AN ANALOGUE OF DEMAILLY'S INEQUALITY FOR STRICTLY PSEUDOCONVEX CR MANIFOLDS

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Recently, Bismut has reformulated Demailly's results on the asymptotic dimension of the $\overline{\partial}$-cohomology of a line bundle L^m, where $m \to \infty$ [2], [3]. His main result is the following:

$$\lim_{m \to \infty} (2\pi/m)^n \operatorname{Tr}|_{\Omega^{0,q}(M;L^m)} e^{-t\Delta/m} = \int_M \det\left(\frac{tF}{1 - e^{-tF}}\right) \operatorname{Tr}|_{\Lambda^{0,q}T_x^*M} e^{-tF}.$$

In this formula, n is the complex dimension of the compact complex manifold M, F is the curvature of the holomorphic line bundle L on M, and Δ is the Laplacian $(\overline{\partial} + \overline{\partial}^*)^2$ acting on the space $\Omega^{0,q}(M;L^m)$. Using the bound

$$\dim H^q(M;L^m) = \dim \ker \Delta_{\Omega^{0,q}(M;L^m)} \leq \operatorname{Tr}|_{\Omega^{0,q}(M;L^m)} e^{-t\Delta/m},$$

we obtain Demailly's original inequality from Bismut's formula:

$$\dim H^q(M;L^m) \leq \left(\frac{m}{2\pi}\right)^n \inf_{t < \infty} \int_M \det\left(\frac{tF}{1 - e^{-tF}}\right) \operatorname{Tr}|_{\Lambda^{0,q}T_x^*M} e^{-tF} + o(m^n)$$

$$= \left(\frac{m}{2\pi}\right)^n \int_{\{F \text{ has } q \text{ negative eigenvalues}\}} \det(F) + o(m^n).$$

In particular, if F is positive semidefinite, combining this inequality with the Hirzebruch-Riemann-Roch theorem, we see that

$$\dim H^q(M;L^m) = \begin{cases} (m/2\pi)^n \int_M \det(F) + o(m^n) & \text{if } q = 0, \\ o(m^n) & \text{if } q > 0. \end{cases}$$

For the applications of this inequality, see Demailly's original paper [3].

Bismut proved this formula (and a similar one for the Dirac operator) using much the same proof as he had earlier given of the index theorem for Dirac operators. One of the goals of this paper is to show how these sorts of results may be proved using the symbol calculus technique of our earlier paper [6]. We will apply this method to study an analogue of Demailly's asymptotic result for the $\square_b$ operator on strictly pseudoconvex CR manifolds (which we shall refer to as Heisenberg manifolds, in the interest of brevity of terminology).

This problem is a simple version of the sort of formulas that one should try to prove for more general CR manifolds, and also for the $\overline{\partial}$-Neumann problem. However, we have not been able to find any simplification for the coefficient of the leading order of $\dim \ker \square_b|_{\Omega^{0,q}(M;L^m)}$ in the special case in which the curvature F of L is positive semidefinite.

It is a pleasure to thank Alan Nadel for proposing this problem.

1. Heisenberg manifolds

Roughly speaking, a Heisenberg manifold is a manifold modeled on the Heisenberg group—recall that this is the $2n + 1$-dimensional nilpotent group

$$H_n = \mathbf{C}^n \times \mathbf{R}$$

with the multiplication

$$(a_0, s_0) \cdot (a_1, s_1) = (a_0 + a_1, s_0 + s_1 + 4 \operatorname{Im} a_0 \cdot a_1).$$

This group has the Lie algebra $\mathfrak{h}_n$ with underlying vector space $\mathbf{C}^n \oplus \mathbf{R}$ and Lie bracket

$$[(a_0, s_0), (a_1, s_1)] = (0, 4 \operatorname{Im} a_0 \cdot a_1).$$

In fact, the Lie algebra element (a, s) corresponds to the left invariant vector field on H_n:

$$(a, s) \mapsto X(a) + sT$$

where $X(a) = Z(a) + \overline{Z}(a)$, and the vector fields Z, $\overline{Z}$ and T are defined by the formulas

$$Z(a) = \sum_{i=1} a_i \left(\frac{\partial}{\partial z_i} + \frac{i\overline{z}_i}{4} \frac{\partial}{\partial t} \right),$$

$$\overline{Z}(t) = \sum_{i=1}^{n} \overline{b_i} \left(\frac{\partial}{\partial \overline{z}_i} - \frac{iz_i}{4} \frac{\partial}{\partial t} \right),$$

$$T = \frac{\partial}{\partial t}.$$

The exponentiation map is a diffeomorphism between the two spaces $\mathfrak{h}_n$ and H_n.

The unitary group $\mathrm{U}(n)$ acts by automorphisms on both H_n and $\mathfrak{g}_n$ by the formula

$$g \cdot (a, s) = (g \cdot a, s).$$

Definition 1.1. An *almost Heisenberg manifold* M is a $2n+1$-dimensional manifold with a $\mathrm{U}(n)$-principal bundle $P \xrightarrow{\pi} M$ and an isomorphism of the tangent bundle of M with the associated bundle $P \times_{U(\mathfrak{n})} \mathfrak{h}_n$.

In other words, a point $p \in P$ corresponds to an isomorphism from $\mathfrak{h}_n$ to $T_{\pi(p)}M$, and the collection of all such frames are related one to another by an element of $U(n)$.

Since the complexification of $\mathfrak{h}_n$ naturally splits into three subspaces, spanned respectively by $\{Z_i | 1 \leq i \leq n\}$, $\{\overline{Z}_i | 1 \leq i \leq n\}$ and T, and this splitting is preserved by the action of $U(n)$ on $\mathfrak{h}_n$, the same holds for the complexification of TM.

$$T_\mathbf{C}M = T^{1,0}M \oplus T^{0,1}M \oplus \mathbf{C}T.$$

Definition 1.2. A *Heisenberg manifold* M is an integrable almost-Heisenberg manifold, that is, such that the sub-bundle $T^{1,0}M$ is involutive:

$$[T^{1,0}M, T^{1,0}M] = T^{1,0}M.$$

The most important example of a Heisenberg manifold is a strictly pseudo-convex hypersurface in $\mathbf{C}^{n+1}$. Such a hypersurface does not have a canonical Heisenberg structure, but obtains one after the choice of a strictly plurisubharmonic defining function ϕ such that M is the manifold $\phi = 0$. Then the 1-form θ obtained by restricting $d\phi$ to TM satisfies:

1. θ is a contact form, that is, $\theta \wedge (d\theta)^n$ is a volume form, and

2. the Levi form on $\ker\theta$ defined by $L(X,Y) = d\theta(X,\overline{Y})$ is positive definite.

Such a hypersurface is a Heisenberg manifold: the admissible frames are those which are an isometry from $\ker\theta$ with the Levi form to $\mathbf{C}^n$ with its standard inner product, and which send the vector field T dual to θ to $1 \in \mathbf{R}$. It is clear that the integrability of this Heisenberg structure follows from the integrability of the complex structure of $\mathbf{C}^{n+1}$.

There is a certain canonical connection, called the Webster connection (Webster [11]) on the principal bundle P of a Heisenberg manifold M. Using this connection, we obtain a canonical set of $n^2 + (2n+1)$ vector fields on the principal bundle P which frame TP globally, identifying T_pP with $\mathfrak{h}_n \times u(n)$ for each p. If (a,b,s) represents an element of $\mathfrak{h}_n \otimes_\mathbf{R} \mathbf{C}$, where a and b are in $\mathbf{C}^n$ and s is in $\mathbf{C}$, we will denote the corresponding vector field on P by $Z(a)+\overline{Z}(b)+sT$; if g is an element of $u(n)\otimes_\mathbf{R}\mathbf{C} \cong gl(n,\mathbf{C})$, the corresponding vertical vector field on P is $\operatorname{ad} g$.

To obtain the nontrivial commutation relations among these vector fields, we make use of the integrability of the Heisenberg structure and the fact that the relations must transform correctly under $gl(n,\mathbf{C})$ to derive the following

formulas:

$$[Z(a), \overline{Z}(v)] = -2ia \cdot \bar{b}T + \operatorname{ad} R(a, b),$$

(1)
$$[Z(a), Z(b)] = \operatorname{ad}(a \otimes \overline{A(b)} - b \otimes \overline{A(a)}),$$

$$[Z(a), T] = \operatorname{ad} W(a) + Z(A(a)).$$

The torsion A and the curvatures R and W are tensors on M:

$$R \in C^\infty(P; \operatorname{Hom}(\mathbf{C}^n \otimes (\mathbf{C}^n)^*, \mathrm{u}(n)))^{\mathrm{U}(n)} \cong \Gamma(\Lambda^{1,1}M \otimes \Lambda^{1,1}M),$$

$$W \in C^\infty(P; \operatorname{Hom}(\mathbf{C}^n, \mathrm{u}(n)))^{\mathrm{U}(n)} \cong \Gamma(\Lambda^{0,1}M \otimes \Lambda^{1,1}M),$$

$$A \in C^\infty(P; \operatorname{End}(\mathbf{C}^n))^{\mathrm{U}(n)} \cong \Gamma(\operatorname{End}(\Lambda^{0,1}M)).$$

If E is a Hermitian vector bundle on a Heisenberg manifold, there is a notion of integrability of E analogous to that for a holomorphic vector bundle on a complex manifold.

Definition 1.3. A *Hermitian bundle* is a Heisenberg bundle if its connection ∇ satisfies:

if X and Y are sections of $T^{1,0}M$, then $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$;

that is, the components of the curvature of the bundle in $\Lambda^{2,0}M$ (and in $\Lambda^{0,2}M$, since the connection preserves the Hermitian structure of E) vanish.

As an example, there is the restriction of a holomorphic vector bundle with Hermitian connection on $\mathbf{C}^{n+1}$ to a strictly pseudoconvex hypersurface.

If E is a Heisenberg bundle on M, then its pullback $\pi^* E$ to the frame bundle P of M has a connection ∇ which satisfies the following commutation relations, generalizing those of (1):

$$[\nabla_{Z(a)}, \nabla_{\overline{Z}(b)}] = -i2a \cdot \bar{b}\nabla_T + \operatorname{ad} R(a, b) + F(a, b),$$

(2)
$$[\nabla_{Z(a)}, \nabla_{Z(b)}] = \operatorname{ad}(a \otimes \overline{A(b)} - b \otimes \overline{A(a)}),$$

$$[\nabla_{Z(a)}, \nabla_T] = \nabla(A(a)) + \operatorname{ad} W(a) + K(a).$$

Here, F and K are the curvatures of the bundle E lying respectively in $\Gamma(\Lambda^{1,1}M \otimes \operatorname{End}(E))$ and $\Gamma(\Lambda^{0,1}M \otimes \operatorname{End}(E))$.

2.   The Cauchy-Riemann operator

If the bundle of antiholomorphic differential forms on a Heisenberg manifold M is pulled back to its frame bundle P, it becomes canonically trivialized:

$$\pi^*(\Lambda^{0,*}M) \cong P \times \Lambda^{0,*}\mathbf{C}^n.$$

From the viewpoint of this trivialization, antiholomorphic forms on M are easily described.

Proposition 2.1.   *There is a canonical isomorphism between the space of antiholomorphic differential forms $\Omega^{0,*}(M) = \Gamma(\Lambda^{0,*}M)$ and the space of $U(n)$-equivariant maps from P to the exterior algebra $\Lambda^{0,*}\mathbf{C}^n$.*

The endomorphism algebra of $\Lambda^{0,*}\mathbf{C}^n$ is a Clifford algebra with generators (annihilation and creation operators) a_i and a_i^*, $1 \le i \le n$, adjoint to each other and given by the formulas

$$a_i^* d\bar{z}_I = d\bar{z}_{i \cup I}, \quad \text{where } I \subset 1, \cdots, n,$$

$$a_i d\bar{z}_I = d\bar{z}_{I \setminus \{i\}}.$$

The Cauchy-Riemann operator $\bar{\partial} \colon \Omega^{0,q}(M) \to \Omega^{0,q+1}(M)$ is a generalization to the splitting of Heisenberg manifolds of the operator $\bar{\partial}_b$ on a strictly pseudoconvex CR manifold. It is most easily defined on the principal bundle P, making use of the representation of Proposition 2.1:

$$\bar{\partial}\alpha = \sum_{i=1}^{n} a_i^* \bar{Z}_i(\alpha).$$

This operator satisfies the following properties:

1. (Leibniz's rule) $\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \bar{\partial}\beta$,
2. (integrability of M) $\bar{\partial}^2 = 0$.

In fact, there is a generalization of the Cauchy-Riemann operator which acts on $\Omega^{0,*}(M;E) = \Gamma(\Lambda^{0,*}M \otimes E)$, where E is a Heisenberg bundle. First we have the following straightforward generalization of Proposition 2.1.

Proposition 2.2.   *There is a canonical isomorphism between $\Omega^{0,*}(M;E)$ and the space of $U(n)$-equivariant sections of the bundle $\Lambda^{0,*}\mathbf{C}^n \otimes \pi^*E$ on P.*

The Cauchy-Riemann operator on $\Omega^{0,*}(M;E)$ is defined on P as before as

$$\bar{\partial}_E = \sum_{i=1}^{n} a_i^* \nabla_{\bar{Z}_i}.$$

This satisfies Leibniz's rule, and $\bar{\partial}_E^2 = 0$, since E is a Heisenberg bundle.

Consider the Laplacian of the complex (all operators are to be understood as operating on E-valued forms),

$$\square = (\bar{\partial} + \bar{\partial}^*)^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

There is a Bochner-type formula for $\square$, generalizing the well-known formula for $\square_b$ on a strictly pseudoconvex CR manifold [5].

Theorem 2.3.   *Let $\nabla^*\nabla$ be the composition of*

$$\nabla \colon \Gamma(\Lambda^{0,*}M \otimes E) \to \Gamma(\Lambda^{0,*}M \otimes E \otimes T^*M)$$

with its adjoint. Then we have the formula

$$\Box = \nabla^*\nabla - i(n - 2q)\nabla_T + \sum_{ij}(a_j^*a_i - a_ia_j^*)\bigg(\sum_{kl} R_{ijkl}a_k^*a_l + F_{ij}\bigg).$$

Proof. This calculation is most easily performed upstairs on P. The adjoint of $\overline{\partial}$ is given by the formula

$$\overline{\partial}^* = -\sum_{i=1}^n a_i\nabla_{Z_i}$$

(since $[a_i^*, \nabla_{\overline{Z}_i}] = 0$), and we obtain

$$\begin{aligned}
(\overline{\partial} + \overline{\partial}^*)^2 &= -\frac{1}{2}\sum_{ij}\{(a_ia_j^* + a_j^*a_i)(\nabla_{Z_i}\nabla_{\overline{Z}_j} + \nabla_{\overline{Z}_j}\nabla_{Z_i}) \\
&\qquad + (a_ia_j^* - a_j^*a_i)(\nabla_{Z_i}\nabla_{\overline{Z}_j} - \nabla_{\overline{Z}_j}\nabla_{Z_i})\} \\
&= \nabla^*\nabla - i(n - 2q)\nabla_T + \dots.
\end{aligned}$$

Here, we have used the commutation relations (2) for ∇_{Z_i} and $\nabla_{\overline{Z}_j}$, and the following formula for the action of u(n) on $\Lambda^{0,*}\mathbf{C}^n$:

$$\mathrm{ad}\,g \cdot \alpha = \sum_{ij} g_{ij}a_i^*a_j. \qquad\qquad \text{q.e.d.}$$

Unlike on a complex manifold, the $\overline{\partial}$-complex is not elliptic on a Heisenberg manifold, since its symbol is not invertible when evaluated on the contact form of M. However, it is subelliptic on $\Omega^{0,q}(M; E)$, at least if $0 < q < n$, as was shown by Kohn [8] (see also Folland and Stein [5]); we will see that this follows from the pseudodifferential symbol calculus on Heisenberg manifolds in the next section.

3.   Symbol calculus on Heisenberg manifolds

Just as on a Riemannian manifold, the psuedodifferential symbol calculus is modeled on the algebra of Fourier multipliers on $\mathbf{R}^n$, so the calculus of pseudodifferential operators on a Heisenberg manifold is modeled on the algebra of left-invariant operators on the Heisenberg group H_n. This insight, due to Stein [9], has proved to be very useful in understanding the Laplacian on Heisenberg manifolds, although the symbol calculus has only been fully developed quite recently (Dynin [4], Taylor [10], Beals et al. [1]). Because of this, and also because we will require an extension of their calculus, we present here a brief summary of the main theorems involved.

We start with left-invariant pseudodifferential operators on the Heisenberg group. (In what follows, we will denote a typical point in $\mathfrak{h}_n$ by (a, s), where

$a \in \mathbf{C}^n$ and $s \in \mathbf{R}$; likewise, elements of $\mathfrak{h}_n^*$ will be written (ξ, σ).) These may be written in the form

$$(3) \qquad p(D) = (2\pi)^{-n-1} \int_{\mathfrak{h}_*} \widehat{p}(a, s) \exp(X(a) + sT),$$

where $\widehat{p}$ is the Fourier transform of the symbol p, which is a function on the vector space $\mathfrak{h}_n^*$. Thus, if p is a polynomial function, then the corresponding operator is a differential operator—all elements of the enveloping algebra of $\mathfrak{h}_n$ are describable in this way.

Our first task is to find a nice class of symbols suitably generalizing the polynomials, such that the corresponding operators form an algebra. The usual choice is the following.

Definition 3.1. *A smooth function on the space $\mathfrak{h}_n^*$, less the origin, is homogeneous of degree m if*

$$p(\lambda\xi, \lambda^{1/2}\sigma) = \lambda^m p(\xi, \sigma).$$

The symbol class $\mathscr{S}^m(\mathfrak{h}_n^*)$ is now defined as the space of smooth functions on $\mathfrak{h}_n^*$ having an asymptotic expansion in homogeneous functions of degree less than or equal to m:

$$p \sim \sum_{i \geq 0} p_{m-i}, \quad \text{where } p_{m-i} \text{ is homogeneous of degree } m - i.$$

In this definition, the asymptotic sum has the following sense: for each $N \geq 0$,

$$\left\| \partial^\alpha \left(p - \sum_{i < N} p_{m-i} \right) \right\| \leq c_{N,\alpha} R^{m-i-|\alpha|},$$

where $R(\xi, \sigma) = (1 + |\xi|^2 + |\sigma|)^{1/2}$. For example, the symbol of a left-invariant differential operator ∂^α on $\mathfrak{h}_n$ lies in $\mathscr{S}^m(\mathfrak{h}_n^*)$, where m is the positive integer obtained by adding 1 for each X in the expression for ∂^α, and 2 for each power of T. (This integer will be denoted by $\|\alpha\|$.)

Define a product on the space $\mathscr{S}^\infty(\mathfrak{h}_n^*) = \bigcup_{m \in \mathbf{Z}} \mathscr{S}^m(\mathfrak{h}_n^*)$ by the following formula:

$$(p * q)(D) = p(D)q(D).$$

This product $p * q$ is bounded from $\mathscr{S}^k(\mathfrak{h}_n^*) \times \mathscr{S}^l(\mathfrak{h}_n^*)$ to $\mathscr{S}^{k+l}(\mathfrak{h}_n^*)$. It is not so hard to demonstrate the following formula for $p * q$ (Hörmander [7, p. 374]):

$$(p * q)(\xi, \sigma) = (4\pi\sigma)^{-2n} \int p(\xi + \alpha, \sigma)q(\xi + \beta, \sigma)e^{i \operatorname{Im} \alpha \cdot \overline{\beta}/2\sigma} \, d\alpha \, d\beta.$$

In particular, if p is a polynomial symbol, then there is a more explicit formula:

$$(p * q)(\xi, \sigma) = \exp 4i\sigma(\partial_\eta \partial_{\overline{\varsigma}} - \partial_\varsigma \partial_{\overline{\eta}}) \cdot p(\varsigma, \sigma)(\eta, \sigma)|_{\xi = \varsigma = \eta}.$$

Having defined the model for our pseudodifferential calculus, we will now define the pseudodifferential operator algebra of a Heisenberg manifold, keeping things as similar as possible to the case of the Heisenberg group. It is easiest to do this by working on the principal frame bundle P, where we have the horizontal vector fields $X(a)$ and T which behave as a kind of ersatz Heisenberg algebra. We say that $p(x, \xi, \sigma)$ is a symbol, and write $p \in \mathscr{S}^m(M)$ if it is a $\mathrm{U}(n)$-equivariant map from P to $\mathscr{S}^m(\mathfrak{h}_n^*)$. We can now imitate the definition; we gave for the Heisenberg group in formula (3):

$$(4) \qquad p(x, D)f = (2\pi)^{-2n-1} \int_{\mathfrak{h}_n} \widehat{p}(x, a, s) \exp(X(a) + sT)_* f.$$

Here, of course, the Fourier transform is only with respect to the $\mathfrak{h}_n^*$ variable of p.

This definition is explicitly invariant under the action of $\mathrm{U}(n)$, so that the operator $p(x, D)$ descends to an operator on M. The collection of all operators on M spanned by these operators and the infinitely smoothing operators is called Ψ^m. We let $\Psi^{-\infty}$ denote the algebra of smoothing operators on M, while $\mathscr{S}^{-\infty}$ denotes $\bigcap_{m \in \mathbf{Z}} \mathscr{S}^m$. (We will write $\mathscr{S}^m$ instead of $\mathscr{S}^m(M)$.) The following theorem summarizes the main properties of this class of operators, and a proof may be found in [1].

Theorem 3.2. *1. The map which sends a symbol $p(x, \xi, \sigma)$ to its quantization $p(x, D)$ induces an isomorphism between the spaces $\mathscr{S}^m/\mathscr{S}^{-\infty}$ and $\Psi^m/\Psi^{-\infty}$. The inverse of this isomorphism is called the symbol map.*

2. Composition defines a bounded map from $\Psi^k \times \Psi^l$ to Ψ^{k+l}, and thus from $\mathscr{S}^k/\mathscr{S}^{-\infty} \times \mathscr{S}^l/\mathscr{S}^{-\infty}$ to $\mathscr{S}^{k+l}/\mathscr{S}^{-\infty}$, which we will denote by $p \circ q$.

3. There is an asymptotic expansion for $p \circ q$, of the form

$$p \circ q \sim p * q + \sum_{i > 0} \phi_i(p, q),$$

where ϕ_i is a bilinear map of the form

$$\phi_i(p, q) = \sum_{||\alpha|| + ||\beta|| = i} c(\alpha, \beta) \partial^\alpha p * \partial^\beta q,$$

the coefficients $c(\alpha, \beta)$ being universal polynomials in the curvatures R, W and A, and their derivatives.

4. The trace of an operator $p(x, D)$, $p \in \mathscr{S}^k$ where $k < -2(n+1)$, is given by the following formula:

$$\mathrm{Tr}\, p(x, D) = (2\pi)^{-(2n+1)} \int_{T^*M} p(x, \xi, \sigma)\, dx\, d\xi\, d\sigma.$$

5. The space Ψ^{-1} is contained in Hörmander's class $\mathrm{Op}\, S_{1/2,1/2}^{-1/2}$; it follows that operators in Ψ^{-1} are compact on $L^2(M)$ if the manifold M is compact.

This theorem has an obvious generalization to operators on a bundle E on M—the symbol is now an element of $\mathscr{S}^m(E) = \mathscr{S}^m(M) \otimes_{C^\infty(M)} \mathrm{End}(E)$, and the asymptotic expansion for the composition of symbols involves the curvatures of the bundle E and their derivatives as well.

We say that a symbol $p \in \mathscr{S}^m(E)$ is elliptic if there is a symbol $h \in \mathscr{S}^{-m}(E)$ such that $p \circ h$ and $h \circ p$ equal 1 plus an element of $\mathscr{S}^{-1}(E)$. From the compactness of elements of $\Psi^{-1}(E)$ on the Hilbert space $L^2(E)$, it follows that if the manifold M is compact, then a pseudodifferential operator $p(x, D)$ with elliptic symbol is Fredholm.

We shall close this section by showing how this theory applies to the Laplacian on a Heisenberg manifold.

Proposition 3.3. *The symbol of the Laplacian $\Box_q$ on $\Omega^{0,q}(M; E)$ is equal to*

$$|\xi|^2 + (n - 2q)\tau + \sum_{ij}(a_j^* a_i - a_i a_j^*)\left(\sum_{kl} R_{ijkl} a_k^* a_l + R_{ij}\right).$$

Proof. The most elementary way to prove this from Bochner's formula is to use the fact that the Heisenberg connection vanishes at the origin in a normal coordinate system, just as for a Riemannian manifold. It follows immediately from this that the symbol of $\nabla^* \nabla$ is $|\xi|^2$.

We will now show that if $0 < q < n$, the symbol of $\Box_q$ is elliptic, and thus, $\Box_q$ is Fredholm. To do this, we use the fact that the solution to the initial value problem

(5) $$(d/dt + |\xi|^2 + (n - 2q)\sigma) * p_t = 0$$

with initial condition $p_0 = 1$ is given by the function

$$p_t(\xi, \sigma) = (\cosh t\sigma)^{-n} e^{-(\tanh t\sigma)|\xi|^2/\sigma - (n-2q)t\sigma}.$$

It follows that the function

$$h(\xi, \sigma) = \int_0^\infty p_t\, dt = \int_0^\infty (\cosh t\sigma)^{-n} e^{-(\tanh t\sigma)|\xi|^2/\sigma - (n-2q)t\sigma}\, dt,$$

which is a well-defined element of $\mathscr{S}^{-1}(\Lambda^{0,q} M \otimes E)$ if $0 < q < n$, is a parametrix for $\Box_q$, showing that for this range of q, the operator $\Box_q$ is Fredholm if M is compact.

4.   The heat equation

One of the oldest applications of the pseudodifferential calculus of the last section is to obtain an approximation for the heat kernel of the Laplacian. The idea is very simple: as was shown in the last section, the leading symbol

of the Laplacian equals $|\xi|^2 + (n - 2q)\sigma$, so that the symbol k_t of the heat kernel $e^{-t\square}$ should in some sense be well approximated by the solution p_t to the initial value problem (4) of §3.

Let us recall how this symbol calculation is made use of in Beals et al. [1] to obtain an approximation to the heat kernel of the Laplacian. First, one extends the pseudodifferential calculus to include the heat operator $\partial_t + \square$ on $\mathbf{R}_+ \times M$. If the conjugate variable to t is called τ, then we consider the symbols of the form $p(x, \xi, \sigma, \tau)$ holomorphic for τ in the lower half-plane, having an asymptotic expansion

$$p \sim \sum_{i \geq 0} p_{m-i},$$

where p_{m-i} is homogeneous of degree $m-i$, it being understood that $\deg(\xi) = 1$ and $\deg(\sigma) = \deg(\tau) = 2$. For example, the symbol of the heat operator is

$$a(x, \xi, \sigma, \tau) = i\tau + |\xi|^2 + (n - 2q)\sigma + \sum_{ij}(a_j^* a_i - a_i a_j^*)\left(\sum_{kl} R_{ijkl} a_k^* a_l + F_{ij}\right).$$

It is now fairly straightforward to extend the pseudodifferential operator calculus of the last section to this setting; we obtain an algebra of pseudodifferential operators on $\mathbf{R}_+ \times M$ invariant under time translation, and a symbol calculus satisfying the analogue of all of the properties listed in Theorem 3.2. As an application of this calculus, we have the following result, proved in [1].

Theorem 4.1. *If M is a compact Heisenberg manifold, there is an asymptotic expansion for the trace of the heat kernel of the Laplacian acting on the space $\Omega^{0,q}(M; E)$, $0 < q < n$, of the form*

$$\operatorname{Tr} e^{-t\square} \sim 2(4\pi t)^{-n-1} \operatorname{rank}(E) \cdot \operatorname{vol}(M) \int_{-\infty}^{\infty} \left(\frac{\sigma}{\sinh \sigma}\right)^n e^{-(n-2q)\sigma} \, d\sigma$$

$$+ \sum_{i > 0} c_i t^{-n-1+i/2}.$$

In this expansion, the coefficients c_i are integrals over M of polynomials in the curvatures of M and E and their derivatives.

Proof. We start by obtaining an asymptotic expansion for the symbol of $(\partial_t + \square_q)^{-1}$ by the same method as is used to calculate the symbol of the parametrix of an elliptic operator. To do this, we invert the leading symbol of $\partial_t + \square_q$ (which is equal to $i\tau + |\xi|^2 + (2q - n)\sigma$), by taking the Fourier transform of $p_t(x, \xi, \sigma)$ in t:

$$h(x, \xi, \sigma, \tau) = \int_0^{\infty} (\cosh t\sigma)^{-n} e^{-(\tanh t\sigma)|\xi|^2/\sigma - (n-2q)t\sigma - it\tau} \, dt.$$

Note that this integral only converges for $0 < q < n$. Actually, it will prove more convenient to replace this symbol by another with the same leading symbol but possessing better regularity properties at $\xi = 0$:

$$h(x, \xi, \sigma, \tau) = \int_0^\infty (\cosh t\sigma)^{-n} e^{-(\tanh t\sigma)|\xi|^2/\sigma - (n-2q)t\sigma - it\tau - t} \, dt.$$

Note that $h(x, \xi, \sigma, \tau)$ is holomorphic for τ in the lower half-plane.

We define the symbol $r(x, \xi, \sigma, \tau) \in \mathscr{S}^{-1}(\Lambda^{0,q} M \otimes E)$ by the formula $a \circ h = 1 + r$, where $a(x, \xi, \sigma, \tau)$ is the full symbol of $\partial_t + \Box_q$, and the symbol $H \in \mathscr{S}^{-2}(\Lambda^{0,q} M \otimes E)$, for N some very large integer, by

$$H = \sum_{i=0}^N (-1)^i h \circ r^{\circ i}.$$

We see that $a \circ H$ and $H \circ a$ lie in $1 + \mathscr{S}^{-N-1}(\Lambda^{0,q} M \otimes E)$; furthermore, the symbol calculus shows that H has an asymptotic expansion whose coefficients can in principle be calculated, and are polynomials in the curvatures and torsion of M and E.

Taking the Fourier transform of the symbol $H(x, \xi, \sigma, \tau)$ with respect to r gives the symbol $H_t(x, \xi, \sigma)$ of a parametrix for the heat kernel of the Laplacian, in the sense that the family of pseudodifferential operators $H_t(x, D)$ obtained by quantizing the family of symbols $H_t(x, \xi, \sigma)$ satisfies the following two properties:

1. The family $H_t(x, D)$ vanishes for $t < 0$, and is bounded as a function from t to $\mathscr{S}^0(\Lambda^{0,q} M \otimes E)$, for small t—this uses the fact that $H(x, \xi, \sigma, \tau)$ can be chosen to be holomorphic as a function of τ in the lower half-plane.

2. The family $R_t(x, D) = (d/dt + \Box_q) \circ H_t(x, D)$ is $O(t^k)$ as a function from t to $\mathscr{S}^{-M}(\Lambda^{0,q} M \otimes E)$ for $k \le N - M - n$ and small t.

Furthermore, the function $\operatorname{Tr} H_t(x, D)$ admits an asymptotic expansion of the form that we are seeking for $\operatorname{Tr} e^{-t\Box_q}$.

As is well known, the heat kernel of the operator $\Box_q$ is given by the sum of the Neumann series

$$K_t = \sum_{n=0}^\infty (-1)^n \int_{s+0+\cdots+s_n=1} H_{s_0} r_{s_1} \ldots R_{s_n} \, ds_0 \cdots ds_n.$$

But by what we have seen above it follows that this sum converges absolutely for small t if N is chosen sufficiently large, and $K_t - H_t$ is $O(t^k)$ in $\mathscr{S}^{-M}(\Lambda^{0,q} M \otimes E)$ for $k \le N - M - n$. Thus, the asymptotic expansion for $\operatorname{Tr} e^{-t\Box_q}$ is implied by the asymptotic expansion of H_t for small t.   q.e.d.

It is this scheme for calculating the asymptotic expansion of $\operatorname{Tr} e^{-t\Box_r}$ that we will imitate in the next section to prove an analogue of Demailly's inequality for Heisenberg manifolds.

5. Demailly's inequality

Consider a compact Heisenberg manifold M with a Heisenberg line bundle L. By Bochner's formula, the Laplacian on the bundle $\Omega^{0,q}(M; L^m)$ is given by the formula

$$\Box_q = \nabla^*\nabla - i(n - 2q)\nabla_T + \sum_{ij}(a_j^*a_i - a_ia_j^*)\left(\sum_{kl} R_{ijkl}a_k^*a_l + F_{ij}\right).$$

In this section, we will obtain a formula for

$$\lim_{m\to\infty} m^{-n-1}\operatorname{Tr}|_{\Omega^{0,q}(M;L^m)}e^{-t\Box_q/m}.$$

The method that we use is based on a symbol calculus for pseudodifferential operators on the line bundle $\mathscr{L}$ on $\mathbf{R}_+ \times M \times N$, whose fiber over the set $\mathbf{R}_+ \times M \times \mathbf{m}$ ($\mathbf{m} \in \mathbf{N}$) is equal to L^m; this bundle carries a connection $\nabla^{\mathscr{L}}$, obtained by piecing together the canonical connections on the line bundles L^m. Let $\mathscr{T}^m$ denote the class of symbols $p(\xi, \sigma, \tau, \mathbf{m})$ on $\mathbf{R} \times \mathfrak{h}_n^* \times \mathbf{N}$, which have asymptotic expansions in homogeneous functions as follows:

$$p \sim \sum p_{m-i},$$

where p_{m-i} is homogeneous in the sense that

$$p_{m-i}(\lambda\xi, \lambda^{1/2}\sigma, \lambda^{1/2}\tau, \lambda^{1/2}\mathbf{m}) = \lambda^{m-i}p_{m-i}(\xi, \sigma, \tau, \mathbf{m}).$$

If p is a $\mathrm{U}(n)$-equivariant map from P to $\mathscr{T}^m$, then we define the corresponding pseudodifferential operator by the formula, analogous to formula (4),

$$p(x, D, \partial_t, \mathbf{m})f_t(x, \mathbf{m})$$
$$= (2\pi)^{-2n-1}\int_{\mathfrak{h}_*}\widehat{p}(x, v, t', \mathbf{m})\exp(v\cdot\nabla^{\mathscr{L}})_*f_{t+t'}(x, \mathbf{m})\,dv\,dt'.$$

There is an asymptotic expansion for the symbol of the composition of two pseudodifferential operators with symbols of the above form, but the leading order of this expansion is more complicated than the corresponding expression $p*q$ of §3. This is seen by looking at the formula for the commutator of $\nabla_{Z(a)}$ and $\nabla_{Z(b)}$ on $M \times \mathbf{N}$:

$$[\nabla_{Z(a)}, \nabla_{\overline{Z}(b)}] = -2i\nabla_T + \mathbf{m}F(a, b) + \operatorname{ad}R(a, b).$$

On the right-hand side of this formula, the first two terms have degree equal to 2, while the third, acting on equivariant sections, has degree equal to 0. In some sense, the leading order symbol calculus on the bundle $\mathscr{L}$ is obtained from that of §3 by replacing σ by $\sigma + F$ in the formula for the leading symbol of $p \circ q$.

With the change, the analogue of Theorem 3.2 holds for this class of pseudodifferential operators on $\mathbf{R} \times M \times \mathbf{N}$. Thus, there is an asymptotic expansion for the symbol of the operator $p(x, D, \partial_t, \mathbf{m}) \circ q(x, D, \partial_t, \mathbf{m})$, with leading order given by the oscillatory integral

$$p*q = (4\pi\sigma)^{2n} \int p(x, \xi+\alpha, \sigma, \tau, \mathbf{m}) q(x, \xi+\beta, \sigma, \tau, \mathbf{m}) e^{i \operatorname{Im} \alpha \cdot (\sigma+\mathbf{m}F)^{-1}\overline{\beta}/2} \, d\alpha \, d\beta.$$

We have the following formula for the trace of $p(x, D, \tau, \mathbf{m})$, which is a function of $\tau \in \mathbf{R}$ and $\mathbf{m} \in \mathbf{N}$:

$$(6) \qquad \operatorname{Tr} p(x, D, \tau, \mathbf{m}) = (2\pi)^{-(2n+1)} \int_{T^*M} p(x, \xi, \sigma, \tau, \mathbf{m}) \, dx \, d\xi \, d\sigma.$$

We can now prove our main result, an analogue of Demailly's asymptotic formula (in Bismut's formulation) for Heisenberg manifolds.

Theorem 5.1. *If M is an $(2n+1)$-dimensional Heisenberg manifold and L is a Heisenberg line bundle on M, then*

$$\lim_{m \to \infty} (4\pi t/m)^{n+1} \operatorname{Tr} |_{\Omega^{0,q}(M;L^m)} e^{-t\square_q/m}$$

$$= 2 \int_M \int_{-\infty}^{\infty} \det\left(\frac{\sigma + tF}{\sinh(\sigma + tF)}\right) \operatorname{Tr} |_{\Lambda^{0,q}T_x^*M} e^{-(n-2q)\sigma - tF} \, d\sigma \, dx.$$

Proof. First, we must calculate the leading symbol of the heat operator $\partial_t + \square_q$ on $\mathbf{R}_+ \times M \times \mathbf{N}$. It is clear that this equals

$$i\tau + |\xi|^2 + \sum (a_j^* a_i - a_i a_j^*)(\sigma + \mathbf{m}F_{ij}).$$

The next step is to solve the equation for the leading symbol of the parametrix of $\partial_t + \square_q$:

$$\left(i\tau + |\xi|^2 + \sum (a_j^* a_i - a_i a_j^*)(\sigma + \mathbf{m}F_{ij})\right) * H(x, \xi, \sigma, \tau, \mathbf{m}) = 1.$$

The solution to this equation is equal to

$$\int_{-\infty}^{\infty} \det(\cosh t(\sigma + mF))^{-1}$$

$$\times \exp\left\{-\left(\xi, \frac{\tanh t(\sigma + \mathbf{m}F)}{\sigma + \mathbf{m}F}\right) - (n - 2q)\sigma - \mathbf{m}F - it\tau\right\} dt$$

The proof now proceeds in exactly the same way as that of Theorem 4.1, using formula (6) to calculate the trace of $p(x, D, \tau, \mathbf{m})$.

References

[1] R. Beals, P. C. Greiner & N. K. Stanton, *The heat equation on a* CR *manifold*, J. Differential Geometry **20** (1984) 343–387.

[2] J.-M. Bismut, *Demailly's asymptotic inequalities: a heat equation proof*, Orsay preprint.

[3] J.-P. Demailly, *Champs magnétiques et inégalités de Morse pour a la $\bar{\partial}$-cohomologie*, Ann. Inst. Fourier (Grenoble) **35** (1984) 189–229.

[4] A. S. Dynin, *An algebra of pseudodifferential operators on the Heisenberg group*, Dokl. Akad. Nauk. SSSR **227** (1976) 729–795.

[5] G. B. Folland & E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974) 429–522.

[6] E. Getzler, *Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem*, Comm. Math. Phys. **92** (1983) 167–178.

[7] L. Hörmander, *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979) 359–443.

[8] J. J. Kohn, *Boundaries of complex manifolds* (Proc. Conf. on Complex Analysis, Minneapolis, 1964), Springer, Berlin, 1965.

[9] E. M. Stein, *Some problems in harmonic analysis suggested by symmetric spaces and semi-simple groups*, Actes du Congrès Internat. Math. (Nice, 1970), Tome 1, Gauthier-Villars, Paris, 1971, 173–179.

[10] M. E. Taylor, *Noncommutative microlocal analysis*, Part I, Mem. Amer. Math. Soc. No. 313 (1984).

[11] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geometry **13** (1978) 25–41.

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