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THE LORENTZIAN SPLITTING THEOREM WITHOUT THE COMPLETENESS ASSUMPTION

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1. Introduction

A number of papers ([6], [3], [4], [2]) have been published which address the problem posed by Yau [10] of establishing a Lorentzian analogue of the Cheeger-Gromoll splitting theorem of Riemannian geometry. A very satisfactory Lorentzian analogue has recently been obtained by Eschenburg. In [4], he proves that a globally hyperbolic, timelike geodesically complete space-time satisfying the "strong energy condition", $\mathrm{Ric}(X,X) \geq 0$, X timelike, which contains a (complete) timelike line, "splits" in a sense made precise below. Prior to Eschenburg's work, Beem et al. [3] proved a Lorentzian splitting theorem assuming a more stringent sectional curvature condition (analogous to nonnegative sectional curvature in the Riemannian case). One interesting feature of their result is that the full assumption of timelike geodesic completeness is not needed; it is only required that the given timelike line be complete. Timelike geodesic completeness is then derived as a consequence of the assumption of global hyperbolicity, the sectional curvature condition, and the completeness of the line. This suggests that there may be some redundancy in the hypotheses of Eschenburg's theorem.

The purpose of this paper is to prove the Lorentzian splitting theorem for globally hyperbolic space-times obeying the strong energy condition, without the assumption of timelike geodesic completeness; i.e. our aim is to prove the following

Theorem. *Let (M,g) be a connected globally hyperbolic space-time which satisfies $\mathrm{Ric}(X,X) \geq 0$ for all timelike vectors X. If (M,g) contains a complete timelike line γ then it is isometric to $(\mathbb{R} \times S, -dt^2 \oplus h)$, where (S,h) is a complete Riemannian manifold, and the factor $(\mathbb{R}, -dt^2)$ is represented by γ.*

Eschenburg uses the assumption of timelike completeness in a number of crucial ways. Consequently, the proof of the above theorem requires some new observations and techniques. At the same time, in devising a method of proof, we were strongly influenced by Eschenburg's work. In particular, our proof

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relies on several results obtained in [4] which do not require the completeness assumption.

As in [3] and [4], our proof involves an analysis of the Lorentzian Busemann functions $b^{\pm}$ associated to the given line γ. The key step in the proof of the Cheeger-Gromoll splitting theorem of Riemannian geometry is to establish the subharmonicity of the Busemann function associated to a ray. The proof of this makes use of the theory of elliptic operators as applied to the Riemannian Laplacian. This approach does not carry over to the Lorentzian case because the Lorentzian Laplacian is hyperbolic, not elliptic. The papers listed in the first paragraph overcome this difficulty in different ways. (The methods of [6] and [2] do not involve Busemann functions at all.) The approach taken here is to consider $b^{\pm}$ restricted to a maximal spacelike hypersurface Σ having edge(Σ) contained in the level set $\{b^+ = 0\}$. Since the induced Laplacian along Σ is elliptic, maximum principle techniques apply. Using such techniques, we establish a basic convexity result (Lemma 2.4) which enables us to show that the level sets $\{b^{\pm} = 0\}$ are smooth spacelike hypersurfaces which agree near $\gamma(0)$. From this fact and a second (related) convexity result (Lemma 2.1) we are able to establish the splitting of a tubular neighborhood of γ. To obtain a global splitting, one can then proceed as in [4] with only minor modifications. The existence of the maximal hypersurface Σ is guaranteed by new results of Bartnik [1] concerning the existence and regularity of solutions to the Dirichlet problem for the prescribed mean curvature equation with rough boundary data.

In the next section we summarize some known, and establish some new, properties of the Lorentzian Busemann function. The proof of the splitting theorem is presented in §3. We refer the reader to Hawking and Ellis [8] for causal theoretic notions used but not defined below.

2. The Lorentzian Busemann function

Throughout this section let (M, g) be a globally hyperbolic space-time. The Lorentzian distance function $d\colon M \times M \to \mathbb{R}$ is continuous and obeys the reverse triangle inequality ("RTI"): for all $p, q, r \in M$ with $p \le q \le r$,

$$d(p,r) \ge d(p,q) + d(q,r). \tag{2.1}$$

A *future directed timelike ray* is a future directed, future inextendible unit speed timelike geodesic $\gamma\colon [a, l) \to M$ which realizes the distance between any two of its points. If $l = \infty$, we say γ is future complete. Although the results of this section are stated for *future directed* rays, obvious analogues of these results hold for *past directed* rays, as well.

Let $\gamma: [a, \infty) \to M$ be a future complete timelike ray, and set $I(\gamma) = I^+(\gamma(a)) \cap I^-(\gamma)$. The Busemann function $b: I(\gamma) \to M$ associated to γ is defined as follows. For each $r > a$, define the function $b_r: M \to \mathbb{R}$ by,

$$b_r(x) = r - d(x, \gamma(r)).$$

For $x \in I^+(\gamma(a)) \cap I^-(\gamma(r))$, $b_r(x)$ is decreasing in r and bounded below by $d(\gamma(a), x) + a$. Thus, $\lim_{r\to\infty} b_r(x)$ exists and, by definition, is $b(x)$. From the RTI one easily derives

$$b_r(q) \geq b_r(p) + d(p,q) \quad \forall \gamma(a) \ll p \leq q \ll \gamma(r), \tag{2.2}$$

which in the limit $r \to \infty$ gives

$$b(q) \geq b(p) + d(p,q) \quad \forall p, q \in I(\gamma),\ p \leq q. \tag{2.3}$$

An *asymptote* to γ from $p \in I(\gamma)$ is a future inextendible causal geodesic ray $\alpha: [0, l) \to M$, with $\alpha(0) = p$, which arises as a limit as $r_n \to \infty$ of maximal timelike geodesic segments from p to $\gamma(r_n)$. In general, an asymptote may be null, and need not be future complete, even though the segments defining it become arbitrarily long. Assume $\alpha: [0, l) \to M$ is a (unit speed) timelike asymptote to γ. Using the RTI, one can derive the inequality

$$b(x) \leq s - d(x, \alpha(s)) + b(\alpha(0)), \qquad 0 < s < l, \tag{2.4}$$

for all $x \in I(\alpha)$. Setting $x = \alpha(t)$ in (2.4), $t < s$, the resulting inequality, used in conjunction with (2.3), gives

$$b(\alpha(t)) = t + b(\alpha(0)) \quad \forall t \in [0, l). \tag{2.5}$$

Moreover, it can be shown (see [4]) that b is differentiable at $\alpha(t)$ if $t > 0$, and $\nabla b(\alpha(t)) = -\alpha'(t)$. It follows that timelike asymptotes to γ emanating from different points cannot intersect (unless one is contained in the other).

In the proof of some of the lemmas to follow, and in the next section we will have occasion to use the notion of *local support functions.* Given a continuous real valued function f, g is said to be an upper (respectively, lower) support function of f if g is continuous in a neighborhood of p, $g \geq f$ (respectively, $g \leq f$), and $g(p) = f(p)$.

Although not used explicitly here, we note that b_r is a continuous time function on $I^-(\gamma(r))$ and, for $c < r$, the level set $\{b_r = c\}$ is a partial Cauchy surface (i.e. is acausal and edgeless) in $I^-(\gamma(r))$. We will need to make use of some causal properties of the Busemann function b. Observe from (2.3) that the level sets $\{b = c\}$ are achronal in $I(\gamma)$. Later we will show that near γ the level sets of the Busemann function are acausal and edgeless, and hence, in a weak sense, spacelike. Our first lemma establishes one of the two basic convexity results alluded to in the Introduction. For this lemma we assume b is smooth and has spacelike level sets.

Lemma 2.1. *Assume M obeys the strong energy condition,* $\mathrm{Ric}(X,X) \geq 0$, *X timelike. Let b be the Busemann function associated with the future complete ray γ. Assume b is smooth on an open set $U \subset I(\gamma)$ with unit timelike gradient. Then $\Delta b \leq 0$ on U.*

This lemma is a Lorentzian analogue of the well-known result of Riemannian geometry that in a complete Riemannian manifold with nonnegative Ricci curvature, the Busemann function associated to a ray is subharmonic (in the sense of continuous functions). We give a geometric proof based on the relationship between the Laplacian of a function and the mean curvature of its level sets.

Before proceeding to the proof, we recall a basic fact concerning the Laplacian of the Lorentzian distance function (see [4]). Define $d_q: I^-(q) \to \mathbb{R}$ by $d_q(x) = d(x,q)$.

Lemma 2.2. *d_q is smooth on $I^-(q)$ outside the cut locus of q and, provided M obeys the strong energy condition, $\Delta d_q \geq -(n-1)/d_q$, where $n = \dim M$.*

We remark that along each level set $\{d_q = c\}$ (and away from the cut locus of q), Δd_q is the negative of the mean curvature of the level set with respect to the future directed normal. (We use the sign convention in which positive mean curvature corresponds to mean contraction of the spacelike hypersurface.)

Proof of Lemma 2.1. Suppose $\Delta b(p) = H > 0$ for some $p \in U$. For some $r_0 > 0$, $p \in I^-(\gamma(r_0))$. Set $U_0 = U \cap I^-(\gamma(r_0))$. Let $c = b(p)$ and consider the smooth spacelike hypersurface $\Sigma = \{b = c\} \cap U_0$. Since $b_r \downarrow b$, $\Sigma \subset \{b_r \geq c\}$ for all $r \geq r_0$. Let H_Σ denote the mean curvature of Σ with respect to the future pointing normal. Since ∇b has unit length and (as follows from (2.3)) is past directed, we have $\Delta b = H_\Sigma$ along Σ.

Choose a point $q \in I^+(p) \cap U_0$ sufficiently close to p so that $H_\Sigma(x) \geq H/2$ for all $x \in \Sigma \cap I^-(q)$. Deform Σ in a small neighborhood of p to obtain a smooth spacelike hypersurface Σ' with the following properties:

1. $A = \Sigma' \backslash \Sigma \subset I^-(q)$.
2. $A \cap I^-(p) \neq \emptyset$.
3. $H_{\Sigma'}(x) \geq H/3$ for all $x \in A$.

Since $b_r(p) \downarrow c$ one easily shows using (2.2) that A meets $\{b_r < c\}$ for all sufficiently large r. Thus, for such r, $b_r|_{\Sigma'}$ achieves an interior minimum $c' < c$ at some point $z \in A$, and hence $\Sigma' \subset \{b_r \geq c'\}$.

Note that $d(z, \gamma(r)) = r - c'$. Let $\eta_r: [0, r-c'] \to M$ be a maximal geodesic segment from z to $\gamma(r)$. Set $y_r = \eta_r((r-c')/2)$, and define the function

$\beta_r: I^-(y_r) \to \mathbb{R}$ by

$$\beta_r(x) = r - [(r - c')/2 + d(x, y_r)].$$

β_r is smooth near z, and $\beta_r(z) = c'$. Furthermore, the RTI implies that $\beta_r \geq b_r$. Thus, near z, $\Sigma_r = \{\beta_r = c'\}$ is a smooth spacelike hypersurface which meets Σ' tangentially at z, and lies to the past of Σ'. The maximum principle then requires

$$H_{\Sigma_r}(z) \geq H_{\Sigma'}(z) \geq H/3.$$

On the other hand, by Lemma 2.2,

$$H_{\Sigma_r}(z) = \Delta\beta_r(z) \leq 2(n-1)/(r - c') < 2(n-1)/(r - c).$$

Noting that this last inequality is valid for all large r, we obtain the desired contradiction. Thus, $\Delta b \leq 0$ on U.

We now consider some properties of b which are valid near the ray γ. An open set $U \subset I(\gamma)$ is said to be *nice* (with respect to γ) if there exist constants $K > 0$ and $T > 0$ such that for each $q \in U$ and $r > T$, any maximal unit speed geodesic segment σ from q to $\gamma(r)$ satisfies

$$g_0(\sigma'(0), \sigma'(0)) \leq K,$$

where g_0 is some fixed Riemannian metric on M. We summarize some facts concerning nice neighborhoods:

1. For each $t > a$, $\gamma(t)$ is contained in a nice neighborhood.
2. Asymptotes to γ from points in nice neighborhoods are timelike.
3. $\{b_r\}$ converges locally uniformly to b on nice neighborhoods, and hence b is continuous on nice neighborhoods.

Properties 1 and 3 are proved in [4], and property 2 is a simple consequence of property 1. The next lemma establishes some causal properties of the Busemann function in nice neighborhoods which will be needed in the proof of the splitting theorem.

Lemma 2.3. *Let U be a nice neighborhood. Then the level set $\{b = c\}$, if it meets U, is a partial Cauchy surface in U, i.e., is closed, acausal and edgeless in U.*

Proof. Let $\Sigma_c = \{b = c\} \cap U$. Since b is continuous in U, Σ_c is closed in U. Equation (2.3) implies that Σ_c is at least achronal. Indeed, (2.3) shows that b is strictly increasing along future directed timelike curves. This fact and the continuity of b in U easily implies that Σ_c is edgeless in U. It remains to show that Σ_c is actually acausal.

If Σ_c is not acausal then there exists a pair of points $p, q \in \Sigma_c$ and a null geodesic η joining p to q. Since q is in a nice neighborhood, there exists a sequence of maximal segments α_n from q to $\gamma(r_n)$ ($r_n \uparrow \infty$) which converges to a timelike geodesic ray α to q. By cutting the corner of the broken geodesic $\eta \cup \alpha_n$ (and comparing with the corner of $\eta \cup \alpha$) we see there exists an $\varepsilon > 0$ such that for all n sufficiently large,

$$b_{r_n}(q) - b_{r_n}(p) = d(p, \gamma(r_n)) - d(q, \gamma(r_n)) > \varepsilon.$$

This contradicts $b(p) = b(q)$.

The next lemma is the key convexity result used in the proof of the splitting theorem.

Lemma 2.4. *Assume M obeys the strong energy condition,* $\mathrm{Ric}(X, X) \geq 0$ *for all X timelike. Let Σ be a connected smooth spacelike hypersurface contained in a "sufficiently small" nice neighborhood of $\gamma(t)$, $t > a$. Assume the mean curvature of Σ is nonnegative, $H_\Sigma \geq 0$. If b achieves a minimum along Σ then b is constant along Σ.*

What is meant by "sufficiently small" will be explained in a moment. Our proof of Lemma 2.4 makes use of a technical lemma, whose proof we relegate to an appendix. For the purpose of stating the technical lemma we introduce the following notation. Suppose $q \in I^-(\gamma(r))$. Let $\eta_{q,r}: [0, l_{q,r}] \to M$ be any unit speed maximal geodesic segment from q to $\gamma(r)$. Define the function $d^s_{q,r}: I^-(\eta_{q,r}(s)) \to \mathbb{R}$ by

$$d^s_{q,r}(x) = d(x, \eta_{q,r}(s)).$$

The Technical Lemma. *For $r_0 > t > a$, there exists a nice neighborhood $U \subset I^-(\gamma(r_0))$ of $\gamma(t)a$ with the following property: For any compact spacelike hypersurface with boundary S contained in U, there exist constants $C > 0$, $\tau > 0$, and $r_1 > r_0$, such that for each $q \in S$ and $r \geq r_1$, there exists a maximal geodesic segment $\eta_{q,r}$ from q to $\gamma(r)$ such that*

$$\mathrm{Hess}\, d^s_{q,r}(w, w) \geq -C \tag{2.6}$$

for all $w \in T_qS$ with $g(w, w) \leq 1$, and for all s, $\tau \leq s \leq l_{q,r}$.

It is well known (see e.g. [4]) that $s \to \mathrm{Hess}\, d^s_{q,r}(w, w)$ is an increasing function of s. The significance of the lemma is that the quantity $\mathrm{Hess}\, d^s_{q,r}(w, w)$ can be bounded from below uniformly in q and r.

Proof of Lemma 2.4. The proof is a maximum principle type argument. Let $U \subset I^-(\gamma(r_0))$ be the nice neighborhood of $\gamma(t)$ guaranteed by the technical lemma. Take "sufficiently small" to mean that $\Sigma \subset U$. Assume b achieves a minimum $b(q) = a$ at $q \in \Sigma$. We claim that $b = a$ in a neighborhood of q in Σ. Suppose not. Then there exists an open coordinate ball $B \subset \Sigma$ centered at q such that $\partial B \neq \partial^0 B$, where

$$\partial^0 B = \{x \in \partial B : b(x) = a\}.$$

The technical lemma applied to $S = \overline{B}$ ensures the existence of a constant $C = C(\overline{B})$ such that (2.6) holds for all $x \in B$, for all r large, for all $w \in T_qB$, $g(w,w) \leq 1$, and for all relevant s.

Note that $b > a$ on $\partial B \backslash \partial^0 B$. Choosing B sufficiently small, we can construct a smooth function h on Σ such that

(i) $h(q) = 0$,

(ii) $|\nabla_\Sigma h| \leq 1$ on B, where ∇_Σ is the gradient operator on Σ,

(iii) $\Delta_\Sigma h \leq -D$ on B, where D is a positive constant and Δ_Σ is the induced Laplacian on Σ, and

(iv) $h > 0$ on $\partial^0 B$.

For details of this construction, see [5].

Consider the function $f_\varepsilon = b + \varepsilon h$. Observe that $f_\varepsilon(q) = a$, and for ε sufficiently small, $f_\varepsilon > a$ on ∂B. Now consider for r large the function $f_{\varepsilon,r} = b_r + \varepsilon h$. Since $f_{\varepsilon,r} \geq f_\varepsilon$ and $f_{\varepsilon,r}(q) \downarrow a$, $f_{\varepsilon,r}(q) < f_{\varepsilon,r}|_{\partial B}$ for all large r. Thus, for such r, $f_{\varepsilon,r}$ achieves a minimum on B, at p, say.

Let $\eta_r\colon [0,l] \to M$ be the maximal geodesic segment from p to $\gamma(r)$ guaranteed by the technical lemma. We get from the RTI that $l \geq r - r_0$, and hence for r sufficiently large, $l/2 > \tau$. Set $y_r = \eta_r(l/2)$. From the RTI we see that the function $\beta_{p,r}\colon I^-(y_r) \to \mathbb{R}$ defined by

$$\beta_{p,r}(x) = r - [l/2 + d(x, y_r)]$$

is an upper support function of b_r at p. The function $\varphi_{\varepsilon,r} = \beta_{p,r} + \varepsilon h$ is thus an upper support function of $f_{\varepsilon,r}$ at p. Hence, in some neighborhood of p, $\varphi_{\varepsilon,r}$ is smooth and achieves a minimum at p. We obtain a contradiction by computing $\Delta_\Sigma \varphi_{\varepsilon,r}(p)$ and showing it is negative for ε sufficiently small and r sufficiently large. To begin, we have,

$$\Delta_\Sigma \varphi_{\varepsilon,r}(p) = \Delta_\Sigma \beta_{p,r}(p) + \varepsilon \Delta_\Sigma h(p). \tag{2.7}$$

The formula relating Δ_Σ to the space-time Laplacian Δ gives

$$\Delta_\Sigma \beta_{p,r} = \Delta \beta_{p,r} - H_\Sigma \langle \nabla \beta_{p,r}, N \rangle + \operatorname{Hess} \beta_{p,r}(N,N), \tag{2.8}$$

where N is the future directed normal to Σ. From Lemma 2.2 we have

$$\begin{aligned} \Delta \beta_{p,r}(p) &\leq (n-1)/d(p,y_r) = 2(n-1)/l \\ &\leq 2(n-1)/(r - r_0), \end{aligned} \tag{2.9}$$

since $l \geq r - r_0$. The equations $\nabla_\Sigma \varphi_{\varepsilon,r}(p) = 0$ and $\nabla \beta_{p,r}(p) = \eta_r'(0)$ imply that $N = \langle N, \eta_r'(0) \rangle^{-1} \cdot [-\eta_r'(0) + \varepsilon \nabla_\Sigma h]$ at p. A computation, using this expression for N, shows

$$\begin{aligned} \operatorname{Hess} \beta_{p,r}(N,N)|_p &= \varepsilon^2 \langle N, \eta_r'(0) \rangle^{-2} \cdot \operatorname{Hess} \beta_{p,r}(\nabla_\Sigma h, \nabla_\Sigma h)|_p \\ &\leq C\varepsilon^2, \end{aligned} \tag{2.10}$$

where the inequality follows from the technical lemma, property (ii) of h, and the reverse Schwarz inequality (which implies $|\langle N, \eta_r'(0)\rangle| \geq 1$). By substituting (2.9) and (2.10) in (2.8) and using the mean curvature assumption we obtain

$$\Delta_\Sigma \beta_{p,r}(p) \leq 2(n-1)/(r-r_0) + C\varepsilon^2.$$

Substituting this inequality into (2.7) and using property (iii) of h gives

$$\Delta_\Sigma \varphi_{\varepsilon,r}(p) \leq 2(n-1)/(r-r_0) + C\varepsilon^2 - D\varepsilon.$$

For ε sufficiently small and r sufficiently large, the right-hand side of the above inequality is negative, and hence $\Delta_\Sigma \varphi_{\varepsilon,r}(p) < 0$, which contradicts the fact that $\varphi_{\varepsilon,r}$ has a minimum at p. Thus, we have shown $b = a$ in a neighborhood of q in Σ, and hence, by connectivity, $b = a$ along Σ.

In the next section we also make use of the following immediate consequence of Lemma 2.4.

Corollary 2.5. *Let Σ be a smooth maximal spacelike hypersurface whose closure is contained in a sufficiently small nice neighborhood U of $\gamma(t)$, $t > a$. Assume Σ is achronal in U and $\overline{\Sigma}$ is compact. If* $\mathrm{edge}(\Sigma) \subset \{b \geq c\}$, *then* $\Sigma \subset \{b \geq c\}$.

This corollary is a Lorentzified rigid generalization of a result of Schoen and Yau [9, Lemma 4],

3. Proof of the Splitting Theorem

A *timelike line* $\gamma\colon (c, d) \to M$ is an inextendible timelike geodesic which realizes the distance between each pair of its points. If $c = -\infty$ and $d = \infty$, we say γ is complete. Let $\gamma\colon \mathbb{R} \to M$ be the given complete timelike line. Without loss of generality we may assume that γ is future directed. Let $-\gamma$ denote γ with the opposite orientation, $-\gamma(t) = \gamma(-t)$ for all $t \in \mathbb{R}$. For each r, define the functions b_r^+, b_r^- by

$$b_r^+(x) = r - d(x, \gamma(r)), \qquad b_r^-(x) = r - d(-\gamma(r), x).$$

On $I(\gamma) = I^+(\gamma) \cap I^-(\gamma)$, we define the Busemann functions b^+ and b^- associated to γ and $-\gamma$, respectively, by $b^\pm = \lim_{r\to\infty} b_r^\pm$. (As follows from the discussion in §2, these limits exist.) Since $\gamma|_{[a,\infty)}$ and $-\gamma|_{[a,\infty)}$ are rays for any a, it is clear that the results of §2 concerning the Busemann function b apply in an obvious way to b^+ and b^-. The RTI can be used to show

$$b^+ + b^- \geq 0 \quad \text{on } I(\gamma), \tag{3.1}$$

with equality holding along γ.

Let U be a sufficiently small nice neighborhood of $\gamma(0)$ (in the sense of §2) with respect to both γ and $-\gamma$. Consider the level sets $S^{\pm} = \{b^{\pm} = 0\} \cap U$. By Lemma 2.3, S^+ is a partial Cauchy surface in U. In particular, S^+ is an imbedded topological submanifold of codimension one. Let W be a small coordinate ball in S^+ centered at $\gamma(0)$, whose closure is contained in S^+. We now invoke Bartnik's fundamental existence result [1, Theorem 4.1]. This result implies that there exists a smooth maximal spacelike hypersurface Σ such that Σ is achronal in U, $\overline{\Sigma}$ is compact, $\operatorname{edge}(\Sigma) = \operatorname{edge}(W)$, and Σ meets γ. (The acausality of S^+ ensures that the *singularity set* of Σ as defined in [1], where Σ can fail to be smooth, is empty.)

We have $\operatorname{edge}(\Sigma) \subset \{b^- \geq 0\}$. Applying Corollary 2.5 to both b^+ and b^-, we conclude $\Sigma \subset \{b^+ \geq 0\} \cap \{b^- \geq 0\}$. This forces Σ to meet γ at $\gamma(0)$. Since $b^+(\gamma(0)) = b^-(\gamma(0)) = 0$, Lemma 2.4 then implies that

$$b^+ = b^- = 0 \quad \text{along } \Sigma. \tag{3.2}$$

Let $B \subset \Sigma$ be a geodesic ball of radius R in Σ centered at $\gamma(0)$. We presently establish a series of claims which leads to the conclusion that a neighborhood of γ is isometric to $(\mathbb{R} \times B, -dt^2 \oplus g|_B)$. From each point of B there exist timelike asymptotes $\alpha^+ : [0, d) \to M$ and $\alpha^- : [0, c) \to M$ to γ and $-\gamma$, respectively. Let $\alpha: (-c, d) \to M$ be the (possibly) broken geodesic defined by

$$\alpha(t) = \begin{cases} \alpha^-(-t), & -c < t \leq 0, \\ \alpha^+(t), & 0 \leq t < d. \end{cases}$$

Claim 1. $b^+(\alpha(t)) = t$, $b^-(\alpha(t)) = -t$, and α is a line.

Proof. This is essentially proved in [4]. For the sake of completeness (and because it is not too hard!) we briefly sketch the proof here. Using (2.3) and (3.2) we obtain $b^+(\alpha^-(t)) \leq -t$. From (2.5) and (3.2) we get $b^-(\alpha^-(t)) = t$. Adding this equation to the previous inequality and using (3.1) gives $(b^+ + b^-)(\alpha^-(t)) = 0$. If $t \geq 0$, (2.5) and (3.2) imply $b^+(\alpha(t)) = t$. If $t \leq 0$, then

$$b^+(\alpha(t)) = b^+(\alpha^-(-t)) = -b^-(\alpha^-(-t)) = t.$$

Thus, $b^+(\alpha(t)) = t$ and, similarly, $b^-(\alpha(t)) = -t$ for all $t \in (-c, d)$. For $t_1 < 0 < t_2$, the length of

$$\alpha|_{[t_1, t_2]} = t_2 - t_1 = b^+(\alpha(t_2)) - b^+(\alpha(t_1)) \geq d(\alpha(t_1), \alpha(t_2))$$

(by (2.3)), which shows that α is a line.

Claim 2. α is a normal geodesic to B.

Proof. As follows from (2.3), the functions $b^{\pm}_{q,r}$ defined by

$$b^+_{q,r}(x) = r - d(x, \alpha(r)), \qquad b^-_{q,r}(x) = -r + d(\alpha(-r), x)$$

are upper and lower support functions, respectively, of b^+ at $q = \alpha(0)$ for $r > 0$ sufficiently small. It follows that b^+ is differentiable at q, and $\nabla b^+(q) = -\alpha'(0)$. The claim follows by noting that $\nabla b^+(q)$ is perpendicular to B.

Consider the normal exponential map $E: U \to M$ of B defined by

$$E(t, q) = \exp tN_q,$$

where N is the future directed unit normal field along B. Here it is understood that U is the largest possible subset of $\mathbb{R} \times B$ on which E can be defined. U is of the form

$$U = \{(t, q) \in \mathbb{R} \times B: -c_q < t < d_q\},$$

where $\alpha_q: (-c_q, d_q) \to M$ is the future directed normal geodesic to B such that $\alpha_q(0) = q$. U is necessarily open.

Claim 3. $U' = E(U)$ is open and $E: U \to U'$ is a diffeomorphism.

Proof. We need to show E is injective and nonsingular. E is injective iff the normal geodesics to B do not intersect. The future directed normal half-geodesics issuing from B are asymptotes to γ (by Claim 2) and hence, as observed in §2, do not intersect. Similarly, the past directed normal half-geodesics to B do not intersect. Also, future and past directed normal half-geodesics cannot intersect without violating the achronality of $\{b^+ = 0\}$. Hence, E is injective.

E is nonsingular iff there are no focal points to B along its normal geodesics. Let $\alpha: [0, d) \to M$ be the future directed normal geodesic to B issuing from $p \in B$. Suppose there is a focal point to B along α. Let $\alpha(a)$ be the first focal point to B along α. Then there exists a neighborhood $V \subset \mathbb{R} \times B$ of $[0, a) \times \{p\}$ such that $E: V \to V'$, $V' = E(V)$, is a diffeomorphism. By Claim 1, $b^+|_V$ satisfies $b^+(\exp tN_q) = t$, i.e., b^+ is just the time coordinate associated with the normal exponential map. In particular, b^+ is smooth on V, and Δb^+ along $\Sigma_t = \{b^+ = t\} \cap V$ is the mean curvature H_{Σ_t} of Σ_t. Hence, by Lemma 2.1, $H_{\Sigma_t} \leq 0$ along $\alpha|_{[0,a)}$. But since $\alpha(a)$ is a focal point to B along α, we must have $\limsup_{t \to a} H_{\Sigma_t} = \infty$. Thus, there are no focal points to B, and E is nonsingular.

Claim 4 (*Local Splitting*). $U = \mathbb{R} \times B$ and $E: U \to U'$ is an isometry.

Proof. We have shown that $E: U \to U'$ is a diffeomorphism. By Claim 1, $b^+: U \to \mathbb{R}$, $b^-: U \to \mathbb{R}$ are given by

$$b^\pm(\exp tN_q) = \pm t.$$

In particular, b^+ and b^- are smooth. Hence Lemma 2.1 implies that $\Delta b^+ \leq 0$ and $\Delta b^- \leq 0$ on U. But since $b^+ = -b^-$, we conclude that $\Delta b^+ = 0$ on U. Since ∇b^+ is the (past directed) unit tangent vector field to the normal geodesics of B, b^+ obeys the well-known formula

$$-\nabla b^+(\Delta b^+) = \mathrm{Ric}(\nabla b^+, \nabla b^+) + |\mathrm{Hess}\, b^+|^2.$$

This equation, together with the strong energy condition and the vanishing of Δb^+, implies that $\mathrm{Hess}\, b^+ = 0$ on U. Thus, ∇b^+ is parallel on U, and it follows that $E: U \to U'$ is an isometry.

We now show that $U = \mathbb{R} \times B$, i.e., that the normal geodesics to B are complete. Introduce the coordinates (t, x) on U, where the coordinates $x = (x^1, \cdots, x^{n-1})$ come from normal coordinates in B in the obvious manner, and t is the time coordinate associated with the normal exponential map (i.e., $t = b^+$). In what follows we will freely confuse a point in U with its (t, x) coordinates.

Fix $r > R$, where, recall, R is the radius of B. Let $\sigma: [0, R) \to B$ be any radial geodesic in B. Let $\alpha_s: [0, l_s) \to M$ be the unit speed future inextendible normal geodesic starting at $\sigma(s)$; hence $\alpha_s(0) = \sigma(s)$ and the length of α_s is l_s. α_s has the trivial coordinate representation $\alpha_s(u) = (u, \sigma(s))$, $u \in [0, l_s)$. To show that the normal geodesics are future complete, it is enough to establish the inequality

$$(*) \qquad l_s > r - s$$

for all $s \in [0, R)$. Indeed, since r can be taken arbitrarily large, it must be that $l_s = \infty$, i.e. α_s must be future complete.

To establish $(*)$ for all $s \in [0, R)$, introduce the set

$$A = \{t \in [0, R) : (*) \text{ holds } \forall s \in [0, t]\}.$$

Since $\alpha_0 = \gamma|_{[0,\infty)}$, $0 \in A$, and hence A is nonempty. Let $a = \sup A$. The aim is to show that $a = R$. Suppose to the contrary $a < R$. We show that $a \in A$. In showing this we may assume $a > 0$, since we have already observed that $0 \in A$. By the definition of a, $l_s > r - s$ for all $s \in [0, a)$. Thus the curve $\eta: [0, a) \to M$, given in coordinates by $\eta(u) = (r - u, \sigma(u))$, is defined. Using the local product structure of U' one easily verifies that η is a past directed null geodesic starting at $\eta(0) = \gamma(r)$.

We claim that $\alpha_a: [0, l_a) \to M$ has length $l_a > r - a$. If not, then for $s \in [0, a)$, $l_s > l_a$. Hence, for all $s \in [0, a)$ and for all $t \in [0, l_a)$, $\alpha_s(t)$ is defined, and by the continuity of the normal exponential map, $\alpha_a(t) = \lim_{s \to a} \alpha_s(t)$. Now, for all $s \in [0, a)$,

$$\alpha_s(t) \ll \alpha_s(r - s) = (r - s, \sigma(s)) = \eta(s) \leq \gamma(r).$$

Since $J^-(\gamma(r))$ is closed, we conclude that $\alpha_a(t) \in J^-(\gamma(r))$ for all $t \in [0, l_a)$. This implies that α_a is imprisoned in the compact set $J^+(\sigma(a)) \cap J^-(\gamma(r))$, a contradiction to strong causality.

Hence, we must have $l_a > r - a$, and so $a \in A$. But then, by continuity properties of the normal exponential map, there exists a $\delta > 0$ such that for all $s \in [a, a+\delta]$, $l_s > r-a > r-s$, which contradicts the definition of a. Thus, we must have $a = R$, and $(*)$ holds for all $s \in [0, R)$. As discussed above, this implies that the normal geodesics to B are future complete. A similar argument shows that the normal geodesics are also past complete, and hence Claim 4 is established.

To achieve a global splitting from the local splitting result, one can proceed as in §7 in [4] with only minor modifications. In fact, all the results of that section remain valid without the timelike completeness assumption, and only the proof of Proposition 7.1 is affected.

Given two (complete future directed timelike) lines γ_1 and γ_2, Eschenburg defines what it means for γ_1 and γ_2 to be *parallel*, and then shows that every point of M lies on a unique line parallel to the given line γ. The global splitting is then easily obtained. Eschenburg's notion of parallel lines is defined in terms of flat strips. A ***flat strip*** is defined to be a totally geodesic isometric immersion f of $(\mathbb{R} \times I, -dt^2 + ds^2)$ into M, for some interval I, such that $f|_{\mathbb{R}\times\{s\}}$ is a complete timelike line for each $s \in I$.

Eschenburg's Proposition 7.1 asserts that if γ is a complete timelike line in a globally hyperbolic space-time M and $\sigma\colon [0,1] \to M$ is any geodesic with $\sigma(0) = \gamma(0)$, then there exists a flat strip containing γ and σ. His proof works without the assumption of timelike geodesic completeness provided one makes the following additional observation: Suppose there is a flat strip $f\colon \mathbb{R} \times [0,a) \to M$ containing γ and $\sigma|_{[0,v)}$ such that for all t, $\gamma(t) = f(t,0)$, and for all $u \in [0,v)$, $\sigma(u) = f(ku, mu)$, where $k, m \in \mathbb{R}$ and $a = mv$. Parallel translate $\gamma'(0)$ along σ to $\sigma(v)$, and call the resulting vector X. Let $\gamma_v\colon (a,b) \to M$ be the inextendible geodesic satisfying $\gamma_v'(0) = X$. Then γ_v is complete, i.e., $a = -\infty$, $b = \infty$. Indeed, suppose $b < \infty$. Then for each $t \in (a,b)$, $\gamma_v(t)$ can be reached in the limit by a sequence of points of the form $f(t_n, s_n)$, where $t_n < kv + b$. Using the geometry of $(\mathbb{R} \times [0,a), -dt^2 + ds^2)$, we see there exists a number $\tau > kv + b$ such that $\gamma_v \subset J^-(\gamma(\tau))$, which contradicts the global hyperbolicity of M: either $J^+(\gamma_v(0)) \cap J^-(\gamma(\tau))$ is noncompact or there is a strong causality violation. Hence, $b = \infty$, and similarly $a = -\infty$, i.e., γ_v is complete. With this observation, Eschenburg's proof of Proposition 7.1 goes through unaltered. Thus, having established the local splitting result (Claim 4), we obtain, via §7 in [4], the desired global splitting.

We make some concluding remarks. In [7] a splitting theorem is obtained for space-times which contain a maximal hypersurface S and an S-*ray*, i.e., a ray which realizes the distance to S from each of its points. The techniques of the present paper can be used there, as well, to eliminate the assumption of future timelike completeness.

We wish to point out that the Lorentzian splitting problem in the form posed by Yau [10] is still open. The reason for this is that Yau does not make the assumption of global hyperbolicity, but instead assumes that space-time is geodesically complete. Although the assumption of global hyperbolicity in Lorentzian geometry is perfectly natural (and, in some sense, more natural than the assumption of geodesic completeness), Yau's statement of the Lorentzian splitting problem parallels more closely the statement of the Riemannian splitting theorem. We believe Yau's statement of the Lorentzian splitting problem to be true. Some evidence for this is provided in [7]. Finally, the physical motivation for establishing a Lorentzian splitting theorem was to study rigidity phenomena within the singularity theory of general relativity. More specifically, the aim is to prove some of the singularity theorems without having to invoke conditions like the so-called "generic condition". The papers [6], [2] address this issue more directly. Some of the questions relating to singularity theory in this context are still open (see e.g. Conjecture 2 in [2]). To apply the Lorentzian splitting theorem to these questions, one needs to establish the existence of a timelike line in space-time under sufficiently general circumstances.

Appendix

Here we present the proof of the technical lemma stated in §2. The proof makes use of the following proposition.

Proposition. *Let $\gamma\colon [a,\infty) \to M$ be a timelike ray in a globally hyperbolic space-time M. For each positive integer n, let β_n be a maximal geodesic segment from $p_n \in I^-(\gamma(r_n))$ to $\gamma(r_n)$. If $p_n \to \gamma(0)$ and $r_n \to \infty$, then $\beta_n'(0) \to \gamma'(0)$.*

The proposition follows from the RTI: assuming the conclusion is false, one can use the RTI to establish the existence of a cut point to $\gamma(0)$ along $\gamma|_{[0,\infty)}$. We omit the details.

Proof of the technical lemma. Fix a Riemannian metric g_0 on M. Let U_0 be a nice neighborhood of $\gamma(t)$ contained in $I^-(\gamma(r_0))$. Let $p = \gamma(t)$, and for some $\tau > 0$, let $p' = \gamma(t+\tau)$. We have $p' = \exp X$, where $X = \tau\gamma'(t)$. Let $V \subset TU_0$ be a neighborhood of X such that exp is defined on V. By choosing τ, U_0, and V sufficiently small, we can ensure that the map $D\colon U_0 \times V \to \mathbb{R}$ defined by

$$D(x,v) = d(x, \exp v)$$

is smooth and positive. For $v \in V$, let $\rho_v: U_0 \to \mathbb{R}$ be defined by $\rho_v(x) = D(x, \exp v)$. Then the map $V \times TU \to \mathbb{R}$ defined by $(v, w) \to \operatorname{Hess}\rho_v(w, w)$ is smooth. Thus, there exist $C > 0$, a neighborhood $V_1 \subset V$ of X and a neighborhood $U_1 \subset U_0$ of p such that,

$$\text{(A1)} \qquad \operatorname{Hess}\rho_v(w, w) \geq -C$$

$\forall v \in V_1$ and $\forall w \in TU_1$ with $g_0(w, w) \leq 1$.

Consider the function $F: [r_0, \infty] \times U_1 \to TU_1$ defined as

$$F(r, q) = \begin{cases} \tau\eta'_{q,r}(0), & r < \infty, \\ X, & r = \infty, \end{cases}$$

where $\eta_{q,r}: [0, l_{q,r}] \to M$ is any maximal geodesic segment from q to $\gamma(r)$. The proposition implies that F is continuous at (∞, p). Thus, there exist a neighborhood $U \subset U_1$ of p and $r_1 \geq r_0$ such that $F([r_1, \infty] \times U) \subset V_1$. In other words, we have shown that $\forall r \geq r_1$ and $\forall q \in U$, $\exists$ a maximal segment $\eta_{q,r}$ from q to $\gamma(r)$ such that (A1) holds for $v = \tau n'_{q,r}(0)$ and $\forall w \in TU$ with $g_0(w, w) \leq 1$. But for this value of v, $\rho_v(x) = d(x, \eta_{q,r}(\tau))$. Thus, we have $\forall r \geq r_1$ and $\forall q \in U$, $\exists$ a maximal segment $\eta_{q,r}$ from q to $\gamma(r)$ such that (2.6) holds for $s = \tau$ and $\forall w \in TU$ with $g_0(w, w) \leq 1$. Since, for fixed $w \in T_qM$, $s \to \operatorname{Hess} d^s_{q,r}(w, w)$ is increasing on $[\tau, l_{q,r})$, we conclude that $\forall r \geq r_1$ and $\forall q \in U$, $\exists$ a maximal segment $\eta_{q,r}$ from q to $\gamma(r)$ such that (2.6) holds $\forall w \in T_qM$ with $g_0(w, w) \leq 1$ and $\forall s \in [\tau, l_{q,r})$. The technical lemma now follows by restricting to S and observing that the Riemannian norms induced by $g_0|_S$ and $g|_S$ are uniformly equivalent on S.

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