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MINIMAL SURFACES IN KÄHLER SURFACES AND RICCI CURVATURE

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Introduction

Let M be a surface immersed in a Kähler surface N. In [2] the author and S. S. Chern defined a function α on M which measures the deviation of the tangent plane T_*M of M from a complex line of T_*N. For example, for $p, q \in M \subset N$ if $\alpha(p) = 0$ then T_pM is a complex line of T_pN and if $\alpha(q) = \pi$ then T_qM is an anticomplex line of T_qN. The point p is called a complex tangent point of M and the point q an anticomplex tangent point of M. The analysis of [2] shows that when the immersion $M \to N$ is minimal, the complex and anticomplex tangent points of M are isolated. Also, though α is continuous everywhere on M, it fails to be differentiable at the complex and anticomplex tangent points.

Assume that the immersion $M \to N$ is minimal and let P denote the number of complex tangent points and Q denote the number of anticomplex tangent points both counted according to multiplicity. In [6] S. Webster showed that

$$-P - Q = \chi(M) + \chi(\nu), \tag{0.1}$$

where $\chi(M)$ is the Euler characteristic of M, and $\chi(\nu)$ is the Euler characteristic of the normal bundle of M in N. Webster then used (0.1) to show that there are no nonholomorphic minimally embedded two-spheres in $\mathbf{C}P^2$. In [7] Webster showed that

$$Q - P = c_1(N)([M]), \tag{0.2}$$

where $c_1(N)$ is the first Chern class of N and $[M] \in H_2(N; \mathbf{Z})$ is the homology class of M in N. Both (0.1) and (0.2) are proved using global arguments.

The present paper began, in 1984, with an attempt to derive a local version of (0.1). This attempt led to an upper bound on the first eigenvalue of a totally real minimal surface in $\mathbf{C}P^2$ (see Theorem 2.3). In §1 we derive local versions

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of (0.1) and (0.2) as follows. Set $f(\alpha) = \ln(\tan^2 \alpha/2)$ and $g(\alpha) = \ln(\sin^2 \alpha)$. Then away from the complex and anticomplex tangent points,

$$i\partial\overline{\partial} f(\alpha) = \mathrm{Ric}, \tag{0.3}$$

$$i\partial\overline{\partial} g(\alpha) = (K + K_\nu)\, d\,\mathrm{vol}_M, \tag{0.4}$$

where Ric is the pull-back to M of the Ricci 2-form of N, K is the Gauss curvature of M, K_ν is the normal curvature of M in N and $d\,\mathrm{vol}_M$ is the volume form on M. Taking the complex and anticomplex tangent points into account leads to the equations of currents on M (1.26) and (1.27). Integration then yields formulas (0.2) and (0.1) respectively. Our derivation of Webster's formulas is very much in the spirit of such classical algebraic geometric formulas as the Poincaré-Lelong equation and the Plücker formulas. More recently similar techniques have been employed in the study of harmonic maps of surfaces by R. Schoen and S. T. Yau [4] and by D. Toledo [5]. Classically given a meromorphic section σ of a hermitian line bundle L over a Riemann surface, a formula relating the curvature of L and the divisor of σ can be found by computing $\partial\overline{\partial} \ln |\sigma|$. Our derivations can be seen from this perspective by considering the line bundles $\det(T_*N)$ and $T_*M \otimes \nu$ over M.

The local formulas (1.26) and (1.27) are of interest themselves. For example, in §2 we let N be a Kähler-Einstein surface and show that if N has negative scalar curvature and M is a totally real minimal surface in N then M is, in fact, Lagrangian. If N is a Ricci flat Kähler surface and M is a totally real minimal surface we show that there is a compatible complex structure on N such that M is a holomorphic curve for this complex structure.

In §3 we use (0.1) and (0.2) to derive restrictions on homology classes that can be represented by embedded minimal surfaces of genus g.

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1. The equations

Let M be a compact, connected, oriented surface, let N be a compact Kähler surface and let $F\colon M \to N$ be a minimal immersion. Throughout this paper we assume, unless stated otherwise, that F is neither a holomorphic nor an antiholomorphic map. All the results of this paper extend without difficulty to branched minimal immersions. We will consequently generally leave these considerations to the reader. The metric on M induced by F can be written

$$ds^2_M = \phi \circ \bar{\phi}, \tag{1.1}$$

where ϕ is a complex valued 1-form defined up to a complex factor of norm one. As in [2] we can choose a unitary coframe $\{\omega_1, \omega_2\}$ for N such that

$$\omega_1 = s\phi, \qquad \omega_2 = t\bar{\phi}, \tag{1.2}$$

where s and t are complex-valued functions on M. Since ds^2_M is the induced metric, we have

$$\omega_1 \circ \bar{\omega}_1 + \omega_2 \circ \bar{\omega}_2 = \phi \circ \bar{\phi},$$

which yields $|s|^2 + |t|^2 = 1$. By setting

$$|s| = \cos\alpha/2, \qquad |t| = \sin\alpha/2, \tag{1.3}$$

where α is a function on M with values between 0 and π, we can choose a unitary coframe $\{\omega_1, \omega_2\}$ satisfying

$$\omega_1 = \cos\frac{\alpha}{2}\phi, \qquad \omega_2 = \sin\frac{\alpha}{2}\bar{\phi}. \tag{1.4}$$

At a point $p \in M$ with $\alpha(p) = 0$ the tangent plane T_pM of M at p is a complex line in $T_{F(p)}N$. Similarly at a point q with $\alpha(q) = \pi$ the tangent plane T_qM is an anticomplex line in $T_{F(q)}N$. Such points are called *complex* (resp. *anticomplex*) *tangent points.* On a minimal surface the complex and anticomplex tangent points are isolated. A surface which has no complex or anticomplex tangent points is called *totally real.* We remark that while α is a smooth function away from the complex and anticomplex tangent points it is only continuous at these points. The nature of the singularities of α at the complex and anticomplex tangent points is of fundamental interest to us. However to begin we compute where α is smooth.

From (1.4) we have

$$\sin\frac{\alpha}{2}\omega_1 - \cos\frac{\alpha}{2}\bar{\omega}_2 = 0. \tag{1.5}$$

Taking the exterior derivative of (1.5) and using Cartan's Lemma we have

$$\tfrac{1}{2}[d\alpha + \sin\alpha(\omega_{1\bar{1}} + \omega_{2\bar{2}})] = a\phi + b\bar{\phi}, \qquad \omega_{1\bar{2}} = b\phi + c\bar{\phi}, \tag{1.6}$$

where $(\omega_{\alpha\bar{\beta}})$, $1 \le \alpha, \beta \le 2$, is the connection 1-form of N and the complex valued functions a, b and c are the components of the second fundamental form of M in N (see [8]). The condition that F is a minimal immersion is expressed by

$$b = 0. \tag{1.7}$$

So

$$\tfrac{1}{2}[d\alpha + \sin\alpha(\omega_{1\bar 1} + \omega_{2\bar 2})] = a\phi. \tag{1.8}$$

Adding (1.8) and its conjugate we have

$$d\alpha = a\phi + \bar a\bar\phi. \tag{1.9}$$

Thus

$$\bar\partial\alpha = \bar a\bar\phi = \tfrac{1}{2}[d\alpha - \sin\alpha(\omega_{1\bar 1} + \omega_{2\bar 2})]. \tag{1.10}$$

Taking the exterior derivative of (1.10) gives

$$\begin{aligned}\partial\bar\partial\alpha = d\bar\partial\alpha &= -\frac{1}{2}\cos\alpha\, d\alpha \wedge (\omega_{1\bar 1} + \omega_{2\bar 2}) - \frac{1}{2}\sin\alpha(\Omega_{1\bar 1} + \Omega_{2\bar 2}) \\ &= \frac{\cos\alpha}{\sin\alpha}\partial\alpha \wedge \bar\partial\alpha - \frac{i}{2}\sin\alpha\,\mathrm{Ric},\end{aligned} \tag{1.11}$$

where $-i(\Omega_{1\bar 1} + \Omega_{2\bar 2}) = \mathrm{Ric}$ denotes the Ricci 2-form of N pulled back by F to M. Set

$$f(\alpha) = \ln(\tan^2(\alpha/2)). \tag{1.12}$$

Then

$$\partial\bar\partial f(\alpha) = f''(\alpha)\partial\alpha \wedge \bar\partial\alpha + f'(\alpha)\partial\bar\partial\alpha = -i\,\mathrm{Ric}. \tag{1.13}$$

Set

$$g(\alpha) = \ln(\sin^2\alpha). \tag{1.14}$$

Then

$$\begin{aligned}\partial\bar\partial g(\alpha) &= -2csc^2\alpha\partial\alpha \wedge \bar\partial\alpha + 2\cot\alpha\partial\bar\partial\alpha \\ &= -2\partial\alpha \wedge \bar\partial\alpha - i\cos\alpha\,\mathrm{Ric}.\end{aligned} \tag{1.15}$$

(1.13) and (1.15), as equations of 2-forms, are valid away from the complex and anticomplex tangent points of M. To simplify (1.15) we note that by (1.4),

$$\phi = \cos\frac{\alpha}{2}\omega_1 + \sin\frac{\alpha}{2}\bar\omega_2.$$

So

$$d\phi = \left(\cos^2\frac{\alpha}{2}\omega_{1\bar 1} + \sin^2\frac{\alpha}{2}\omega_{2\bar 2}\right) \wedge \phi.$$

And thus since $d\phi = -i\rho \wedge \phi$,

$$i\rho = \sin^2\frac{\alpha}{2}\omega_{2\bar 2} - \cos^2\frac{\alpha}{2}\omega_{1\bar 1}, \tag{1.16}$$

where ρ is the connection 1-form of ds_M^2. Similarly,

$$i\rho_\nu = \sin^2\frac{\alpha}{2}\omega_{1\bar 1} - \cos^2\frac{\alpha}{2}\omega_{2\bar 2}, \tag{1.17}$$

where ρ_ν is the connection 1-form of the normal bundle ν. The Gauss curvature K of M is uniquely determined by the equation

$$d\rho = -K\frac{i}{2}\phi \wedge \bar{\phi}. \tag{1.18}$$

The normal curvature K_ν of ν is uniquely determined by the equation

$$d\rho_\nu = -K_\nu\frac{i}{2}\phi \wedge \bar{\phi}. \tag{1.19}$$

Taking the exterior derivative of (1.16) and (1.17) and using (1.18) and (1.19) we have

$$\tfrac{1}{2}(K + K_\nu)\phi \wedge \bar{\phi} = -2\partial\alpha \wedge \bar{\partial}\alpha - i\cos\alpha\,\mathrm{Ric}\,. \tag{1.20}$$

Combining (1.15) and (1.20) we have

$$\partial\bar{\partial}g(\alpha) = -i(K + K_\nu)\,d\,\mathrm{vol}_M, \tag{1.21}$$

where $d\,\mathrm{vol}_M = (i/2)\phi \wedge \bar{\phi}$ is the volume form of M.

To analyse (1.13) and (1.21) at the singularities of α we return to (1.2). Taking the exterior derivative of the first equation of (1.2) we have

$$(ds - i\rho s - s\omega_{1\bar{1}}) \wedge \phi = 0. \tag{1.22}$$

Let ς be a local complex coordinate on M. Then (1.22) gives

$$\frac{\partial s}{\partial \bar{\varsigma}} = sh, \tag{1.23}$$

where h is the complex-valued function such that $hd\bar{\varsigma}$ is the (0,1) part of $i\rho + \omega_{1\bar{1}}$. A well-known result of Bers [1] then implies that the zeros of s are isolated and that if the complex coordinate ς is centered at a zero q of s, then near q

$$s = \tilde{s}\varsigma^\sigma, \tag{1.24}$$

where $\tilde{s}(q) \neq 0$ and σ is a positive integer. Similarly if ς is centered at a zero p of t then near p

$$t = \tilde{t}\bar{\varsigma}^\tau, \tag{1.25}$$

where $\tilde{t}(p) \neq 0$ and τ is a positive integer. The point p is a complex tangent point of *order* τ (write, $\mathrm{ord}(p) = \tau$). The point q is an anticomplex tangent point of order σ ($\mathrm{ord}(q) = \sigma$). We remark that the index of a complex or anticomplex tangent point as defined by Webster [1] is the negative of the order of the point as defined here.

For $x \in M$ let δ_x denote the Dirac delta function at x.

Theorem 1.1. *Let $F\colon M \to N$ be a minimal immersion of the surface M into the Kähler surface N. Denote the complex tangent points of M by*

$\{p_k\}$ and the anticomplex tangent points of M by $\{q_l\}$. Then the equations of currents

$$\partial\bar{\partial} f(\alpha) + i\,\mathrm{Ric} = 2\pi i \left\{ \sum_l \mathrm{ord}(q_l)\delta_{q_l} - \sum_k \mathrm{ord}(p_k)\delta_{p_k} \right\}, \tag{1.26}$$

$$\begin{aligned} &\partial\bar{\partial} g(\alpha) + i(K + K_\nu)\, d\,\mathrm{vol}_M \\ &\qquad = -2\pi i \left\{ \sum_l \mathrm{ord}(q_l)\delta_{q_l} + \sum_k \mathrm{ord}(p_k)\delta_{p_k} \right\} \end{aligned} \tag{1.27}$$

hold on M.

Proof. We prove (1.26), the proof of (1.27) being entirely similar.

Choose $\varepsilon > 0$ so small that the ε-balls $B_\varepsilon(p_k)$ and $B_\varepsilon(q_l)$ centered at the p_k and the q_l are pairwise disjoint. For any $h \in C^\infty(M)$ by (1.13)

$$\begin{aligned} \int_M h(\partial\bar{\partial} f(\alpha) + i\,\mathrm{Ric}) = \lim_{\varepsilon \to 0} \Bigg\{ &\sum_k \int_{B_\varepsilon(p_k)} h(\partial\bar{\partial} f(\alpha) + i\,\mathrm{Ric}) \\ &+ \sum_l \int_{B_\varepsilon(q_l)} h(\partial\bar{\partial} f(\alpha) + i\,\mathrm{Ric}) \Bigg\}. \end{aligned} \tag{1.28}$$

Now for each p_k, by Stokes' theorem,

(1.29)

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{B_\varepsilon(p_k)} h(\partial\bar{\partial} f(\alpha) + i\,\mathrm{Ric}) \\ &\quad = \lim_{\varepsilon \to 0} \left\{ \int_{\partial B_\varepsilon(p_k)} h\bar{\partial} f(\alpha) - \int_{B_\varepsilon(p_k)} \partial h \wedge \bar{\partial} f(\alpha) + \int_{B_\varepsilon(p_k)} hi\,\mathrm{Ric} \right\}. \end{aligned}$$

Let ς be a local complex coordinate centered at p_k. Then t satisfies (1.25) with $\tau = \mathrm{ord}(p_k)$ and $s(p_k) \neq 0$. Using (1.3) we obtain

$$\bar{\partial} f(\alpha) = \bar{\partial} \ln\left(\frac{|t|^2}{|s|^2}\right) = \bar{\partial} \ln\left(\frac{|\tilde{t}|^2}{|s|^2}\right) + \bar{\partial} \ln(\varsigma^{\mathrm{ord}(p_k)}) + \bar{\partial} \ln(\bar{\varsigma}^{\mathrm{ord}(p_k)}). \tag{1.30}$$

Hence,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p_k)} h\bar{\partial} f(\alpha) &= \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(p_k)} h\, \mathrm{ord}(p_k) \frac{d\bar{\varsigma}}{\bar{\varsigma}} \\ &= -2\pi i\, \mathrm{ord}(p_k) h(p_k), \end{aligned} \tag{1.31}$$

since the first term in the right-hand side of (1.30) is bounded and the second term is zero. Also using (1.30) the other two integrals in the right-hand side of (1.29) go to zero as $\varepsilon \to 0$. Similarly,

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon(q_l)} h(\partial\bar{\partial} f(\alpha) + i\,\mathrm{Ric}) = 2\pi i\, \mathrm{ord}(q_l) h(q_l). \tag{1.32}$$

(1.26) follows.

Remarks. (1) If $F: M \to N$ is a branched minimal immersion, equation (1.26) remains true. (1.27) must be altered by the addition of branching index terms to the right-hand side. We leave the details to the reader.

(2) Using (1.24) and (1.25) it is straightforward to verify that both $f(\alpha)$ and $g(\alpha)$ are in $L^p(M)$ for $0 < p < \infty$, and in $L_1^p(M)$ for $1 \le p < 2$. If $f(\alpha) \in L_1^2(M)$ (or $g(\alpha) \in L_1^2(M)$), then the singularities of $f(\alpha)$ (or $g(\alpha)$) are removable and the immersion is totally real.

Corollary 1.2. *With the hypotheses of the theorem, let P denote the sum of the orders of all the complex tangent points, and Q denote the sum of the orders of all the anticomplex tangent points. Then*

$$Q - P = F^* c_1(N)[M], \tag{1.33}$$

$$-(Q + P) = \chi(M) + \chi(\nu), \tag{1.34}$$

where $c_1(N)$ is the first Chern class of N, $[M]$ is the fundamental homology class of M, $\chi(M)$ is the Euler characteristic of M, and $\chi(\nu)$ is the Euler characteristic of ν.

Proof. Recall that the first Chern form of N satisfies $c_1(N) = (1/2\pi)\,\mathrm{Ric}$. Thus integrating the left-hand side of (1.26) against the test function $h \equiv 1$ gives

$$i \int_M \mathrm{Ric} = 2\pi i \int_M c_1(N) = 2\pi i F^* c_1(N)[M],$$

and (1.33) follows. Integrating the left-hand side of (1.27) against $h \equiv 1$ gives

$$i \int K \, d\,\mathrm{vol}_M + i \int K_\nu \, d\,\mathrm{vol}_M = 2\pi i(\chi(M) + \chi(\nu)),$$

and (1.34) follows. q.e.d.

As mentioned in the Introduction Webster proved (1.33) and (1.34) using global arguments.

Remark. If $H: M \to N$ is a holomorphic curve, then using the above notation

$$H^* c_1(N)[M] = \chi(M) + \chi(\nu).$$

(1.33) and (1.34) can be regarded as measuring the "global" deviation of this formula for a minimal immersion.

2. Applications: Kähler-Einstein surfaces

Let ω denote the Kähler form of N, and let Ric denote the Ricci 2-form of N. In this section we will assume that N is a Kähler-Einstein surface, i.e., there is a constant c such that

$$\text{Ric} = c\omega. \tag{2.1}$$

The existence of many examples of such Kähler surfaces is assured by Yau's proof of the Calabi conjecture [10]. The scalar curvature of N, denoted R, is, by definition,

$$R = \text{tr}(\text{Ric}). \tag{2.2}$$

It follows from (2.1) that R is constant and that $c = R$. On M the Kähler form can be written

$$\omega = \frac{i}{2}(\omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2) = \frac{i}{2} \cos \alpha \phi \wedge \bar{\phi}. \tag{2.3}$$

Also,

$$\partial \bar{\partial} f(\alpha) = \tfrac{1}{4} \Delta f(\alpha) \phi \wedge \bar{\phi}, \tag{2.4}$$

where $-\Delta$ is the Laplace-Beltrami operator on M. Setting

$$u = f(\alpha), \tag{2.5}$$

(1.13) becomes

$$\Delta u = 2Rh(u), \tag{2.6}$$

where $h(x) = \cos \circ f^{-1}(x)$. From (1.12) we get $f^{-1}(x) = 2 \arctan(e^{x/2})$ and so,

$$h(x) = -\tanh(x/2), \tag{2.7}$$

and (2.6) becomes

$$\Delta u = -R \tanh(u). \tag{2.8}$$

A surface immersed in N, $j\colon M \to N$, is called Lagrangian if $j^*\omega = 0$. By (2.3) this is equivalent to $\alpha = \pi/2$.

Theorem 2.1. *Let N be a Kähler-Einstein surface of negative scalar curvature. If $F\colon M \to N$ is a totally real (branched) minimal immersion, then $F\colon M \to N$ is Lagrangian.*

Proof. Note that $u = f(\alpha)$ is a C^∞ function since $F\colon M \to N$ is totally real. (u extends smoothly across branch points.) The maximum principle applied to (2.8) implies that u attains its maximum when $u \leq 0$. On the other hand the minimum principle implies that u attains its minimum when $u \geq 0$. Hence $u = 0$ or $\alpha = \pi/2$.

A Ricci-flat $K3$ surface (i.e., a Ricci-flat simply-connected Kähler surface) admits a family of complex structures, parametrized by the two-sphere, with the property that each complex structure of this family together with the metric of the surface determines a distinct Kähler structure. We say each of these complex structures is "compatible with the metric."

Theorem 2.2. *Let N be a Ricci-flat $K3$ surface and let $F\colon M \to N$ be a totally real (branched) minimal immersion. With respect to one of the complex structures on N compatible with the metric, $F\colon M \to N$ is a holomorphic map.*

Proof. The equation $\Delta u = 0$ implies that, for any of the compatible complex structures, $\alpha =$ constant. Choose a point $p \in M$ and consider the tangent plane T_pM as a subspace of $T_{F(p)}N$. One of the compatible complex structures on N gives a complex structure on the vector space $T_{F(p)}N$ such that T_pM is a complex line. For this complex structure $\alpha(p) = 0$. But α is constant on M. Consequently $\alpha \equiv 0$ and $F\colon M \to N$ is a holomorphic curve for this complex structure.

Theorem 2.3. *Let N be an Kähler-Einstein surface of positive scalar curvature R. Let $F\colon M \to N$ be a totally real (branched) minimal immersion which is not Lagrangian. Then $\lambda_1(M) < R$, where $\lambda_1(M)$ is the first eigenvalue of the Laplace-Beltrami operator on M with the induced metric.*

Proof. The assumptions of the theorem imply that (2.6) has a nonconstant solution u. This solution satisfies $\int_M h(u)\,d\mathrm{vol} = 0$. The Poincaré equality gives that

$$\lambda_1 = \inf_k \frac{\int_M |\nabla k|^2\,d\mathrm{vol}}{\int_M k^2\,d\mathrm{vol}},$$

where the inf is taken over functions k satisfying $\int_M k\,d\mathrm{vol} = 0$. Applying this to $h(u)$ we have

$$\int_M (h(u))^2\,d\mathrm{vol} \le \frac{1}{\lambda_1}\int_M |\nabla h(u)|^2\,d\mathrm{vol}.$$

From (2.7) we have $|h'(u)| \le \frac{1}{2}$ and so

$$|\nabla h(u)|^2 = |h'(u)|^2|\nabla u|^2 \le \tfrac{1}{2}|h'(u)|\,|\nabla u|^2$$

with equality if and only if $u = 0$. Hence

$$\int_M (h(u))^2\,d\mathrm{vol} < \frac{1}{\lambda_1}\int_M \frac{1}{2}|h'(u)|\,|\nabla u|^2\,d\mathrm{vol}.$$

From (2.6) we have

$$\int_M [\Delta u \cdot h(u) - 2R(h(u))^2]\,d\mathrm{vol} = 0.$$

Integrating by parts,

$$\int_M [h'(u)|\nabla u|^2 + 2R(h(u))^2]\,d\mathrm{vol} = 0,$$

so

$$0 < \int_M |\nabla u|^2 \left[h'(u) + \frac{1}{\lambda_1} R|h'(u)| \right] d\,\mathrm{vol}.$$

As u is not constant, $|\nabla u|^2$ is not identically zero. Thus for some point $p \in M$

$$h'(u(p)) + \frac{1}{\lambda_1} R|h'(u(p))| > 0.$$

But $h'(u(p)) < 0$, so we have $\lambda_1 < R$. q.e.d.

The proof of this theorem is the result of discussions with C. Croke. It is a pleasure to thank him.

Remark. If N is $\mathbf{C}P^2$ equipped with the Fubini-Study metric of constant holomorphic bisectional curvature 4, then $R = 6$. The eigenvalue estimate becomes $\lambda_1 < 6$. In $\mathbf{C}P^2$ there is a minimal Lagrangian torus with $\lambda_1 = 6$, namely the Clifford torus (see [6] or [9] for details). Thus the requirement that M not be Lagrangian is necessary.

Theorem 2.4. *Let N be an Kähler-Einstein surface of scalar curvature R, and let $F\colon M \to N$ be a (branched) minimal immersion. Then*

$$Q - P = \tfrac{1}{2} R \deg F.$$

Proof. The proof is left to the reader.

Corollary 2.5. *$Q = P$ when N is Ricci flat.*

This corollary was first observed by Al Vitter by showing that when N is Ricci flat, the function $s/\bar{t}$ of (1.2) is meromorphic.

Now consider (1.15) under the assumption that N is a Kähler-Einstein surface. (2.1)–(2.4) imply that (1.15) becomes

$$\Delta g(\alpha) = -2|\nabla \alpha|^2 + 2R\cos^2\alpha, \tag{2.9}$$

where the gradient, $\nabla\alpha$, of α satisfies

$$\partial\alpha \wedge \bar{\partial}\alpha = \tfrac{1}{4}|\nabla\alpha|^2 \phi \wedge \bar{\phi}. \tag{2.10}$$

A point $p \in M$ is called a *Lagrangian point* if $T_pM \subset T_{F(p)}N$ is a Lagrangian plane or, equivalently, if $\alpha(p) = \pi/2$.

Proposition 2.6. *If N is a Kähler-Einstein surface with positive scalar curvature, and $F\colon M \to N$ is a minimal immersion, then the set of Lagrangian points on M equals the set of local maximum points of $g(\alpha)$ on M. Consequently every such minimal surface admits Lagrangian points.*

Proof. Clearly at the Lagrangian points the function $g(\alpha) = \ln(\sin^2\alpha)$ assumes its maximum value. Suppose that $p \in M$ is a local maximum point of $g(\alpha)$. At p,

$$0 = \nabla g(\alpha) = 2\cot\alpha \cdot \nabla\alpha.$$

So either $(\nabla\alpha)(p) = 0$ or $\alpha(p) = \pi/2$. If the latter holds we are done, so suppose $(\nabla\alpha)(p) = 0$. Then (2.9) gives

$$0 \geq (\Delta g(\alpha))(p) = 2R\cos^2\alpha(p).$$

This implies $\alpha(p) = \pi/2$.

3. Applications: Global results

Let N be a compact Kähler surface equipped with any Kähler metric, and $F\colon M \to N$ be an immersion. If g is the genus of M then

$$\chi(M) = 2 - 2g. \tag{3.1}$$

Let $F_*[M]^\# \in H^2(N;\mathbf{Z})$ denote the Poincaré dual of $F_*[M]$. The self-intersection number, I_F, of M is

$$I_F = (F_*[M]^\# \cup F_*[M]^\#)([N]). \tag{3.2}$$

Let D_F denote the number of double points of F. Then

$$\chi(\nu) = I_F - 2D_F. \tag{3.3}$$

(3.1) and (3.3) together with (1.33) and (1.34) show that P and Q are determined by the homology class of $F_*[M]$ and the number of double points of F. In particular if F is an embedding, then P and Q are determined by homology only. Setting

$$c_1(N)(F_*[M]) = c_1(F), \tag{3.4}$$

we have

Theorem 3.1. *Let $F\colon M \to N$ be a minimal immersion. Then*

$$(2 - 2g) + |c_1(F)| + I_F - 2D_F \leq -2\min(P, Q) \leq 0. \tag{3.5}$$

If F is, in addition, an embedding, then

$$(2 - 2g) + |c_1(F)| + I_F \leq -2\min(P, Q) \leq 0. \tag{3.6}$$

Proof. The proof is left to the reader.

Theorem 3.1 has many consequences. For example we have

Corollary 3.2. *A homology class $\beta \in H_2(N;\mathbf{Z})$ satisfying $(\beta^\# \cup \beta^\#)[N] \geq 2g_0 - 1$ cannot be represented by an embedded minimal surface of genus $g \leq g_0$.*

Corollary 3.3. *If N is Ricci flat, and $\beta \in H_2(N;\mathbf{Z})$ is a homology class satisfying $(\beta^\# \cup \beta^\#)([N]) \geq 2g_0 - 2$, then an embedded minimal surface of genus $g \leq g_0$ must have genus g_0 and must be holomorphic for one of the compatible complex structures on N.*

Proof. As N is Ricci flat, $c_1(N) = 0$. Thus $(2-2g)+I_F \leq 0$, for a minimal embedding $F\colon M \to N$, with equality if and only if M is totally real. The result follows from Corollary 3.2 and Theorem 2.2.

Let N be $\mathbf{C}P^2$ equipped with any Kähler metric. We normalize this metric so that its Kähler form ω satisfies $(1/\pi)\int_{\mathbf{C}P^1} \omega = 1$. If $F\colon M \to N$ has degree d, then

$$Q - P = c_1(N)(F_*[M]) = 3d. \tag{3.7}$$

Thus Theorem 3.1 becomes

Corollary 3.4. *Let $F\colon M \to \mathbf{C}P^2$ be a minimal immersion of degree d. Then*

$$(2-2g) + 3|d| + d^2 - 2D_F \leq -2\min(P, Q) \leq 0. \tag{3.8}$$

If F is, in addition, an embedding, then

$$(2-2g) + 3|d| + d^2 \leq -2\min(P, Q) \leq 0. \tag{3.9}$$

Corollary 3.4 is due to Webster [7]. Its consequences when $g = 0$ or 1 are investigated in [6] and when $g = 2$ in [7]. We discuss the case $g = 3$ to illustrate the use of this result. (3.7) and (3.9) imply that an embedded minimal surface of genus 3 either has degree zero and two complex and two anticomplex tangent points or has degree 1 and three anticomplex tangent points. (The case degree -1 and three complex tangent points is the latter case with the orientation reversed.) The reader can continue this analysis and apply similar reasoning to (3.5) and (3.6). Note that, except in genus one, there are no totally real embedded minimal surfaces in $\mathbf{C}P^2$.

Example. *Superminimal surfaces and the Plücker formulas.* Let N be $\mathbf{C}P^2$ equipped with the Fubini-Study metric. We consider the superminimal surfaces as described in [2]. Take a holomorphic curve $h_0\colon M \to \mathbf{C}P^2$, where M is a Riemann surface of genus g. Its first associated curve $h_1\colon M \to G(2,3) \approx \mathbf{C}P^2$ is given by $h_1(\varsigma) = h_0(\varsigma) \wedge h_0'(\varsigma)$ for $\varsigma \in M$. The line in $h_0(\varsigma) \wedge h_0'(\varsigma)$ orthogonal to $h_0(\varsigma)$ describes a map $h\colon M \to \mathbf{C}P^2$ which is minimal. Such minimal maps are called *superminimal.* We remark that when M has genus zero, every minimal map is superminimal.

The geometry of a holomorphic curve and its associated curve is classically described in the Plücker formulas [3]. Let β_l, $l = 0, 1$, be the total ramification index of h_l, and let d_l, $l = 0, 1$, denote the degree of h_l. Then the Plücker formulas are

$$-d_0 + 2d_1 = -(2g-2) + \beta_1, \qquad -2d_0 + d_1 = (2g-2) - \beta_0. \tag{3.10}$$

We can relate the invariants β_l and d_l to the superminimal surface h. Choose a unitary framing $\{Z_0, Z_1, Z_2\}$ of $\mathbf{C}^3$ along M adapted to our situation as follows. Let $Z_0(\varsigma)$ be a vector in $\mathbf{C}^3$ representing $h_0(\varsigma)$, and let $Z_0(\varsigma)\wedge Z_1(\varsigma)$ represent $h_1(\varsigma)$ for $\varsigma \in M$. Then the vector Z_1 is a homogeneous coordinate vector for the map h. We have

$$d\begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \psi_{0\bar{0}} & \psi_{0\bar{1}} & 0 \\ -\bar{\psi}_{0\bar{1}} & \psi_{1\bar{1}} & \psi_{1\bar{2}} \\ 0 & -\bar{\psi}_{1\bar{2}} & \psi_{2\bar{2}} \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}, \tag{3.11}$$

where the 1-forms $\psi_{0\bar{1}}$ and $\psi_{1\bar{2}}$ are of type (1,0). Write

$$\psi_{1\bar{2}} = s\phi, \qquad -\bar{\psi}_{0\bar{1}} = t\bar{\phi}. \tag{3.12}$$

These equations are equivalent to (1.2) for the minimal immersion h. Consequently the number of zeros of t counted according to multiplicity is P, and the number of zeros of s counted according to multiplicity is Q. On the other hand by the definition of the ramification index of h_0 and h_1, β_0 is the number of zeros of $\psi_{0\bar{1}}$, and β_1 is the number of zeros of $\psi_{1\bar{2}}$ both counted according to multiplicity. Hence,

$$\beta_0 = P, \qquad \beta_1 = Q. \tag{3.13}$$

The degree of a map $g\colon M \to \mathbf{C}P^2$ can be computed by

$$\deg g = \frac{1}{\pi}\int_M g^*\omega,$$

where ω is the Kähler form. Thus

$$\begin{aligned} d = \deg h &= \frac{1}{2\pi}\int_M \psi_{1\bar{2}} \wedge \bar{\psi}_{1\bar{2}} + \bar{\psi}_{0\bar{1}} \wedge \psi_{0\bar{1}} \\ &= \deg h_1 - \deg h_0 = d_1 - d_0. \end{aligned} \tag{3.14}$$

Adding the Plücker formulas (3.10) we have

$$3(d_1 - d_0) = \beta_1 - \beta_0. \tag{3.15}$$

From (3.13) and (3.14) this is (3.7). For superminimal surfaces in $\mathbf{C}P^2$ formula (1.33) is a consequence of the classical Plücker formulas. We also note that the superminimal surfaces provide solutions with singularities of (2.8) with $R > 0$.

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