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SO(3)-CONNECTIONS AND RATIONAL HOMOLOGY COBORDISMS

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1. Introduction

The main purpose of this paper is the study of rational homology cobordisms of rational homology 3-spheres. In particular it is shown that the ρ_α-invariants of Atiyah-Patodi-Singer [2] which can be defined as spectral invariants are, under some extra conditions, integral homology cobordism invariants of rational homology spheres.

Related to this study is the question of when a rational homology sphere Σ bounds a rational homology ball. This can also be answered in some cases in terms of ρ_α-invariants. In turn, these invariants can then be used to answer questions concerning sliceness of knots. Casson and Gordon [3] have constructed an invariant that detects when a two-bridge knot is not ribbon. This invariant is actually the ρ_α-invariant for the double branched cover Σ of S^3 branched over $K \subset S^3$ and character $\alpha: H_1(\Sigma) \to \mathrm{U}(1)$. For characters of prime power order Casson and Gordon [3] also show that this is a slice invariant. Namely, if $\rho_\alpha(\Sigma) = \sigma(K, \alpha) \neq \pm 1$ then K is not ribbon, and if α is of prime power order they can also conclude that it is not slice. In the case that Σ is a spherical space form, Fintushel and Stern [7] remove the condition that α be a prime-power order. In this paper the condition that Σ be a spherical space form is replaced by a weaker condition that $H^1(\Sigma, \mathbf{L}_\alpha) = 0$, where $\mathbf{L}_\alpha$ is the flat complex line bundle induced by the character α.

More specifically, let X be a compact, smooth 4-manifold with boundary ∂X. Let α: $\pi_1(X) \to \mathrm{U}(1)$ be a nontrivial character. It defines a complex line bundle $\mathbf{L}_\alpha$ on X by $\mathbf{L}_\alpha = \tilde{X} \times_\alpha \mathbf{C}$ where $\tilde{X}$ is the universal cover of W. Let $\operatorname{sign}_\alpha(X)$ denote the signature of the hermitian form induced on $H^2(X, \mathbf{L}_\alpha)$ by the cup-product. Then $\rho_\alpha(\partial X) = \operatorname{sign}(X) - \operatorname{sign}_\alpha(X)$ is a differential invariant of the boundary [2].

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Theorem 1.1. *Let X be a compact, smooth 4-manifold with boundary components $\partial X_1, \cdots, \partial X_n$ which are rational homology spheres. Assume that X has rational homology of an n-punctured 4-sphere. Let $\alpha: H_1(X) \to \mathrm{U}(1)$ be a nontrivial character, and let α_i be the induced characters on the boundary, $\alpha_i: H_1(\partial X_i) \to \mathrm{U}(1)$. Denote by m the number of ∂X_i for which α_i is nontrivial. Suppose that, for some i, $H^2(X, \partial X_i; \mathbf{Z})$ has no 2-torsion and α_i has order greater than 2. Let $e_\alpha \in H^2(X, \mathbf{Z})$ be the Euler class of the bundle $\mathbf{L}_\alpha$ and define*

$$\mu(X, \alpha) = \#\left[\left\{ e \in H^2(X, \mathbf{Z}) \mid j_i^*(e) = \pm j_i^*(e_\alpha)\ \forall i \right\}/e \sim -e\right].$$

If $H^1(\partial X, \mathbf{L}_\alpha) = 0$, then the following are true:

(a) $m \equiv \rho_\alpha(\partial X) \pmod 2$.

(b) *If* $|\rho_\alpha(\partial X)| > 3 - m$, *then* $\mu(X, \alpha) = 0 \pmod 2$.

The above theorem was first proved in [7] for the case when boundary components are spherical space forms. The technique used was to study the moduli spaces of solutions to perturbed self-duality and anti-self-duality equations in a V-manifold setting. The V-manifold used was obtained by coning off the boundary components of W. In the case when boundary components are not spherical space forms this procedure does not give us a V-manifold. In order to use the same idea of applying the gauge-theoretic type arguments ([4], [5]) we need to be able to deal with manifolds with boundary. To do that, we elongate the manifold by adding cylinders $\partial X_i \times [0, \infty)$ along the boundary and use the Fredholm theory and gauge theories on end-periodic 4-manifolds as introduced by C. Taubes in [15].

§2 gives a sketch of the proof of Theorem 1.1 and will, it is hoped, lead the reader through the more technical §§2–5 in which the theorems needed to translate the formalism from the compact manifold case to cylindrical end case are given. §6 gives the complete proof of the theorem. In §7 the applications to homology cobordisms and sliceness of knots questions are described. The Appendix gives details of some technical points which were deferred to the end so as not to interrupt the flow of the argument.

I would like to express my gratitude to my advisor, Professor Ronald J. Stern, for advice, help, and encouragement during the course of my studies and work on this paper. I also wish to thank Professor Clifford H. Taubes for patiently explaining his work to me and for suggesting a proof of Theorem 3.1.

The author has been informed that D. Ruberman has obtained similar results.

2. Strategy of proof

Let X be a 4-manifold with boundary components $\partial X_1, \cdots, \partial X_n$. A character $\alpha : H_1(X) \to \mathrm{U}(1)$ defines a flat complex line bundle $\mathbf{L}_\alpha$ with a canonical flat connection ∇_α. On each boundary component, α induces a representation $\alpha_i : H_1(\partial X_i) \to \mathrm{U}(1)$ by setting $\alpha_i = \alpha \circ j_{i*}$, where $j_{i*} : H_1(\partial X_i) \to H_1(X)$ is the inclusion induced homomorphism. The restrictions of $\mathbf{L}_\alpha$ to boundary components ∂X_i are exactly the flat bundles $\mathbf{L}_{\alpha_i}$ defined by characters α_i.

Define $M = X \cup_{\partial X} (\partial X \times [0, \infty))$. Give X a Riemannian metric which is product near the boundary and extend it to M as a product metric on End $M = \partial X \times [0, \infty)$. Extend the bundle $\mathbf{L}_\alpha$ to End M as product $\mathbf{L}_{\alpha_i} \times [0, \infty)$ and the connection ∇_α as $\nabla_\alpha = d/dt + \nabla_{\alpha_i}$ on $\partial X_i \times [0, \infty)$. Note that the same bundle and connection could be obtained by considering $\alpha : H_1(M) = H_1(X) \to \mathrm{U}(1)$.

To prove Theorem 1.1 we will study as in [7] the space of SO(3)-connections on $\mathbf{E} = \mathbf{L}_\alpha \oplus \varepsilon$, where ε is the trivial real line bundle. We give $\mathbf{E}$ the natural Riemannian metric coming from the real part of the hermitian metric on $\mathbf{L}_\alpha$ and the canonical Riemannian metric on ε. Connection $\nabla_0 = \nabla_\alpha \oplus \varepsilon$ is an SO(3)-connection giving $\mathbf{E}$ a structure of a flat SO(3)-bundle. We will use this connection as the center of an affine space of SO(3)-connections on $\mathbf{E}$.

Let $\mathbf{P}$ denote the principal SO(3)-bundle associated to $\mathbf{E}$ and let $\mathfrak{g}_{\mathbf{E}}$ denote the associated bundle of Lie algebras. Let $\mathrm{Ad}\,\mathbf{P}$ denote the bundle of groups associated to $\mathbf{P}$ by conjugation, $\mathrm{Ad}\,\mathbf{P} = \mathbf{P} \times_{\mathrm{Ad}} \mathrm{SO}(3)$. Then both $\mathrm{Ad}\,\mathbf{P}$ and $\mathfrak{g}_{\mathbf{E}}$ can be viewed as subbundles of the bundle $\mathfrak{gl}(\mathbf{E})$. $\mathrm{Ad}\,\mathbf{P}$ has fibers consisting of orthogonal transformations and $\mathfrak{g}_{\mathbf{E}}$ has fibers consisting of skew-symmetric transformations. Sections of $\mathrm{Ad}\,\mathbf{P}$ are naturally identified with automorphisms of $\mathbf{P}$ (and $\mathbf{E}$). These sections are called gauge transformations. They form a group, called the gauge group, under pointwise multiplication, which is denoted by $\mathscr{G}^\infty = C^\infty(\mathrm{Ad}\,\mathbf{P})$. The space of smooth SO(3)-connections on $\mathbf{P}$ has an affine space structure. Two connections ∇_0, ∇_1 differ by a 1-form with values in $\mathfrak{g}_{\mathbf{E}}$, $\nabla_1 - \nabla_0 \in C^\infty(T^*M \otimes \mathfrak{g}_{\mathbf{E}})$. The space of connections can therefore be identified as $\mathscr{C}^\infty = \nabla_0 + C^\infty(T^*M \otimes \mathfrak{g}_{\mathbf{E}})$. The gauge group $\mathscr{G}^\infty$ acts on the space of connections by $\nabla^g = g^{-1} \circ \nabla \circ g$. If described in affine coordinates $C^\infty(T^*M \otimes \mathfrak{g}_{\mathbf{E}})$ centered at ∇_0, the action of $\mathscr{G}^\infty$ is given by $e(g,a) = g^{-1} \circ \nabla_0(g) + g^{-1} \circ a \circ g$. ∇_0 here denotes a connection on $\mathfrak{gl}(\mathbf{E})$ defined by $\nabla_0(\phi)(\sigma) = \nabla_0(\phi(\sigma)) - \phi(\nabla_0(\sigma))$ for $\phi \in C^\infty(\mathfrak{gl}(\mathbf{E}))$, $\sigma \in C^\infty(\mathbf{E})$.

The Riemannian metric on M defines an involution on the space of 2-forms, the Hodge $*$-operator $* : \Lambda^2(T^4M) \to \Lambda^2(T^*M)$. It extends to the space of 2-forms with values in a bundle by acting by $*$ on the form part and by identity on the bundle part. Since $*$ is an involution it defines a splitting of Λ^2

into a direct sum of ± 1 eigenspaces $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. For a 2-form ϕ we will denote the projections on Λ^2_+ and Λ^2_- by ϕ_+ and ϕ_-. Every SO(3)-connection ∇ on $\mathbf{E}$ has a curvature form $R^\nabla \in C^\infty(\Lambda^2(T^*M) \otimes \mathfrak{g}_{\mathbf{E}})$ associated to it. We will call a connection self-dual if $*R^\nabla = R^\nabla$, i.e. $R^\nabla_- = 0$, and anti-self-dual if $*R^\nabla = -R^\nabla$, i.e. $R^\nabla_+ = 0$. Connection is flat if $R^\nabla = 0$.

On a compact closed manifold, the Pontrjagin class of the bundle $\mathbf{E}$ is independent of the connection and the formula

$$(2.1) \qquad p_1(\mathbf{E}) = \frac{1}{4\pi^2}\int_M \operatorname{tr}(R^\nabla \wedge R^\nabla) = \frac{1}{4\pi^2}\int_M |R^\nabla_+|^2 - |R^\nabla_-|^2$$

implies that a connection in a bundle with $p_1 = 0$ is flat if and only if it is self-dual, and if and only if it is anti-self-dual. On a noncompact manifold the integral

$$p_1(\nabla) = \frac{1}{4\pi^2}\int_M \operatorname{tr}(R^\nabla \wedge R^\nabla)$$

depends on ∇. For example, on $\mathbf{R}^4$ any integer can be obtained as $p_1(\nabla)$. To be able to make the same argument as in the closed manifold case we need to resrict ourselves to a smaller space of connections. An appropriate space to consider is

$$\mathscr{C}_0^\infty = \nabla_0 + C_0^\infty(T^*M \otimes \mathfrak{g}_{\mathbf{E}}),$$

the space of connections that differ from the base connection ∇_0 by a compactly supported form. It is easy to show that $p_1(\nabla) = p_1(\nabla_0) = 0$ for $\nabla \in \mathscr{C}_0^\infty$.

In the closed manifold case the right space of connections for gauge theory is obtained by completing $\mathscr{C}_0^\infty = \mathscr{C}^\infty$ in appropriate Sobolev norms, which make the fundamental complexes of Yang-Mills theory

$$0 \to \Gamma(\mathfrak{g}_{\mathbf{E}}) \xrightarrow{d^{\nabla_0}} \Gamma(T^*M \otimes \mathfrak{g}_{\mathbf{E}}) \xrightarrow{d^{\nabla_0}_\pm} \Gamma(\Lambda^2_\pm(T^*M) \otimes \mathfrak{g}_{\mathbf{E}}) \to 0$$

Fredholm. The space of connections becomes an affine Banach space and the gauge group a Banach Lie group which acts smoothly on the space of connections.

In the case of a noncompact manifold, completing the compactly supported sections in ordinary Sobolev norms will not produce a Fredholm complex. In [15] Taubes constructed a Fredholm theory for manifolds with periodic ends by generalizing a theory of Lockhart and McOwen [13] for manifolds with cylindrical ends. The appropriate norms turn out to be weighted Sobolev norms. They are defined as follows.

Let $\tau : M \to [0, \infty)$ be a smooth function such that $\tau(y, t) = t$ for $y \in \partial X$ and $t > \varepsilon > 0$ and $\tau(x) = 0$ for $x \in X$, i.e., τ is a smoothing of the cylinder coordinate t to a function that is 0 on the compact part X. For $\delta \in \mathbf{R}$ define the weighted L^p spaces of sections of $\mathbf{E}$, denoted $L^p_\delta(\mathbf{E})$, as completions of $C_0^\infty(\mathbf{E})$ in the norm

$$\|\sigma\|_{L^p_\delta} = \left(\int_M e^{\tau\delta}|\sigma|^p\right)^{1/p}.$$

Space $L^p_{k,\delta}(\mathbf{E})$ are defined as completions of $C_0^\infty(\mathbf{E})$ in the norm

$$\|\sigma\|_{L^p_{k,\delta}} = \left[\int_M e^{\tau\delta}\left(|\nabla_0^{(k)}\sigma|^p + \cdots + |\nabla_0\sigma|^p + |\sigma|^p\right)\right]^{1/p}$$

where $\nabla_0^{(k)} = \nabla_0 \circ \cdots \circ \nabla_0$ (k times) is the kth differential defined as composition of

$$\nabla_0 : C_0^\infty\Big(\bigotimes_l T^*M \otimes \mathbf{E}\Big) \to C_0^\infty\Big(\bigotimes_{l+1} T^*M \otimes \mathbf{E}\Big).$$

It is actually possible to choose a different weight δ for each boundary component ∂X_i of ∂X. In that case we think of δ as an n-tuple of real numbers $\delta = (\delta^1, \cdots, \delta^n)$, one for each component, and think of τ as a $\mathbf{R}$-valued function $(\tau^1, \cdots, \tau^n)$ where

$$\tau_i(x, t) = \begin{cases} t, & x \in \partial X_i, \\ 0, & x \notin \partial X_i, \end{cases}$$

and $\tau\delta$ is the scalar product of δ and τ.

Theorem 3.2 of [15] applied to the fundamental complexes of Yang-Mills theory implies

Theorem 2.1. *There is a discrete set $\mathscr{D} \subset \mathbf{R}^n$ without accumulation points such that*

$$0 \to L^2_{3,\delta}(\mathfrak{g}_\mathbf{E}) \overset{\nabla_0}{\to} L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_\mathbf{E}) \overset{d_-^{\nabla_0}}{\to} L^2_{1,\delta}(\Lambda^2_-(T^*M) \otimes \mathfrak{g}_\mathbf{E}) \to 0$$

and

$$0 \to L^2_{3,\delta}(\mathfrak{g}_\mathbf{E}) \overset{\nabla_0}{\to} L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_\mathbf{E}) \overset{d_+^{\nabla_0}}{\to} L^2_{1,\delta}(\Lambda^2_+(T^*M) \otimes \mathfrak{g}_\mathbf{E}) \to 0$$

are Fredholm for $\delta \in \mathbf{R}^n \setminus \mathscr{D}$.

In §3 we describe the gauge theory on M as developed by C. Taubes in [15]. The space of connections $\mathscr{C}$ and the gauge group $\mathscr{G}$ are chosen so that the arguments in [7] can be applied to the present situation. The orbit space $\mathscr{B} = \mathscr{C}/\mathscr{G}$ is given a Banach manifold structure. The set of $\mathscr{G}$-reducible connections is described in §4. In §5 the indices of two fundamental Fredholm complexes needed for further discussion are computed. In §6 we prove the following

Proposition 2.2. *Suppose* $d_{-} = -3 + m - \rho_\alpha(\partial X) > 0$. *Under the assumptions of Theorem* 1.1, *there is a compact submanifold* $\mathcal{M}_{-} \subset \mathcal{B}$ *which is a smooth* d_{-}*-dimensional manifold with a finite number*

$$\mu = \mu(X, \alpha) = \#\left[\left\{e \in H^2(W, X) \mid j_i^*(e) = \pm j_i^*(e_\alpha)\ \forall i\right\}/e \sim -e\right]$$

of singular points. Each singularity has a structure of a cone on a complex projective space, and it corresponds to a reducible flat connection in $\mathcal{C}$. *Suppose* $d_{+} = -3 + m + \rho_\alpha(\partial X) > 0$. *Then there is a compact submanifold* $\mathcal{M}_{+} \subset \mathcal{B}$ *of dimension* d_{+} *with* μ *singular points whose neighborhoods are cones on complex projective spaces.*

The manifold $\mathcal{M}_{-}$ is obtained by perturbing the self-duality equations and $\mathcal{M}_{+}$ is obtained by perturbing the anti-self-duality equations. Both of them are compact as perturbations of the compact moduli space of flat connections $\mathcal{F} \subset \mathcal{B}$ which can be viewed as the zero set of both the self-duality and anti-self-duality equations. Dimensions d_{+} and d_{-} of these perturbed moduli spaces are obtained from indices to two fundamental complexes computed in §7. As in [5], [6] a cobordism argument shows that $\mathcal{M}_{+}$ and $\mathcal{M}_{-}$ cannot exist inside $\mathcal{B}$ unless μ is even. Therefore, if μ is odd both d_{+} and d_{-} have to be $\leqslant 0$ which proves statement (b) in Theorem 1.1. Statement (a) follows from a simple observation that both d_{-} and d_{+} have to be odd.

3. Gauge theory on M

Recall that we have defined a bundle $\mathbf{E} = \mathbf{L}_\alpha \oplus \varepsilon$ on M and a connection $\nabla_0 = \nabla_\alpha \oplus \varepsilon$. We complete the space of connections

$$\mathcal{C}_0^\infty = \nabla_0 + C_0^\infty\left(T^*M \otimes \mathfrak{g}_{\mathbf{E}}\right)$$

in the Sobolev $L^2_{2,\delta}$ norm and denote the new space by

$$\mathcal{C} = \nabla_0 + L^2_{2,\delta}\left(T^*M \otimes \mathfrak{g}_{\mathbf{E}}\right) = \nabla_0 + \mathcal{A}.$$

To define the appropriate gauge group we first consider the following space of sections of $\mathfrak{gl}(\mathbf{E})$. Define

$$\mathcal{R} = \left\{\phi \in L^2_{2,\mathrm{loc}}\left(\mathfrak{gl}(\mathbf{E})\right) \mid \|\nabla_0\phi\|_{L^2_{2,\delta}} < \infty\right\}.$$

Here ∇_0 is the connection defined by ∇_0 on $\mathfrak{gl}(\mathbf{E})$ and $L^2_{2,\delta}$ is the weighted Sobolev norm on the bundle $T^*M \otimes \mathfrak{gl}(\mathbf{E})$. The following theorem, which gives a Banach space structure to $\mathcal{R}$, will be proved in the Appendix.

Theorem 3.1. *Let $\mathscr{H}$ denote the subspace of $\mathscr{R}$ consisting of harmonic sections, i.e.,*

$$\mathscr{H} = \left\{ \phi \in \mathscr{R} \mid e^{-\tau\delta} \nabla_0^* e^{\tau\delta} \nabla_0 \phi = 0 \right\}.$$

Then there is a direct sum decomposition

$$\mathscr{R} = L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E})) \oplus \mathscr{H}.$$

Furthermore, there is a well-defined map $r: \mathscr{R} \to \operatorname{Ker} \nabla_\partial$ given by

$$r(\phi)(y) = \lim_{t \to \infty} \phi(y, t).$$

Here $\operatorname{Ker} \nabla_\partial$ denotes the set of parallel sections of the bundle $\mathfrak{gl}(\mathbf{E}|_{\partial X})$ with respect to the connection ∇_∂ (recall that $\nabla_0 = d/dt + \nabla_\partial$ on $\operatorname{End}(M)$). Also $r^{-1}(0) = L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$ and $r: \mathscr{H} \to \operatorname{Ker} \nabla_\partial$ is an isomorphism. The norm

$$\|\phi\|^2_{\mathscr{R}} = \|\nabla_0(\phi)\|^2_{L^2_{2,\delta}} + \int_{\partial X} |r(\phi)|^2$$

gives $\mathscr{R}$ a Banach space structure in which the projections $\pi_0: \mathscr{R} \to L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$ and $\pi: \mathscr{R} \to \mathscr{H}$ are continuous.

In the Appendix we will also show that pointwise multiplication $\cdot: \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ is well defined and continuous, and that $r: \mathscr{R} \to \operatorname{Ker} \nabla_\partial$ is a continuous homomorphism.

$$\mathscr{G} = \left\{ \phi \in \mathscr{R} \mid \phi \cdot \phi^* = \mathrm{id}, \det \phi = 1 \right\}$$

is a closed submanifold of $\mathscr{R}$ which is a Banach Lie group with Lie algebra

$$\mathfrak{g} = \left\{ \phi \in \mathscr{R} \mid \phi^* = -\phi \right\} = \left\{ \phi \in L^2_{3,\mathrm{loc}}(\mathfrak{g}_{\mathbf{E}}) \mid \|\nabla_0 \phi\|_{L^2_{2,\delta}} < \infty \right\}.$$

The restriction of $r: \mathscr{R} \to \operatorname{Ker} \nabla_\partial$ to $\mathscr{G}$ is a smooth homomorphism from $\mathscr{G}$ to

$$\begin{aligned} \bar{G} &= \left\{ g \in \ker \nabla_\partial \mid g \circ g^* = 1, \det g = 1 \right\} \\ &= \text{the stabilizer of } \nabla_\partial \text{ in the gauge group for } \mathbf{E}|_{\partial X}. \end{aligned}$$

On each component on which $\alpha_i: H_1(\partial X_i) \to \mathrm{U}(1)$ is trivial, ∇_{∂_i} is a trivial flat connection on a trivial bundle and the stabilizer is an SO(3) of constant gauge transformation. For components ∂X_i for which α_i is nontrivial, ∇_{∂_i} is a nontrivial reducible flat connection $\nabla_{\partial_i} = \nabla_{\alpha_i} \oplus d$ and the stabilizer is an S^1 of rotations for a constant angle in the $\mathbf{L}_{\alpha_i}$ component of $\mathbf{E}/\partial X_i = \mathbf{L}_{\alpha_i} \oplus \varepsilon$ if α_i does not have order 2, and an O(2) if α_i has order 2. Let $\Gamma_{\nabla_{\partial_i}}$ denote the stabilizer of ∇_{∂_i}. Then

$$r: \mathscr{G} \to \bar{G} = \bigtimes_{i=1}^{n} \Gamma_{\nabla_{\partial_i}}.$$

Denote by $\mathfrak{G}_0$ the Lie algebra Lie($\overline{G}$). Then $\mathfrak{G}_0 = \operatorname{Ker} \nabla_\partial$ on sections of $\mathfrak{g}_{\mathbf{E}}/\partial X$. Since r: $\mathfrak{g} \to \overline{G}$ is a restriction of the linear map r: $\mathscr{R} \to \operatorname{Ker} \nabla_\partial$, its differential is $Dr = r$: $\mathfrak{g} \to \mathfrak{G}_0$. Let β be a cutoff function on End M, namely a smooth function β: $M \to [0,1]$ such that $\beta(x) = 0$ for $x \in X$ and $\beta(x) = 1$ for $x \in \partial X \times [\varepsilon, \infty)$ for some $\varepsilon > 0$. Let $\sigma \in \mathfrak{G}_0$. Than $\beta \cdot \sigma$ defines a section of $\mathfrak{gl}(\mathbf{E})$ which is in $\mathfrak{g}$ and

$$r(\beta \cdot \sigma)(y) = \lim_{t \to \infty} \beta \cdot \sigma(y, t) = \sigma(y).$$

Therefore Dr is surjective and r: $\mathscr{G} \to \overline{G}$ is a submersion. Denote by $\mathscr{G}_0$ the kernel of r, $\mathscr{G}_0 = r^{-1}(e)$. $\mathscr{G}_0$ is a closed Lie subgroup of $\mathscr{G}$ with Lie algebra

$$\mathfrak{g}_0 = r^{-1}(0) = \mathfrak{g} \cap r^{-1}(0) = L^2_{3,\delta}(\mathfrak{g}_{\mathbf{E}}).$$

Let $G = \operatorname{im}(r\colon \mathscr{G} \to \overline{G})$. G contains the identity component of $\overline{G}$ and has the same Lie algebra. If we assume that $H^2(X, \mathbf{Z})$ has no 2-torsion or that all α_i have order greater than 2 then

$$G = \overline{G} = \left(\bigtimes_{i=1}^{k} \mathrm{SO}(3) \right) \times \left(\bigtimes_{j=1}^{n-k} S^1 \right),$$

where k is the number of boundary components for which the representation α_i is trivial.

In the Appendix we will also show that the Lie group $\mathscr{G}$ acts smoothly on the space of connections $\mathscr{C}$ by the usual action $(g, \nabla) \to \nabla^g = g^{-1} \circ \nabla \circ g$. Lemma 7.5 in [15] shows that the definition of $\mathscr{C} = \nabla_0 + L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}})$ does not depend on the choice of $\nabla \in \mathscr{C}$. In other words, the norms induced on $C_0^\infty(\mathbf{E})$ by ∇ and ∇_0 are equivalent, and hence define equivalent norms on $L^2_{2,\delta}$. Theorem 3.1 then shows that the norms $\| \ \|_{\mathscr{R}}$ defined on $\mathscr{R}$ are equivalent.

It is important to note that the action of $\mathscr{G}_0$ on $\mathscr{C}$ is free. Suppose, namely, that $\nabla^g = \nabla$ for $\nabla \in \mathscr{C}$, $g \in \mathscr{G}_0$. Then $\nabla g = 0$. Equation (7.13) in [15] says that

$$\text{(3.1)} \qquad \int_{\mathrm{End}\, M} e^{\tau\delta} |\nabla g|^2 \geqslant \zeta \cdot \int_{\mathrm{End}\, M} e^{\tau\delta} |g - r(g)|^2, \qquad \zeta > 0.$$

It follows that $g = r(g) = \mathrm{id}$ on End M. But then $g = \mathrm{id}$ since $\nabla(g) = 0$.

Let $\mathscr{C}^*$ denote the set of irreducible connections, i.e., connections $\nabla \in \mathscr{C}$ such that $\nabla^g = \nabla$ implies $g = \mathrm{id}$ for $g \in \mathscr{G}$. $\mathscr{G}$ acts freely on $\mathscr{C}^*$. We want to give a Banach manifold structure to the quotient spaces $\mathscr{B}_0 = \mathscr{C}/\mathscr{G}_0$ and $\mathscr{B}^* = \mathscr{C}^*/\mathscr{G}$. Note that $\mathscr{B}^* = \mathscr{C}^*/\mathscr{G} = \mathscr{C}^*/\mathscr{G}_0/\mathscr{G}/\mathscr{G}_0 = \mathscr{B}_0^*/G$ where G acts on $\mathscr{B}_0^*$ as the quotient group $\mathscr{G}/\mathscr{G}_0 \cong G$.

Theorem 3.2. *$\mathscr{B}_0 = \mathscr{C}/\mathscr{G}_0$ is a Banach manifold. The tangent space to $[\nabla] \in \mathscr{B}_0$ is isomorphic to*

$$\chi_\nabla = \left\{ a \in L^2_{2,\delta}\left(T^*M \otimes \mathfrak{g}_{\mathbf{E}}\right) \mid e^{-\tau\delta} d^*_\nabla e^{\tau\delta} a = 0 \right\}$$
$$= L^2_\delta\text{-orthogonal complement of } d_\nabla(\mathfrak{g}_0) \text{ in } L^2_{2,\delta}\left(T^*M \otimes \mathfrak{g}_{\mathbf{E}}\right).$$

The map π_0: $\mathscr{C} \to \mathscr{B}_0$ is a principal $\mathscr{G}_0$-bundle.

Proof. The proof is a direct translation of the proof in the compact case (cf. [12], [8]). The use of the group $\mathscr{G}_0$ makes the standard boot strapping argument work. It is important to be able to estimate the norm $\|g\text{-id}\|_{\mathscr{R}}$ in terms of $\|\nabla g\|_{L^2_{2,\delta}}$ in order to show that local slices $\chi_{\nabla,\varepsilon} = \{a \in \chi_\nabla \mid \|a\|_{L^2_{2,\delta}} < \varepsilon\}$ are actually slices. This is the reason we cannot use $\mathscr{G}$ in this standard argument.

To give a Banach manifold structure to $\mathscr{B}^* = \mathscr{C}^*/\mathscr{G}$ we consider the residual $\mathscr{G}/\mathscr{C}_0 \cong G$ action on $\mathscr{B}^*_0 = \mathscr{C}^*/\mathscr{G}_0$ defined by $[e]([g],[a]) = [e(g,a)]$. Identify the tangent spaces $T_{[\mathrm{id}]}G$ and $T_{[\nabla]}\mathscr{B}^*_0$ with $\mathfrak{g}/\mathfrak{g}_0$ and $\mathscr{A}/d_\nabla(\mathfrak{g}_0)$ respectively. With this identification the differential of the action map $[e]$ at ([id], $[\nabla]$) is

$$D_{([\mathrm{id}],[\nabla])}[e]([\gamma],[a]) = [d_\nabla\gamma + a] \in \mathscr{A}/d_\nabla(\mathfrak{g}_0).$$

A neighborhood of $[\nabla] \in \mathscr{B}_0$ is given by

$$\mathscr{U}_{\nabla,\varepsilon} = \pi(\chi_{\nabla,\varepsilon}) = \left\{[a] \mid a \perp d_\nabla(\mathfrak{g}_0),\ \|a\| < \varepsilon\right\}.$$

Let $\mathscr{O}_{\nabla,\varepsilon} \subset \mathscr{U}_{\nabla,\varepsilon}$ be a closed submanifold defined by

$$\mathscr{O}_{\nabla,\varepsilon} = \left\{[a] \in \mathscr{U}_{\nabla,\varepsilon} \mid [a] \text{ has a representative } a \perp d_\nabla(\mathfrak{g})\right\}.$$

Then $T_{[\nabla]}\mathscr{O}_{\nabla,\varepsilon} = \{[a] \mid a \perp d_\nabla(\mathfrak{g})\}$. The differential of $[e]$: $G \times \mathscr{O}_{\nabla,\varepsilon} \to \mathscr{B}_0$ at ([id], $[\nabla]$) is an isomorphism. Assume, namely, that

$$D_{([\mathrm{id}],[\nabla])}[e]([\gamma],[a]) = [d_\nabla\gamma + a] = 0 \in \mathscr{A}/d_\nabla(\mathfrak{g}_0)$$

for $[a] \in T_{[\nabla]}\mathscr{O}_{\nabla,\varepsilon}$. Then there is $\gamma' \in \mathfrak{g}_0$ such that $d_\nabla\gamma + a = d_\nabla\gamma'$. Let a be the representative of $[a]$ perpendicular to $d_\nabla(\mathfrak{g})$. Since $a = d_\nabla(\gamma' - \gamma) = d_\nabla(\tilde{\gamma})$ we have $a = 0$, $d_\nabla(\tilde{\gamma}) = 0$. ∇ is irreducible, which implies $\tilde{\gamma} = 0$. Therefore $\gamma' = \gamma \in \mathfrak{g}_0$, ($[\gamma]$, $[a]$) = (0, 0), and $D[e]$ is injective. It is obviously surjective. The open mapping theorem and inverse function theorem imply that $[e]$ is a local diffeomorphism. So, for small enough ε, $\mathscr{O}_{\nabla,\varepsilon}$ is a local slice for the G action. Since G is compact, we can find $\varepsilon' < \varepsilon$ small enough such that $\mathscr{O}_{\nabla,\varepsilon'}$ is a slice for the G action. This proves

Proposition 3.3. *$\mathscr{B}^* = \mathscr{B}^*_0/G = \mathscr{C}^*/\mathscr{G}$ has a manifold structure with coordinates given by $\mathscr{O}_{\nabla,\varepsilon}$. The quotient map π_G: $\mathscr{B}^*_0 \to \mathscr{B}^*$ is a principal G-bundle.*

Note that we have actually found the slice for the $\mathscr{G}$ action on $\mathscr{C}^*$. Let $a_1, a_2 \in \tilde{\mathscr{O}}_{\nabla,\varepsilon} = \pi_0^{-1}(\mathscr{O}_{\nabla,\varepsilon})$, and let $a_2 = e(g, a_1)$. Then on the quotient space $\mathscr{B}_0^*$ we have $[a_2] = [e]([g],[a_1])$ and since $\mathscr{O}_{\nabla,\varepsilon}$ is a slice for the $G = \mathscr{G}/\mathscr{G}_0$ action, $[g] = \mathrm{id} \in G$ and hence $g \in \mathscr{G}_0$. Now $a_1, a_2 \in \pi_0^{-1}(\mathscr{O}_{\nabla,\varepsilon}) \subset \chi_{\nabla,\varepsilon}$ which is a slice of the $\mathscr{G}_0$ action. Therefore $g = \mathrm{id}$ and $a_2 = a_1$. We can summarize the above results in Theorem 3.4.

Theorem 3.4. *Let $\mathscr{C}^* \subset \mathscr{C}$ denote the space of irreducible connections in $\mathscr{C}$. Than the quotient spaces $\mathscr{B}_0 = \mathscr{C}/\mathscr{G}_0$ and $\mathscr{B}^* = \mathscr{C}^*/\mathscr{G}$ are Banach manifolds with tangent spaces at $[\nabla] \in \mathscr{B}_0$ and $[\nabla] \in \mathscr{B}^*$ described as*

$$T_{[\nabla]}\mathscr{B}_0 = \chi_\nabla = \left\{a \in L^2_{2,\delta}\left(T^*M \otimes \mathfrak{g}_{\mathbf{E}}\right) \mid a \perp d_\nabla(\mathfrak{g}_0)\right\}$$

and

$$T_{[\nabla]}\mathscr{B}^* = \mathscr{O}_\nabla = \left\{a \in L^2_{2,\delta}\left(T^*M \otimes \mathfrak{g}_{\mathbf{E}}\right) \mid a \perp d_\nabla(\mathfrak{g})\right\}.$$

We still need to show that the graph $\Gamma = \{(\nabla, \nabla^g) \mid \nabla \in \mathscr{C}, g \in \mathscr{G}\}$ is closed in order to prove that the quotient space $\mathscr{B}^*$ is Hausdorff. Assume that a sequence $(\nabla_n, \tilde{\nabla}_n) \in \Gamma$ converges in $\mathscr{C} \times \mathscr{C}$ to $(\nabla, \tilde{\nabla})$. We need to show that there is $g \in \mathscr{G}$ such that $\tilde{\nabla} = \nabla^g$. Write $\tilde{\nabla}, \nabla_n$, and $\tilde{\nabla}_n$ in coordinates centered at ∇ as $\tilde{\nabla} = \nabla + a$, $\tilde{\nabla}_n = \nabla + a_n$, and $\nabla_n = \nabla + b_n$. Since $(\nabla_n, \tilde{\nabla}_n) \in \Gamma$ there is a sequence $\{g_n\}$ in $\mathscr{G}$ such that $\tilde{\nabla}_n = \nabla_n^{g_n}$, i.e., $a_n = e_\nabla(g_n, b_n) = g_n^{-1} \circ \nabla g_n + g_n^{-1} \circ b_n \circ g_n$. A standard boot strapping argument (cf. [12], [8]) gives an estimate

$$\|\nabla g_n\|_{L^2_{2,\delta}} \leqslant P\left(\|a_n\|_{L^2_{2,\delta}}, \|b_n\|_{L^2_{2,\delta}}\right) \tag{3.2}$$

and since $a_n \to a$, $b_n \to b$ this shows that $\{\|\nabla g_n\|_{L^2_{2,\delta}}\}$ is uniformly bounded. Therefore, the sequence g_n is uniformly bounded in $\| \ \|_{L^2_{3,\delta}(M_N)}$ for each compact $M_N = \tau^{-1}([0, N]) \subset M$. The Sobolev embedding theorems for compact domains give a weakly convergent subsequence $\{g_n^N\}$ converging to $g_N \in L^2_{3,\delta}(M_N)$ in the $L^2_{2,\delta}(M_N)$ norm. Since $\nabla g_n = g_n a_n - b_n g_n$, Sobolev multiplication theorems say that $\nabla g_n^N \to \nabla g_N$ in $L^2_{2,\delta}(M_N)$, hence $g_n^N \to g_N$ in $L^2_{3,\delta}(M_N)$. Using standard diagonalization procedure we obtain a subsequence $\{g_k\}$ of the original sequence converging to g_N in $L^2_{3,\delta}(M_N)$. Since $L^2_{3,\delta}$ convergence implies pointwise convergence, $g_{N+1} \mid M_N = g_N$ and we can define a gauge transformation g by setting $g \mid M_N = g_N$. Continuity of multiplication on M_N implies that $\tilde{\nabla} = \nabla^g$ on M_N. Therefore, $\tilde{\nabla} = \nabla^g$. Equation (3.2) gives $g \in \mathscr{G}$.

As in the case of compact closed M, we can identify the neighborhood of a reducible connection in $\mathscr{B} = \mathscr{C}/\mathscr{G}$ with

$$\mathscr{O}_{\nabla,\varepsilon}/\Gamma_\nabla = \left\{ a \in L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}}) \left| \begin{matrix} a \perp d_\nabla(\mathfrak{g}) \\ \|a\|_{L^2_{2,\delta}<\varepsilon} \end{matrix} \right. \right\} \Big| \Gamma_\nabla .$$

4. Reducible connections

The singular points in $\mathscr{B} = \mathscr{C}/\mathscr{G}$ are the gauge equivalence classes of connections with nontrivial isotropy group $\Gamma_\nabla = \{ g \in \, | \nabla^g = \nabla \}$. The situation here is parallel to the compact case and most of the proofs are the same. The difference is in the extra information about ∇ and g on the cylindrical part of M.

We will say that ∇ is a topologically reducible connection on $\mathbf{E}$ if there is a splitting $\mathbf{E} = \mathbf{L} \oplus \varepsilon$ of $\mathbf{E}$ as a sum of a complex line bundle $\mathbf{L}$ and a trivial real line bundle ε such that $\nabla = \nabla_{\mathbf{L}} \oplus d$, where $\nabla_{\mathbf{L}}$ is an SO(2) connection on $\mathbf{L}$ and d is the trivial connection on ε.

The classification of SO(3)-bundles over a 4-complex tells us that $\mathbf{E}$ topologically reduces to $\mathbf{L} \oplus \varepsilon$ if and only if

$$p_1(\mathbf{E}) = p_1(\mathbf{L} \oplus \varepsilon) = c_1(\mathbf{L}) \cup c_1(\mathbf{L}),$$
$$w_2(\mathbf{E}) = w_2(\mathbf{L} \oplus \varepsilon) = c_1(\mathbf{L}) \pmod 2.$$

By assumption, X has the rational homology of an n-punctured 4-sphere. The second cohomology $H^2(X, \mathbf{Z})$ is therefore torsion and $H^4(X, \mathbf{Z}) = 0$. So $\mathbf{L} \oplus \varepsilon = \mathbf{E} = \mathbf{L}_\alpha \oplus \varepsilon$ if and only if

$$c_1(\mathbf{L}) = c_1(\mathbf{L}_\alpha) \pmod 2. \tag{4.1}$$

In order to identify the singular points in $\mathscr{B}$ we need to identify the topological reductions that result from a nontrivial isotropy in $\mathscr{G}$. We will call connections $\nabla \in \mathscr{C}$ such that $\Gamma_\nabla = \{ g \in \, | \nabla^g = \nabla \} \neq \mathrm{id}$ $\mathscr{G}$-reducible. Then, as in the proof of Proposition 3.1 in [5], the splitting $\mathbf{E} = \mathbf{L}_g \oplus \eta_g$ can be constructed where η_g is the 1-eigenspace bundle for $g \in \mathrm{Aut}(\mathbf{E})$. If there is a component such that the order of α_i is greater than 2, then $r_i(g) \in \ker \nabla_{\partial_i} \cong S^1$ and, in the limit at that end, η_g coincides with ε. This is because ε is the unique line fixed by a nontrivial element of $\ker \nabla_{\partial_i}$. Therefore, if $j_i \colon \partial X_i \to X$ denotes the inclusion and j_i^* the induced homomorphism on cohomology, $j_i^*(w_1(\eta_g))$ $= 0$. Since $H^2(X, \partial X_i, \mathbf{Z})$ odd torsion implies (by universal coefficient theorem) that $H^1(X, \partial X_i, \mathbf{Z}_2) = 0$, the exact sequence

$$0 \to H^1(X, \partial X_i, \mathbf{Z}_2) \to H^1(X, \mathbf{Z}_2) \xrightarrow{\partial_i^*} H^1(\partial X_i, \mathbf{Z}_2)$$

implies that ∂_i^* is 1-1 and therefore $w_1(\eta_g) = 0$. Hence, no reduction coming from $g \in \mathcal{G}$ can be nonorientable and we have the following proposition.

Proposition 4.1. *For $\nabla \in \mathscr{C}$ the following are equivalent*:

(a) $\Gamma_\nabla = \{ g \in \operatorname{Aut} \mathbf{P} \mid \nabla^g = \nabla \} \neq \mathrm{id}$.

(b) ∇ *is topologically reducible.*

(c) $\Gamma_\nabla \cong S^1$.

(d) $d_\nabla \colon \Gamma(\mathfrak{g}_\mathbf{E}) \to \Gamma(T^*M \otimes \mathfrak{g}_\mathbf{E})$ *has nontrivial kernel.*

We are interested in identifying the set of gauge equivalence classes of reducible flat connections in $\mathscr{C}$. It is a well-known fact that a flat connection is determined, up to gauge equivalence, by its holonomy representation

$$[\alpha_\nabla] \in \operatorname{Hom}(\pi_1(X), \mathrm{SO}(3))/\mathrm{SO}(3)$$

which is independent of the choice of $x_0 \in X$. Let $j_i^t \colon \partial X_i \to \partial X_i \times \{t\} \subset M$ be the inclusion $j_i^t(x) = (x, t)$. j_i^t pulls back ∇ to a flat connection ∇_i^t on $\mathbf{E}/\partial X_i$. The holonomy representation of ∇_i^t is given by

$$\alpha_{\nabla_i^t} = \alpha_\nabla \circ j_i^{t*} \colon \pi_1(\partial X_i, y_0) \to \mathrm{SO}(3)$$

since it is given by a parallel transport along the loops, which are naturally in X. For a different cylinder coordinate we get a conjugate representation, since we are only changing the base point for π_1. Therefore the connections ∇_i^t give the same $[\alpha_{\nabla_i^t}] \in \operatorname{Hom}(\pi_1(\partial X_i), \mathrm{SO}(3))/\mathrm{SO}(3)$. Since $\nabla \in \mathscr{C}$, $\nabla_i^t \to \nabla_{\partial_i}$ as $t \to \infty$ and hence $[\alpha_{\nabla_i^t}] = [\alpha_{\nabla_{\partial_i}}] = [\nu_i]$, where ν_i is the composition $\pi_1(\partial X_i) \xrightarrow{\alpha_i} S^1 \to \mathrm{SO}(3)$.

This shows that on the cylindrical end, any flat connection is reducible and gauge equivalent to the reducible connection $\nabla_0 = \nabla_{\alpha_0} + d$. Therefore, for any reducible flat connection reducing to $\nabla = \nabla_\mathbf{L} + d$ on $\mathbf{E} = \mathbf{L} + \varepsilon$, we know that $c_1(\mathbf{L}|_{\partial X_i}) = \pm c_1(\mathbf{L}_\alpha|_{\partial X_i})$ ($\pm$ sign results from the fact that holonomy representations are conjugate to each other by an element of SO(3) which may change the orientation of line bundles). This shows that a topological reduction $\mathbf{E} = \mathbf{L} + \varepsilon$ associated with a reducible flat connection in $\mathscr{C}$ satisfies

$$j_i^*(c_1(\mathbf{L}_\alpha)) = \pm j_i^*(c_1(\mathbf{L})). \tag{4.2}$$

Note that since $H^2(X, \partial X_i, \mathbf{Z})$ has no 2-torsion, a reduction $\mathbf{E} = \mathbf{L} + \varepsilon$ satisfying (4.2) automatically satisfies (4.1). We can thus prove the following theorem.

Theorem 4.2. *Under the conditions of Theorem* 1.1 *the set*

$$S = \left\{ c \in H^2(X, \mathbf{Z}) \mid j_i^*(c) = \pm j_i^*(c_1(\mathbf{L}_\alpha)),\ i = 1, \cdots, n \right\} / c \sim -c$$

is in 1-1 *correspondence with the set of $\mathcal{G}$-equivalence classes of $\mathcal{G}$-reducible flat connections in $\mathscr{C}$.*

Proof. Let ∇ be a $\mathscr{G}$-reducible flat connection. We have shown that there is a $c \in S$ such that $\mathbf{E} = \mathbf{L}_c + \varepsilon$ and $\nabla = \nabla_c + d$. Each complex line bundle on X supports a unique gauge equivalence class of flat connections since

$$\begin{aligned}\{\alpha: \pi_1(X) \to S^1\}/\text{conjugation} &\simeq \operatorname{Hom}(H_1(X, \mathbf{Z}), S^1) \simeq H_1(X, \mathbf{Z})\\ &\simeq \operatorname{Tor} H^2(X, \mathbf{Z}) \simeq H^2(X, \mathbf{Z}).\end{aligned}$$

Therefore, if there is a reducible flat connection producing a splitting corresponding to a $c \in S$, it is unique up to gauge equivalence.

On the other hand, let $c \in S$ and let ∇_c be a flat connection on $\mathbf{L}_c$. The same argument as before shows that there is a unique gauge equivalence class of flat connections on any complex line bundle over $\partial X_i \times [0, \infty)$. Therefore, $\nabla_c | \partial X_i$ is gauge equivalent to ∇_{α_i} on $\partial X_i \times [0, \infty)$ since $c \in S$ implies that $\mathbf{L}_c \cong \mathbf{L}_{\alpha_i}$ up to orientation. Let $\nabla_{\alpha_i} = g_i^{-1} \circ \nabla_c \circ g_i$ on $\partial X_i \times [0, \infty)$, $g_i \in C^\infty(\operatorname{Ad} \mathbf{L}_c |_{\partial X_i \times [0, \infty)})$. Note that $\operatorname{Ad} \mathbf{L} = M \times S^1$ since S^2 is commutative and the Ad action is trivial. Therefore we can think of g_i as functions g_i: $\partial X_i \times [0, \infty) \to S^1$. Let us consider the restriction $g_{i,\partial}$: $\partial X_i \to S^1$. Recall that ∂X_i is a rational homology sphere. Since $(g_{i,\partial})_*$: $\pi_1(\partial X_i) \to \pi_1(S^1) = \mathbf{Z}$ factors through $H_1(\partial X_i, \mathbf{Z})$ and $H_1(\partial X_i, \mathbf{Z})$ is torsion, $(g_{i,\partial})_*$ is a 0-map. Therefore, there is a lifting $\tilde{g}_{i,\partial} \to \mathbf{R}$ of $g_{i,\partial}$ which shows that $g_{i,\partial}$ is homotopic to a constant. Using this homotopy we can extend the gauge transformation g_i to a gauge transformation on all of $\mathbf{L}_c$, and define $\nabla_1 = g^{-1} \circ \nabla_c \circ g$. The connection $\nabla = \nabla_1 + d$ is then a reducible flat connection in $\mathscr{C}$ which corresponds to $c \in S$.

Hence, for every $c \in S$ there is a unique $\mathscr{G}$-equivalence class of reducible flat connections such that $\mathbf{L}_c \oplus \varepsilon$ is the associated topological splitting. On the other hand, if ∇ is $\mathscr{G}$-reducible, it induces a splitting corresponding to $c \in S$.

5. Index computations

In this section we will compute indices of Fredholm complexes which we will use for the proof of Theorem 1.1 in §6, and relate these indices to ρ_α invariant of ∂X.

Recall that $\mathfrak{g} = \{\phi \in L^2_{3,\mathrm{loc}}(\mathfrak{g}_{\mathbf{E}}) \,|\, \|\nabla_0 \phi\|_{L^2_{2,\delta}} < \infty\}$ can be split as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathscr{H}_{\mathfrak{g}}$, where $\mathfrak{g}_0 = L^2_{3,\delta}(\mathfrak{g}_{\mathbf{E}})$ and $\mathscr{H}_{\mathfrak{g}} = \mathscr{H} \cap \mathfrak{g}$. We have also shown that there is an isomorphism

$$\begin{aligned}\mathscr{H}_{\mathfrak{g}} &\simeq \operatorname{Ker}\left(d^{\nabla_\partial}: \Gamma(\mathfrak{g}_{\mathbf{E}/\partial X}) \to \Gamma(T^*(\partial X) \otimes \mathfrak{g}_{\mathbf{E}/\partial X})\right)\\ &\simeq \mathbf{R}^{(n-k)} \times \mathbf{R}^{3k} = \mathbf{R}^{n+2k},\end{aligned}$$

where n is the number of boundary components and k is the number of boundary components such that α_i is trivial on $H_1(\partial X_i, \mathbf{Z})$. Therefore, the complex

$$(\mathrm{FC})_\pm \quad 0 \to \mathfrak{g} \xrightarrow{d^{\nabla_0}} L^2_{2,\delta}(T^*M \times \mathfrak{g}_\mathbf{E}) \xrightarrow{d^{\nabla_0}} L^2_{1,\delta}(\Lambda^2_\pm(T^*M) \otimes \mathfrak{g}_\mathbf{E}) \to 0$$

is Fredholm if and only if the complex

$$(5.1)_\pm \quad 0 \to L^2_{3,\delta}(\mathfrak{g}_\mathbf{E}) \xrightarrow{d^{\nabla_0}} L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_\mathbf{E}) \xrightarrow{d^{\nabla_0}_\pm} L^2_{1,\delta}(\Lambda_\pm(T^*M) \otimes \mathfrak{g}_\mathbf{E}) \to 0$$

is Fredholm. This can, by Theorem 2.2, be obtained for a proper choice of δ. Note that, since $\mathfrak{g} = L^2_{3,\delta}(\mathfrak{g}_\mathbf{E}) \oplus \mathbf{R}^{n+2k}$ we have $\operatorname{index}(\mathrm{FC})_\pm = (n+2k) + \operatorname{index}(5.1)_\pm$.

Since $\mathfrak{g}_\mathbf{E} \simeq \mathbf{E} = \mathbf{L}_\alpha \oplus \varepsilon$ and $\nabla_0 = \nabla_\alpha + d$, the complexes $(\mathrm{FC})_\pm$ split into the direct sum

$$(1)_\pm \quad 0 \to \mathscr{R}(\varepsilon) \xrightarrow{d} L^2_{2,\delta}(T^*M) \xrightarrow{d_\pm} L^2_{1,\delta}(\Lambda^2_\pm(T^*M)) \to 0$$

$$(2)_\pm \quad 0 \to \mathscr{R}(\mathbf{L}_\alpha) \xrightarrow{d^{\nabla_\alpha}} L^2_{2,\delta}(T^*M \otimes \mathbf{L}_\alpha) \xrightarrow{d^{\nabla_\alpha}_\pm} L^2_{1,\delta}(\Lambda^2_\pm(T^*M) \otimes \mathbf{L}_\alpha) \to 0$$

and $\operatorname{Index}(\mathrm{FC})_\pm = \operatorname{index}_\mathbf{R}(1)_\pm + 2\operatorname{index}_\mathbf{C}(2)_\pm$. Here

$$\mathscr{R}(\varepsilon) = \left\{ f \in L^2_{3,\mathrm{loc}}(M) \mid \|df\|_{L^2_{2,\delta}} < \infty \right\},$$

$$\mathscr{R}(\mathbf{L}_\alpha) = \left\{ \sigma \in L^2_{3,\mathrm{loc}}(\mathbf{L}_\alpha) \mid \|\nabla_\alpha \sigma\|_{L^2_{2,\delta}} < \infty \right\}.$$

Denote by $H^i(1)_\pm$ the cohomology groups of complex $(1)_\pm$ and by $H^i(2)_\pm$ the cohomology groups of complex $(2)_\pm$.

Lemma 5.1. $\dim H^0(1)_\pm = 1$, $\dim H^0(2)_\pm = 0$.

Proof. Let $f \in L^2_{3,\mathrm{loc}}(M)$, $df = 0$. Then $f = \mathrm{const} \in L^2_{3,\mathrm{loc}}(M)$ so $H^0(1)_\pm = \{ f_c(x) = c \mid c \in \mathbf{R} \}$. Let $\sigma \in L^2_{3,\mathrm{loc}}(\mathbf{L}_\alpha)$. Since $\mathbf{L}_\alpha$ is a nontrivial 1-dimensional complex bundle, there are no parallel sections, hence $H^0(2)_\pm = 0$.

Lemma 5.2. *Let n be the number of boundary components for X. Assume each boundary component ∂X_i is a rational homology sphere. Let k be the number of ∂X_i such that α_i: $H_1(\partial X_i, \mathbf{Z}) \to S^1$ is trivial. Assume also that $H_1(\partial X_i, \mathbf{L}_\alpha) = 0$, $\forall i$. Then*

(a) $\dim_\mathbf{R} H^1(1)_\pm = \dim_\mathbf{R} H^1_{\mathrm{comp}}(M) - n + 1$,

(b) $\dim_\mathbf{C} H^1(2)_\pm = \dim_\mathbf{C} H^1_{\mathrm{comp}}(M, \mathbf{L}_\alpha) - k$.

Proof. We will do the computation for $H^1(1)_-$ first. The computation for $H^1(1)_+$ is exactly the same. Let $[\omega] \in H^1(1)_-$. Then $(d\omega)_- = 0$ and since $d\omega$ is integrable, as in the case of compact manifolds, $0 = \int |(d\omega)_-|^2 = \frac{1}{2}\int |d\omega|^2$ and hence $d\omega = 0$. Therefore

$$H^1(1)_- = H^1_\delta(H) = \frac{\ker\big(d : L^2_{2,\delta}(T^*M) \to L^2_{1,\delta}(\Lambda^2 T^*M)\big)}{\operatorname{im}\big(d : \mathscr{R}(\varepsilon) \to L^2_{2,\delta}(T^*M)\big)}.$$

Since ∂X_i is a rational homology sphere, for each $i = 1, \cdots, n$,

$$H^1_{\mathrm{DR}}(\partial X_i \times [0, \infty)) = 0$$

and $\omega|_{\partial X_i \times [0,\infty)} = df_i$. Let β_i be smooth cutoff functions on ends, $\beta_i : M \to [0, 1]$, such that

$$\beta_i(x) = \begin{cases} 1, & x \in \partial X_i \times [1, \infty), \\ 0, & x \in X, \end{cases}$$

and define $f = \sum_i \beta_i \cdot f_i$. Since $df|_{\partial X \times [1,\infty)} = \omega|_{\partial X \times [1,\infty)}$ we see that $f \in \mathscr{R}(\varepsilon)$. Therefore, $\omega' = \omega - df$ is another representative of $[\omega]_{H^1(1)}$ but ω' is compactly supported. This shows that the natural map $i : H^1_{\mathrm{comp}}(M) \to H^1(1)$ given by $i[\omega]_{H^1_{\mathrm{comp}}} = [\omega]_{H^1(1)}$ is surjective.

Define the set $K = \{f_c = \sum \beta_i c_i \mid c = (c_1, \cdots, c_n) \in \mathbf{R}^n\}$ and let j: $K \to H^1_{\mathrm{comp}}(M)$ be given by $j(f_c) = [df_c]_{H^1_{\mathrm{comp}}}$. This is well defined since $df_c(x) = 0$ for $x \in \partial X \times [1, \infty)$. We will prove that the sequence

$$K \xrightarrow{j} H^1_{\mathrm{comp}}(M) \xrightarrow{i} H^1(1)_- \to 0 \tag{5.2}$$

is exact and that $\dim(\ker j) = 1$. Then

$$\begin{aligned} \dim H^1(1)_- &= \dim H^1_{\mathrm{comp}}(M) - \dim(\operatorname{im} j) \\ &= \dim H^1_{\mathrm{comp}}(M) - \dim K + \dim(\ker j) \\ &= \dim H^1_{\mathrm{comp}}(M) - n + 1. \end{aligned}$$

Since $f_c \in \mathscr{R}(\varepsilon)$ and $i \circ j(f_c) = i[df_c] = [df_c]_{H^1(1)} = 0$, $\operatorname{im} j \subset \ker i$. On the other hand, let $[\omega]_{H^1_{\mathrm{comp}}} \in \ker i$. Then $\omega = df$ for $f \in \mathscr{R}(\varepsilon)$. Since ω is compactly supported, $df = 0$ outside a compact set C and for large enough C' (such that $M/C' \subset \operatorname{End} M$) there is an $f_c \in K$ such that $f = f_c$ on M/C'. Let $g = f_c - f$. g is compactly supported and

$$j(f_c) = [df_c]_{H^1_{\mathrm{comp}}} = [dg + df]_{H^1_{\mathrm{comp}}} = [\omega + dg]_{H^1_{\mathrm{comp}}} = [\omega]_{H^1_{\mathrm{comp}}}.$$

Therefore $\operatorname{Ker} i \subset \operatorname{im} j$, and (5.2) is exact.

Let $f_c \in \ker j$. $[df_c]_{H^1_{\mathrm{comp}}} = 0$ implies that $df_c = dg$ for a compactly supported g. Therefore $f_c - g = \text{const} = k$. Since g is compactly supported this implies that $c = (k, \cdots, k)$ and hence $\dim(\ker j) = 1$. We have proved statement (a) of Lemma 5.2.

Proof of statement (b) follows in exactly the same way. Let i_α: $H^1_{\text{comp}}(M, \mathbf{L}_\alpha) \to H^1(2)$ be given as before. The assumption $H^1(\partial X_i; \mathbf{L}_{\alpha_i}) = 0$ replaces the fact that $H^1_{\text{DR}}(\partial X_i, \mathbf{Z}) = 0$ in proving that i_α is onto. Let

$$K_\alpha = \left\{ \sigma = \sum_{i=1}^{n} \beta_i \sigma_i \,|\, \sigma_i \in \Gamma(\mathbf{L}|_{\partial X_i}),\ d^{\nabla_{\alpha_i}}(\sigma_i) = 0 \right\}.$$

Let k denote the number of boundary components ∂X_i such that $\alpha_i : H_1(\partial X_i, \mathbf{Z}) \to S^1$ is trivial. Than $\dim K_\alpha = k$. Defining $j_\alpha : K_\alpha \to H^1_{\text{comp}}(M, \mathbf{L}_\alpha)$ as before, we get an exact sequence

$$K_\alpha \xrightarrow{j_\alpha} H^1_{\text{comp}}(M, \mathbf{L}_\alpha) \xrightarrow{i_\alpha} H^1(2) \to 0, \tag{5.3}$$

which is exact. The proof is the same as for (5.2).

For $\sigma \in K_\alpha$, $j_\alpha(\sigma) = [d\sigma]_{H^1_{\text{comp}}} = 0$ implies $d^{\nabla_\alpha}(\alpha) = d^{\nabla_\alpha}(\rho)$ for a compactly supported ρ. Therefore $d^{\nabla_\alpha}(\sigma - \rho) = 0$ and since $\mathbf{L}_\alpha$ is nontrivial, $\sigma = \rho$. Hence, $\sigma \in K_\alpha \cap C^\infty_{\text{comp}}(\mathbf{L}_\alpha) = 0$ and $\ker j_\alpha = 0$. The computation

$$\begin{aligned}
\dim_{\mathbf{C}} H^1(2) &= \dim_{\mathbf{C}} H^1_{\text{comp}}(M, \mathbf{L}_\alpha) = \dim_{\mathbf{C}}(\ker i_\alpha) \\
&= \dim_{\mathbf{C}} H^1_{\text{comp}}(M, \mathbf{L}_\alpha) - \dim_{\mathbf{C}}(\operatorname{im} j_\alpha) \\
&= \dim_{\mathbf{C}} H^1_{\text{comp}}(M, \mathbf{L}_\alpha) - k
\end{aligned}$$

proves statement (b).

Lemma 5.3. *Under the assumptions of Lemma* 5.2,

(a) $\dim_{\mathbf{R}} H^2(1)_\pm = b^2_\pm(M)$,

(b) $\dim_{\mathbf{C}} H^2(2)_\pm = b^2_\pm(M, \mathbf{L}_\alpha)$,

where $b^2_\pm(M) = \dim(\mathscr{H}^2(M) \cap \Omega^2_\pm)$ *and* $b^2_\pm(M, \mathbf{L}_\alpha) = \dim(\mathscr{H}^2(M, \mathbf{L}_\alpha) \cap \Omega^2_\pm(\mathbf{L}_\alpha))$. *Here* $\mathscr{H}^2(M)$ *and* $\mathscr{H}^2(M, \mathbf{L}_\alpha)$ *denote the spaces of harmonic* 2*-forms and* $\Omega^2_\pm$ *the space of anti-self-dual* (*self-dual*) 2*-forms.*

Proof. Let $[\omega] \in H^2(1)_\pm$. There is a unique representative such that

$$\left\{ \begin{array}{l} *\omega = \pm\omega \\ d^*(e^{\tau\delta}\omega) = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} *\omega = \pm\omega \\ d(e^{\tau\delta}\omega) = 0 \end{array} \right\}. \tag{5.4}$$

Since the boundary components are rational homology spheres, the proof of Lemma 5.3 in [15] applies to our situation. Therefore, for small enough choice of δ, for a representative of $[\omega] \in H^2(1)_\pm$ which satisfies (5.4) we have $\int |e^{\tau\delta}\omega|^2 < \infty$. We can define a mapping $\mathscr{k}$: $H^2(1)_\pm \to \mathscr{H}^2(M) \cap \Omega^2_\pm$ by setting $\mathscr{k}[\omega]_{H^2(1)_\pm} = e^{\tau\delta}\omega$, ω being a representative of $[\omega]$ satisfying (5.4).

Injectivity of the map is clear from the definition. Surjectivity follows from the fact that for $\phi \in \mathscr{H}^2(M) \cap \Omega^2_\pm$ the form $\omega = \varepsilon^{-\tau\delta}\phi$ satisfies equation (5.4), $\omega \in L^2_{1,\delta}(\Lambda^2_\pm T^*M)$, and $\phi = \mathscr{k}(\omega)$.

Exactly the same proof works for complexes $(2)_{\pm}$. The assumption that $H^1(\partial X_i, \mathbf{L}_{\alpha_i}) = 0$ makes the proof of Lemma 5.3 in [15] work for the forms with values in $\mathbf{L}_\alpha$.

Atiyah, Patodi, and Singer show in [1], [2] that there is a connection between L^2-harmonic forms on M and cohomology of X. The space $\mathscr{H}^*(M)$ of L^2-harmonic forms on a manifold M obtained from a compact X by adding cylinders along the boundary is naturally isomorphic with the image $\hat{H}^*(X) = \mathrm{im}(H^*(X, \partial X, \mathbf{R}) \to H^*(X, \mathbf{R}))$. Since the kernel $\ker(H^2(X, \partial X, \mathbf{R}) \to H^2(X, \mathbf{R}))$ is exactly the radical of the cup-product induced pairing on the second cohomology, $b^2_{\pm}(M)$ are exactly the dimensions of positive and negative definite parts of $\hat{H}^2(X)$. Hence $\mathrm{sign}(X) = b^2_+(M) - b^2_-(M)$ and $\dim \hat{H}^2(X) = b^2_+ + b^2_-$. The statements are exactly the same for cohomology with local coefficients (see [1, §4] and [2, §2]). Since $H^1(\partial X, \mathbf{R}) = 0$ and $H^1(\partial X_i, \mathbf{L}_\alpha) = 0$, the pairings on $H^2(X, \mathbf{R}) \cong H^2(X, \partial X, \mathbf{R})$ and $H^2(X; \mathbf{L}_\alpha) \cong H^2(X, \partial X; \mathbf{L}_\alpha)$ are nondegenerate and

$$b^2(X) = \dim H^2(X, \mathbf{R}) = \dim \mathscr{H}^2(M) = b^2_+(M) + b^2_-(M),$$

$$b^2(X, \mathbf{L}_\alpha) = \dim H^2(X, \mathbf{L}_\alpha) = \dim \mathscr{H}^2(\mathbf{L}_\alpha, M) = b^2_+(M, \mathbf{L}_\alpha) + b^2_-(M, \mathbf{L}_\alpha).$$

We can now prove the following

Theorem 5.4. *For small enough $\delta > 0$, the indices of the complexes*

$$(\mathrm{FC})_{\pm} \quad 0 \to \mathfrak{g} \xrightarrow{d^{\nabla_0}} L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}}) \xrightarrow{d^{\nabla_0}_{\pm}} L^2_{1,\delta}(\Lambda^2_{\pm}(T^*M) \otimes \mathfrak{g}_{\mathbf{E}}) \to 0$$

are

$$\mathrm{index}(\mathrm{FC})_+ = 3 - m - \rho_\alpha(\partial X),$$

$$\mathrm{index}(\mathrm{FC})_- = 3 - m + \rho_\alpha(\partial X),$$

where $\rho_\alpha(\partial X) = \mathrm{sign}\, X - \mathrm{sign}_\alpha(X)$ is a diffeomorphism invariant of ∂X and α: $H^1(\partial X, \mathbf{Z}) \to S^1$. m is the number of boundary components for which α_i: $H^1(\partial X_i) \to S^1$ is nontrivial.

Proof.

$$\begin{aligned} \mathrm{index}_{\mathbf{R}}(\mathrm{FC})_{\pm} &= \mathrm{index}_{\mathbf{R}}(1)_{\pm} + 2 \cdot \mathrm{index}_{\mathbf{C}}(2)_{\pm} \\ &= 1 - \left(\dim_{\mathbf{R}} H^1_{\mathrm{comp}}(M) - n + 1\right) + b^2_{\pm}(M) \\ &\quad + 2\left\{-\left(\dim_{\mathbf{C}} H^1_{\mathrm{comp}}(M, \mathbf{L}_\alpha) - (n - m)\right) + b^2_{\pm}(M, \mathbf{L}_\alpha)\right\}. \end{aligned}$$

Since $H^1_{\text{comp}}(M) \cong H^1(X, \partial X, \mathbf{R})$, the exact sequence of the pair gives

$$\begin{array}{cccccccccccc}
H^0(X,\partial X,\mathbf{R}) & \longrightarrow & H^0(X,\mathbf{R}) & \longrightarrow & H^0(\partial X,\mathbf{R}) & \longrightarrow & H^1(X,\partial X,\mathbf{R}) & \longrightarrow & H^1(X,\mathbf{R}) & \longrightarrow & H^1(\partial X,\mathbf{R}) \\
\| & & \| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{R}^n & \longrightarrow & H^1(X,\partial X,\mathbf{R}) & \longrightarrow & H^1(X,\mathbf{R}) & \longrightarrow & 0
\end{array}$$

and, since X has rational homology of an n-punctured 4-sphere, $\dim H^1(X, \partial X, \mathbf{R}) = n - 1$. Also, $H^2(X, \mathbf{R}) = 0$ and hence $b^2_{\pm}(M) = 0$. If m denotes the number of boundary components for which α_i: $H_1(\partial X_i, \mathbf{Z}) \to S^1$ is nontrivial, $H^0(\partial X, \mathbf{L}_\alpha) = \oplus H^0(\partial X_i, \mathbf{L}_{\alpha_i}) \cong \mathbf{C}^{n-m}$. The exact sequence

$$\begin{array}{ccccccccc}
H^0(X;\mathbf{L}_\alpha) & \to & H^0(\partial X;\mathbf{L}_\alpha) & \to & H^1(X,\partial H;\mathbf{L}_\alpha) & \to & H^1(X,\mathbf{L}_\alpha) & \to & H^1(\partial X,\mathbf{L}_\alpha) \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & \mathbf{C}^{n-m} & \longrightarrow & H^1(X,\partial X;\mathbf{L}_\alpha) & \to & H^1(X,\mathbf{L}_\alpha) & \longrightarrow & 0
\end{array}$$

gives that

$$\dim_{\mathbf{C}} H^1(X, \partial X; \mathbf{L}_\alpha) = \dim H^1_{\mathbf{C}}(X, \mathbf{L}_\alpha) + n - m.$$

Since $H^1(X, \partial X; \mathbf{L}_\alpha) \cong H^1_{\text{comp}}(M, \mathbf{L}_\alpha) \cong H^3(M, \mathbf{L}_\alpha)^*$,

$$\begin{aligned}
\operatorname{index}_{\mathbf{R}}(\mathrm{FC})_{\pm} &= 1 - 2 \cdot \dim\mathbf{C}\, H^1_{\text{comp}}(M, \mathbf{L}_\alpha) + 2(n - m) + 2b^2_{\pm}(M, \mathbf{L}_\alpha) \\
&= 1 - \dim_{\mathbf{C}} H^1(X, \mathbf{L}_\alpha) - n + m - \dim_{\mathbf{C}} H^3(X, \mathbf{L}_\alpha) \\
&\quad + 2n - 2m + 2b^2_{\pm}(M, \mathbf{L}_\alpha) \\
&= 1 + n - m + \chi(X, \mathbf{L}_\alpha) - \dim_{\mathbf{C}} H^2(X, \mathbf{L}_\alpha) + 2b^2_{+}(M, \mathbf{L}_\alpha) \\
&= 1 + n - m + \chi(X) \pm \operatorname{sign}_\alpha(X) \\
&= 1 + n - m + (2 - n) \mp (\operatorname{sign} X - \operatorname{sign}_\alpha(X)) \\
&= 3 - m \mp \rho_\alpha(\partial X).
\end{aligned}$$

Note that $\operatorname{index}_{\mathbf{R}}(\mathrm{FC})_{\pm}$ is odd since $\operatorname{index}_{\mathbf{R}}(\mathrm{FC})_{\pm} = 1 + 2I$.

6. The proof of Theorem 1.1

Using the explicit formula for the difference of Pontrjagin forms (found for example in [18, Chapter III, §3]) it can be shown that $p_1(\nabla) = p_1(\nabla_0) = 0$ for every $\nabla \in \mathscr{C}$. Therefore, as in the compact manifold case, $R^\nabla = 0 \Leftrightarrow R^\nabla_+ = 0 \Leftrightarrow R^\nabla_- = 0$ and hence the set of flat connections can be described as the set of solutions of either of the two equations

$$d^\nabla_- a + [a, a]_- = 0, \tag{6.1}$$

or

$$d^\nabla_+ a + [a, a]_+ = 0. \tag{6.2}$$

It is important to note that, for $\nabla \in \mathscr{C}$, the complex

$$(6.3) \qquad 0 \to \mathfrak{g}_0 \xrightarrow{d^\nabla} L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}}) \xrightarrow{d^{\nabla\pm}} L^2_{1,\delta}(\Lambda^2_\pm(T^*M) \otimes \mathfrak{g}_{\mathbf{E}}) \to 0$$

is Fredholm. Since d^∇ has closed image, the complex (6.3) is Fredholm if and only if the operator

$$(6.4)_\pm \qquad D^\pm(\nabla)\colon L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}}) \begin{array}{c} \xrightarrow{\qquad d^{\nabla\pm} \qquad} L^2_{1,\delta}(\Lambda^2_\pm(T^*M) \otimes \mathfrak{g}_{\mathbf{E}}) \\ \oplus \qquad\qquad\qquad \oplus \\ \xrightarrow[e^{\tau\delta}(d^\nabla)^* e^{\tau\delta}]{\qquad\qquad} L^2_{1,\delta}(\mathfrak{g}_{\mathbf{E}}) \end{array}$$

is Fredholm. Also, index(6.3) = –index($D^\pm(\nabla)$). Theorem 2.2 shows that the corresponding operators $D^\pm(\nabla_0)$ are Fredholm. Since $\nabla = \nabla_0 + a$ for $a \in L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}})$, the two maps differ by a compact operator and therefore $D^\pm(\nabla)$ is Fredholm and index $D^\pm(\nabla) =$ index $D^\pm(\nabla^0)$. The complex

$$(6.5)_\pm \qquad 0 \to \mathfrak{g} \xrightarrow{d^\nabla} L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}}) \xrightarrow{d^\nabla_\pm} L^2_{1,\delta}(\Lambda^2_\pm(T^*M) \otimes \mathfrak{g}_{\mathbf{E}}) \to 0$$

is then also Fredholm with index$(6.3)_\pm = 3 - k \mp \rho_\alpha(\partial X)$.

Let $\mathscr{F} \subset \mathscr{B}$ denote the space of $\mathscr{G}$-equivalence classes of flat connections in $\mathscr{C}$. Considering $\mathscr{F}$ as the set of solutions of (6.1) and (6.2) and applying the standard Kuranishi argument (see proof of Theorem III, 2.1 in [12]) we get the following theorem.

Theorem 6.1. *Let $\nabla \in \mathscr{C}$ be a flat connection. Then there are neighborhoods $\mathscr{O}_\pm$ of 0 in the first cohomology $H^1_{\nabla,\pm}$ of $(6.5)_\pm$ and differentiable maps ϕ_+ and ϕ_-,*

$$\phi_\pm\colon \mathscr{O}_\pm \to H^2_{\nabla,\pm},$$

with $\phi_\pm(0) = 0$ which are Γ_∇-equivariant if ∇ is reducible such that

(a) *$\mathscr{F} \cap \mathscr{O}_{\nabla,\varepsilon} \overset{\rho}{\cong} \phi^{-1}_\pm(0)$ if ∇ is irreducible,*

(b) *$\mathscr{F} \cap (\mathscr{O}_{\nabla,\varepsilon}/\Gamma_\nabla) \overset{\rho}{\cong} \phi^{-1}_\pm(0)/S^1$ if ∇ is reducible.*

Here $\overset{\rho}{\cong}$ denotes a homeomorphism defined by some ambient diffeomorphism ρ of $(d^\nabla(\mathfrak{g}))^\perp \subset L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}})$ which contains both $\mathscr{O}_{\nabla,\varepsilon}$ and $H^1_{\nabla,\pm}$ interpreted as $\ker d^\nabla_\pm \cap (d^\nabla(\mathfrak{g}))^\perp$.

For a $\mathscr{G}$-reducible connection ∇ the bundle and the connection split as $\mathbf{E} = \mathbf{L} \oplus \varepsilon$, $\nabla = \nabla_{\mathbf{L}} + d$, and Γ_∇ is the group of rotations for a constant

$z \in S^1$ inside $\mathbf{L}$. The complexes $(6.5)_\pm$ split as a direct sum

$$(6.6)_\pm \qquad 0 \to \mathscr{R}(\varepsilon) \xrightarrow{d} L^2_{2,\delta}(T^*M) \xrightarrow{d_\pm} L^2_{1,\delta}\big(\Lambda^2_\pm(T^*M)\big) \to 0$$

$$\oplus \qquad\qquad \oplus \qquad\qquad \oplus$$

$$(6.7)_\pm \qquad 0 \to \mathscr{R}(\mathbf{L}) \xrightarrow{d^{\nabla_{\mathbf{L}}}} L^2_{2,\delta}(T^*M \otimes \mathbf{L}) \xrightarrow{d^{\nabla_{\mathbf{L}}}_\pm} L^2_{1,\delta}\big(\Lambda^2_\pm(T^*M) \otimes \mathbf{L}\big) \to 0.$$

In the proof of Theorem 5.4 we have shown that the first and second cohomologies of complexes $(6.6)_\pm$ vanish. Therefore $H^1_{\nabla,\pm}$ and $H^2_{\nabla,\pm}$ have a structure of a complex vector space and the Γ_∇ action induces an action of $S^1 = \mathrm{U}(1)$ by multiplication.

If $H^2_{\nabla,\pm} = 0$ this gives $\mathscr{F}$ a structure of a smooth manifold of dimension $d_\pm = -\mathrm{index(FC)}_\pm = -3 + m \pm \rho_\alpha(\partial X)$ in a neighborhood of an irredudible ∇, and a structure of a cone on $\mathbf{C}P^{l_\pm}$, $l_\pm = (-2 + m \pm \rho_\alpha(\partial X))/2$, in a neighborhood of a $\mathscr{G}$-reducible ∇. This, however, is not possible if $H^2(X, \mathbf{L}_\alpha) \neq 0$. In order to obtain a manifold we need to perturb the equations to regularize $\mathscr{F}$. Before we do that we need to make the following observation.

Theorem 6.2. *The space $\mathscr{F}$ of $\mathscr{G}$-equivalence classes of flat connections in $\mathscr{C}$ is compact.*

Proof. It is a standard result that the space of equivalence classes of flat G-connections on a principal G bundle can be identified with an open and closed subset of $\mathrm{Hom}(\pi_1(X), G)/G$ via the holonomy representation (see [10, §1]). Since $\pi_1(M)$ is finitely generated and SO(3) is compact, this is a compact space. If we denote by ν_i the composition $\pi_1(\partial X_i) \xrightarrow{\alpha_i} S^1 \to \mathrm{SO}(3)$ the discussion in §4 shows that the gauge equivalent classes of flat connections in $\mathscr{C}$ are in 1-1 correspondence with the closed subset

$$\bigcap_{i=1}^{n} (j_i^*)^{-1}([\nu_i]) \subset \mathrm{Hom}(\pi_1(X), \mathrm{SO}(3))/\mathrm{SO}(3),$$

where j_i^*: $\mathrm{Hom}(\pi_1(X)_1\mathrm{SO}(3))/\mathrm{SO}(3) \to \mathrm{Hom}(\pi_1(\partial X_i), \mathrm{SO}(3))/\mathrm{SO}(3)$ is given by $j_i^*([\alpha]) = [\alpha \circ j_*]$ and j_*: $\pi_1(\partial X_i) \to \pi_1(X)$ is the inclusion induced morphism. Therefore, $\mathscr{F}$ is compact.

Note that by using Theorem 6.2 we can avoid the use of compactness theorems of K. Uhlenbeck [17], [18] in the rest of the argument.

We have shown in Theorem 4.2 that

$$\mu = \#\big[\{c \in H^2(X, \mathbf{Z}) \mid j_i^*(c) = \pm j_i^*(c(\mathbf{L}_\alpha))\}_{c\sim -c}\big]$$

is the number of $\mathscr{G}$-equivalence classes of $\mathscr{G}$-reducible flat connections in $\mathscr{C}$.

Since $\mathscr{G}$ acts on $\mathscr{C}$ and $L^2_1(\Lambda^2_\pm(T^*M) \otimes \mathfrak{g}_{\mathbf{E}})$ by conjugation and $R^{\nabla^g} = g^{-1} \circ R^\nabla \circ g$, the maps $\nabla \to R^\nabla_\pm$ define sections $\mathscr{P}_\pm$ of the bundles $\mathscr{L}_\pm \to \mathscr{B}$ where $\mathscr{L}_\pm = \mathscr{C} \times_{\mathscr{G}} L^2_1(\Lambda^2_\pm(T^*M) \otimes \mathfrak{g}_{\mathbf{E}})$. By Theorem 6.1 $\mathscr{F} \subset \mathscr{B}$ is the zero set of both sections $\mathscr{P}_\pm$: $\mathscr{B} \to \mathscr{L}_\pm$.

We start perturbing the equations $\mathscr{P}_{\pm}([\nabla]) = 0$, as in [12, IV], by first perturbing them in the neighborhood of a $\mathscr{G}$-reducible flat connection $\nabla \in \mathscr{C}$. Compact perturbations can be chosen Γ_∇-equivariantly so that the zero set of the perturbed section $S^0_{\pm}$ has a structure of a cone on $\mathbf{C}P^{l_\pm}$ in the neighborhood of $[\nabla]$.

After doing this perturbation around each of the μ classes of $\mathscr{G}$-reducible connections we get a compact perturbation $S^1_{\pm}$ of $\mathscr{P}_{\pm}$. We still need to regularize $S^1_{\pm}$ on the complement of the cones. Since $\mathscr{F}$ was compact, the zero sets $\mathscr{M}^1_{\pm}$ of $S^1_{\pm}$ are also compact. We need to regularize them outside the μ cones. Since complement of the open cones is a compact set, we can cover it by finitely many $\mathscr{O}_{\nabla,\varepsilon}$ and as in [12, IV, §4] produce a compact perturbation $S_{\pm}$ of $S^1_{\pm}$ such that the zero sets $\mathscr{M}_{\pm}$ of $S_{\pm}$ are compact $d_{\pm}$-dimensional manifolds with μ singular points.

Note that we have just proved Proposition 2.2.

We now follow an argument of Fintushel and Stern ([5], [6]) to show that if either one of d_+ and d_- is positive, the number μ has to be even. Define $\mathscr{G}_1 = \{g \in \mathscr{G} \mid r_1(g) = \mathrm{id} \in \mathfrak{g}(\mathbf{E})/\partial X_i\}$, where ∂X_i is a boundary component for which α_i has order greater than 2. Then $\mathscr{G}_0 \subset \mathscr{G}_1$ and $\mathscr{G}_1/\mathscr{G}_0 = G/\Gamma_{\nabla_{\partial_1}} = G_1$ is a compact group. As in the proof of Theorem 3.4, we can show that $\mathscr{B}^*_1 = \mathscr{B}^*/\mathscr{G}_1$ has a Banach manifold structure, and that $\pi_1\colon \mathscr{C}^* \to \mathscr{B}^*_1$ is a principal $\mathscr{G}_1$-bundle. Moreover, the projection $\pi_{G_1}\colon \mathscr{B}^*_1 \to \mathscr{B}^*$ is a principal $G/G_1 = \Gamma_{\nabla_{\partial_1}} \simeq S^1$-bundle.

If either d_+ or d_- is positive, the above perturbation argument produces a manifold $\mathscr{M} \subset \mathscr{B}$ with isolated conical singularities. Let $\overline{\mathscr{M}} = \mathscr{M} \setminus$ cones. $\overline{\mathscr{M}}$ has μ boundary components each of which is $\mathbf{C}P^l$. The bundle $\pi_{G_1}\colon \mathscr{B}^*_1 \to \mathscr{B}^*$ restricts to a bundle with $w_1 = 1 \in H^2(\mathbf{C}P^l, \mathbf{Z}_2)$ on each $\mathbf{C}P^l$ (see [5], [6]). Since an odd number of such bundles cannot bound, μ has to be even.

Since $d_{\pm} = -3 + m \pm \rho_\alpha(\partial X)$ the condition that $d_{\pm} \leqslant 0$ translates into $|\rho_\alpha(\partial X)| \leqslant 3 - m$. Therefore if $|\rho_\alpha(\partial X)| > 3 - m$, μ has to be even, which proves statement (b) in Theorem 1.1. Since $d_{\pm}$ is odd, $m \pm \rho_\alpha(\partial X) = 0 \pmod 2$ which shows statement (a) in Theorem 1.1.

7. Applications

An immediate corollary of Theorem 1.1 is the integral homology invariance of the invariants $\rho_\alpha(\Sigma)$.

Theorem 7.1. *Suppose that Σ_1 and Σ_2 are rational homology spheres which are $\mathbf{Z}$-homology cobordant. Let $\alpha\colon H_1(\Sigma_i, \mathbf{Z}) \to S^1$ be a character of order greater than 2 such that $H^1(\Sigma_i, \mathbf{L}_\alpha) = 0$. Then*

$$\rho_\alpha(\Sigma_1) = \rho_\alpha(\Sigma_2).$$

Proof. Let X denote the $\mathbf{Z}$-homology cobordism. Since $H_2(M^3)$ is free for any 3-manifold, $H_2(\Sigma_i) = 0$ and hence $H_2(X) = 0$ (all homologies are with $\mathbf{Z}$ coefficients). The exact homology sequence of the pair shows that $H_2(X, \Sigma) = H_1(X, \Sigma) = 0$. The universal coefficient theorem implies that $H^2(X, \Sigma) = 0$. Therefore $j_i^*\colon H^2(X) \to H^2(\Sigma_i)$ is injective and $\mu(X, \alpha) = 1$ for every α. If $H^1(\Sigma_i, \mathbf{L}_\alpha) = 0$, Theorem 1.1. applies and shows that $|\rho_\alpha(\partial X)| \leqslant 3 - 2 = 1$. Statement (a) in Theorem 1.1 shows that $\rho_\alpha(\partial X) = 2 \pmod 2$, and therefore $\rho_\alpha(\partial X) = 0$. Since $\partial X = \Sigma_1 \cup (-\Sigma_2)$, $0 = \rho_\alpha(\partial X) = \rho_\alpha(\Sigma_1) - \rho_\alpha(\Sigma_2)$.

Remark. This theorem was originally proved by Gilmer and Livingston in [9] in the case that α is of prime power order. This condition was removed in the case of spherical space forms in [7].

Remark. Suppose that the α-induced cyclic cover $\tilde{\Sigma}$ of Σ is also a rational homology sphere. Let the image of α be $\mathbf{Z}_d \subset S^1$. Then $\mathbf{Z}_d$ acts on $\tilde{\Sigma}_i$ freely and

$$H^1(\tilde{\Sigma}_i, \mathbf{C}) = \sum_{\substack{\rho \text{ character} \\ \text{of } \mathbf{Z}_d}} H^1(\Sigma, \mathbf{L}_\rho) \tag{7.1}$$

(see Lemma 2.5 in [2]). Therefore, if $\tilde{\Sigma}$ is a rational homology sphere, then $H^1(\Sigma, \mathbf{L}_\rho) = 0$ for every ρ.

Example. It is clear from the above remark that lens spaces satisfy the condition $H^1(L, \mathbf{L}_\alpha) = 0$ for any representation $\alpha\colon H_1(L) \to S^1$. Another family of examples can be obtained by looking at Brieskorn spheres $\Sigma(a_1, \cdots, a_n)$. They are Seifert fibered manifolds with exceptional fibers of orders $a_1, \cdots, a_n$ (cf. [14]). Take an integer d prime to each $a_1, \cdots, a_n$. Then $\mathbf{Z}_d \subset S^1$ acts freely on $\Sigma(a_1, \cdots, a_n)$ and the quotient $\Sigma = \Sigma(a_1, \cdots, a_n) | \mathbf{Z}_d$ is a rational homology sphere with $H_1(\Sigma) = \mathbf{Z}_d$ of the type described above.

We next discuss the question of when a rational homology sphere bounds a rational homology ball. Let X be a rational homology ball and $\partial X = \Sigma$ be a rational homology sphere. Since $H_2(\Sigma) = 0$, the exact homology sequence of the pair (X, Σ)

$$0 \to H_2(X) \to H_2(X, \Sigma) \to H_1(\Sigma) \to H_1(X) \to H_1(X, \Sigma) \to 0$$

shows that

$$|H_1(\Sigma)| = \frac{|H_1(X)|}{|H_1(X, \Sigma)|} \cdot \frac{|H_2(X, \Sigma)|}{|H_2(X)|}.$$

Duality and universal coefficients theorems show that

$$|H_1(X)| = |H^3(X, \Sigma)| = |\mathrm{Ext}(H_2(X, \Sigma), \mathbf{Z})| = |H_2(X, \Sigma)|,$$

$$|H_1(X, \Sigma)| = |H^3(X)| = |\mathrm{Ext}(H_2(X), \mathbf{Z})| = |H_2(X)|.$$

Therefore,

$$|H_1(\Sigma)| = \left(\frac{|H_1(X)|}{|H_1(X,\Sigma)|} \right)^2 = |\mathrm{im}(H_1(\Sigma) \to H_1(X))|^2.$$

The necessary condition for Σ to bound a rational homology ball is that $|H_1(\Sigma)| = m^2$ and $m = |\mathrm{im}(H_1(\Sigma) \to H_1(X))|$.

We are interested in representations α_Σ: $H_1(\Sigma) \to S^1$ which will factor through α: $H_1(X) \to S^1$:

$$\text{(7.2)} \qquad \begin{tikzcd} H_1(\Sigma) \arrow[rr, "\alpha_\Sigma"] \arrow[d] & & \mathbf{Z}_d & \varsigma & S^1 \\ \mathrm{im}(H_1(\Sigma) \arrow[rr] & & H_1(X)) \arrow[u] & \varsigma & H_1(X) \arrow[u, "\alpha"] \end{tikzcd}$$

If we assume that $H_1(\Sigma) = \mathbf{Z}_{m^2}$, then, for any d dividing m, all representations α_Σ: $H_1(\Sigma) \to \mathbf{Z}_d \subset S^1$ factor through $H_1(X)$ in the sense of (7.2) for any rational homology ball X such that $\partial X = \Sigma$. Therefore, the following theorem is true.

Theorem 7.2. *Let Σ be a rational homology sphere with cyclic first homology group $H_1(\Sigma) = \mathbf{Z}_{m^2}$. Assume that Σ bounds a rational homology ball with $H^2(X,\Sigma)$ having no 2-torsion. Then, for every nontrivial character α: $H_1(\Sigma) \to S^1$ of order $d > 2$ dividing m for which $H^1(\Sigma, \mathbf{L}_\alpha) = 0$,*

$$\rho_\alpha(\Sigma) = \pm 1.$$

Proof. Theorem 1.1 applies and shows that $|\rho_\alpha(\Sigma)| \leqslant 2$ and $\rho_\alpha(\Sigma) \equiv 1 \pmod 2$.

Remark. Theorem 7.2 was originally proved in [3] for d a prime power. In the case of a spherical space from Σ it was proved in [7] by applying their version of Theorem 1.1.

The question of when a rational homology sphere bounds a rational homology ball is closely connected with the question about sliceness of knots. We say that a knot $K \subset S^3$ is slice if there is a disc $D^2 \subset B^4$ such that $\partial D^2 = K$. Let Σ denote the double cover of S^3 branched over K and X the double cover of B^4 branched over D^2. Then Σ is a $\mathbf{Z}_2$-homology sphere and X is a $\mathbf{Z}_2$-homology ball. Casson and Gordon [3] have defined an invariant $\sigma(K,\alpha)$ for a knot K with the double branched cover Σ satisfying $H_1(\Sigma) = \mathbf{Z}_{m^2}$ and α: $H_1(\Sigma) \to S^1$ by setting $\sigma(K,\alpha) = \rho_\alpha(\Sigma)$. They have shown that, for any α of order d dividing m, the fact that a knot is ribbon (i.e., a special type of slice exists, cf. [3]) implies that $\sigma(K,\alpha) = \pm 1$. They have also shown that, for α of prime power order p^k dividing m, $\sigma(K,\alpha) = \pm 1$ if a knot is slice. Fintushel and Stern in [7] prove that when Σ is a lens space $\mathbf{L}(m^2, q)$, $\sigma(K,\alpha) = \pm 1$ for any character α. Theorem 7.2 has the following corollary.

Corollary 7.3. *Let $K \subset S^3$ be a knot such that the double branched cover Σ satisfies $H_1(\Sigma) = \mathbf{Z}_{m^2}$ and that the m-fold cyclic cover $\tilde{\Sigma}$ of Σ is a rational homology sphere. Then for any character α: $H_1(\Sigma) \to S^1$ of order $d > 2$ dividing m, $\sigma(K, \alpha) = \pm 1$ if K is slice.*

8. Appendix

In this appendix we will, for the sake of completeness, prove Theorem 3.1 which gives a Banach space structure to

$$\mathscr{R} = \left\{ \phi \in L^2_{3,\mathrm{loc}}(\mathfrak{gl}(\mathbf{E})) \mid \|\nabla_0 \phi\|_{L^2_{2,\delta}} \right\} < \infty.$$

Recall that ∇_0 was defined by a connection $\nabla_0 = \nabla_\alpha + d$ on $\mathbf{E} = \mathbf{L}_\alpha \oplus \varepsilon$, where α: $H_1(X) \to S^1 = \mathrm{U}(1)$ and $\mathbf{L}_\alpha = \tilde{X} \times_\alpha \mathbf{C}$ for homology cover $\tilde{X}$. We want to define the limit

$$r(\sigma) = \lim_{n \to \infty} \sigma \Big|_{\partial X \times \{n\}}$$

for a section $\sigma \in \mathscr{R}$. To make this precise we will use Lemma 5.2 in [15].

Let $\phi \in \mathscr{R} \cap C^\infty$. For each boundary component ∂X_i we can consider the homology cover $\partial \tilde{X}_i$. This is a finite cover and the pullback of $\mathbf{L}_\alpha$ from ∂X_i is a trivial complex line bundle on ∂X_i. Therefore pullbacks of $\mathbf{E}$ and $\mathfrak{gl}(\mathbf{E})$ are also trivial. The pullback of the section ϕ gives

$$\tilde{\phi} \in C^\infty(\pi^*(\operatorname{End} \mathbf{E}/\partial X_i \times [0,\infty))) \cong C^\infty(\partial \tilde{X}_i \times [0,\infty), \mathfrak{gl}(3, \mathbf{R}))$$

such that

$$\int_{\partial \tilde{X}_i \times [0,\infty)} e^{\tau\delta} |d\tilde{\phi}|^2 = |H_1(\partial X_i)| \cdot \int_{\partial X_i \times [0,\infty)} e^{\tau\delta} |\nabla_0 \phi|^2 < \infty.$$

Applying Lemma 5.2 in [15] the section $\tilde{\phi} = \beta_i \tilde{\phi}$ on the trivial $\mathfrak{gl}(3, \mathbf{R})$-bundle over the manifold $M_i = Y \bigcup_{\partial \tilde{X}_i} \partial \tilde{X}_i \times [0, \infty)$, where Y is any 4-manifold with $\partial Y = \partial \tilde{X}_i$ and $\beta_i : \partial \tilde{X}_i \times [0, \infty) \to [0, 1]$ is a cutoff function

$$\beta_i(x, y) = \begin{cases} 0, & t < 1/2, \\ 1, & t > 1, \end{cases}$$

we get that there is a constant $\tilde{A}_\phi \in \mathfrak{gl}(3, \mathbf{R})$ such that

$$\lim_{n \to \infty} \tilde{\phi}(x, n) = \lim_{n \to \infty} \tilde{\phi}(x, n) = \tilde{A}_\phi.$$

Passing to the quotient, $\tilde{A}_\phi$ gives a locally constant section of $\mathfrak{gl}(\mathbf{E}/\partial X_i)$ denoted $r_i(\phi) = A_\phi \in \operatorname{Ker} \nabla_{\partial_i}$. Here ∇_{∂_i} is the canonical flat connection on $\mathfrak{gl}(\mathbf{E})$ defined by the canonical flat connection on $\mathbf{E}/\partial X_i = (\partial \tilde{X}_i \times_\alpha \mathbf{C}) \oplus \varepsilon$.

Lemma 8.1. *Let $\phi \in \mathscr{R}$. Let H_0 denote the completion of $C_0^\infty(\mathfrak{gl}(\mathbf{E}))$ in the norm $\|\sigma\| = \int_M e^{\tau\delta}|\nabla_0\sigma|^2$. Then there is a unique $\xi(\phi) \in H_0$ such that $\phi + \xi(\phi) \in \mathscr{R}$ is harmonic, i.e.,*

$$e^{-\tau\delta}\nabla_0^* e^{\tau\delta}\nabla_0(\phi + \xi(\phi)) = 0. \tag{8.1}$$

Proof. Note that $\|\phi\| = \int_M e^{\tau\delta}|\nabla_0\sigma|^2$ is a norm on $C_0^\infty(\mathfrak{gl}(\mathbf{E}))$ since, for a compactly supported section, $\nabla_0\sigma = 0$ implies $\sigma = 0$. For each $\phi \in \mathscr{R}$ we can define a functional a_ϕ on $C_0^\infty(\mathfrak{gl}(\mathbf{E}))$ by

$$a_\phi(\xi) = \int_M e^{\tau\delta}(\langle\nabla_0\xi, \nabla_0\xi\rangle + 2\langle\nabla_0\phi, \nabla_0\xi\rangle)$$

and extend it to H_0 by continuity. It is easy to check that a_ϕ is strictly convex. Therefore, it has a unique minimum point. On the other hand, for a convex differentiable functional, every critical point is an absolute minimum point (cf. [11, IV, §7]). Hence, a_ϕ has a unique critical point $\xi(\phi)$. Since the differential of a_ϕ is

$$\begin{aligned} Da_\phi(\xi)(\sigma) &= \frac{d}{dt}\bigg|_{t=0} a_\phi(\xi + \tau\sigma) = 2\int e^{\tau\delta}\langle\nabla_0\sigma, \nabla_0(\phi + \xi)\rangle \\ &= 2\int e^{\tau\delta}\langle\sigma, e^{-\tau\delta}\nabla_0^* e^{\tau\delta}\nabla_0(\phi + \xi)\rangle \end{aligned}$$

the unique critical point $\xi(\phi) \in H^0$ satisfies (8.1).

The fact that $\|\nabla\phi\|_{L^2_{2,\delta}} < \infty$, $\nabla_0^* e^{\tau\delta}\nabla_0(\phi + \xi(\phi)) = 0$, and integration by parts show that $\xi(\phi) \in \mathscr{R}$. On the other hand, integration by parts shows that there is a constant $c > 0$ such that

$$c^{-1} \cdot \int_M e^{\tau\delta}|\nabla_0\xi|^2 \leqslant \int_M e^{\tau\delta}|\xi|^2 \leqslant c \cdot \int_M e^{\tau\delta}|\nabla_0\xi|^2 \tag{8.2}$$

for $\xi \in H_0$ (cf. equation 5.30 in [15]). Therefore, $\xi(\phi) \in L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$.

Lemma 8.2. *Let $\mathscr{H}$ denote the subspace of $\mathscr{R}$ consisting of harmonic sections, i.e.,*

$$\mathscr{H} = \left\{\phi \in \mathscr{R} \mid e^{-\tau\delta}\nabla_0^* e^{\tau\delta}\nabla_0\phi\right\} = 0.$$

Then $\mathscr{R}$ decomposes into a direct sum

$$\mathscr{R} = L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E})) \oplus \mathscr{H}.$$

Proof. Lemma 8.1 shows that $\mathscr{R} = L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E})) + \mathscr{H}$. Suppose $\phi \in L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E})) \cap \mathscr{H}$. Since on $L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$ we have that the formal L^2_δ-adjoint of ∇_0, $e^{-\tau\delta\nabla_0^*}e^{\tau\delta}$, is the Hilbert adjoint, it follows that

$$\int e^{\tau\delta}\langle\nabla_0\phi, \nabla_0\phi\rangle = \int e^{\tau\delta}\langle\phi, e^{-\tau\delta}\nabla_0^* e^{\tau\delta}\nabla_0\phi\rangle = 0,$$

and by (10.5)

$$\int e^{\tau\delta}|\phi|^2 \leqslant c \cdot \int e^{\tau\delta}|\nabla_0\phi|^2 = 0.$$

Hence, $\phi \in L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E})) \cap \mathscr{R}$ implies $\phi = 0$.

Recall that we have defined r_i: $\mathscr{R} \cap C^\infty \to \operatorname{Ker}\nabla_{\partial_i}$ for each i. Define

$$r = \bigoplus_{i=1}^{n} r_i\colon \mathscr{R} \cap C^\infty \to V = \bigoplus_{i=1}^{n} \ker\nabla_{\partial_i}.$$

Lemma 8.3. *r: $\mathscr{H} \to V$ is an isomorphism with inverse l: $V \to \mathscr{H}$ defined by*

$$l(\sigma) = l(\sigma_1, \cdots, \sigma_n) = \sum_i \beta_i\sigma_i + \xi\Big(\sum_i \beta_i\sigma_i\Big),$$

where β_i: $\partial X_i \times [0, \infty) \to [0, 1]$ is a cutoff function such that

$$\beta_i(x, t) = \beta(t) = \begin{cases} 0, & t \leqslant 1/2, \\ 1, & t \geqslant 1. \end{cases}$$

Proof. By definition, $l(\sigma) \in \mathscr{H}$ since $\nabla\sigma$ is compactly supported. For a section $\xi \in L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$, Lemma 5.2 in [15] and definition of r_i show that $r_i(\xi) = 0\ \forall i$. Hence, for any $\sigma \in V$

$$r(l(\sigma)) = r\Big(\sum \beta_i\sigma_i + \xi\Big(\sum_i \beta_i\sigma_i\Big)\Big) = \oplus r_i(\beta_i\sigma_i) = \sigma.$$

So, r is surjective. On the other hand, suppose that $\psi \in \mathscr{H}$ and $r(\psi) = 0$. Then $e^{\delta n/2}\xi(-, n) \to 0$ in C^0 by the last statement in Lemma 5.2 in [15]. Since ψ is harmonic

$$0 = \int_{\tau \leqslant n} \langle \psi, \nabla_0^* e^{\tau\delta}\nabla_0\psi\rangle = -\int_{\tau \leqslant n} e^{\tau\delta}\langle \nabla_0\psi, \nabla_0\psi\rangle + \int_{\tau = n} e^{\tau\delta}\langle \psi, dt\nabla_0\psi\rangle.$$

Therefore,

$$\int_{\tau \leqslant n} e^{\tau\delta}|\nabla_0\psi|^2 \leqslant \int_{\tau = n} e^{\tau\delta}|\psi|\,|\nabla_0\psi| \leqslant \left(\int_{\tau = n} e^{\tau\delta}|\psi|^2\right)^{1/2}\left(\int_{\tau = n} e^{\tau\delta}|\nabla_0\psi|\right)^{1/2}.$$

Since the restriction to the codimension 1 submanifold induces a continuous map $L^2_1(M) \to L^2(N)$, $\int_{\tau=n} e^{\tau\delta}|\nabla_0\psi|^2$ is a bounded sequence. $\int_{\tau=n} e^{\tau\delta}|\psi|^2 \to 0$ since $e^{\tau\delta/2}|\psi| \to 0$ in C^0 and ∂X_i is compact $\forall i$. This shows that $\int_M e^{\tau\delta}|\nabla_0\psi|^2 = 0$. $\nabla_0\psi = 0$ implies that $|\psi|$ is constant and $r(\psi) = 0$ then shows $\psi = 0$. Therefore, r is injective on $\mathscr{H}$.

We can now define r: $\mathscr{R} \to V$ by setting $r(\phi) = r(\pi_{\mathscr{H}}(\phi))$ where $\pi_{\mathscr{H}}(\phi) = \phi + \xi(\phi) \in \mathscr{H}$, and introduce a norm on $\mathscr{R}$ by

$$\|\phi\|^2_{\mathscr{R}} = \|\nabla_0\phi\|^2_{L^2_{2,\delta}} + \int_{\partial X} |r(\phi)|^2.$$

This is obviously a seminorm. To show that it is actually a norm, suppose $\|\phi\|_{\mathscr{R}} = 0$. Then, $\int_{\partial X}|e(\phi)|^2 = 0$, hence $r(\phi) = 0$. Therefore, $\phi \in L^2_{3,\delta}$, $\|\phi\|_{L^2_{3,\delta}} = \|\nabla_0\phi\|_{L^2_{2,\delta}} + |\phi|_{L^2_\delta} \leqslant 2 \cdot \|\nabla_0\phi\|_{L^2_{2,\delta}} = 0$, and $\phi = 0$.

Lemma 8.4. *The projections*

$$\pi_{\mathscr{H}}\colon (\mathscr{R}, \|\ \|_{\mathscr{R}}) \to (\mathscr{H}, \|\ \|_{\mathscr{R}}) \quad \textit{and} \quad \pi_0\colon (\mathscr{R}, \|\ \|_{\mathscr{R}}) \to (L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E})), \|\ \|_{\mathscr{R}})$$

are continuous. $(\mathscr{R}, \|\ \|_{\mathscr{R}})$ *is a Hilbert space with inner product defined by*

$$\langle\phi, \psi\rangle_{\mathscr{R}} = \int_M e^{\tau\delta}\{\langle\nabla_0\nabla_0\nabla_0\phi, \nabla_0\nabla_0\nabla_0\psi\rangle + \langle\nabla_0\nabla_0\phi, \nabla_0\nabla_0\psi\rangle + \langle\nabla_0\phi, \nabla_0\psi\rangle\}$$
$$+ \int_{\partial X}\langle r(\phi), r(\psi)\rangle.$$

Proof. Note that $\langle\phi,\psi\rangle_{\mathscr{R}}$ is well defined for $\phi, \psi \in \mathscr{R}$ and that $\|\ \|_{\mathscr{R}}$ is the associated norm. Since for $\phi \in L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$ we have $r(\phi) = 0$, equation (8.2) implies that

$$\|\phi\|^2_{\mathscr{R}} = \|\nabla_0\phi\|^2_{L^2_{2,\delta}} \leqslant \|\phi\|^2_{L^2_{3,\delta}} = \|\nabla_0\phi\|^2_{L^2_{2,\delta}} + \int_M e^{\tau\delta}|\phi|^2$$
$$\leqslant \|\nabla_0\phi\|^2_{L^2_{2,\delta}} + C \cdot \int_M e^{\tau\delta}|\nabla_0\phi|^2 \leqslant (1 + C)\|\nabla_0\phi\|^2_{L^2_{2,\delta}}.$$

Therefore, $\|\phi\|_{\mathscr{R}} \leqslant \|\phi\|_{L^2_{3,\delta}} \leqslant 2\|\phi\|$, i.e., norms $\|\ \|_{\mathscr{R}}$ and $\|\ \|_{L^2_{3,\delta}}$ are equivalent on $L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$. Since $\mathscr{H}$ is a finite dimensional vector space, any two norms on $\mathscr{H}$ are equivalent. With this in mind, we proceed to prove continuity of projections:

$$\|\pi_0(\phi)\|_{\mathscr{R}} = \|\phi - l(r(\phi))\|_{\mathscr{R}} \leqslant \|\phi\|_{\mathscr{R}} + \|l(r(\phi))\|_{\mathscr{R}}$$
$$\leqslant \|\phi\|_{\mathscr{R}} + \zeta\left(\int_{\partial X}|r(\phi)|^2\right)^{1/2} \leqslant (1 + \zeta)\|\phi\|_{\mathscr{R}}$$

which shows continuity of π_0. Continuity of $\pi_{\mathscr{H}}$ follows from

$$\|\pi_{\mathscr{H}}(\phi)\|_{\mathscr{R}} = \|\phi + \xi(\phi)\|_{\mathscr{R}} = \|l(r(\phi))\|_{\mathscr{R}} \leqslant \zeta\left(\int_{\partial X}|r(\phi)|^2\right)^{1/2} \leqslant \zeta\|\phi\|_{\mathscr{R}}.$$

The fact that $(\mathscr{R}, \|\ \|_{\mathscr{R}})$ is complete follows from the continuity of projections and completeness of $(L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E})), \|\ \|_{\mathscr{R}})$ and $(\mathscr{H}, \|\ \|_{\mathscr{R}})$.

Note that we have just proved Theorem 3.1.

Lemma 8.5. *Pointwise multiplication,* $\mathscr{R} \times \mathscr{R} \to \mathscr{R}$, *is well defined and continuous.*

Proof. Suppose $\phi, \psi \in \mathscr{R}$. We need to show that $\phi \circ \psi \in \mathscr{R}$. The Sobolev multiplication theorems for compact domains show that $\phi \circ \xi \in L^2_{3,\mathrm{loc}}(\mathfrak{gl}(\mathbf{E}))$.

We need to estimate $\|\phi \circ \psi\|_{\mathscr{R}}$ in terms of $\|\phi\|_{\mathscr{R}}$ and $\|\psi\|_{\mathscr{R}}$. This will be done term by term.

$$\begin{aligned}\int_M e^{\tau\delta}|\nabla(\phi\circ\psi)|^2 &= \int_M e^{\tau\delta}|\nabla(\phi)\circ\psi + \phi\circ\nabla(\psi)|^2\\ &\leqslant 2\int_M e^{\tau\delta}\left(|\nabla\phi\circ\psi|^2 + |\phi\circ\nabla\psi|^2\right)\\ &\leqslant c\int_M e^{\tau\delta}|\nabla\phi|^2|\psi|^2 + c\int_M e^{\tau\delta}|\phi|^2|\nabla\psi|^2.\end{aligned}$$

Since ϕ, $\psi \in L^2_{3,\mathrm{loc}}$ and $L^2_3 \hookrightarrow C^0$ on compact 4-manifolds, ϕ and ψ are continuous functions. They are bounded since $e^{\tau\delta/2}\phi(-,t) \to r(\phi)$ and $e^{\tau\delta/2}\psi(-,t) \to r(\psi)$ in C^0. Let $\sigma \in L^2_{3,\delta}(\mathfrak{gl}(\mathbf{E}))$ and denote by M_n the compact 4-manifold $M_n = \tau^{-1}[n, n+1]$. Then

$$\begin{aligned}\|\sigma\|_\infty^2 &= \sup_n \|\sigma|_{M_n}\|^2_{L^\infty(M_n)} \leqslant \sup_u c(M_n)^2\|\sigma\|^2_{L^2_3(M_n)}\\ &\leqslant c^2\cdot \sup_n \|\sigma\|^2_{L^2_3(M_n)} \leqslant c^2\cdot\sum_n \|\sigma\|^2_{L^2_3(M_n)}\\ &= c^2\cdot\|\sigma\|^2_{L^2_3(M)} \leqslant c^2\cdot\|\sigma\|^2_{L^2_{3,\delta}}.\end{aligned}$$

For $\phi \in \mathscr{R}$ we then have

$$\begin{aligned}\|\phi\|_\infty &= \|\pi_0(\phi) + \pi_{\mathscr{R}}(\phi)\|_\infty \leqslant \|\pi_0(\phi)\|_\infty + \|\pi_{\mathscr{H}}(\phi)\|_\infty\\ &\leqslant c\|\pi_0(\phi)\|_{L^2_{3,\delta}} + c'\|\pi_{\mathscr{H}}(\phi)\| \quad \text{(by continuity of projections)}\\ &\leqslant c\cdot c_1\|\phi\|_{\mathscr{R}} + c'\cdot c_2\|\phi\|_{\mathscr{R}} \leqslant K_\infty\|\phi\|_{\mathscr{R}}.\end{aligned}$$

This gives

$$\begin{aligned}\int_M e^{\tau\delta}|\nabla_0(\phi\circ\psi)|^2 &\leqslant c\|\psi\|^2_{\mathscr{R}}\int_M e^{\tau\delta}|\nabla_0\phi|^2 + c\|\phi\|^2_{\mathscr{R}}\cdot\int_M e^{\tau\delta}|\nabla_0\psi|^2\\ &\leqslant Z_i\|\phi\|^2_{\mathscr{R}}\|\psi\|^2_{\mathscr{R}}.\end{aligned}$$

The estimates for the terms $\int_M e^{\tau\delta}|\nabla_0\nabla_0(\phi\circ\psi)|^2$ and $\int_M e^{\tau\delta}|\nabla_0\nabla_0\nabla_0(\phi\circ\psi)|^2$ are done exactly in the same manner

$$\begin{aligned}\int_{\partial X}|r(\phi\circ\psi)|^2 &= \int_{\partial X}|r(\phi)\circ r(\psi)|^2 = \int_{\partial X}|r(\phi)|^2|r(\psi)|^2\\ &\leqslant \left(\int_{\partial X}|r(\phi)|^4\right)^{1/2}\left(\int_{\partial X}|r(\psi)|^4\right)^{1/2} \leqslant \|r(\phi)\|^2_{L^4}\circ\|r(\psi)\|^2_{L^4}\\ &\leqslant K\cdot\|r(\phi)\|^2_{L^2}\cdot\|r(\psi)\|^2_{L^2} \leqslant K\cdot\|\phi\|_{\mathscr{R}}\cdot\|\psi\|_{\mathscr{R}}.\end{aligned}$$

Adding the estimates for all four terms we get that

$$\|\phi \circ \psi\|_{\mathscr{R}}^2 \leqslant \zeta^2 \cdot \|\phi\|_{\mathscr{R}}^2 \cdot \|\psi\|_{\mathscr{R}}^2$$

and therefore the multiplication is well defined and continuous.

Lemma 8.5 shows that $\mathscr{R}$ has a structure of Banach algebra with respect to pointwise multiplication. We are interested in a subgroup of $\mathscr{R}$ defined as

$$\mathscr{G} = \{\phi \in \mathscr{R} \mid \phi \circ \phi^* = \mathrm{id}, \det \phi = 1\}.$$

We want to show that $\mathscr{G}$ is a closed Hilbert submanifold of $\mathscr{R}$. To prove this, we characterize $\mathscr{G}$ as the zero set of a smooth submersion. Note that $\mathscr{G} = F^{-1}(0)$ for $F: \mathscr{R} \to \mathscr{R}$ given by $F(u) = u \circ u^* - \mathrm{id}$. This map itself is not a submersion. However, if we define $S = \{\phi \in \mathscr{R} \mid \phi^* = \phi\}$ it is clear that $F(u) = u \circ u^* - \mathrm{id} \in S$ for every $u \in \mathscr{R}$. Therefore, we can think of F as $F: \mathscr{R} \to S$.

Lemma 8.6. *The smooth map $F: \mathscr{R} \to S$ given by $F(u) = u \circ u^* - \mathrm{id}$ is a submersion at every point $a \in \mathscr{G}$.*

Proof.

$$\begin{aligned} D(F)(a)(h) &= \frac{d}{dt}\Big|_{t=0} F(a + th) = \frac{d}{dt}\Big|_{t=0}\big((a + th)\circ(a + th)^* - \mathrm{id}\big) \\ &= \frac{d}{dt}\Big|_{t=0}\big(a \circ a^* + t(a \circ h^* + h \circ a^*) + t^2 h \circ h^* - \mathrm{id}\big) \\ &= a \circ h^* + h \circ a^*. \end{aligned}$$

To show that $D(F)(a)$ is surjective for every $a \in \mathscr{G}$ we will consider the operator

$$\Psi_a = D(F)(a) \circ r_a : \mathscr{R} \to S$$

where $r_a: \mathscr{R} \to \mathscr{R}$ is the operator of right multiplication by a, $r_a(\phi) = \phi \circ a$. Since multiplication in $\mathscr{R}$ is continuous, r_a is a continuous homomorphism of $\mathscr{R}$. If $a \in \mathscr{G}$, r_{a*} is the continuous inverse of r_a. Therefore, if Ψ_a is surjective, so is $D(F)(a)$. However,

$$\begin{aligned} \Psi_a(h) &= D(F)(a)(h \circ a) = a \circ (h \circ a)^* + (h \circ a) \circ a^* \\ &= a \circ a^* \circ h^* + h \circ a \circ a^* = h^* + h. \end{aligned}$$

Since for any $s \in S$, $\Psi_a(\frac{1}{2}s) = s$, Ψ_a is surjective and so is $D(F)(a)$.

Since $F: \mathscr{R} \to S$ is a smooth mapping between Hilbert manifolds and for every point $a \in F^{-1}(0)$, $D(F)(a)$ is surjective, $\mathscr{G} = F^{-1}(0)$ is a Hilbert submanifold of $\mathscr{R}$ with tangent space

$$T_a(\mathscr{G}) = \ker D(F)(a) = r_a\big(\Psi_a^{-1}(0)\big) = r_a(\mathfrak{g})$$

for

$$\mathfrak{g} = \{\phi \in \mathscr{R} \mid \phi^* = -\phi\} = \ker \Psi_a.$$

The multiplication $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ is a smooth function since it is a restriction of a smooth function on $\mathscr{R} \times \mathscr{R}$. Since for $a \in \mathscr{G}$, $a^{-1} = a^*$ and $(\ )^*$ is a smooth function on $\mathscr{R}$, $(\ )^{-1}$: $\mathscr{G} \to \mathscr{G}$ is also smooth. Therefore, $\mathscr{G}$ is a Hilbert Lie group with Lie algebra $\mathfrak{g} = T_e(\mathscr{G})$. If we denote by $\mathfrak{g}_{\mathbf{E}}$ the bundle of antisymmetric operators $\mathfrak{g}_{\mathbf{E}} \subset \mathfrak{gl}(\mathbf{E})$ it is clear that

$$\mathfrak{g} = \left\{ \phi \in L^2_{3,\mathrm{loc}}(\mathfrak{g}_{\mathbf{E}}) \mid \|\nabla_0 \phi\|_{L^2_{2,\delta}} < \infty \right\}.$$

We have stated in Theorem 3.1 that $\mathscr{G}$ acts smoothly on the space of connections $\mathscr{C}$. We prove this statement in the following lemma.

Lemma 8.7. *The Lie group $\mathscr{G}$ acts smoothly on the space of connections by the usual action*

$$(g, \nabla) \to \nabla^g = g^{-1} \circ \nabla \circ g.$$

Proof. The action of $\mathscr{G}$ on $\mathscr{C}$ is described in coordinates $\mathscr{C} = \nabla_0 + L^2_{2,\delta}(T^*M \otimes \mathfrak{g}_{\mathbf{E}})$ by

$$e(g, a) = g^{-1} \circ \nabla_0 g + g^{-1} \circ a \circ g.$$

Since $g \to g^{-1}$ is smooth as a restriction of $\phi \to \phi^*$ on $\mathscr{R}$, and ∇_0: $\mathscr{R} \to L^2_{2,\delta}(T^*M \otimes \mathfrak{gl}(\mathbf{E}))$ is a continuous $\mathbf{R}$-linear homomorphism, we only need to know that the pointwise products $\circ$: $\mathscr{R} \times L^2_{2,\delta}(T^*M \otimes \mathfrak{gl}(\mathbf{E})) \to L^2_{2,\delta}(T^*M \otimes \mathfrak{gl}(\mathbf{E}))$ and $\circ$: $L^2_{2,\delta}(T^*M \otimes \mathfrak{gl}_{\mathbf{E}}) \times \mathscr{R} \to L^2_{2,\delta}(T^*M \otimes \mathfrak{gl}_{\mathbf{E}})$ are smooth. Estimates done in the proof of Lemma 8.5 actually show that both $\circ$ are continuous. Since continuous bilinear maps are smooth, e is smooth as a composition of smooth maps.

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