J. DIFFERENTIAL GEOMETRY
28 (1988) 341-343

POON'S SELF-DUAL METRICS AND KÄHLER GEOMETRY

CLAUDE LEBRUN

Abstract

It is shown that the self-dual conformal metrics on connected sums of $\mathbf{CP}_2$'s recently produced by Y. S. Poon arise from zero scalar curvature Kähler metrics on blow-ups of $\mathbf{C}^2$ by adding a point at infinity and reversing the orientation.

As noted by many authors ([4], [5], [6]), a complex surface with Kähler metric has anti-self-dual Weyl curvature iff the scalar curvature vanishes. On what would initially appear to be a completely unrelated front, Poon ([8], [9]) has produced positive scalar curvature self-dual metrics on connected sums of two and three complex projective planes. In fact, however, these phenomena are closely related:

Theorem. *Let $M = m\mathbf{CP}_2$, $0 \leq m \leq 3$, be equipped with a self-dual metric g of positive scalar curvature. There exists at least one point $p \in M$ such that $(M - \{p\}, g)$ is conformally isometric to $\mathbf{C}^2$ with m points blown up equipped with an asymptotically flat Kähler metric of zero scalar curvature.*

(**Remark.** The conformal isometry, of course, reverses orientation.)

Proof. Let $\pi\colon Z \to M$ be the canonical projection from the twistor space Z onto M; recall [1] that Z consists of all orthogonal almost-complex structure tensors on M inducing the reverse orientation. There exists ([8], [9]) a complex surface $\Sigma \subset Z$ isomorphic to $\mathbf{CP}_2$ blown up at m points such that $\pi|_\Sigma\colon \Sigma \to M$ is a diffeomorphism away from a projective line $L \subset \Sigma$ sent to a point $p \in M$; e.g. when $m = 0$, $M = S^4$, $Z = \mathbf{CP}_3$, and Σ is a hyperplane. By construction, $(\pi|_\Sigma)^* g$ is a Hermitian metric on $\Sigma - L$ but degenerates at L. Identifying $\Sigma - L$ with $\mathbf{C}^2$ blow up at m points, let

$$\hat{g} = (1 + r^2)^2 (\pi|_\Sigma)^* g,$$

where r is the Euclidean distance from the origin in $\mathbf{C}^2$; this is not only Hermitian, but asymptotically flat, differing from the standard metric only by terms of order $1/r^2$ because the projection $\Sigma \to M$ is standard on the

Received February 10, 1987 and, in revised form, July 20, 1987.

second neighborhood of $L \subset \Sigma$. Notice that the Weyl curvature of $\hat{g}$ is anti-self-dual with respect to the complex orientation.

(The second infinitesimal neighborhood of any twistor line is isomorphic to the second neighborhood of the zero section in $\mathscr{O}(1) \oplus \mathscr{O}(1) \to \mathbf{CP}_1$; this amounts to the fact that any conformal metric is flat to first order, but may be seen more directly from the obstruction theory of [7] via the vanishing of $H^1(\mathbf{CP}_1, T \otimes N^*)$ and $H^1(\mathbf{CP}_1, \hat{T} \otimes \bigodot^2 N^*)$, where $T \cong \mathscr{O}(2)$ is the tangent bundle, $N \cong \mathscr{O}(1) \oplus \mathscr{O}(1)$ is the normal bundle, and $\hat{T} \cong T \oplus N$ is the extended tangent bundle. Since Σ is a blow-up of $\mathbf{CP}_2$, it contains a complex 2-parameter family of projective lines, one of which is L. This implies that the second neighborhood of $L \subset \Sigma$ corresponds to one of the $\mathscr{O}(1)$ factors of $\mathscr{O}(1) \oplus \mathscr{O}(1) \to \mathbf{CP}_1$.)

We now proceed by modifying an argument given for compact surfaces by Boyer [2, p. 522]. Let ω be the $(1,1)$ form associated with $\hat{g}$, and let β be the 1-form defined by

$$-d\omega = \beta \wedge \omega.$$

Because $\hat{g}$ is anti-self-dual, $d\beta$ is an anti-self-dual 2-form. Letting $B_r \subset (\Sigma - L)$ be the blow up of the ball of radius r, and letting dV denote the metric volume form of $(\Sigma - L, \hat{g})$, we have

$$\begin{aligned} -\int_{B_r} \|d\beta\|^2 \, dV &= -\int_{B_r} d\beta \wedge {}^* d\beta = \int_{B_r} d\beta \wedge d\beta \\ &= \int_{B_r} d(\beta \wedge d\beta) = \int_{\partial B_r} \beta \wedge d\beta. \end{aligned}$$

But β and $d\beta$ are of order $1/r^3$ and $1/r^4$, respectively, so

$$-\int_{\Sigma - L} \|d\beta\|^2 \, dV = \lim_{r \to \infty} \int_{\partial B_r} \beta \wedge d\beta = 0.$$

Hence $d\beta = 0$. But $\Sigma - L$ is simply connected, so $\beta = df$. Hence $d(e^f \omega) = 0$, so $h = e^f \hat{g}$ is Kähler and anti-self-dual, and thus has scalar curvature zero. Since f differs from a constant by terms of order $1/r^2$, h is asymptotically flat. q.e.d.

These Kähler metrics may be written down explicitly for $m = 0, 1$. For $m = 0$, h is the standard flat metric on $\mathbf{C}^2$. For $m = 1$, h is the metric with Kähler potential on $\mathbf{C}^2 - \{0\}$ given by $\|\vec{z}\|^2 + \log \|\vec{z}\|^2$; this metric, first pointed out in this context by Burns [3], is the restriction of the standard product metric on $\mathbf{CP}_1 \times \mathbf{C}^2$ to the blow-up

$$\tilde{\mathbf{C}}^2 = \{[\vec{w}], \vec{z}) \in \mathbf{CP}_1 \times \mathbf{C}^2 \mid \vec{w} \wedge \vec{z} = 0\}.$$

Its remarkable conformal isometry with the Fubini-Study metric is obtained by extending the map $\Phi\colon \mathbf{C}^2 - \{0\} \to \mathbf{C}^2 - \{0\}\colon \vec{z} \to \vec{z}/\|\vec{z}\|^2$ to a diffeomorphism between $\tilde{\mathbf{C}}^2$ and $\mathbf{CP}_2 - \{\text{point}\}$.

The explicit form of the potentials for $m = 2, 3$ remains a topic for further investigation. One may hope to produce an ansatz for the necessary potentials for arbitrary values of m, thereby producing self-dual metrics on all connected sums of $\mathbf{CP}_2$'s; but while such an ansatz may give the general solution for $m \leq 3$, one should only expect to produce special solutions in this manner for larger values of m. The reason is that the Kähler form gives rise to a solution of the twistor equation $\nabla_{A'}{}^{(A}\omega^{BC)} = 0$ and thus to an element of $H^0(Z, K^{-1/2})$. The Riemann-Roch formula predicts that sections of $K^{-1/2}$ must exist for small values of m, but gives no information when m is large; similarly our complex surface Σ need not exist. Nonetheless, there seems to be much to be learned here.

References

[1] M. F. Atiyah, N. J. Hitchin & I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.

[2] C. Boyer, *Conformal duality and compact complex surfaces*, Math. Ann. **274** (1986) 517–526.

[3] D. Burns, *Twistors and harmonic maps*, Amer. Math. Soc. conference talk, Charlotte, NC, October 86.

[4] A. Derdzinski, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983) 405–433.

[5] M. Itoh, *Self-duality of Kähler surfaces*, Compositio Math. **51** (1984) 265–273.

[6] C. LeBrun, *On the topology of self-dual manifolds*, Proc. Amer. Math. Soc. **98** (1986) 637–640.

[7] ——, *Fattening complex manifolds*, preprint, 1987.

[8] Y. S. Poon, *Compact self-dual manifolds with positive scalar curvature*, J. Differential Geometry **24** (1986) 97–132.

[9] ——, *Small resolutions and twistor spaces*, J. Differential Geometry, in press.

STATE UNIVERSITY OF NEW YORK, STONY BROOK