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COMPLETE MINIMAL SURFACES AND MINIMAL HERISSONS

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Let W denote complete minimal surfaces (c.m.s.'s) in R^3 of finite total curvature. We allow the surfaces in W to have a finite number of branch points. Let H be the surfaces M of W of total curvature 4π ($c(M) = 4\pi$). By convention a point is in H. Surfaces in H are called minimal herissons; they can be parametrized by their Gauss maps.

In [3], Langevin, Levitt and Rosenberg introduce a sum operation in W:

$$M_1 + M_2 = \bigcup_{z\in S^2} \left\{ \sum_i x_i + \sum_j y_j / g_1^{-1}(z) = \{x_i\},\ g_2^{-1}(z) = \{y_j\} \right\},$$

where $g_i \colon M_i \to S^2$ is the Gauss map. Then $M_1 + M_2 \in H$ and the sum operation induces a group structure on H; indeed H is an infinite dimensional vector space where cM is the homothety of M by the real number c [3].

In this paper we will discuss some geometric properties of H, and supply details of some of the results announced in [3].

First we establish some elementary properties of W. The classical theory of Osserman of immersed surfaces in W extends to W. Each M in W has finite conformal type (i.e., M is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at a finite number of points) and the Weierstrass representation (g, ω) of M extends to $\overline{M}$ meromorphically. The Gauss map of $M \in W$ can miss at most $(4N + b)/(N + 1)$ points, where b is the total branching order and N is the degree of the Gauss map. This is sharp. When $b = 0$, this is Osserman's theorem that for an immersed $M \in W$, g misses at most three points. It is still unknown if three is sharp. We show that a c.m.s. with a finite number of branch points has a dense Gaussian image. This is false in the presence of an infinite number of branch points [5].

In §2 we establish the Weierstrass representation of $M_1 + M_2$ and completeness of $M_1 + M_2$. We construct an infinite family in H: surfaces that have n catenoid type ends for each $n \geq 2$. We establish equations such as $M_{2n} + M_{2n} = C_1 + \cdots + C_n$, where M_{2n} is the immersed Meeks-Jorge surface:

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$g(z) = z^{2n-1}$, $\omega = dz/(z^{2n}-1)^2$, and C_k is a catenoid with axes parallel to $e^{2\pi ik/n}$, $k = 1, 2, \cdots, n$. We prove that if $M \in H$ has n-catenoid type ends and if M is invariant by a rotation by $2\pi/n$, then $M = M_{2n} + M_{2n}$.

We prove $M + M$ is a point if $M \in W$ has only bounded ends (i.e., asymptotic to planes). Also $M + M$ is a point where M is the three-punctured torus discovered by Costa and shown to be embedded by Hoffman and Meeks [1].

In §3 we define a sum operation in W which produces a nonorientable c.m.s. of total curvature 2π (they are parametrized by P^2). We add all points on M_1 and M_2 having the same (unoriented) tangent plane. This yields a simple method of constructing nonorientable minimal surfaces. We show how to write the Weierstrass representation of $M_1 \tilde{+} M_2$. It follows immediately that Enneper $\tilde{+}$ Enneper = Henneberg's surface.

Let M be the surface of Meeks-Jorge with three catenoid type ends:

$$g(z) = z^2, \qquad \omega = dz/(z^3-1)^2.$$

Then $N = M \tilde{+} M$ is a nonorientable c.m.s. with all catenoid type ends. There is still no example known of an immersed nonorientable c.m.s. having only catenoid type ends.

In §4, we discuss deformations, (in the sense of [6]) of $M \in W$.

We prove ε deformations of $M \in W$ are also in W and have the same number of branch points.

Finally we prove minimal herissons have no deformations, if the ends are of catenoid type.

We remark that if M is a (small) piece of a minimal surface in R^3 and if M^* is the conjugate surface, then $M(t) = \cos(t)M + \sin(t)M^*$ is the usual Weierstrass deformation of M. So, at least locally, the sum operation was already known to Weierstrass.

1. Properties of W

Let M be a Riemann surface and $X\colon M \to R^3$ a continuous map which is conformal and harmonic except at a finite number of points $y_1, \cdots, y_m$. If the total curvature $(c(M))$ of M is finite and M is complete in the induced metric then we say M is a c.m.s. in W; the y_j are called the branch points.

Proposition 1.1. *Each c.m.s. M in W has finite conformal type; i.e., there is a compact Riemann surface $\overline{M}$ and M is conformally equivalent to $\overline{M}$ punctured at a finite number of points (called the punctures).*

Proof. The coordinate functions x_i of X are continuous and harmonic in punctured discs about each branch point, so they are harmonic at the branch points too. Let $z = u + iv$ be a conformal parameter about a branch point

$u = v = 0$. Then $\phi_k = \partial x_k/\partial u - i\,\partial x_k/\partial y$ extends analytically to the branch point and $\sum_{k=1}^3 \phi_k^2 = 0$ at the branch points as well.

Therefore the globally defined one-form $\omega = \phi_1 - i\phi_2$ is analytic on M and if M is not a flat plane, $g = \phi_3/(\phi_1 - i\phi_2)$ is a meromorphic map on M; at the branch points as well. Rotate M (i.e., $X(M)$) so that g has no poles at the branch points. Then

$$\sum_{k=1}^{3} |\phi_k|^2 = X_u \cdot X_u + X_v \cdot X_v,$$

and this is 0 at the branch point O if and only if $\omega(0) = 0$. Thus M has a Weierstrass representation (g, ω) where ω is an analytic one-form on M whose zeros are precisely the branch points of M, and poles of g.

Let N be a Riemannian manifold obtained from M by removing a small disc about each branch point and attaching another disc to obtain a smooth Riemannian surface, i.e., the new metric ds_N has no singularities. This can be done so that N is complete and $c(N) < \infty$. Thus by Huber's theorem [2], N is of finite conformal type. Hence M is topologically a compact Riemann surface punctured at a finite number of points. The metric on M is nonsingular and complete at each annular end of M. Then exactly as in Osserman [5], each end of M is conformally a punctured disc.

Remark. Clearly g and ω extend meromorphically to $\overline{M}$, just as in [5].

Proposition 1.2. *Let $M \in W$, and let b be the total branching order of M. Let N be the degree of the extended Gauss map $g\colon \overline{M} \to S^2$. Then $g\colon M \to S^2$ misses at most $(4N + b)/(N + 1)$ points.*

Proof. We rotate M so that g has only simple poles and takes finite nonzero values at the punctures. Let $q_1, \cdots, q_k \in S = S^2$ be the points missed by g and let $p_1, \cdots, p_r \in \overline{M}$ be the punctures; $g^{-1}\{q_1, \cdots, q_k\} \subset \{p_1, \cdots, p_r\}$.

Let $1 + a_j$ be the number of times g takes its value at p_j. Then

$$k \cdot N \le \sum_{j=1}^{r} (1 + a_j) = r + \sum_{j=1}^{r} a_j \le r + n,$$

where n is the total ramification order.

Riemann's relation for $\Omega = g'(z)\,dz$ yields $2N - n = 2 - 2s$, where s is the genus of $\overline{M}$. Therefore

$$k \cdot N \le r + 2N + 2s - 2.$$

Now ω has double zeros at the poles of g and b zeros at the branch points. So

$$\sum_{j=1}^{r} c_j - 2N = b + 2 - 2s,$$

where c_j is the multiplicity of the pole of ω at p_j. We know $c_j \geq 2$ and $k \leq r$, hence

$$r + k - 2N \leq 2r - 2N \leq \sum_{j=1}^{r} c_j - 2N = 2 - 2s + b.$$

Combining this relation with $k \cdot N \leq r + 2N + 2s - 2$, we obtain the result.

Remark. When $b = 0$, this is Osserman's theorem that the Gauss map of an immersed $M \in W$ can miss at most three points. Although it is not known if 3 is sharp in Osserman's theorem, it is true that $(4N + b)/(N + 1)$ is sharp. The reader can check this for the minimal herissons with catenoid type ends constructed in §2.

Proposition 1.3. *Let M be a c.m.s. with a finite number of branch points. Then $g(M)$ is dense in S.*

Proof. Assume the contrary, and rotate M so that $g(M)$ misses a neighborhood of the north pole; i.e., $g\colon M \to \mathbf{C}$ is bounded. The induced metric on M is $ds = |\omega|(1 + |g|^2/2)$ and it is singular at the branch points, the zeros of ω. The metric $d\tilde{s} = |\omega| = |f(z)|\,|dz|$ (where $\omega = f(z)\,dz$ in a local conformal parameter) is then also complete on M, singular at the branch points, and flat elsewhere, since its curvature is given by $-\Delta \log |f(z)|^2/|f(z)|^2$ and $\log |f|$ is harmonic where $f \neq 0$.

As in 1.2, we attach a disc about each branch point to obtain a new Riemannian surface of finite total curvature. Then M is of finite topological type and by Osserman's techniques, M is of finite conformal type. More precisely, each end of M is conformally an annulus $\{z \in \mathbf{C} \mid 0 < r_1 < |z| < r_2\}$. Write $\omega = f(z)\,dz$ on the end, $f(z) \neq 0$ for $r_1 < |z| < r_2$. Then $\Delta \log |f(z)| = 0$ and $\int_\gamma |f(z)|\,|dz| = \infty$ for all paths $\gamma(t)$ such that $\lim_t |\gamma(t)| = r_2$ (by completeness at the end and boundedness of g). Therefore $r_2 = \infty$ and the end is a punctured disc [5].

However g extends to the compact Riemann surface $\overline{M}$ since if a puncture were an essential singularity of g, the image of the end would be dense in S. Now $g\colon \overline{M} \to S$ is bounded, hence constant and M is a plane.

Remark. Osserman proved 1.3 assuming M simply connected and showed 1.3 is false in the presence of an infinite number of branch points [5].

2. Minimal herissons

A minimal herisson is an element of W of total curvature 4π. We denote by H the minimal herissons and we agree a point is in H.

Let M, M^1 be in W and have limiting normal vectors at the ends $v_1, \cdots, v_n$, $v_1^1, \cdots, v_m^1$ respectively. Let $M + M^1$ be the surface parametrized by $X\colon S - \{v_1, \cdots, v_n, v_1^1, \cdots, v_m^1\} \to R^3$ where $X(v)$ is the sum in R^3 of all points on M and M^1 having v as normal vector.

Theorem 2.1. *$M + M^1$ is in H.*

Proof. Clearly it suffices to show $M + M$ is in H for $M \in W$. We will do this by constructing a Weierstrass pair $(\tilde{g}, \tilde{\omega})$ for $M + M$. Naturally $\tilde{g}(z) = z$, so the problem is to construct $\tilde{\omega}$. Let $v_1, \cdots, v_k$ be the limiting normals at the ends of M and g be the Gauss map of M, degree $g = n$. Let $X\colon M \to R^3$ parametrize M. For $v \in S^2 - \{v_1, \cdots, v_k\} = S'$ we define $\tilde{X}(v) = \sum_{i=1}^n X(z_i)$, where $g^{-1}(v) = \{z_1, \cdots, z_n\}$; the z_i are not necessarily distinct. We will see $\tilde{X}$ parametrizes a c.m.s.

Let $u_1, \cdots, u_l$ be the images of the ramification points of g (here we mean $g'(z) = 0$ and $g(z) =$ some u_i).

Let $v \in S^2 - \{v_1, \cdots, v_k, u_1, \cdots, u_l\}$, $z_1, \cdots, z_n$ the preimages S of g. Let D_i be a small disc on M at z_i, $D \subset S^2$ a disc at v and $h_i\colon D \to D_i$ an inverse of g.

Let $\tilde{x}_k$ be coordinate functions of $\tilde{X}$; then

$$\tilde{x}_3(v) = \sum_{i=1}^n x_3(h_i(v)), \qquad v \in D,$$

hence

$$(*) \qquad \frac{\partial \tilde{x}_3}{\partial v}(v)\, dv = \sum \frac{\partial h_i(v)}{\partial v} \frac{\partial x_3}{\partial z_i}(h_i(v))\, dv.$$

On D_i, write $\omega = f_i(z_i)\, dz_i$, and on D write $\tilde{\omega} = f(v)\, dv$. Then $(*)$ yields

$$\tilde{g}(v) f(v)\, dv = \sum \frac{\partial h_i}{\partial v}(v) \frac{\partial x_3}{\partial z_i}(h_i(v))\, dv;$$

it is always true that $\partial x_k/\partial z = \phi_k$, $\tilde{g}(v) = v$ and $g(h_i(v)) = v$ $\forall v \in D$, hence

$$\begin{aligned} f(v)\, dv &= \sum_{i=1}^n \frac{\partial h_i(v)}{\partial v} f_i(h_i(v))\, dv \\ &= \sum f_i(h_i(v)) d(h_i(v)) = \sum f_i(z_i) dz_i. \end{aligned}$$

Hence $\tilde{\omega}/D = \sum_{i=1}^n h_i^*(\omega/D_i)$.

Clearly this gives a globally defined form $\tilde{\omega}$ on $S^2 - \{v_i, u_j\}$. Now we will extend $\tilde{\omega}$ to $u_1, \cdots, u_l$. We will suppose ω is holomorphic at $g^{-1}(u_j)$; the same proof works if ω has a pole there.

Let $u \in \{u_1, \cdots, u_l\}$ and $z_1, \cdots, z_r$ be the distinct points of M, $g(z_i) = u$. For each point $z_i \in \{z_1, \cdots, z_r\}$, we will define a holomorphic $\tilde{\omega}_i$ in a neighborhood of u and $\tilde{\omega}$ will be the sum of the $\tilde{\omega}_i$ in the neighborhood, hence holomorphic at u as well.

Let h = degree of g at z_i. Let (D, z) be a conformal disc at z_i where $g(z) = z^h$.

Write $\omega = (\sum_0^\infty a_i z^i)\, dz$ in D, $g(D) = E$, and choose a conformal parameter v in E with $u = 0$. Let $v \in E - 0$. v has h distinct roots in D, $\{x_0, x_1, \cdots, x_{h-1}\}$. Let j be a generator of the group of hth roots of 1, so $x_m = j^m x_0$, $m = 0, \cdots, h-1$. Let $x_0 = v^{1/h}$ be a root chosen once and for all.

Define $\tilde{\omega}$ in a neighborhood of v by

$$\tilde{\omega}_i = \sum_{m=0}^{h-1} f(x_m)\, dx_m,$$

where $f(x) = \sum_0^\infty a_i x^i$ and x_m is in a small disc at $j^m x_0$. Then

$$\begin{aligned}
\tilde{\omega}_i &= \sum_{m=0}^{h-1} f(j^m x_0) d(j^m x_0) = \sum_{0}^{h-1} j^m f(j^m x_0)\, dx_0 \\
&= \sum j^m (a_0 + a_1 j^m x_0 + \cdots + a_i j^{mi} x_0^i + \cdots +)\, dx_0 \\
&= \sum (a_0 j^m + a_1 j^{2m} x_0 + \cdots + a_i j^{m(i+1)} x_0^i + \cdots)\, dx_0 \\
&= \sum_{i=0}^{\infty} \left(a_i x_0^i \sum_{m=0}^{h-1} j^{m(i+1)} \right) dx_0.
\end{aligned}$$

Since

$$\sum_{i=0}^{h-1} (j^m)^i = \begin{cases} h & \text{if } m \equiv O(h), \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned}
\sum_{m=0}^{h-1} (j^{(i+1)})^m = h &\Leftrightarrow i + 1 \equiv O(h) \\
&\Leftrightarrow i \equiv -1(h) \\
&\Leftrightarrow i = lh - 1, \qquad l \in Z^+.
\end{aligned}$$

Hence

$$\tilde{\omega}_i = \sum_{l=1}^{\infty} h a_{lh-1} x_0^{lh-1}\, dx_0,$$

$$x_0 = v^{1/h}, \quad dx_0 = \frac{1}{h} v^{(1-h)/h}\, dv, \quad x_0^{lh-1} = v^{(lh-1)/h},$$

$$\begin{aligned}
h a_{lh-1} x_0^{lh-1}\, dx_0 &= h a_{lh-1} v^{(lh-1)/h} \frac{1}{h} v^{(1-h)/h}\, dv \\
&= a_{lh-1} v^{(lh-1+1-h)/h}\, dv = a_{lh-1} v^{l-1}\, dv.
\end{aligned}$$

Thus

$$\tilde{\omega}_i = \sum_{l-1}^{\infty} a_{lh-1} v^{l-1}\, dv,$$

and $\tilde{\omega}_i$ is in fact a form defined for all $v \in E$ and is holomorphic. By construction $\sum \tilde{\omega}_i = \tilde{\omega}$ so indeed $\tilde{\omega}$ is holomorphic at u as well.

Now if ω had a pole at z_i, the same proof shows that $\tilde{\omega}$ can be extended meromorphically to z_i. It may happen that the pole of ω does not give rise to a pole of $\tilde{\omega}$, in which case $\tilde{\omega}$ is holomorphic at z_i, so $M + M$ does not have a puncture at z_i.

An example. Let $g(z) = z^2$, $\omega = dz/(z^3 - 1)^2$. This is the surface of Jorge-Meeks modelled on S minus the cube roots of unity. Then $M + M = \tilde{M}$ has the Weierstrass representation:

$$\begin{aligned}\tilde{g}(u) = u, \qquad \tilde{\omega}(u) &= \frac{d(u^{1/2})}{(u^{3/2}-1)^2} + \frac{d(-u^{1/2})}{(u^{3/2}+1)^2} \\ &= \frac{1}{2}u^{-1/2}\left[\frac{(u^{3/2}+1)^2 - (u^{3/2}-1)^2}{(u^{3/2}-1)^2(u^{3/2}+1)^2}\right] du \\ &= \frac{2u}{(u^3-1)^2}\, du.\end{aligned}$$

Another example is a catenoid plus Enneper's surface: $g_1(z) = z$, $\omega_1(z) = dz/z^2$, $g_2(z) = z$, $\omega_2(z) = dz$. Hence the sum is given by

$$\tilde{g}(z) = z, \qquad \tilde{\omega}(z) = \frac{dz}{z^2} + dz = \left(\frac{1+z^2}{z}\right) dz.$$

Theorem 2.2. *Let $M \in W$ have all bounded ends (i.e., each end is asymptotic to a plane). Then $\tilde{M} = M + M$ is a point.*

Proof. We will show $\tilde{\omega}$ has no poles at the ends of $\tilde{M}$; this means $\tilde{\omega}$ is holomorphic on a compact Riemann surface, hence constant. Let $p_1, \cdots, p_l$ be the punctures of M. We know the poles of $\tilde{\omega}$ can only be at the points $g(p_1), \cdots, g(p_l)$; we will see that $\tilde{\omega}$ is holomorphic at each such point.

Let $D \subset M$ be a conformal disc where the puncture p corresponds to 0 and $g(z) = z^k$ in D. Let $\omega = (a_{-n}/z^n + \cdots + a_{-1}/z + F(z))\, dz$ in D, where F is holomorphic in D. Since each end is bounded, we have $k \geq n$; x_3 is bounded on D. We will first sum the points in D with the same normal; then $\tilde{\omega}$ is obtained by adding the $\tilde{\omega}$'s so obtained at each puncture to holomorphic $\tilde{\omega}$'s at interior points. So it suffices to show the $\tilde{\omega}$ obtained at the puncture is holomorphic.

We have

$$\tilde{\omega}/D = \sum_{m=0}^{k-1} \left(\frac{a_{-n}}{(j^m z)^n} + \frac{a_{-n+1}}{(j^m z)^{n-1}} + \cdots + \frac{a_{-1}}{j^m z} + F(j^m z) \right) d(j^m z),$$

where $z^k = v$, and j is a kth root of unity. Using the fact that $k \geq n$ and

$$\sum_{m=0}^{k-1} j^{-m(g-1)} = \begin{cases} k & \text{if } g-1 \equiv O(k), \\ 0 & \text{otherwise,} \end{cases}$$

one obtains $\tilde{\omega}/D = (ka_{-1}/z)\,dz + G(z)\,dz$, G holomorphic at 0.

Therefore it suffices to show $a_{-1} = 0$. The coordinate functions $x_1 = \operatorname{Re}\int \phi_1$ and $x_2 = \operatorname{Re}\int \phi_2$ are well defined on M where $\phi_1 = \omega(1-g^2)$ and $\phi_2 = i\omega(1+g^2)$. A direct calculation yields

$$\operatorname{Re}(i \operatorname{Res}(\phi_1, 0)) = \operatorname{Re}(i \operatorname{Res}(\phi_2, 0)) = 0 = \operatorname{Re}(a_{-1}) = \operatorname{Re}(ia_{-1}).$$

Hence $a_{-1} = 0$.

Remarks. 1. Many examples of M with bounded ends exist. For example

$$g(z) = -\frac{1}{2}\frac{z}{z^3-1}, \qquad \omega = \frac{1}{4}\frac{(z^3-1)^2}{(z^3+1/2)^2}\,dz,$$

modelled on $\mathbf{C}$ less the cube roots of $-1/2$ [7].

2. The calculation of $\tilde{\omega}$ near bounded ends can yield global results. For example, one has $M + M$ is a point where M is the Costa example of a three-punctured torus with two catenoid type ends and one bounded end (M is embedded [1]). $\overline{M}$ is the torus $\mathbf{C}$ *modulo* Z^2, $g(z) = 2a\sqrt{2\pi}/P'(z)$, $\omega = P(z)\,dz$ where P is the Weierstrass P function and $a = P(1/2)$. The punctures are at $1/2$, $i/2$ and 0 and the total curvature is 12π. The limiting normals are the same at the catenoid type ends and g is 3 to 1 near the bounded end, having limiting normal the negative of the catenoid end normal. Thus the bounded end in M becomes regular in $M+M$ and the catenoid ends become one end in $M+M$ (one adds two points near ∞ on the catenoid ends to one point in a compact part of M). This one end is either bounded or a catenoid type end and both of these cases are impossible (just apply the maximum principle to the one ended c.m.s. $M+M$). Therefore $M+M$ is a point.

Theorem 2.3. *Let M be the surface of Jorge-Meeks: $g(z) = z^{2n-1}$, $\omega = dz/(z^{2n}-1)^2$, modelled on $S - \{x^{2n} = 1\}$. Then $M + M = C_1 + \cdots + C_n$, where each C_i is a catenoid; the axes of C_j are parallel to $e^{2\pi ij/n}$.*

Proof. We calculate $\tilde{\omega}$ of $M+M$: Let $v \in C^*$, and j be a generator of the $2n-1$ roots of unity. Choose $z_0, z_0^{2n-1} = v$ and let $z = z_0$ also denote a local parameter in a neighborhood of z_0 where $v^{1/(2n-1)}$ is analytic. Then $z_m = j^m z_0$, $m = 0, \cdots, 2n-2$, are local parameters for the other inverses of g, in a neighborhood of v.

We know

$$\tilde{\omega} = \sum_{m=0}^{2n-2} \frac{dz_m}{(z_m^{2n}-1)^2} = \sum \frac{j^m\,dz}{(j^{m2n}z^{2n}-1)^2}$$
$$= \sum \frac{j^m\,dz}{j^{2m}(z^{2n}-j^{-m})^2} = \sum_{m=0}^{2n-2} \frac{j^{-m}\,dz}{(z^{2n}-j^{-m})^2},$$

since $j = j^{2n} \Rightarrow j^m = j^{2nm}$. Moreover

$$\tilde{\omega} = \frac{1}{(z^{2n(2n-1)}-1)^2} \sum_{m=0}^{2n-2} j^{-m} \frac{(z^{2n(2n-1)}-1)^2}{(z^{2n}-j^{-m})^2}\,dz$$
$$= \frac{1}{(z^{2n(2n-1)}-1)^2} \sum_{0}^{2n-2} j^{-m}[z^{2n(2n-2)} + j^{-m}z^{2n(2n-3)} + \cdots + j^{-m(2n-2)}]^2\,dz,$$

in consequence of

$$\frac{x^{(2n-1)}-1}{x-j^{-m}} = x^{2n-2}+j^{-m}x^{2n-3}+j^{-2m}x^{2n-4}+\cdots+j^{-m(2n-3)}x+j^{-m(2n-2)}.$$

Thus

$$\tilde{\omega} = \frac{1}{(z^{2n(2n-1)}-1)^2} \sum_{m=0}^{2n-2} (2n-1)z^{2n(2n-2)}\,dz$$
$$= \frac{(2n-1)^2}{(z^{2n(2n-1)}-1)^2} z^{2n(2n-2)}\,dz,$$

where we have used $\sum_{m=0}^{2n-2}(j^k)^m = 0$ for $0 < k < 2n-1$. Now we write $\tilde{\omega}$ in terms of v:

$$v = z^{2n-1}, \quad dz = \frac{1}{2n-1} v^{1/(2n-1)-1}\,dv, \quad \tilde{\omega} = \frac{(2n-1)v^{2n-2}}{(v^{2n}-1)^2}\,dv.$$

Since $\tilde{\omega}$ is holomorphic on S punctured at the $2n$th roots of unity, we see that $M+M$ is modelled on this space. Next observe:

$$\frac{v^{2n-2}}{(v^{2n}-1)^2} = \frac{1}{n^2}\left[\frac{1}{(v^2-1)^2} + \frac{\alpha}{(v^2-\alpha)^2} + \cdots + \frac{\alpha^m}{(v^2-\alpha^m)^2} + \cdots + \frac{\alpha^{n-1}}{(v^2-\alpha^{n-1})^2}\right],$$

where α is a root generator of $X^n = 1$. So

$$\tilde{\omega} = \frac{2n-1}{n^2} \sum_{k=0}^{n-1} \frac{\alpha^k}{(v^2-\alpha^k)^2}\,dv,$$

the surfaces C_k parametrized by $S^2 - \{z^2 = \alpha^k\}$, $g(z) = z$, and $\omega = ((2n-1)/n^2)\alpha^k\, dz/(z^2-\alpha^k)^2$ are catenoids, and hence $M+M = C_1+\cdots+C_n$.

Theorem 2.4. *Let $M \in H$ have exactly n ends, each of catenoid type and suppose M is invariant by a rotation by $2\pi/n$. Then $M = M_n+M_n$ where M_n is the c.m.s. of Jorge-Meeks: $g(z) = z^{n-1}$, $\omega = dz/(z^n-1)^2$, modelled on S punctured at the nth roots of unity.*

Proof. Let R be the rotation leaving M invariant. R permutes the ends of M, so permutes the limiting normals $z_1, \cdots, z_n$, as well. The limiting normals are on a circle orthogonal to the axis of rotation, so taking this axis to be the x_3-axis, we have $z_1 = \rho, z_2 = \rho j, \cdots, z_n = \rho j^{n-1}$, where ρ is a positive real number and j is a generator of the nth roots of unity.

We have M represented by (g,ω): $g(z) = z$,

$$\begin{aligned}
\omega &= \sum_{p=1}^{n}\left(\frac{a_p}{(z-z_p)^2}+\frac{b_p}{z-z_p}\right)dz\\
&= \sum_{p=1}^{n}\left(\frac{a_p}{(z-\rho j^{p-1})^2}+\frac{b_p}{z-\rho j^{p-1}}\right)dz,
\end{aligned}$$

where a_p are real, $b_p = -2a_p\bar{j}^{p-1}\rho/(1+\rho^2)$, and $\sum_{p=0}^{n} a_p\pi^{-1}(z_p) = 0$, π being a stereographic projection. These relations among a_p and b_p will be derived in the proof of 2.5.

Now M is modelled on $S - \{\rho, \rho j, \cdots, \rho j^{n-1}\}$, and $R(M)$ as well. A Weierstrass representation of $R(M)$ is given by (g^1, ω^1) where $g^1(z) = jg(z) = jz$ and $\omega^1 = \bar{j}\omega$.

Let $z = \bar{j}u$, $dz = \bar{j}\,du$. Then if (g_0, ω_0) denotes the Weierstrass pair of $R(M)$ in this new coordinate, we have: $g_0(u) = u$,

$$\begin{aligned}
\omega_0 &= \bar{j}^2\sum_{p=1}^{n}\left(\frac{a_p}{(\bar{j}u-\rho j^{p-1})^2}+\frac{b_p}{(\bar{j}u-\rho j^{p-1})}\right)du\\
&= \sum_{p=1}^{n}\left(\frac{\alpha_p}{(u-\rho j^p)^2}+\frac{b_p\bar{j}}{u-\rho j^p}\right)du\\
&= \sum_{p=1}^{n}\left(\frac{a_p}{(z-\rho j^p)^2}+\frac{b_p\bar{j}}{z-\rho j^p}\right)dz.
\end{aligned}$$

Now $R(M) = M$ and $g(z) = g_0(z) = z$, so $\omega = \omega_0$. This yields $a_p = a_{p+1}$, $p = 1, \cdots, n-1$, and $b_{p+1} = b_p\bar{j}$. Denote a_p by a, so

$$\bar{j}b_p = -\bar{j}\frac{2a\rho\bar{j}^{p-1}}{1+\rho^2} = \frac{-2a\rho\bar{j}^p}{1+\rho^2} = b_{p+1}.$$

Thus $\sum_{p=1}^{n} \pi^{-1}(\rho j^p) = 0$ yields $\rho = 1$ and $z_i = j^i$. We have shown that M has the representation $g(z) = z$, $\omega = a\sum_{p=1}^{a}(1/(z - j^p)^2 - \bar{j}^p/(z - j^p))\,dz$, where a is real and M is modelled on S punctured at the nth roots of unity. Now $M_n + M_n$ is also modelled on $S - \{x^n = 1\}$, is invariant by R, and has n catenoid type ends. Thus it has the same (g, ω) (up to multiplication by a real number in ω) and $M_n + M_n$ is a homothety of M.

Theorem 2.5. *Let $z_1, \cdots, z_n$ be distinct points of* $\mathbf{C}$. *A necessary and sufficient condition for the existence of $M \in H$ having n ends, each of catenoid type, and with limiting normals $\pi^{-1}(z_i)$, $i = 1, \cdots, n$, is the existence of real numbers $a_1, \cdots, a_n$, such that $\sum_{i=1}^{n} a_i \pi^{-1}(z_i) = 0$.*

Proof. We shall assume such a surface exists and derive what ω must be. Then it will be clear this ω works.

We assume M is parametrized by $S - \{\pi^{-1}(z_1), \cdots, \pi^{-1}(z_n)\}$ and $g(z) = z$. The ends of this type catenoid imply ω has a double pole at each end and is holomorphic elsewhere. Hence

$$\omega = \left[\sum_{i=1}^{n}\left(\frac{a_i}{(z - z_1)^2} + \frac{b_i}{z - z_i}\right) + P(z)\right] dz$$

where P is a polynomial. To understand M at ∞, apply the rotation by π about the x_1 axis. This gives

$$\tilde{g}(z) = \frac{1}{g(z)}, \qquad \tilde{\omega} = -\omega g^2,$$

$$\tilde{\omega} = -\left[\sum\left(\frac{a_i z_i^2}{(z - z_i)^2} + \frac{2a_i z_i + z_i^2 b_i}{z - z_i} + a_i + b_i z_i + b_i z\right) + z^2 P(z)\right] dz,$$

and ∞ is not a pole of $\tilde{g}$. Let $u = 1/z$ and g_0, ω_0 be the induced representation:

$$g_0(u) = u,$$

$$\omega_0 = \left[\sum_{i=1}^{n}\left(\frac{a_i z_i^2}{(1 - u z_i)^2} + \frac{2a_i z_i^2 + b_i z_i^3}{1 - u z_i} + \frac{2a_i z_i + b_i z_i^2}{u} + \frac{a_i + b_i z_i}{u^2} + \frac{b_i}{u^3}\right) + \frac{1}{u^4} P\left(\frac{1}{u}\right)\right] du.$$

We have ω holomorphic at ∞, so ω_0 is holomorphic at 0, so that

$$(*) \qquad P \equiv 0, \quad \sum b_i = 0, \quad \sum a_i + b_i z_i = 0, \quad \sum 2a_i z_i + b_i z_i^2 = 0.$$

Therefore

$$\omega = \left(\sum \frac{a_i}{(z - z_i)^2} + \frac{b_i}{z - z_i}\right) dz,$$

where a_i, b_i satisfy $(*)$.

Now we have Re $\int \phi_k$ are period free, so

$$\operatorname{Re}(2\pi i \operatorname{Res}(\phi_k, z_j)) = 0, \qquad j = 1, \cdots, n, \ k = 1, 2, 3.$$

This gives

$$(**) \qquad \begin{aligned} &\operatorname{Im}(b_i(1 - z_i^2) - 2a_i z_i) = 0, \\ &\operatorname{Re}(b_i(1 + z_i^2) + 2a_i z_i) = 0, \qquad i = 1, \cdots, n, \\ &\operatorname{Im}(a_i + z_i b_i) = 0. \end{aligned}$$

Here we use

$$\phi_1 = \frac{1}{2} \sum_{i=1}^{n} \left[z \frac{a_i(1 - z_i^2)}{(z - z_i)^2} + \frac{b_i(1 - z_i^2) - 2a_i z_i}{z - z_i} - a_i - 2b_i z_i - b_i(z - z_i) \right] dz,$$

and analogous expressions for ϕ_2, ϕ_3. Solving (**) gives

$$\operatorname{Im} a_j = 0, \quad b_j = \frac{-2a_j \overline{z}_j}{1 + |z_j|^2}, \qquad j = 1, \cdots, n.$$

So we have three equations (comes from (*)):

$$(1) \qquad \sum_{1}^{n} b_i = 0,$$

$$(2) \qquad \sum_{i}^{n} a_i + b_i z_i = 0,$$

$$(3) \qquad \sum_{i}^{n} 2a_i z_i + b_i z_i^2 = 0 \quad \text{(comes also from (1) and (**))}.$$

Write $z_j = x_j + i y_j$; then $\sum b_j = 0$ and (**) imply

$$\sum_{j=1}^{n} \frac{a_j x_j}{1 + |z_j|^2} = 0 = \sum_{i}^{n} \frac{a_j y_j}{1 + |z_j|^2}; \qquad a_j \text{ real}, \ j = 1, \cdots, n.$$

We know

$$\sum a_j + b_j z_j = 0, \qquad b_j = \frac{-2a_j \overline{z}_j}{1 + |z_j|^2},$$

hence

$$\sum_j \frac{a_j(1 - |z_j|^2)}{1 + |z_j^2|} = 0.$$

So finally

$$a_j \text{ real}, \quad b_j = -2a_j \frac{\overline{z}_j}{1 + |z_j|^2}, \quad \text{and} \quad \sum_j a_j \pi^{-1}(z_j) = 0,$$

where

$$\pi^{-1}(z_j) = \left(\frac{x_j}{1+|z_j|^2}, \frac{y_j}{1+|z_j|^2}, \frac{1-|z_j|^2}{1+|z_j|^2}\right).$$

Thus if we are given real numbers a_j and points $p_j \in S^2$ such that $\sum a_j p_j = 0$, we can define b_j by (**) and get M as desired:

$$g(z) = z, \qquad \omega = \sum_{i=1}^{n} \left(\frac{a_i}{(z-z_i)^2} - \frac{2a_i \bar{z}_i}{1+|z_i|^2} \cdot \frac{1}{z-z_i}\right) dz.$$

There is a relation among catenoid type ends in general (in [8] this relation is derived for two ends).

Proposition 2.6. *Let $M \in W$ have n ends, each of catenoid type, with limiting normals $N_1, \cdots, N_n$ (N_i pointing towards the opening of the ith end). Then there are positive real numbers $a_1, \cdots, a_n$ such that $\sum_{i=1}^n a_i N_i = 0$.*

Proof. Let $b = (b_1, b_2, b_3)$ be a point of R^3 and $X\colon M \to R^3$ a parametrization of M, conformal except at the branch points. The function $h(z) = \langle X(z), b\rangle$ is harmonic on M. Let $S_i(R)$ be a cylinder with axis N_i and radius R, and let $C_i(R)$ be the intersection of $S_i(R)$ with the ith end. For large R, $C_i(R)$ is "almost" a geometric circle. Let $M(R)$ be the compact submanifold of M bounded by $\bigcup_{i=1}^a C_i(R)$. The flux of h across $\partial M(R)$ is 0 by Stokes theorem and harmonicity of h. Hence

$$0 = \sum_{i=1}^{n} \int_{C_i(R)} \langle \nabla_{R^3} h, n_i\rangle\, ds = \sum_{i=1}^{n} \int_{C_i(R)} \langle b, n_i\rangle, ds,$$

where n_i is the interior unit normal field to $C_i(R)$, tangent to M.

We calculate $\int_{C_i(R)} \langle b, n_i\rangle\, ds$. Choose coordinates in R^3 so that the ith end is a graph over the (x_1, x_2) plane and $N_i = (0,0,1)$. Then $x_3 = a_i \log R + O(R^{-1})$ with $R = \sqrt{x_1^2 + x_2^2}$ and $a_i > 0$. A calculation yields

$$n_i = R^{-1}(-x_1, -x_2, -a_i) + O(R^{-2}).$$

It then follows that

$$\int_{C_i(R)} \langle b, n_i\rangle\, ds = 2\pi b_3 a_i + O(R^{-1}) = 2\pi \langle b, a_i N_i\rangle + O(R^{-1}).$$

Hence $2\pi\langle b, \sum_{i=1}^n a_i N_i\rangle + O(R^{-1}) = 0$. Letting $R \to \infty$, we obtain $\langle b, \sum_{i=1}^n a_i N_i\rangle = 0$. Since b was arbitrary, we have $\sum_{i=1}^n a_i N_i = 0$.

3. Nonorientable herissons

We define a nonorientable minimal herisson to be a c.m.s. of total curvature 2π and having a finite number of branch points. Such a surface is parametrized by the projective plane P punctured at a finite number of points.

Let $p\colon S \to P$ be the two-sheeted orientable cover of P. Then a nonorientable herisson lifts by p to an orientable herisson with (g, ω) satisfying: $g(z) = z$, $\omega = f(z)\,dz$ with $f(z) = -(z^4)^{-1}\overline{f(-1/\overline{z})}$ (cf. [4]). For example, Henneberg's surface is parametrized by P minus one point, $g(z) = z$ and $f(z) = 1 - 1/z^4$; the points 1 and i are branch points.

Now for M, N in W, define $M \tilde{+} N$ as before, except for each tangent plane in R^3 add all points of M and N with this tangent plane.

Theorem 3.1. *For M, N in W, $M \tilde{+} N$ is a nonorientable herisson, or a point.*

Proof. $M \tilde{+} N$ can be obtained by first adding all points of M and N having the same oriented tangent plane (to obtain an orientable herisson) and then adding all points on this surface having the same tangent plane. So it suffices to prove the theorem for $M \tilde{+} M$ where M is an orientable herisson.

Let X parametrize M conformally (except at the branch points) and $g(z) = z$, $\omega = f(z)\,dz$ be the Weierstrass pair of M. If $a_1, \cdots, a_p$ are the limiting normals at the ends of M, then $\tilde{X}\colon S - \{\pm a_1, \cdots, \pm a_p\}$ parametrizes $M \tilde{+} M$ where $\tilde{X}(z) = X(z) + X(-1/\overline{z})$. Hence

$$\frac{\partial \tilde{x}_i}{\partial z}(z) = \frac{\partial x_i(z)}{\partial z} + \frac{1}{z^2}\overline{\frac{\partial x_i}{\partial z}\left(-\frac{1}{\overline{z}}\right)} \quad \text{for } i = 1, 2, 3.$$

This gives

$$\begin{aligned}
\frac{\partial \tilde{x}_1}{\partial x} &= \frac{f(z)}{2}(1 - z^2) + \overline{f\left(-\frac{1}{\overline{z}}\right)}\left(1 - \frac{1}{z^2}\right)\cdot\frac{1}{z^2},\\
\frac{\partial \tilde{x}_2}{\partial z} &= i\frac{f(z)}{2}(1 + z^2) - \frac{i}{2}\overline{f\left(-\frac{1}{\overline{z}}\right)}\left(1 + \frac{1}{z^2}\right)\frac{1}{z^2},\\
\frac{\partial \tilde{x}_3}{\partial z} &= f(z)z - \overline{f\left(-\frac{1}{\overline{z}}\right)}\frac{1}{z^3}.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{f}(z) &= f(z) - \frac{1}{z^4}\overline{f\left(-\frac{1}{\overline{z}}\right)},\\
\tilde{g}(z) &= \frac{\partial \tilde{x}_3(z)/\partial z}{\tilde{f}(z)} = z.
\end{aligned}$$

Then $(\tilde{g}, \tilde{f})$ defines a nonorientable c.m.s., $\tilde{f}$ is meromorphic and $\tilde{f}(z) = -(z^4)^{-1}\overline{\tilde{f}(-1/\overline{z})}$. So $M \tilde{+} M$ is parametrized by P if f is not constant and is a point if f is constant.

For example, if M is the catenoid, $f(z) = 1/z^2$ so $\tilde{f}(z) \equiv 0$ and $M \tilde{+} M$ is a point. If M is Enneper's surface, $f(z) = 1$, so $\tilde{f}(z) = 1 - 1/z^4$. Hence $M \tilde{+} M$ is Henneberg's surface.

For the surface $g(z) = z^2$, $\omega = dz/(z^3 - 1)^2$. First we form $M \mathbin{\tilde{+}} M$; this gives the orientable herisson $g(z) = z$, $\omega = (2z/(z^3 - 1)^2)\,dz$. Then $M \mathbin{\tilde{+}} M$ is $g(z) = z$, $\omega = (4z(z^6 + 1)/(z^6 - 1)^2)\,dz$. This surface has three ends, each of catenoid type.

Proposition 3.2. *Let M be a nonorientable c.m.s. with a finite number b of branch points. Let k be the number of points missed by the Gauss map (with values in P) and let N be the degree of the Gauss map (between the orientable covers). Then $k \le (2N + b)/(N + 1)$.*

This follows easily from 1.2 so we leave the proof to the reader. We remark that this inequality is sharp: one has equality for nonorientable minimal herissons having all catenoid type ends. For example, when $g(z) = z$, $\omega = (4z(z^6 + 1)/(z^6 - 1)^2)\,dz$, we have $N = 1$, $b = 4$ and $k = 3$.

4. Deformations of surfaces in W

Let $X\colon M \to R^3$ parametrize an element of W, and let N denote a unit vector field normal to M in R^3. An ε C^2-deformation of M is a minimal surface which is a graph over M and is ε C^2-close to M; i.e., $X_1\colon M \to R^3$ is a minimal surface of the form $X_1(z) = X(z) + h(z)N(z)$ where $h\colon M \to R$ is a smooth function with $\|h\|_{C^2} < \varepsilon$. We do not require X_1 to be conformal. This notion of deformations has been introduced and studied in [6] and [7].

Theorem 4.1. *Let $M \in W$ and M_1 be an ε C^2-deformation of M. Then for ε sufficiently small, $M_1 \in W$, $c(M_1) = c(M)$ and the branch points of M_1 coincide with those of M (with multiplicity).*

Proof. This theorem was proved in [6], when M has no branch points. In the presence of branch points, the same proof adapts to show $M_1 \in W$ and $c(M_1) = c(M)$; the Gauss maps of M and M_1 are smooth and close near the branch points. We need only check the multiplicities are the same.

If D is a small disc at a branch point $p = 0$ of M, then one has a local Gauss Bonnet formula:

$$\int_D K + \int_{\partial D} k_g = 2\pi(n+1),$$

where K and k_g are the Gaussian and geodesic curvatures of M and ∂D respectively, and n is the multiplicity of the branch point. We include a proof of this in an appendix.

Since the left side of this equation for M_1 is close to that of M, it follows the branch points have the same multiplicity.

Theorem 4.2. *Let $M \in H$ have only catenoid type ends. Then M is isolated, i.e., if M_1 is a sufficiently small deformation of M then M_1 is congruent to M.*

Proof. M is parametrized by $S - \{z_1, \cdots, z_n\}$, z_i are the limiting normal vectors at the ends. We have $g(z) = z$ and

$$\omega = \sum_{j=1}^{n} \left(\frac{a_j}{(z - z_j)^2} - \frac{2a_j \bar{z}_j}{1 + |z_j|^2} \cdot \frac{1}{z - z_j} \right) dz,$$

with a_j real and $\sum_{j=1}^{n} a_j \pi^{-1}(z_j) = 0$. These properties of (g, ω) were obtained in the proof of 2.5.

Now for M_1 a small deformation of M, the limiting values of the Gauss maps of M and M_1 are the same at the ends. Thus M_1 is also parametrized by $S - \{z_1, \cdots, z_n\}$, and $g_1(z) = z$,

$$\omega_1 = \sum_{j=1}^{n} \left(\frac{\alpha_j}{(z - z_j)^2} - \frac{2\alpha_j \bar{z}_j}{1 + |z_j|^2} \cdot \frac{1}{z - z_j} \right) dz$$

with α_j real and $\sum_{j=1}^{n} \alpha_j \pi^{-1}(z_j) = 0$.

Let us apply a rotation R to M so that at the jth end we have $g(z_j) = 0$. Then the end is a graph over the (x_1, x_2) plane of the form

$$x_3(z) = a_j K_j \log|z| + O(|z|),$$

where z is a conformal parameter at the end, z_j corresponds to 0, and K_j depends only on R and z_j. M_1 also becomes a graph over the (x_1, x_2)-plane, near the perturbed end, and M_1 is a graph of the form

$$x_3^1(z) = \alpha_j K_j^1 \log|z| + O(|z|).$$

Since K_j^1 depends only on R and z_j, we have $K_j^1 = K_j$. Also the perturbed end is close (in R^3) to the jth end of M, so $\alpha_j K_j^1 = a_j K_j$. Hence $\alpha_j = a_j$ for $j = 1, \cdots, n$, and $\omega = \omega_1$. Thus $M = M_1$.

Appendix

The local Gauss-Bonnet formula will result from the following.

Lemma. *Let D_r be a disc of radius r centered at a branch point o of order n. Then*

$$\lim_{r \to 0} \int_{\partial D_r} k_g \, ds = 2\pi(n + 1).$$

Proof. Choose coordinates so that $g(0) = 0$, $g(z) = a_p z^p + o(z^{p+1})$, $\omega = (b_n z^n + o(z^{n+1}))\, dz$, $b_n \neq 0$, $a_p \neq 0$.

$$\phi_1 = \frac{1}{2}(b_n z^n + o(z^n))\, dz,$$

$$\phi_2 = \frac{i}{2}(b_n z^n + o(z^n))\, dz,$$

$$\phi_3 = (a_p b_n z^{n+p} + o(z^{n+p}))\, dz.$$

Let $z = re^{i\theta}$; then

$$2x_1 = r^{n+1}\operatorname{Re}\left(\frac{b_n}{n+1}e^{i(n+1)\theta}\right) + o(r^{n+1}),$$
$$2x_2 = -r^{n+1}\operatorname{Im}\left(\frac{b_n}{n+1}e^{i(n+1)\theta}\right) + o(r^{n+1}),$$
$$x_3 = r^{n+p+1}\operatorname{Re}\left(\frac{a_p b_n}{n+p+1}e^{i(n+p+1)\theta}\right) + o(r^{n+p+1}).$$

Let $X^r = X/(|z| = r)$, $X^r_\theta = \partial X^r/\partial\theta$; then k_g of $X(S_r)$ is

$$|X^r_\theta, X^r_{\theta\theta}, \overrightarrow{N}|/\langle X^r_\theta, X^r_\theta\rangle^{3/2},$$

where

$$X^r_\theta = \left|\begin{array}{l} -\frac{1}{2}r^{n+1}\operatorname{Im}(b_n e^{i(n+1)\theta}) + o(r^{n+1}) \\ -\frac{r^{n+1}}{2}\operatorname{Re}(be^{i(n+1)\theta}) + o(r^{n+1}) \\ -r^{n+p+1}\operatorname{Im}(a_p b_n e^{i(n+p+1)\theta}) + o(r^{n+p+1}) \end{array}\right|,$$

$$X_{\theta\theta} = \left|\begin{array}{l} -\frac{(n+1)}{2}r^{n+1}\operatorname{Re}(b_n e^{i(n+1)\theta}) + o(r^{n+1}) \\ +\frac{(n+1)}{2}r^{n+1}\operatorname{Im}(b_n e^{i(n+1)\theta}) + o(r^{n+1}) \\ -(n+p+1)r^{n+p+1}\operatorname{Re}(a_p b_n e^{i(n+p+1)\theta}) + o(r^{n+p+1}) \end{array}\right|,$$

$$\overrightarrow{N} = \frac{1}{1+o(r^{2p-1})}\left|\begin{array}{l} 2r^p\operatorname{Re}(a_p e^{ip\theta}) + o(r^p) \\ 2r^p\operatorname{Im}(a_p e^{ip\theta}) + o(r^p) \\ -1 + o(r^{2p-1}) \end{array}\right|,$$

$$|X^r_\theta, X^r_{\theta\theta}, \overrightarrow{N}| = \left(\frac{n+1}{4}\right)|b_n|^2 r^{2n+2} + o(r^{2n+2}),$$

$$k_r = \frac{2(n+1)r^{2(n+1)} + o(r^{2n+2})}{|b_n|r^{3(n+1)}},$$

ds is the arc length on $X(S_r), d = |X^r_\theta|\,d\theta$. Hence

$$\begin{aligned} k_r\,ds &= k_r|X^r_\theta|\,d\theta \\ &= \frac{2(n+1)r^{2(n+1)} + o(r^{2(n+1)})}{|b_n|r^{3(n+1)}}\left(\frac{r^{n+1}}{2}|b_n| + o(r^{n+1})\right)d\theta \\ &= (n+1+o(1))\,d\theta. \end{aligned}$$

This proves the lemma.

Now if D is a disc about a branch point, apply the lemma to $E_r = D - \{z \mid |z| < r\}$. Then

$$\int_{E_r} K\,dA + \int_{\partial D} k_g\,ds - \int_{|z|=r} k_g\,ds = 0.$$

Thus, letting $r \to 0$, we obtain

$$\int_D K\,dA + \int_{\partial D} k_g\,ds = 2\pi(n+1).$$

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