1).

6.3. Replacement of α_m by a tangential $(0,q)$ form. Recall that $\overline{\partial}_b\overline{z}_1, \overline{\partial}_b\overline{z}_2, \cdots, \overline{\partial}_b\overline{z}_{n-1}$ generate the germs at p_0 of sections of the vector bundle of tangential $(0,1)$ forms associated to the abstract CR structure. Since the coordinate functions $z_1, \cdots, z_{n-1}$ are almost-CR at p_0, it is also true that $(\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})z_i$ vanishes to infinite order at p_0.

(6.3.1) Proposition. *For every positive k and N there exists $C_{N,k}$ such that the difference between I_1, defined by* (5.3.3), *and the integral obtained by substituting $\langle\alpha_m^{(k)}\rangle_b^{0,q}$ for $\langle\alpha_m^{(k)}\rangle_{\mathbf{M}}^{0,q}$, is no larger than $C_{N,k}r_m^N$ uniformly in m.*

Proof. Clearly the error integral is dominated by the norm in $C^0(R_m \cap \mathbf{M})$ of

$$\begin{aligned}\langle\alpha_m^{(k)}\rangle_b^{0,q} &- \langle\alpha_m^{(k)}\rangle_{\mathbf{M}}^{0,q} \\ &= [\overline{\partial}_b\overline{z}_1 \wedge \overline{\partial}_b\overline{z}_2 \wedge \cdots \wedge \overline{\partial}_b\overline{z}_q - \overline{\partial}_{\mathbf{M}}\overline{z}_1 \wedge \cdots \wedge \overline{\partial}_{\mathbf{M}}\overline{z}_q] f_m^{(k)}\chi_{1,m}.\end{aligned} \tag{6.3.2}$$

The norm in $C^0(R_m \cap \mathbf{M})$ of the bracketed factor on the right of (6.3.2) is majorized by $\sup_{z\in R_m\cap\mathbf{M}} C_N(\|z\|^N)$ for every choice of $N \geq 0$. However, from the construction of R_m it is clear that $\sup_{z\in R_m}\|z\| \leq \mathscr{O}(r_m^p)$. Thus the bracketed factor is $\mathscr{O}(r_m^{Np}) = \mathscr{O}(r_m^{N'})$ for every $N' \geq 0$.

The right-hand factors in (6.3.2) can be majorized using (6.1.3) by some negative power of r_m which is fixed once k is chosen. But since N' is arbitrary, the product in (6.3.2) is majorized in the norm of $C^0(R_m \cap \mathbf{M})$ by $\mathscr{O}(r_m^N)$ for every $N \geq 0$.

7. Introduction of $\Box_b$

7.1. Notation. Given a Hermitian metric $\langle\ ,\ \rangle$ on the complexified cotangent bundle of M, we set

$$\Box_b^{(q)} = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b,$$

where $\overline{\partial}_b^*$ is the adjoint of the operator $\overline{\partial}_b$ relative to the L^2 norm $\int_M \langle\ ,\ \rangle\, d\sigma_M$ obtained by extending the Hermitian metric to tangential $(0,j)$ forms on M for $j \in \{q-1, q, q+1\}$. Note that $\Box_b^{(q)}$ is defined for every $0 < q < n-1$ as an operator which maps $(0,q)$ forms to $(0,q)$ forms.

7.2. Hypotheses on $\Box_b$. We shall assume the following:

(7.2.1) $\mathscr{H}$, the orthogonal projection onto $\ker \Box_b$, has a finite-dimensional range consisting of C^∞ tangential $(0,q)$ forms on M;

(7.2.2) there exists $C < \infty$ such that for every tangential $(0,q)$ form α which is smooth on M, there exists $u \in L^2(M)$ satisfying $\alpha - \mathscr{H}\alpha = \Box_b u$ with $\|u\|_{L^2(M)} \le C\|\alpha\|_{L^2(M)}$, so that $\alpha = \mathscr{H}\alpha + \overline{\partial}_b^*\overline{\partial}_b u + \overline{\partial}_b\overline{\partial}_b^* u$ is an orthogonal decomposition of α in $L^2(M)$; and

(7.2.3) $\Box_b$ satisfies at p_0 a subelliptic Sobolev-space estimate of order $\varepsilon > 0$ in the sense that for some neighborhood U of p_0, $\|\varphi\|_\varepsilon \le C(\|\Box_b\varphi\|^2_{L^2(M)} + \|\varphi\|_{L^2(M)})$ for all smooth test forms of tangential type $(0,q)$ which are compactly supported within U.

Here $\|\varphi\|_\varepsilon$ denotes the L^2-Sobolev norm of order ε applied to the coefficients of the form φ, defined a local chart for M at p_0.

7.3. Introduction of $\Box_b$ into the boundary integrals. The boundary integral I_1 defined by (5.3.4) can be linked to $\Box_b$.

(7.3.1) Lemma. *There exists an e independent of k and m such that for every k*

$$|I_1| = \left| \int_{\mathbf{M}} \langle \alpha_m^{(0)} \rangle_b^{0,q} \wedge (\mathscr{L}_T^k \beta_m) \wedge dz)^n \right| + \mathscr{O}(r_m^{-[k(1+p)+e]}) \tag{7.3.2}$$

uniformly in m.

Proof. From Proposition (6.3.1),

$$|I_1| = \left| \int \langle \alpha_m^{(k)} \rangle_b^{0,q} \wedge \beta_m \wedge (dz)^n \right| + \mathscr{O}(r_m^N) \quad \forall N \ge 0, \tag{7.3.3}$$

where

$$\begin{aligned} \langle \alpha_m^{(k)} \rangle_b^{0,q} &= [\mathscr{L}_T^k f_m]\chi_{1,m} F_m^*(\chi_{2,m})\overline{\partial}_b \overline{z}_1 \wedge \cdots \wedge \overline{\partial}_b \overline{z}_q \\ &= \mathscr{L}_T^k [f_m \chi_{1,m} F_m^*(\chi_{2,m})\overline{\partial}_b \overline{z}_1 \wedge \cdots \wedge \overline{\partial}_b \overline{z}_q] \\ &\quad - f_m \chi_{1,m} \mathscr{L}_T^k [F_m^*(\chi_{2,m})\overline{\partial}_b \overline{z}_1 \wedge \cdots \wedge \overline{\partial}_b \overline{z}_q], \\ &\qquad\qquad (\text{since } \mathscr{L}_T \chi_{1,m}(z_1, \cdots, z_2) = 0) \\ &= \mathscr{L}_T^k \langle \alpha_m^{(0)} \rangle_b^{0,q} - f_m \chi_{1,m} \mathscr{L}_T^k [F_m^*(\chi_{2,m})\overline{\partial}_b \overline{z}_1 \wedge \cdots \wedge \overline{\partial}_b \overline{z}_q]. \end{aligned} \tag{7.3.4}$$

Inserting (7.3.4) into (7.3.3) and integrating parts to simplify the penultimate term on the right side of (7.3.4) yields

$$|I_1| = \left| \int_{\mathbf{M}} \langle \alpha_m^{(0)} \rangle_b^{0,q} \wedge \mathscr{L}_T^k \beta_m \wedge (dz)^n \right| + \mathscr{O}(r_m^N) + \left| \int_{\mathbf{M}} f_m \chi_{1,m} [\mathscr{L}_T^k F_m^*(\chi_{2,m}) \overline{\partial}_b \overline{z}_1 \wedge \cdots \wedge \overline{\partial}_b \overline{z}_q] \wedge \beta_m \wedge (dz)^n \right|.$$

The last expression is majorized by

$$\|f_m \chi_{1,m}\|_{L^2(M)} \|F_m^*(\chi_{2,m})\|_{\mathbf{C}^k(\mathbf{M}\cap R_m)} \|\beta_m\|_{\mathbf{C}^0(\mathbf{M}\cap \operatorname{supp} \chi_{1,m})} \leq \mathscr{O}(r_m^{-e_1}) \cdot \mathscr{O}(r_m^{-(1+p)k}) \cdot \mathscr{O}(r_m^{-e_2})$$

by (6.1.7), (6.1.6), and (6.1.8); where e_1 and e_2 are independent of k and m. This completes the proof of (7.3.1).

Note that (7.2.3) implies that $(I-\mathscr{H})\langle \alpha_m^{(0)} \rangle_b^{0,q} = \Box_b u_m$ with $\|u_m\|_{L^2(M)} \leq C\|\alpha_m^{(0)}\|_{L^2(\mathbf{M})}$. Substitute the decomposition

$$\begin{aligned} \langle \alpha_m^{(0)} \rangle_b^{0,q} &= \mathscr{H}\langle \alpha_m^{(0)} \rangle_b^{0,q} + \Box_b u_m = \mathscr{H}\langle \alpha_m^{(0)} \rangle_b^{0,q} + \overline{\partial}_b \overline{\partial}_b^* u_m + \overline{\partial}_b^* \overline{\partial}_b u_m \\ &= \mathscr{H}\langle \alpha_m^{(0)} \rangle_b^{0,q} + (\overline{\partial}_b - \overline{\partial}_{\mathbf{M}}) \overline{\partial}_b^* u_m + \overline{\partial}_{\mathbf{M}} \overline{\partial}_b u_m + \overline{\partial}_b^* \overline{\partial}_b u_m \end{aligned}$$

into the right side of (7.3.2); multiply the integrand by cut-off functions $\chi_{1,m}'$ and $F_m^*(\chi_{2,m}')$ which are identically 1 on $\operatorname{supp} \chi_{1,m}$ and $\operatorname{supp} F_m^*(\chi_{2,m})$ respectively; and integrate by parts to obtain

$$I_1 = \int_{\mathbf{M}} J_1 + J_2 + J_3 + J_4 + \mathscr{O}(r_m^{-[(1+p)k+e]}), \tag{7.3.5}$$

where

$$J_1 = J_{1,m,k} = (\mathscr{L}_T^k [\chi_{1,m}' F_m^*(\chi_{2,m}') \mathscr{H}\langle \alpha_m^{(0)} \rangle_b^{0,q}]) \wedge \beta_m \wedge (dz)^n, \tag{7.3.6}$$

$$\begin{aligned} J_2 &= J_{2,m,k} \\ &= (-1)^k (\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})(\overline{\partial}_b^* u_m) \wedge \mathscr{L}_T^k [\chi_{1,m}' F_m^*(\chi_{2,m}') \beta_m] \wedge (dz)^n, \end{aligned} \tag{7.3.7}$$

$$J_3 = J_{3,m,k} = \mathscr{L}_T^k [\overline{\partial}_{\mathbf{M}} \overline{\partial}_b^* u_m] \wedge \beta \wedge (dz)^n, \tag{7.3.8}$$

$$J_4 = J_{4,m,k} = \mathscr{L}_T^k [\chi_{1,m}' F_m^*(\chi_{2,m}') \overline{\partial}_b^* \overline{\partial}_b u_m] \wedge \beta_m \wedge (dz)^n. \tag{7.3.9}$$

Note that $\chi_{1,m}'$ and $\chi_{2,m}'$ can be defined so that $\operatorname{supp} \chi_{1,m}' F_m^*(\chi_{2,m}') \Subset R_m$ and so that (6.1.3)–(6.1.9) are valid for the primed functions too.

7.4. Estimation of $\int_{\mathbf{M}} J_1$.

(7.4.1) Proposition. *There exists an exponent e independent of k and m such that for every k*

$$\left| \int_{\mathbf{M}} J_1 \right| \leq C_k r_m^{-(k+e)}.$$

Proof. Clearly (7.2.1) implies that $\mathscr{H}$ is a compact mapping from $L^2(M)$ into $C^k(M)$. Thus

$$\begin{aligned}\|\mathscr{H}\langle\alpha_m^{(0)}\rangle_b^{0,q}\|_{C^k(M)} &\le C_k\|\alpha_m^{(0)}\|_{L^2(M)} \le C_k\|f_m\|_{L^2(\mathbf{M}\cap R_m)}\\ &\le \mathscr{O}(r_m^e) \quad \text{with } e \text{ independent of } k \text{ by (6.1.7).}\end{aligned}$$

Also, $\|\beta_m\|_{C^0(M\cap \operatorname{supp}\chi'_{1,m})} \le \mathscr{O}(r_m^{-e'})$ with e' independent of k by (6.1.8). Thus the only dependence of k arises from differentiation of $\chi'_{1,m}$, which is estimated by (6.1.3) with $\nu = k$.

7.5. Estimation of $\int_{\mathbf{M}} J_2$.

(7.5.1) Proposition. *For every k and N there exists $C_{N,k}$ such that*

$$\left|\int_{\mathbf{M}} J_2\right| \le C_{N,k} r_m^N.$$

Proof. Since the coefficients of $\overline{\partial}_b - \overline{\partial}_{\mathbf{M}}$ vanish to infinite order at $p_0 \in M$, the coefficients of the operator $(\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})^\dagger$ also vanish to infinite order at $p_0 \in M$. Here $(\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})^\dagger$ is the adjoint operator which satisfies

$$\begin{aligned}\left|\int_{\mathbf{M}} J_2\right| &= \left|\int_{\mathbf{M}} [(\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})\overline{\partial}_b^* u_m] \wedge \mathscr{L}_T^k \chi'_{1,m} F_m^*(\chi'_{2,m} b_{n-q})\right|\\ &= \left|\int_{\mathbf{M}} (\overline{\partial}_b^* u_m) \wedge (\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})^\dagger \mathscr{L}_T^k \chi'_{1,m} F_m^*(\chi'_{2,m} b_{n-q})\right|\\ &\le \|\overline{\partial}_b^* u_m\| \|(\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})^\dagger \mathscr{L}_T^k \chi'_{1,m} F_m^*(\chi'_{2,m} b_{n-q})\|_{C^0(M)}\\ &\le \|\alpha_m^{(0)}\|_{L^2(\mathbf{M})} \cdot \|\text{ coefficients of } (\overline{\partial}_b - \overline{\partial}_{\mathbf{M}})^\dagger\|_{C^0(\mathbf{M}\cap R_m)}\\ &\quad \cdot \|\chi'_{1,m} F_m^*(\chi'_{2,m} b_{n-q})\|_{C^{k+1}(\mathbf{M})}.\end{aligned} \tag{7.5.2}$$

The first factor on the right side of (7.5.2) is $\mathscr{O}(r_m^{-e_1})$ where e_1 is independent of k. The middle factor is $\mathscr{O}(d^N)$ for every $N \ge 0$, where $d = \sup_{z\in\mathbf{M}\cap R_m} \operatorname{dist}(z, p_0) \le s_m = \mathscr{O}(r_m^p)$. Since $p > 0$, the middle factor is $\mathscr{O}(r_m^{N'})$ for every $N' \ge 0$. Finally, the third factor is $\mathscr{O}(r_m^{-e(k)})$, where $e(k)$ depends polynomially on k. Since N' can be chosen arbitrarily larger than $e_1 + e(k)$, the proposition is established.

7.6. Consequences of a subelliptic estimate. It follows from the local subelliptic estimate (7.2.3), the elliptic regularization results of Kohn and Nirenberg [7], and (7.2.1) that the solution u of (7.2.2) is smooth on a neighborhood U if α is smooth on U and if U is contained within some neighborhood on which (7.2.3) is valid. To obtain quantitative bounds on the derivatives of u in terms of derivatives of $\Box_b u = (I - \mathscr{H})\alpha$, one can iterate the localized estimate (7.2.3). In particular, one can perform such iterations on a subset U where $\alpha \equiv 0$, in which case one expects strong estimates on the derivatives of u because one has strong estimates on derivatives of the harmonic term $-\mathscr{H}\alpha$. However, complications arise when this technique is applied to a sequence $\{\alpha_m\}_{m=1}^{\infty}$ in which $\operatorname{supp}\alpha_m$ depends significantly on m. If the localization of the subelliptic estimate to a region outside $\operatorname{supp}\alpha_m$ requires cut-off functions whose supports shrink as $m \to \infty$, then the derivatives of these cut-off functions will degrade the estimates obtained by iteration. The details have been carried out in [1] for the $\overline{\partial}$-Neumann problem on open subdomains of $\mathbf{C}^n$, and have been applied to $\Box_b$ on boundaries of subdomains in $\mathbf{C}^n$ in [3]. The a priori estimates in [3] require no changes when applied to abstract CR structures. The key results are summarized below in Proposition (7.6.2).

(7.6.1) Notation. In terms of fixed local coordinates $x_1, \cdots, x_{2n-1}$ on $M \cong R^{2n-1}$ let Λ^s be the pseudodifferential operator whose symbol is $(1+|\xi|^2)^{s/2}$. For fixed N, let $\varsigma_1(x) > \varsigma_2(x) > \cdots > \varsigma_N(x)$ be cut-off functions concentrically supported within a ball of radius 1 centered at the origin in R^{2n-1} such that $\varsigma_j \equiv 1$ on $\operatorname{supp}\varsigma_{j+1}$. For $t \in (0,1]$ and for $y \in K \Subset U \subset M$, define

$$\varsigma_{j,t,y}(\cdot) = \varsigma_j\left(\frac{\cdot - y}{t}\right).$$

Because the operator Λ^s fails to transform homogeneously under dilations of R^{2n-1}, we introduce the auxiliary operator Λ_t^s whose symbol is $\psi(t|\xi|)|\xi|^s$. Here ψ is a cut-off function satisfying $\psi(x) = 1$ if $|x| \geq 1$ and $\psi(x) = 0$ if $|x| \leq 1/2$. Set $A_{j,t,y} = \varsigma_{j,t,y}\, \Lambda_t^{j\varepsilon}\, \varsigma_{j+1,t,y}$, where ε is given by (7.2.3). For generic $x \in R^{2n-1}$, consider the dilation by a factor t which fixes x. If $[\varphi]_x^t(y) = \varphi(x + t(y-x))$, then the homogeneity property of $A_{j,t,x}$ can be expressed as

$$(A_{j,t,x}\varphi)_x^t = t^{-j\varepsilon}A_{j,1,x}(\varphi_x^t).$$

Also

$$\|A_{j,t,x}\varphi\|_{L^2(M)}^2 \approx t^{2n-1-2j\varepsilon}\|A_{j,1,x}(\varphi_x^t)\|_{L^2(M)}^2,$$

where $2n-1 = \dim M$.

(7.6.2) Proposition. *For every $N \geq 1$ there exists C_N such that for all $t \in (0,1]$,*

$$\left\|\Lambda^{(N+1)\varepsilon}\varsigma_{N+1,t,x}v\right\|^2 \leq C_N\sum_{j=1}^{N}\|A_{j,t,x}\Box_b v\|^2 t^{-2(N-j)} + C_N t^{-2(N-1)}\|\varsigma_{1,t,x}\Box_b v\|^2 + C_N t^{-2(N+1)}\|v\|^2,$$

where $\|\ \| = \|\ \|_{L^2(M)}$.

7.7. Estimation of $\int_{\mathbf{M}} J_3$. In this section we prove that

$$\left|\int_{\mathbf{M}} J_3\right| < \mathscr{O}(r_m^{-(k/\varepsilon)+e}) + \mathscr{O}(r_m^{-(1+p)k+e'}).$$

Note that

$$\left|\int_{\mathbf{M}} J_3\right| = \int_{\mathbf{M}} (\mathscr{L}_T^k(\overline{\partial}_{\mathbf{M}}\overline{\partial}_b^* u_m) \wedge \chi'_{1,m} F_m^*(\chi'_{2,m} b_{n-q}) \wedge (dz)^n|$$
$$= \left|\int_{\mathbf{M}} (\mathscr{L}_T^k \overline{\partial}_b^* u_m) \wedge \overline{\partial}_{\mathbf{M}}(\chi'_{1,m} F_m^*(\chi'_{2,m} b_{n-q})) \wedge (dz)^n\right|.$$

Using the anti-derivation property of $\overline{\partial}_{\mathbf{M}}$ to expand the term in parentheses, and then integrating by parts in one of the resulting expressions yields

(7.7.1)

$$\left|\int_{\mathbf{M}} J_3\right| \leq \left|\int_{\mathbf{M}} (\mathscr{L}_T^k \overline{\partial}_b^* u_m) \wedge \overline{\partial}_{\mathbf{M}} \chi'_{1,m} \wedge F_m^*(\chi'_{2,m} b_{n-q}) \wedge (dz)^n\right|$$
$$+ \left|\int_{\mathbf{M}} \overline{\partial}_b^* u_m \wedge \mathscr{L}_T^k [\chi'_{1,m} F_m^*(\overline{\partial}_{\mathbf{M}} \chi'_{2,m} \wedge b_{n-q})] \wedge (dz)^n\right|$$
$$\leq \|u_m\|_{C^{k+1}(\mathbf{M}\cap R_m \cap \operatorname{supp}\operatorname{grad}\chi'_{1,m})} \|\overline{\partial}_{\mathbf{M}} \chi_{1,m} \wedge F_m^*(\chi'_{2,m} b_{n-q})\|_{C^0(M)}$$
$$+ \|\overline{\partial}_b^* u_m\|_{L^2(\mathbf{M})} \|\chi'_{1,m} F_m^*(\overline{\partial} \chi_{2,m} \wedge b_{n-q})\|_{C^k(\mathbf{M})}.$$

We shall majorize every term on the right side of (7.7.1).

(7.7.2) Proposition. *There exists e, independent of k and m, such that for every k the solution u_m of $\Box_b u_m = (I - \mathscr{H})\langle \alpha_m^{(0)} \rangle_b^{0,q}$ given by* (7.2.2) *satisfies*

$$\|u_m\|_{C^{k+1}(\mathbf{M}\cap R_m \cap \operatorname{supp}\operatorname{grad}\chi'_{1,m})} \leq \mathscr{O}(r_m^{-(k/\varepsilon+e)})$$

uniformly in m. Here ε is given by (7.2.3).

Proof.

$$\|u_m\|_{C^{k+1}(\mathbf{M}\cap R_m \cap \operatorname{supp}\operatorname{grad}\chi'_{1,m})} = \sup_{x\in \mathbf{M}\cap R_m \cap \operatorname{supp}\operatorname{grad}\chi_{1,m}} \ \sup_{|\alpha|\leq k+1} |D^\alpha u_m(x)|.$$

For every x, the Sobolev imbedding theorem implies that if ς is any smooth, compactly supported cut-off function which is identically 1 on some neighborhood of x, then

$$\sup_{|\alpha|\leq k+1} |D^\alpha u_m(x)| = \sup_{|\alpha|\leq k+1} D^\alpha[\varsigma u_m(x)]| \leq C_{k,n}\|\varsigma u_m\|_{\wedge^{k+1+n}}.$$

Here $C_{k,n}$ is independent of ς, and $\wedge^{k+1+n}$ is the Sobolev space of order $k+1+n$ on R^{2n-1}. Thus for each x,

$$\sup_{|\alpha|\leq k+1} |D^\alpha u_m(x)| \leq \inf_{\varsigma=1 \text{ at } x} C_{k,n}\|\varsigma u_m\|_{\wedge^{k+1+n}}. \tag{7.7.3}$$

We shall majorize this last expression for each fixed x by invoking (7.6.2). The arbitrary parameters (N, t) which occur in (7.6.2) can be advantageously specified as follows. Let N be the smallest integer satisfying $(N+1)\varepsilon \geq k+1+n$, and let $t = \operatorname{dist}(x, \operatorname{supp}\chi_{1,m})$. Note that by the construction of $\chi_{1,m}$ and $\chi'_{1,m}$

$$t = t_m \geq \operatorname{dist}(\operatorname{supp}\operatorname{grad}\chi'_{1,m}, \operatorname{supp}\chi_{1,m}) > r_m$$

uniformly in $x \in \operatorname{supp}\operatorname{grad}\chi'_{1,m}$. Because the cut-off functions $\varsigma_{1,t,x} \geq \varsigma_{2,t,x} \geq \cdots \geq \varsigma_{N+1,t,x}$ are supported on balls of radius $\leq t$ centered at x, each $\varsigma_{j,t,x}$ vanishes on $\operatorname{supp}\chi_{1,m} \subset \operatorname{supp}\langle\alpha_m^{(0)}\rangle_b^{0,q}$. Therefore,

$$\varsigma_{j,t,x}\Box_b u_m = \varsigma_{j,t,x}(I - \mathscr{H})\langle\alpha_m^{(0)}\rangle_b^{0,q} = -\varsigma_{j,t,x}\mathscr{H}\langle\alpha_m^{(0)}\rangle_b^{0,q}.$$

Also,

$$A_{j,t}\Box_b u_m = -A_{j,t}\mathscr{H}\langle\alpha_m^{(0)}\rangle_b^{0,q}.$$

The homogeneity property of $A_{j,t}$ implies that

$$\begin{aligned}\|A_{j,t}\Box_b u_m\|^2_{L^2(M)} &= \|A_{j,t}\mathscr{H}\langle\alpha_m^{(0)}\rangle_b^{0,q}\|^2_{L^2(M)} \\ &\approx t^{2n-1}t^{-2j\varepsilon}\|A_{j,1}[\mathscr{H}\langle\alpha_m^{(0)}\rangle_b^{0,q}]^t\|^2_{L^2(M)}.\end{aligned}$$

The continuity of the operator $\mathscr{H}$ from $L^2(M)$ into $C^\infty(M)$ implied by (7.2.1), the uniform boundedness in $C^k(M)$ of the dilation operators $[\ ]^t$ for all $0 < t \leq 1$, the continuity of $A_{j,1}$ from $C^\infty(\mathbf{M})$ into $L^2(\mathbf{M})$, and the fact that $\varepsilon \leq 1$ imply that the right-hand side is majorized by $t^{2n-2j-1}\|\langle\alpha_m^{(0)}\rangle_b^{0,q}\|^2_{L^2(M)}$ uniformly in m. Substitution of this result into (7.6.2) yields

$$\begin{aligned}\left\|\bigwedge^{(N+1)\varepsilon}\varsigma_{N+1,t,x}u_m\right\|^2 < t^{-2N+2n-1}&\|\langle\alpha_m^{(0)}\rangle_b^{0,q}\|^2_{L^2(M)} \\ &+ t^{-2(N-1)}\|\varsigma_{1,t,x}\mathscr{H}\langle\alpha_m^0\rangle_b^{0,q}\|^2_{L^2(M)} \\ &+ t^{-2(N-1)}\|u_m\|^2_{L^2(M)}.\end{aligned}$$

It is clear from (7.2.1) and (7.2.2) that $\|\mathscr{H}\langle\alpha_m^0\rangle_b^{0,q}\|_{L^2(M)}$ and $\|u_m\|_{L^2(M)}$ can be dominated by $\|\langle\alpha_m^{(0)}\rangle\|_{L^2(M)}$. Thus

$$\left\|\bigwedge^{(N+1)\varepsilon}\varsigma_{N+1,t,x}u_m\right\|^2 \leq t^{-2(n+1)}\|\langle\alpha_m^{(0)}\rangle_b^{0,q}\|^2_{L^2(M)}.$$

From the choice of N and t, it follows that

$$\inf_{\varsigma=1 \text{ at } x}\|\varsigma u_m\|^2_{\wedge^{k+1+n}} \leq r_m^{-2[2+(k+1+n)/\varepsilon]}\|\langle\alpha_m^0\rangle_b^{0,q}\|^2_{L^2(M)}. \tag{7.7.4}$$

The rightmost factor is majorized by a power of r_m which is independent of k. Substituting this into (7.7.4), and substituting (7.7.4) into (7.7.3) yields (7.7.2).

(7.7.5) Proposition. $\|\bar\partial_b^* u_m\|^2_{L^2(M)} < \mathscr{O}(r_m^{-e})$ *where e is independent of k and m.*

Proof.

$$\begin{aligned}\|\overline{\partial}_b^* u_m\|^2_{L^2(\mathbf{M})} &\leq \|\overline{\partial}_b^* u_m\|^2_{L^2(\mathbf{M})} + \|\overline{\partial}_b u_m\|^2_{L^2(\mathbf{M})} = (\Box_b u_m, u_m)_{L^2(\mathbf{M})} \\ &\leq \|\Box_b u_m\|_{L^2(\mathbf{M})}\|u_m\|_{L^2(\mathbf{M})} \leq \|\langle \alpha_m^{(0)}\rangle_b^{0,q}\|^2 \quad \text{by (7.2.2)}, \\ &\leq \mathscr{O}(r_m^{-e}) \quad \text{by (6.1.7)}.\end{aligned}$$

All quantities in (7.7.1) which remain to be estimated are independent of u_m. The quantities $\|\chi_{1,m}\|_{C^1(\mathbf{M})}$ and $\|F_m^*(b_{n-q})\|_{C^0(\mathbf{M})}$ are bounded by powers of $1/r_m$ which do not depend on k, because of (6.1.3) and (6.1.8). The quantity

$$\|\chi'_{1,m}F_m^*(\overline{\partial}\chi_{1,m} \wedge b_{n-q})\|_{C^k(\mathbf{M})}$$

is majorized using (6.1.3), (6.1.6), and (6.1.9). The resulting bound is $\mathscr{O}(r_m^{-(1+p)k+e})$ where e is independent of k and m. Substituting these bounds and the results of (7.7.5) and (7.7.2) into (7.7.1) yields the following.

(7.7.6) Proposition. *There exist e and e' independent of k such that for every k we have*

$$\left|\int_{\mathbf{M}} J_3\right| \leq \mathscr{O}(r_m^{-[(k/\varepsilon)+e]}) + \mathscr{O}(r_m^{-[(1+p)k+e']})$$

uniformly in m.

7.8. Estimation of $\int_{\mathbf{M}} J_4$. Recall that

$$\left|\int_{\mathbf{M}} J_4\right| = \left|\int_{\mathbf{M}} \mathscr{L}_T^k[\chi'_{1,m}F_m^*(\chi_{2,m})'\overline{\partial}_b^*\overline{\partial}_b u_m] \wedge \beta_m \wedge (dz)^n\right|,$$

where $\beta_m = F_m^*(b_{n-q})$. It is advantageous to integrate by parts to shift derivatives of order $\leq k$ onto β_m, at least when working on the region where the strong bound (6.1.10) for β_m is valid. Outside that region, we shall be forced to estimate derivatives of $\overline{\partial}_b^*\overline{\partial}_b u_m$. The principal problem in applying (7.6.2) will be the estimation of derivatives of $\Box_b v_m$.

(7.8.1) Definition. Let $\chi''_{2,m}(z_{q+1},\cdots,z_n) = \chi((z_{q+1}\cdots z_n)/c'r_m^p)$ be such that $\chi''_{2,m}$ is supported within the ball in $\mathbf{C}^{n-q}$ on which $\chi_{2,m} \equiv 1$. Here p is the small exponent constrained by (3.3.9), and $\chi_{2,m}$ is defined in (5.1.2). The constant c' can be chosen so that

$$\text{(7.8.2)} \qquad \operatorname{dist}(\operatorname{supp}\chi''_{2,m}, \operatorname{supp}\operatorname{grad}\chi_{2,m}) \approx r_m^p \quad \text{uniformly in } m.$$

Because F_m is smooth on a neighborhood of $\operatorname{supp} J_4$, clearly J_4 smoothly decomposes as

$$J_4 = F_m^*(\chi''_{2,m})J_4 + F_m^*(1-\chi''_{2,m})J_4.$$

(7.8.3) Proposition. $|\int_{\mathbf{M}} F_m^*(1-\chi''_{2,m})J_4| \leq \mathscr{O}(r_m^{-k(1+p)+e})$ *uniformly in m, where e is independent of k and m.*

Proof. Expanding J_4 by means of (7.3.9) and integrating by parts yields

(7.8.4)
$$\begin{aligned}&\left|\int_{\mathbf{M}} F_m^*(1-\chi_{2,m}'')J_4)\right| \\ &\leq \left|\int_{\mathbf{M}} \chi_{1,m}' F_m^*(\chi_{2,m})\overline{\partial}_b^*\overline{\partial}_b u_m \wedge \mathscr{L}_T^k[F_m^*(1-\chi_{2,m}'')\wedge\beta_m\wedge(dz)^n]\right| \\ &\leq \|\overline{\partial}_b^*\overline{\partial}_b u_m\|_{L^2(\mathbf{M})}\cdot\|\chi_{1,m}' F_m^*(1-\chi_{2,m}'')\beta_m\|_{C^k(\mathbf{M})}.\end{aligned}$$

The first factor on the right side of (7.8.4) is bounded by a power of $1/r_m$ that is independent of k. In fact, (7.2.2) implies

$$\|\overline{\partial}_b^*\overline{\partial}_b u_m\|^2_{L^2(\tilde{M})} \leq \|\langle\alpha_m^{(0)}\rangle_b^{0,q}\|^2_{L^2(\mathbf{M})} < \|f_m\|^2_{L^2(M)},$$

which by (6.1.7) is bounded by a polynomial in $1/r_m$.

The rightmost factor in (7.8.4) can be dominated, using (6.1.10), by $r_m^{-(1+p)k+e}$ in which e is independent of k.

(7.8.5) Definition. Let $K_m = \operatorname{supp}[\chi_{1,m}F_m^*(\chi_{2,m}'')]$.

(7.8.6) Lemma. *Let p be the small exponent constrained by* (3.3.9). *Let z_0 be the point on* $\mathbf{M}$ *which is the limit point of the admissible sequence* $\{V_m\}_{m=1}^{\infty}$ *of* (3.3.8). *Then the set K_m satisfies uniformly in m*

$$\operatorname{dist}(K_m, \operatorname{supp}\chi_{1,m}\operatorname{grad}F_m^*(\chi_{2,m})) \geq r_m^p, \tag{7.8.7}$$

$$\sup_{z\in K_m}\operatorname{dist}(z,z_0) \leq r_m^p. \tag{7.8.8}$$

Proof. To prove (7.8.7), let $z\in K_m$ and $w\in\operatorname{supp}\chi_{1,m}\operatorname{grad}F_m^*(\chi_{2,m})$ be arbitrary points. Then

$$\begin{aligned}|z-w| &\geq |(z_{q+1},\cdots,z_n)-(w_{q+1},\cdots,w_n)| \\ &= |[F_m(z)-H_m(z)]-[F_m(w)-H_m(w)]| \\ &\geq |F_m(z)-F_m(w)|-|H_m(z)-H_m(w)| \\ &= \operatorname{dist}(z_*,w_*)-|H_m(z_1\cdots z_q)-H_m(w_1\cdots w_q)|,\end{aligned}$$

where $z_* = F_m(z)\in\operatorname{supp}\chi_{2,m}''$ by definition of K_m, and $w_* = F_m(w)\in \operatorname{supp}\operatorname{grad}\chi_{2,m}$. Now, $\operatorname{dist}(z_*,w_*) > r_m^p$ by (7.8.2). Also,

$$\begin{aligned}|H_m(z_1\cdots z_n)-H_m(w_1\cdots w_q)| &\leq |(z_1\cdots z_q, H_m(z_1\cdots z_q)) \\ &\quad -(w_1\cdots w_q, H_m(w_1,\cdots,w_q))| \\ &= |z'-w'|,\end{aligned}$$

where $z',w'\in V_m$. From (3.1.3), $|z'-w'|\leq\mathscr{O}(r_m)$. Thus $|z-w| > r_m^p - \mathscr{O}(r_m) > r_m^p$ since $p<1$ and $r_m\to 0$.

To prove (7.8.8) we note that for each $z \in K_m$,

$$\operatorname{dist}(z_0, z) \leq \sup_{w \in V_m} (z_0, w) + \inf_{w \in V_m} (w, z) \leq \mathscr{O}(r_m^p) + \inf_{w \in V_m} (w, z).$$

Given $z = (z_1, \cdots, z_q, z_{q+1}, \cdots, z_n)$, set $w = (z_1, \cdots, z_q, H_m(z_1 \cdots z_q)) \in V_m$. Then

$$\inf_{w \in V_m} (w, z) \leq |(z_{q+1}, \cdots, z_n) - H_m(z_1, \cdots, z_1)| = |F_m(z)|.$$

Since $z \in K_m \subset \operatorname{supp} F_m^*(\chi''_{2,m})$, necessarily $|F_m(z)| \leq \mathscr{O}(r_m^p)$. Thus $\operatorname{dist}(z_0, z) \leq \mathscr{O}(r_m^p) + \mathscr{O}(r_m^p) = \mathscr{O}(r_m^p)$.

(7.8.9) Lemma. *For every (N, k) there exists $C_{N,k}$ such that*

$$\|\Box_b \overline{\partial}_b^* \overline{\partial}_b u_m\|_{C^k(K_m)} \leq C_{N,k} r_m^n. \tag{7.8.10}$$

Proof. Note that if φ is any form defined on a neighborhood of z_0 which vanishes to infinite order at z_0, then for every (N, k) there exists $C_{N,k}$ such that $\|\varphi\|_{C^k(K_m)} \leq C_{N,k} r_m^n$. In fact, the vanishing of φ to infinite order at z_0 implies that for every N

$$\|\varphi\|_{C^k(K_m)} \leq C_N \left[\sup_{z \in K_m} \operatorname{dist}(z, z_0)\right]^N ;$$

which by (7.8.8) is majorized by $\mathscr{O}(r_m^{N_p}) = \mathscr{O}(r_m^{N'})$, where $N' = Np$ can be made arbitrarily large. Note also that if $\{\psi_m\}_{m=1}^{\infty}$ is a sequence of functions smooth on K_m such that for every $j \geq 1$ there exists N_j such that $\|\psi_m\|_{C^j(K_m)} \leq C_j r_m^{-N_j}$ then for all $n \geq 0$, $\|\psi \varphi_m\|_{C^j(K_m)} \leq C_{j,n} r_m^n$.

To prove (7.8.10), we shall expand $\Box_b \overline{\partial}_b^* \overline{\partial}_b u_m$ as a finite sum of terms each of the form $\varphi \psi_m$. Now,

$$\Box_b \overline{\partial}_b^* \overline{\partial}_b u_m = \overline{\partial}_b^* \overline{\partial}_b \Box_b u_m = \overline{\partial}_b^* \overline{\partial}(I - \mathscr{H}) \langle \alpha_m^{(0)} \rangle_b^{0,q} = \overline{\partial}_b^* \overline{\partial}_b \langle \alpha_m^{(0)} \rangle_b^{0,q},$$

where $\langle \alpha_m^{(0)} \rangle_b^{0,q} = \chi_{1,m} f_m F_m^*(\chi_{2,m}) \overline{\partial}_b \bar{z}_1 \wedge \overline{\partial}_b \bar{z}_2 \wedge \cdots \wedge \overline{\partial}_b \bar{z}_q$. Thus

$$\begin{aligned} \Box_b \overline{\partial}_b^* \overline{\partial}_b u_m = {} & \overline{\partial}_b^* \{ f_m F_m^*(\chi_{2,m}) \overline{\partial}_b \chi_{1,m} \wedge \overline{\partial}_b \bar{z}_1 \wedge \cdots \wedge \overline{\partial}_b \bar{z}_q \} \\ & + \overline{\partial}_b^* \{ \chi_{1,m} F_m^*(\chi_{2,m}) \overline{\partial}_b f_m \wedge \overline{\partial}_b \bar{z}_1 \wedge \cdots \wedge \overline{\partial}_b \bar{z}_q \} \\ & + \overline{\partial}_b^* \{ \chi_{1,m} f_m \overline{\partial}_b F_m^*(\chi_{2,m}) \wedge \overline{\partial}_b \bar{z}_1 \wedge \cdots \wedge \overline{\partial}_b \bar{z}_q \}. \end{aligned} \tag{7.8.11}$$

The first term on the right side of (7.8.11) contains the expression $\overline{\partial}_b \chi_{1,m} \wedge \overline{\partial}_b \bar{z}_1 \wedge \cdots \wedge \overline{\partial}_b \bar{z}_q$. However,

$$\overline{\partial}_b \chi_{1,m}(z_1 \cdots z_q, \bar{z}_1 \cdots \bar{z}_q) = \sum_{j=1}^{q} \frac{\partial \chi_{1,m}}{\partial \bar{z}_j} \overline{\partial}_b \bar{z}_j + \sum_{j=1}^{q} \frac{\partial \chi_{1,m}}{\partial z_j} \overline{\partial}_b z_j.$$

The first q terms are annihilated by exterior multiplication by $\overline{\partial}_b \bar{z}_1 \wedge \cdots \wedge \overline{\partial}_b \bar{z}_q$. The last q terms are multiples of $\overline{\partial}_b z_j$, which vanish to infinite order at p_0

by the construction of the almost-CR imbedding functions $(z_1, z_2, \cdots, z_n)$. It follows that the first term on the right side of (7.8.11) can be expanded as a sum of terms of the form $\varphi\psi_m$ in which

$$\begin{aligned}
\varphi &= \text{a derivative of order 0 or 1 of a coefficient of } \overline{\partial}_b z_j \\
&= \text{a term that vanishes to infinite order at } p_0, \\
\psi_m &= \text{a derivative of order 1 or 0 of} \\
&\quad \frac{\partial \chi_{1,m}}{\partial z_j} f_m F_m^*(\chi_{2,m}) \overline{\partial}_b \overline{z}_1 \wedge \cdots \wedge \overline{\partial}_b \overline{z}_z \\
&= \text{a term satisfying the estimates (6.1.2) which} \\
&\quad \text{are polynomial in } 1/r_m.
\end{aligned}$$

The second term on the right side of (7.8.11) contains the quantity

$$\overline{\partial}_b f_m = \sum_{j=1}^{n} (\partial f / \partial z_j) \overline{\partial}_b z_j,$$

in which $\overline{\partial}_b z_j$ vanishes to infinite order at p_0.

(Note that the holomorphicity of f_m with respect to $(z_1, \cdots, z_n)$ has been used here.) Thus the second term in (7.8.11) can be expanded as a sum of expressions of the form $\varphi\psi_m$ in which

$$\begin{aligned}
\varphi &= \text{derivatives of order 0 or 1 of } \overline{\partial}_b z_j \\
&= \text{a term that vanishes to infinite order at } p_0, \\
\psi_m &= \text{derivative of order 1 or 0 of} \\
&\quad \frac{\partial f}{\partial z_j} \chi_{1,m} F_m^*(\chi_{2,m}) \\
&= \text{a term that satisfies the polynomial bounds (6.1.2) in } 1/r_m.
\end{aligned}$$

The third term on the right side of (9.8.11) actually vanishes on K_m. This proves (7.8.10).

(7.8.12) Lemma. *There exists e independent of k and m such that for every k we have uniformly in m*

$$\|\overline{\partial}_b^* \overline{\partial}_b u_m\|_{C^k(K_m)} \leq \mathscr{O}(r_m^{-[(kp/\varepsilon)+e]}). \tag{7.8.13}$$

Proof. As in (7.7.3), is suffices to majorize uniformly in $x \in K_m$ the expression

$$\inf_{\varsigma=1 \text{ at } x} \|\varsigma \overline{\partial}_b^* \overline{\partial}_b u_m\|_{\wedge^{k+n}(M)} = \left\| \bigwedge{}^{k+n} \varsigma \overline{\partial}_b^* \overline{\partial}_b u_m \right\|_{L^2(M)}.$$

We use (7.6.2) with $v = \overline{\partial}_b^* \overline{\partial}_b u_m$, $N =$ least integer greater than $(k+n)/\varepsilon$, and

$$t = \operatorname{dist}(x, \operatorname{supp} \chi_{1,m} \operatorname{grad} F_m^*(\chi_{2,m})).$$

Thus

$$\begin{aligned} t = t_m &\geq \operatorname{dist}(K_m, \operatorname{supp} \chi_{1,m} \operatorname{grad} F_m^*(\chi_{2,m})) \\ &\geq \mathscr{O}(r_m^p) \quad \text{by (7.8.7)}. \end{aligned}$$

This choice of t ensures that the cut-off functions $\varsigma_{1,t,x} > \varsigma_{2,t,x} > \cdots > \varsigma_{N+1,t,x}$ centered at x all vanish on $\operatorname{supp} \chi_{1,m} \operatorname{grad} F_m^*(\chi_{2,m})$. Substituting $t_m > r_m^p$ into (7.6.2) yields

(7.8.14)

$$\begin{aligned} \left\| \bigwedge^{k+n} \varsigma_{N+1,t,x} \overline{\partial}_b^* \overline{\partial}_b u_m \right\|_{L^2(M)}^2 &< r_m^{-2Np} \sum_{j=1}^{N} \| A_{j,t,x} \Box_b \overline{\partial}_b^* \overline{\partial}_b u_m \|_{L^2(M)}^2 \\ &\quad + r_m^{-2(N-1)p} \| \varsigma_{1,t,x} \Box_b \overline{\partial}_b^* \overline{\partial}_b u_m \|_{L^2(M)}^2 \\ &\quad + r_m^{-2(N+1)p} \| \overline{\partial}_b^* \overline{\partial}_b u_m \|_{L^2(M)}^2 . \end{aligned}$$

The majorization of every term on the right of (7.8.14) begins by noting that from the homogeneity property (7.6.1), the first terms on the right side of (7.8.14) satisfy

(7.8.15)

$$\begin{aligned} \| A_{j,t,x} \Box_b \overline{\partial}_b^* \overline{\partial}_b u_m \|_{L^2(M)}^2 &\approx t^{-2j\varepsilon + 2n - 1} \| A_{j,1,x} [\Box_b \overline{\partial}_b^* \overline{\partial}_b u_m]^t \|_{L^2(M)}^2 \\ &\leq t^{-2j\varepsilon + 2n - 1} \| \varsigma_{j,1,x} [\Box_b \overline{\partial}_b \overline{\partial}_b u_m]^t \|_{C^{j-n+1}(M)}^2 \\ &\leq \mathscr{O}(r_m^{N'}) \end{aligned}$$

for every N' by (7.8.9) and by the uniform boundedness of the dilation operators $[\ ]^t$ for $t \leq 1$.

Next, we note that the term on the second line on the right side of (7.8.14) satisfies

$$\text{(7.8.16)} \qquad \| \varsigma_{1,t,x} \Box_b \overline{\partial}_b^* \overline{\partial}_b u_m \|_{L^2(M)} \leq \mathscr{O}(r_m^{N'}) \quad \text{for every } N' \text{ by (7.8.10)}.$$

Finally, the last term in (7.8.14) satisfies

$$\text{(7.8.17)} \qquad \| \overline{\partial}_b^* \overline{\partial}_b u_m \|_{L^2(M)}^2 < \| \langle \alpha_m^{(0)} \rangle_b^{0,q} \|_{L^2(M)}^2 \leq \mathscr{O}(r_m^e)$$

by (7.2.2). Here e is independent of k and m.

Substituting (7.8.15)–(7.8.17) into (7.8.14) yields

$$\left\| \bigwedge^{k+1} \varsigma_{N+1,t,x} \overline{\partial}_b^* \overline{\partial}_b u_m \right\|_{L^2(M)}^2 \leq r_m^{-2[(N+1)p + e]} \leq r_m^{-2[(kp/\varepsilon) + e']}$$

by the choice of N.

(7.8.18) Proposition. *There exists e independent of k and m such that for every k*

$$\left| \int_{\mathbf{M}} F_m^*(\chi_{2,m}'') J_4 \right| \leq \mathscr{O}(r_m^{-k+e})$$

uniformly in m*, provided that* p *is chosen to satisfy*

$$p < \min(\alpha, 1, \eta - 1, \varepsilon). \tag{7.8.19}$$

Proof.

$$\begin{aligned}
\left|\int_{\mathbf{M}} F_m^*(\chi''_{2,m}) J_4\right| &= \left|\int_{\mathbf{M}} F_m^*(\chi''_{2,m}) \mathscr{L}_T^k[\chi'_{1,m} F_m^*(\chi'_{2,m}) \overline{\partial}_b^* \overline{\partial}_b u_m] \wedge \beta_m\right| \\
&\leq \|\chi'_{1,m} F_m^*(\chi'_{2,m}) \overline{\partial}_b^* \overline{\partial}_b u_m\|_{C^k(\operatorname{supp} \chi_{1,m} F_m^*(\chi''_{2,m})} \\
&\quad \cdot \|\beta_m\|_{C^0(\operatorname{supp} \chi_{1,m} F_m^*(\chi''_{2,m})} \\
&\leq [\mathscr{O}(r_m^{-k}) + \mathscr{O}(r_m^{-kp}) + \mathscr{O}(r_m^{-[(kp/\varepsilon)+e]})] \mathscr{O}(r_m^{e'}) \\
&\qquad\qquad \text{by (6.1.3), (6.1.6), (7.8.13), and (6.1.8)} \\
&\leq \mathscr{O}(r_m^{-k+e'}) \quad \text{from the choice of } p.
\end{aligned}$$

(7.8.20) Proposition. *There exists* e *independent of* k *and* m *such that for every* k *we have uniformly in* m

$$\left|\int_{\mathbf{M}} J_4\right| \leq \mathscr{O}(r_m^{[-k(1+p)+e]}),$$

provided that p *is chosen small enough to satisfy* (7.8.19).

Proof. Combine (7.8.3) and (7.8.18).

7.9. Final estimates.

(7.9.1) Proposition. *There exist* e *and* e' *independent of* k *and* m *such that for every* k *we have uniformly in* m

$$|I_1| \leq \mathscr{O}(r_m^{-[(k/\varepsilon)+e]}) + \mathscr{O}(r_m^{-[(1+p)k+e']}),$$

provided p *satisfies* (7.8.19).

Proof. Substitute into (7.3.5) the results of (7.4.1), (7.5.1), (7.7.6), and (7.8.20).

(7.9.2) Proposition. *For each* k*, the right side of* (5.1.5) *is majorized by* $\mathscr{O}(r_m^{-[(k/\varepsilon)+e]}) + \mathscr{O}(r_m^{-[(1+p)k+e']})$ *provided that* p *satisfies* (7.8.19).

Proof. Combine (5.3.3) with (7.9.1) and (6.2.1).

(7.9.3) Theorem. *Let* M *be a compact pseudoconvex CR manifold of dimension* $2n-1$ *which satisfies hypotheses* (7.2.1)–(7.2.3) *for tangential* $(0,q)$ *forms at* $p_0 \in M$*. If any germ of an almost-CR imbedded image* $\mathbf{M}$ *of* M *into* $\mathbf{C}^n$ *based at* $p_0 \in M$ *has the property that there exists an admissible sequence* $\{V_m\}_{m=1}^{\infty}$ *whose order of contact with* $\mathbf{M}$ *at* $z(p_0)$ *is at least* η*, then* $\eta \leq 1/\varepsilon$.

Proof. After constructing the test forms (5.1.4) and the currents in (3.3.8), the basic pairing formula (5.1.5) can be minorized for each k using (5.2.1), and majorized for each k using (7.9.2). The result is that

$$r_m^{-\eta(k+1)+2q} \leq \mathscr{O}(r_m^{-[(k/\varepsilon)+e]}) + \mathscr{O}(r_m^{-[(1+p)k+e']}),$$

where $p > 0$ is arbitrary save for the constant (7.8.19). Choose p so small that $(1+p) < 1/\varepsilon$, which is possible since $\varepsilon < 1$. The inequality $r_m^{-\eta(k+1)+2q} \le \mathscr{O}(r_m^{-[(k/\varepsilon)+e]})$ valid for $r_m \to 0^+$ implies that $\eta(k+1)+2q \le (k/\varepsilon)+e$.

Since e is independent of k, the validity of this inequality for all k forces $\eta \le 1/\varepsilon$.

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