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A DIFFERENTIAL COMPLEX FOR POISSON MANIFOLDS

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0. Introduction

In this article, we consider Poisson manifolds M, that is, manifolds (say C^∞) for which there is a bracket operation $\{\ ,\ \}$ on smooth functions which has the usual properties of Poisson brackets. Poisson manifolds, apparently first considered by Lie, have been recently studied by Lichnérowicz [18] and by Weinstein [23]. Our main object here is the *canonical complex*

$$\cdots \to \Omega^{n+1}(M) \xrightarrow{\delta} \Omega^n(M) \xrightarrow{\delta} \Omega^{n-1}(M) \to \cdots,$$

where δ is given by the formula

$$\begin{aligned}\delta(f_0 f_1 \wedge \cdots \wedge df_n) &= \sum_{i=1}^{n} (-1)^{i+1}\{f_0, f_i\}\, df_1 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge df_n \\ &\quad + \sum_{1 \le i < j \le n} (-1)^{i+j} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df}_i \\ &\qquad\qquad \wedge \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_n.\end{aligned}$$

This differential coincides with the one introduced by Koszul [17] which he denotes Δ.

The homology of the canonical complex is called the *canonical homology* of M. From its definition, it is clear there is a map from the *Lie algebra homology* $H_*(L, L)$, where L is the Lie algebra of C^∞-functions on M, with bracket $\{\ ,\ \}$, to the canonical homology of M.

The relation $d\delta + \delta d = 0$, proven by Koszul (where d denotes exterior differentiation), allows us to introduce a double complex, studied in §1.3.

In the case of symplectic manifolds, we prove in §§2.2 that δ is equal, up to sign, to $*d*$, where $*$ is the symplectic analog of the $*$ operator for Riemannian manifolds. We then conjecture that any de Rham cohomology class has a representative α such that $d\alpha = \delta\alpha = 0$. Some evidence for this conjecture is presented in §§2.2 and 2.3. We prove the conjecture for a compact Kähler

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manifold by proving that on (p,q)-forms, the symplectic $*$ operator is equal to a constant times the Riemannian $*$ operator (cf. §2.4).

In §3, we use the canonical complex as a tool for studying the Hochschild homology of noncommutative algebras which admit an algebra filtration with commutative graded algebra. Then this graded algebra has a Poisson structure. One may use the filtration of the algebra to construct a spectral sequence which has E^∞ term equal to the Hochschild homology of the algebra, and E^1 term to the Hochschild homology of the graded algebra. Theorem 3.1.1 identifies the E^1 term, with its differential, with the canonical complex of the Poisson manifold given by the graded algebra.

Examples show that this spectral sequence tends to degenerate at E^2. For the algebra of differential operators on a manifold M, the E^2 term is equal to the de Rham cohomology of M (with an inversion of degrees). We relate this to a result of Kassel and Mitschi [15], who computed the Hochschild homology for that algebra in the algebraic and complex-analytic cases. We thus obtain the degeneracy at E^2 of the spectral sequence. Let us note that this degeneracy has been now obtained directly [4], [5], [24].

We also study the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$. By imitating a famous trick of Hermann Weyl, one may restrict one's attention to the invariant part of the canonical complex, which is very easily computed. This trick should afford computations for higher-dimensional reductive Lie algebras (§3.4).

There is a similar spectral sequence converging to the cyclic homology of such algebras, which we investigate (§§3.2, 3.3, 3.4). Further results for general Lie algebras have been obtained by Kassel [14].

We have not studied the canonical complex for general Poisson manifolds.

A local study should be possible by using suitable local coordinate systems as in [18] and [23]. Duals of Lie algebras already have a very rich and intricate Poisson structure, as our joint work with Borho has convinced us.

We could have derived δ from the Hochschild boundary of the first order noncommutative deformation of the algebra of functions on the Poisson manifolds, i.e. we could have used the computations of §3.1 as a way to define δ. However, it would still have been necessary to prove $\delta \circ \delta = 0$ by further computations.

We hope that the canonical homology introduced here will turn out to be useful in the geometry of Poisson manifolds, and in their mechanics.

It is a pleasure to thank Alain Connes, Ezra Getzler, Christian Kassel, André Lichnérowicz and Joseph Oesterlé for their interest and for helpful conversations. In particular, Lichnérowicz explained to me about Poisson manifolds and the Schouten-Nijenjuis bracket, and he suggested that I investigate the case of Kähler manifolds. Christian Kassel told me about his result on the Hochschild and cyclic homology of the algebra of differential operators on an affine variety. I am grateful to the referee for many suggestions, which led in particular to a substantial simplification in §1.

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1. Construction of the canonical complex of a Poisson manifold

1.1. Let M be a C^∞-manifold. A *Poisson bracket* on M is a complex bilinear form $\{\ ,\ \}\colon A \times A \to A$ where $A = C^\infty(M)$ is the algebra of C^∞ real-valued functions on M, satisfying the following properties:

(i) for any $f \in A$, there exists a (uniquely defined) vector field ξ_f on M such that $\{f, g\} = \xi_f(g)$ for all $g \in A$;

(ii) $\{f, g\} = -\{g, f\}$;

(iii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ (Jacobi's identity).

Property (i) implies in particular the relation $\{f, gh\} = h \cdot \{f, g\} + g \cdot \{f, h\}$ which will be used frequently. Properties (ii) and (iii) mean that $\{\ ,\ \}$ endows A with a Lie algebra structure; to avoid confusion when we consider Hochschild and Lie algebra homologies, this Lie algebra will be denoted by L.

Since $\{\ ,\ \}$ is a local operation by (i) and (ii), it also defines a Poisson bracket on any open set of U, and we have a bracket $\{\ ,\ \}\colon \mathscr{C}_M^\infty \times \mathscr{C}_M^\infty \to \mathscr{C}_M^\infty$, where $\mathscr{C}_M^\infty$ is the sheaf of germs of C^∞-functions on M.

The concept of Poisson manifold is due to Lichnérowicz, who has a more compact formulation of it [18, pp. 254–255]. He remarks that the Poisson bracket gives rise to a covariant antisymmetric tensor G of order 2, such that $i(G)(df \wedge dg) = \{f, g\}$ for $f, g \in A$, where $i(G)$ denotes the interior product by G. Such a tensor G gives rise to a bracket $\{\ ,\ \}$ satisfying (i) and (ii); Jacobi's identity (iii) is then equivalent to the condition $[G, G] = 0$, where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket (cf. loc. cit).

Example 1.2. Let (M, ω) be a symplectic manifold. For $f \in A$, let ξ_f be the corresponding Hamiltonian vector field, such that $\xi_f = I(df)$ where $I\colon T^*M \to TM$ is the isomorphism of vector bundles on M such that

$$\langle \alpha, \xi \rangle = \omega(\xi, I(\alpha))$$

for any 1-form α and vector field ξ.

Then $\{f, g\} = \xi_f(g)$ defines a Poisson bracket on M (see for instance [1]).

If $(p_1, \cdots, p_n, q_n, \cdots, q_n)$ are local coordinates on M such that $\omega = \sum_{i=1}^n dp_i \wedge dq_i$, then the Poisson bracket has the form

$$\{f,g\} = \sum_i \left(\frac{\partial g}{\partial p_i} \cdot \frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i} \cdot \frac{\partial f}{\partial p_i} \right).$$

Therefore the tensor G is given by $G = \sum_1^n \partial/\partial q_i \wedge \partial/\partial p_i$. Hence we have $G = -\Lambda^2(I)(\omega)$, where $\Lambda^2(I)\colon \Lambda^2(T^*M) \to \Lambda^2(TM)$ is the isomorphism deduced from I.

1.2. Let M be a Poisson manifold with bracket $\{\ ,\ \}$, and let G, a section of the vector bundle $\Lambda^2(TM)$, be the associated antisymmetric tensor of §1.1, such that $i(G)(df \wedge dg) = \{f,g\}$. For $\omega \in \Omega^k(M)$ a differential form of degree k, $i(G)(\omega)$ is a differential form of degree $(k-2)$.

Koszul [17, p. 265], introduced a differential $\Delta\colon \Omega^k(M) \to \Omega^{k-1}(M)$ defined by $\Delta = i(G) \circ d - d \circ i(G)$. (However, he uses ω instead of G.) For reasons which will become clear in §2, we prefer to use the notation δ instead of Δ. The following lemma relates this to our original definition of δ, inspired by Lie algebra homology and by Hochschild homology.

Lemma 1.2.1. *$\delta = i(G) \circ d - d \circ i(G)$ is given by the following formula:*

$$\text{(F)} \quad \begin{aligned} \delta(f_0 df_1 \wedge \cdots \wedge df_k) = & \sum_{1 \le i \le p} (-1)^{i+1} \{f_0, f_i\}\, df_1 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge df_k \\ & + \sum_{1 \le i < j \le k} (-1)^{i+j} f_0 d\{f_i f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df}_i \\ & \wedge \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_k. \end{aligned}$$

Proof.

$$\begin{aligned} & \delta(f_0 df_1 \wedge \cdots \wedge df_*) \\ & = i(G)[df_0 \wedge df_1 \wedge \cdots \wedge df_k] \\ & \quad - d\left[\sum_{1 \le i < j \le k} (-1)^{i+j+1} f_0 \{f_i, f_j\} df_1 \wedge \cdots \wedge \widehat{df}_i \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_i \right] \\ & = \sum_{0 \le i < j \le k} (-1)^{i+j+1} \{f_i, f_j\} df_0 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_k \\ & \quad + \sum_{1 \le i < j \le k} (-1)^{i+j} \{f_i, f_j\} df_0 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_k \\ & \quad + \sum_{1 \le i < j \le k} (-1)^{i+j} f_0 d\{f_i, f_j\} df_1 \wedge \cdots \wedge \widehat{df}_i \wedge \cdots \wedge \widehat{df}_j \wedge \cdots \wedge df_k, \end{aligned}$$

which gives the expression in the lemma.

An interesting feature of this formula is that it exhibits a relation of the differential δ with the differential δ of the Chevalley-Eilenberg complex $C_*(L,L)$, where $L = C^\infty(M)$ is viewed as a Lie algebra, and L is viewed as an L-module (L acting on itself by derivations). Indeed recall that $C_k(L,L) = L \underset{\mathbf{C}}{\otimes} (\wedge^k L)$ and the differential δ is given by

$$\begin{aligned}\delta(f_0 \otimes (f_1 \wedge \cdots \wedge f_k)) = &\sum_{1\le i\le k} (-1)^{i+1}\{f_0, f_i\} \otimes (f_1 \wedge \cdots \wedge \hat{f^i} \wedge \cdots \wedge f_k)\\ &+ \sum_{1\le i<j\le k} (-1)^{i+j} f_0 \otimes (\{f_i, f_j\} \wedge f_1 \wedge \cdots \wedge \hat{f_i}\\ &\qquad\qquad \wedge \cdots \wedge \hat{f_j} \wedge \cdots \wedge f_k)\end{aligned}$$

(cf. [8], [7]).

This makes the following proposition self-evident.

Proposition 1.2.2. *The linear maps $\pi_k\colon C_k(L,L) \to \Omega^k(M)$ defined by $\pi_k(f_0 \otimes (f_1 \wedge \cdots \wedge f_k)) = f_0 df_1 \wedge \cdots \wedge df_k$ commute with the differentials, i.e., $\pi_k \circ \delta = \delta \circ \pi_{k+1}$.*

Formula (F) could be used as a definition of δ, but a somewhat painful verification that δ is well defined (independently of local expressions of a differential form) would be necessary; alternately, one might define δ using the boundary of the Hochschild complex of a deformed algebra (cf. Example 3.1.2).

To compare the two approaches to δ, let us give two proofs of the following:

Proposition 1.2.3 [17]. $\delta \circ \delta = 0$.

We reproduce first Koszul's proof. A general property of the Schouten-Nijenhuis bracket is that $[[i(a), d], i(b)] = i([a,b])$. Hence

$$[\delta, i(G)] = [[i(G), d], i(G)] = -i([G,G]) = 0,$$

since $[G,G] = 0$. Hence we have

$$\begin{aligned}\delta \circ \delta &= i(G) \circ d \circ i(G) \circ d + d \circ i(G) \circ i(G) \circ d\\ &= -i(G) \circ \delta \circ d - \delta \circ i(G) \circ d = -[i(G), \delta] \circ d = 0.\end{aligned}$$

The second proof is: the question being local on M, one may check $\delta\circ\delta = 0$ on the image of π_k (since π_k is locally surjective). But $\delta \circ \delta = 0$ in the Chevalley-Eilenberg complex $C_*(L,L)$. So the same holds in $\Omega^*(M)$, using Proposition 1.2.2.

Definition 1.2.4. For $(M, \{\ ,\ \})$ a Poisson manifold, the complex

$$\cdots \to \Omega^{k+1}(M) \xrightarrow{\delta} \Omega^k(M) \xrightarrow{\delta} \Omega^{k-1}(M) \to \cdots$$

is called the *canonical complex* of M. This complex will be denoted $C(M)$, with $C_k(M) = \Omega^k(M)$.

The homology of this complex is denoted $H_*^{\mathrm{can}}(M)$, and called the *canonical homology* of $(M, \{\ ,\ \})$.

Sorite 1.2.5. Let $(M, \{\ ,\ \}_M)$, $(N, \{\ ,\ \}_N)$ be two Poisson manifolds. A morphism from the first Poisson manifold to the second is a C^∞-map $\pi\colon M \to N$ such that for any $g, h \in C^\infty(N)$, one has $\{g,h\}_N \circ \pi = \{g\circ\pi, h\circ\pi\}_N$. Such a morphism induces a Lie algebra morphism from $L_N = C^\infty(N)$ to $L_M = C^\infty(M)$. Hence it follows from Proposition 1.2.3 that the pull-back map gives a morphism of chain complexes $\mathscr{C}.(N) \to \mathscr{C}.(M)$.

Sorite 1.2.6. With the notations of 1.2.5, there is a Poisson bracket $\{\ ,\ \}_{M\times N}$ on $M\times N$ such that the projections $M\times N \xrightarrow{p_1} M$ and $M\times N \xrightarrow{p_2} N$ are morphisms of Poisson manifolds. If f_1, g_1 are C^∞-functions on M, and f_2, g_2 are C^∞-functions on N, then

$$\{f_1 \boxtimes f_2, g_1 \boxtimes g_2\}_{M\times N} = \{f_1, g_1\}_M \boxtimes f_2 g_2 + f_1 g_1 \boxtimes \{f_2, g_2\}_N.$$

We use the notation $-\boxtimes-$ for the canonical algebra morphism $C^\infty(M)\otimes C^\infty(N) \to C^\infty(M\times N)$.

There is an obvious Lie algebra morphism

$$L_M \oplus L_N \xrightarrow{(p_1^{-1}, p_2^{-1})} L_{M\times N}$$

and the product map $\boxtimes$ induces a map $L_M\otimes L_N \to L_{M\times N}$ of modules over these Lie algebras. Hence we obtain a morphism of complexes:

$$\begin{tikzcd} C.(L_M, L_M)\otimes C.(L_N, L_N) \arrow[r, "\sim"] \arrow[dr] & C.(L_M\oplus L_N, L_M\otimes L_N) \arrow[d] \\ & C.(L_{M\times N}, L_{M\times N}). \end{tikzcd}$$

This morphism fits inside a commutative diagram

$$\begin{tikzcd} C.(L_M, L_M)\otimes C.(L_N, L_N) \arrow[r, "\pi_M\otimes\pi_N"] \arrow[d] & \mathscr{C}.(M)\otimes\mathscr{C}.(N) \arrow[d, "m"] \\ C.(L_{M\times N}, L_{M\times N}) \arrow[r, "\pi_{M\times N}"] & \mathscr{C}.(M\times N) \end{tikzcd}$$

where $m(\alpha\otimes\beta) = p_1^*\alpha\wedge p_2^*\beta$ for $\alpha\in\mathscr{C}.(M) = \Omega^\cdot(M)$ and $\beta\in\mathscr{C}.(N) = \Omega^\cdot(N)$. Hence we obtain a morphism of complexes $m\colon \mathscr{C}.(M)\otimes\mathscr{C}.(N)\to\mathscr{C}.(M\times N)$. In concrete terms, this means

$$\delta(p_1^*\alpha\wedge p_2^*\beta) = p_1^*(\delta\alpha)\wedge p_2^*\beta + (-1)^{\deg(\alpha)} p_1^*\alpha\wedge p_2^*(\delta\beta).$$

1.3. The canonical double complex. In this section, M is a Poisson manifold with Poisson bracket $\{\ ,\ \}$, and $\delta\colon \Omega^k(M)\to\Omega^{k-1}(M)$ is the differential defined in 1.2. As usual $d\colon \Omega^l(M)\to\Omega^{l+1}(M)$ denotes the exterior differential.

Theorem 1.3.1 [17, p. 265]. *For any $k\geq 0$, $d\delta+\delta d$ induces the zero map on $\Omega^k(M)$.*

This could also easily be shown, using formula (F).

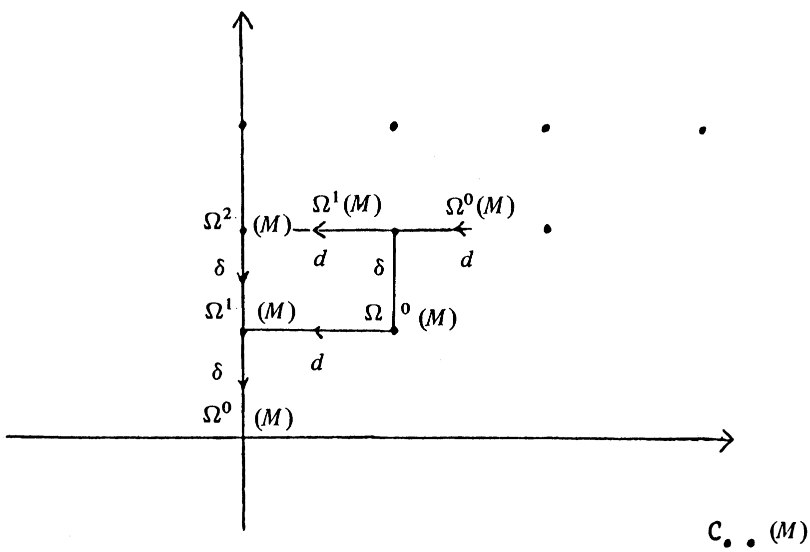
Proposition 1.3.2. *Let α be a closed differential form on M. Then $\delta\alpha$ is an exact differential form.*

Proof. $\delta\alpha = -d[i(G)\alpha]$ from Koszul's definition of δ.

In the case of a differential form of degree 1, we deduce:

Corollary 1.3.3. *Let $\alpha = f_1 dg_1 + \cdots + f_n dg_n$ be a closed 1-form. Then the function $\{f_1, g_1\} + \cdots + \{f_n, g_n\}$ is (identically) zero.*

Construction 1.3.4. As we will see later, the differential δ is analogous (and related) to the Hochschild boundary b, and the differential d is analogous to the operator B of Connes [9]. It is therefore natural in our context to imitate Connes and introduce the double complex $\mathscr{C}_{\cdot\cdot}(M)$ which is defined by $\mathscr{C}_{k,l}(M) = \Omega^{l-k}(M)$ for $k, l \geq 0$, which has d for horizontal differential and δ for vertical differential (both of degree -1).



This Connes-like double complex is concentrated on the first quadrant. As in [11] we introduce the periodic double complex $\mathscr{C}_{\cdot\cdot}^{\mathrm{per}}(M)$, such that $\mathscr{C}_{k,l}^{\mathrm{per}}(M) = \Omega^{l-k}(M)$ for all $k, l \in \mathbf{Z}$.

Problem 1.3.5. (a) Give conditions on a compact Poisson manifold M which ensure that any cohomology class in $H^k(M, \mathbf{C})$ has a representative α such that $d\alpha = \delta\alpha = 0$.

(b) Give conditions on a compact Poisson manifold M which ensure the degeneracy at E_1 of the first spectral sequence for the double complex $\mathscr{C}_{\cdot\cdot}^{\mathrm{per}}(M)$.

It is easily seen that (a) implies (b).

2. Canonical homology of symplectic manifolds

2.1. Let (M,ω) be a symplectic C^∞-manifold of dimension $2m$, and let G be the antisymmetric covariant tensor of order 2 described in 1.2. Let $I\colon T^*M \to TM$ be the isomorphism of vector bundles described in 1.2. We may consider G as an antisymmetric bilinear pairing $G\colon T^*M \times T^*M \to C^\infty(M)$. For any $k \geq 0$, we denote by $\Lambda^k(G)$ the associated pairing $\Lambda^k(G)\colon \Lambda^k(T^*M) \times \Lambda^k(T^*M) \to C^\infty(M)$, which is $(-1)^k$-symmetric. As a volume form on M, we choose the $2m$-form $v_M = \omega^m/m!$.

Imitating the star isomorphism for Riemannian manifolds, we define the $*$ operation $*\colon \Omega^k(M) \to \Omega^{2m-k}(M)$ by the condition $\beta\wedge(*\alpha) = \Lambda^k G(\beta,\alpha)\cdot v_M$ for all $\alpha, \beta \in \Omega^k(M)$.

Let us give some properties of this operator.

Lemma 2.1.1. *Let (M_1,ω_1), (M_2,ω_2) be symplectic manifolds of respective dimensions $2m_1$ and $2m_2$. For α_1 a k_1-form on M_1 and α_2 a k_2-form on M_2, $\alpha_1 \wedge \alpha_2$ is a (k_1+k_2)-form on $M_1 \times M_2$ such that*

$$*(\alpha_1 \wedge \alpha_2) = (-1)^{k_1k_2}(*_1\alpha_1) \wedge (*_2\alpha_2) = (*_2\alpha_2) \wedge (*_1\alpha_1).$$

Proof. The symplectic form on $M_1 \times M_2$ is $\omega_1+\omega_2$, hence $I\colon T^*M \to TM$ is the direct sum of $I_1\colon T^*M_1 \to TM_1$ and of $I_2\colon T^*M_2 \to TM_2$. Hence if G is the covariant antisymmetric 2-tensor on M, we have $G = G_1 + G_2$. Now if β_i $(i = 1,2)$ is a differential form of degree k_i on M_i, we have (for $*$ the $*$ operator on M, $*_i$ the one on M_i)

$$\begin{aligned}\beta_1 \wedge \beta_2 \wedge (\alpha_1 \wedge \alpha_2) &= \Lambda^{k_1+k_2}G(\beta_1 \wedge \beta_2, \alpha_1 \wedge \alpha_2)v_M \\ &= \Lambda^{k_1}G_1(\beta_1,\alpha_1)\Lambda^{k_2}G_2(\beta_2,\alpha_2)v_M.\end{aligned}$$

On the other hand $\beta_i \wedge *_i\alpha_i = \Lambda^{k_i}G_i(\beta_i,\alpha_i)v_{M_i}$. Since $v_M = v_{M_1} \wedge v_{M_2}$, we have

$$\begin{aligned}\beta_1 \wedge \beta_2 \wedge *(\alpha_1 \wedge \alpha_2) &= \beta_1 \wedge (*_1\alpha_1) \wedge \beta_2 \wedge (*_2\alpha_2) \\ &= (-1)^{k_1k_2}\beta_1 \wedge \beta_2 \wedge (*_1\alpha_1) \wedge (*_2\alpha_2),\end{aligned}$$

which proves the lemma.

Lemma 2.1.2. *For $\beta \in \Omega^k(M)$, we have $*(*\beta) = \beta$.*

Proof. If this is known for symplectic manifolds M_1 and M_2, one proves it for $M = M_1 \times M_2$ since for $\alpha_i \in \Omega^{k_i}(M_i)$ $(i = 1,2)$, one obtains

$$**(\alpha_1 \wedge \alpha_2) = *(*_2\alpha_2 \wedge *_1\alpha_1) = (*_1 *_1 \alpha_1) \wedge (*_2 *_2 \alpha_2) = \alpha_1 \wedge \alpha_2,$$

and since differential forms of this type generate $\Omega^k(M)$, we obtain the lemma for M.

Since the statement is local on M, we may assume that $\dim(M) = 2$ and $\omega = dp\wedge dq$ for a coordinate system (p,q). We then have the explicit formulas

(i) $*(f) = f dp \wedge dq$ for $f \in C^\infty(M)$,

(ii) $*\omega = -\omega$ for any 1-form ω on M,

(iii) $*(f dp \wedge dq) = f$ for $f \in C^\infty(M)$,

which prove $*(*\beta) = \beta$ in this case.

Lemma 2.1.3. *For α, β in $\Omega^k(M)$, we have*

$$\beta \wedge (*\alpha) = (-1)^k \alpha \wedge (*\beta).$$

Proof. This is immediate from $\Lambda^k G(\beta, \alpha) = (-1)^k \Lambda^k G(\alpha, \beta)$.

Remark. The formulas concerning the symplectic $*$ operator involve simpler signs than those for the $*$ operator in Riemannian geometry.

2.2. (M, ω) is a symplectic manifold. The operator δ is defined as in §1.2, the $*$ operator as in §2.1.

Theorem 2.2.1. *The relation $\delta = (-1)^{k+1} * d*$ holds on $\Omega^k(M)$ for any $k \geq 0$.*

Proof. First let us treat the case $\dim(M) = 2$. We may then assume $\omega = dp \wedge dq$ for some coordinate system (p, q). We then have:

(i) $\delta f = *d * f = 0$ for $f \in C^\infty(M)$;

(ii) $\delta(f dp) = \{f, p\} = \frac{\partial f}{\partial q} = *(\frac{\partial f}{\partial q} dp \wedge dq) = - * d(f dp) = *d * (f dp)$;

(iii) $\delta(f dq) = \{f, q\} = -\frac{\partial f}{\partial p} = - * (\frac{\partial f}{\partial q} dp \wedge dq) = - * d(f dg) = *d * (f dg)$;

(iv) $\delta(f dp \wedge dq) = \{f, p\} dq - \{f, q\} dp = \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp = df = - * df = - * d * (f dp \wedge dq)$.

Hence the theorem in this case.

The rest of the proof is an induction on $\dim(M)$. Replacing M by a suitable open set, we may assume M is the product of two symplectic manifolds (M_1, ω_1), (M_2, ω_2) of positive dimension. Assume the theorem known for M_1 and M_2. Then using Lemmas 2.1.1 and 2.1.2, and denoting by the symbol $*$ the $*$ operator for M_1, M_2 and $M = M_1 \times M_2$, for $\alpha_i \in \Omega^{k_i}(M_i)$ $(i = 1, 2)$ we have

$$\begin{aligned} *d * (\alpha_1 \wedge \alpha_2) &= *d(*\alpha_2 \wedge *\alpha_1) = *(d * \alpha_2 \wedge *\alpha_1) + (-1)^{k_2}(*\alpha_2 \wedge d * \alpha_1) \\ &= \alpha_1 \wedge (*d * \alpha_2) + (-1)^{k_2}(*d * \alpha_1 \wedge \alpha_2) \\ &= (-1)^{k_2+1} \alpha_1 \wedge \delta \alpha_2 + (-1)^{k_1+k_2+1}(\delta \alpha_1 \wedge \alpha_2) \\ &= (-1)^{k_1+k_2+1} \cdot [\delta \alpha_1 \wedge \alpha_2 + (-1)^{k_1} \alpha_1 \wedge \delta \alpha_2] \\ &= (-1)^{k_1+k_2+1} \delta(\alpha_1 \wedge \alpha_2), \end{aligned}$$

where we use 1.2.6 in the last step.

This finishes the induction, and the proof of the theorem.

Corollary 2.2.2. *The operator $*$ establishes an isomorphism of the canonical homology group $H_i^{\mathrm{can}}(M)$ with the de Rham cohomology group $H^{2m-i}(M)$ for M a symplectic manifold of dimension $2m$.*

This implies a sort of Poincaré lemma for symplectic manifolds. To state it, let us denote by $\underline{\mathscr{C}}_{\cdot,M}$ the complex of sheaves on M such that $P(U,\underline{\mathscr{C}}_{\cdot,M}) = \mathscr{C}.(U)$ for any open set U of M. As usual, $\mathbf{R}_M$ is the constant sheaf on M.

Corollary 2.2.3. *The morphism of complexes of sheaves* $\mathbf{R}_M[-2_m] \to \underline{\mathscr{C}}.$, *which sends* 1 *to the section* v_M *of* $\underline{\mathscr{C}}_{2m} = \Omega^{2m}$, *is a quasi-isomorphism.*

We denote by $C_c^\infty(M)$ the vector space of C^∞-functions on M with compact support, and by $l\colon C_c^\infty(M) \to \mathbf{R}$ the linear form $l(f) = \int_M f.v_M$.

The following result is due to Calabi [6] and Lichnérowicz [18].

Corollary 2.2.4. *The kernel of* l *is the vector space generated by Poisson brackets* $\{f,g\}$, *where* $f,g \in C_c^\infty(M)$.

Proof. $*\colon C_c^\infty(M) \to \Omega_c^{2m}(M)$ induces an isomorphism from $C_c^\infty(M)/\delta\Omega_c^1(M)$ to $\Omega_c^{2m}(M)/d\Omega_c^{2m-1}(M)$. This latter vector space is one-dimensional. We deduce that $\ker(l)$ is equal to $\Omega_c^1(M)$. Using a partition of unity, we see that any $\alpha \in \Omega_c^1(M)$ has a finite expression $\alpha = \sum_1^N f_j\, dg_j$, where f_j and g_j have compact support. q.e.d.

Recall the Lie algebra $L = C^\infty(M)$, endowed with the Poisson bracket. Then $C_c^\infty(M)$ is an L-module. Let $B\colon C_c^\infty(M) \times C_c^\infty(M) \to \mathbf{R}$ be the bilinear form defined by $B(f,g) = l(fg) = \int_M fgv_M$.

Proposition 2.2.5. *The bilinear form* B *is* L-*invariant, i.e.*,

$$B(\{f,h\},g) + B(f,\{g,h\}) = 0,$$

for $f,g \in C_c^\infty(M)$, $h \in L = C^\infty(M)$.

Proof. $B(\{f,h\},g) + B(f,\{g,h\}) = l(\{f,h\}g) + l(f\{g,h\}) = l(\{fg,h\}) = 0$ using 2.2.4.

Remark 2.2.6. This proposition means that the L-module $C_C^\infty(M)$ is self-dual in a certain sense. In particular, there is a duality between the Lie algebra homology $H_*(L,L)$ and the Lie algebra cohomology $H^*(L,L)$ considered in [18] (at least for M compact). One might compare our complex $\to \Omega^k(M) \xrightarrow{\delta} \Omega^{k-1}(M) \to$ with the complex of [18], $\to \Lambda^l TM \xrightarrow{\partial} \Lambda^{l+1}TM$, using the natural isomorphism $\Omega^k M \to \Lambda^{2m-k}TM$. However, it seems more natural to compare Lichnérowicz's complex directly with the de Rham complex. In loc. cit. the cohomology of this complex is shown to be the same as the differentiable Lie algebra cohomology $H_{\text{diff}}(L,L)$. It follows that this differentiable Lie algebra cohomology is none other (for symplectic manifolds) than the de Rham cohomology. For compact symplectic manifolds, we conjecture that Problem 1.3.5(a) always has a positive answer, i.e.,

Conjecture 2.2.7. If M is a symplectic manifold which is compact, any cohomology class in $H^*(M,\mathbf{C})$ has a representative α such that $d\alpha = \delta\alpha = 0$.

Such a form α may be called (symplectically) *harmonic*, by analogy with the Riemannian case. Notice however that Theorem 1.3.1 says that the

"Laplacian" $d\delta + \delta d$ is identically zero. We present below some fragmentary evidence for the conjecture, which might well be true for a large class of noncompact symplectic manifolds (cf. Corollary 2.2.13).

Proposition 2.2.8. *If α is a closed 1-form on M, then $\delta\alpha = 0$.*

(This is just a restatement of Corollary 1.3.2.)

Proposition 2.2.9. *For any j, $\delta(\omega^j) = 0$.*

Proposition 2.2.10. *Conjecture 2.2.7 is true if $M = \mathbf{R}^{2n}/\Gamma$, where $\Gamma \subset \mathbf{R}^{2n}$ is a discrete subgroup, and $\mathbf{R}^{2n}$ is endowed with the standard symplectic structure.*

2.2.9 and 2.2.10 easily follow from the following lemma, where $(q_1, \cdots, q_n; p_1, \cdots, p_n)$ are canonical coordinates such that $\omega = \sum_1^n dp_i \wedge dq_i$.

Lemma 2.2.11. *For $\alpha = dq_{i_1} \wedge \cdots \wedge dq_{ik} \wedge dp_{ji} \wedge \cdots \wedge dp_{jl}$, $\delta\alpha = 0$.*

Proof. Immediate from formula (F).

More evidence for the conjecture is given in the rest of this section. Let us just point out here the elementary

Proposition 2.2.12. *Let M be a Poisson manifold with Poisson bracket $\{\ ,\ \}$. Let N be a manifold, $\pi\colon M \to N$ be a C^∞ map such that:*

(i) *for any $f, g \in C^\infty(N)$, we have $\{f \circ \pi, g \circ \pi\} = 0$;*
(ii) *$\pi^*\colon H^*(N, \mathbf{C}) \to H^*(M, \mathbf{C})$ is surjective.*

Then Conjecture 2.2.7 is true for M.

Corollary 2.2.13. *For any C^∞-manifold N, the symplectic manifold $M = T^*N$ satisfies Conjecture 2.2.7.*

Proof. Apply 2.2.12 to the projection map $T^*N \to N$.

2.3. Degeneration of a spectral sequence. We prove here the degeneracy of the first spectral sequence for the double complex $\mathscr{C}_{\cdot\cdot}(M)$ in case M is a symplectic manifold. This answers Problem 1.3.5(b) in that case.

First we need to explain the meaning of *degeneration at E^1* of the first spectral sequence for a complex $\mathscr{C}_{**}$ which satisfies $\mathscr{C}_{p,q} = 0$ for $p > q$. We denote by d the horizontal differential (of degree -1) and by δ the vertical differential. We remark that for (p, q) fixed, there exists an integer r_0 such that $E^{r+1}_{p,q}$ injects into $E^r_{p,q}$ for $r \geq r_0$ (this follows since $E^r_{p+r,q-r+1}$ is 0 for $r \geq r_0$) and also $E^\infty_{p,q}$ injects into $E^r_{p,q}$. We say that the first spectral sequence degenerates at E^1 if

(a) $d_r = 0$ for $r \geq 1$,
(b) for each (p, q), the injection $E^\infty_{p,q} \hookrightarrow E^{r_0}_{p,q} \simeq E^1_{p,q}$ is an isomorphism.

Theorem 2.3.1. *For M a compact symplectic manifold M, the first spectral sequence of the double complex $\mathscr{C}^{\mathrm{per}}_{\cdot\cdot}(M)$ degenerates at E^1.*

Proof. We simply compute $E^1_{p,q}$ and $E^\infty_{p,q}$, and observe that they are isomorphic. First we have $E^1_{p-q} = H_{q-p}(\mathscr{C}_\cdot(M)) = H^{2m-q+p}(M)$. Next, we use Corollary 2.2.3 to construct, for any $k \in \mathbf{Z}$, a morphism of complexes $\varphi_k : \underline{\mathbf{R}}_M[-2m-2k] \to \mathrm{Tot}(\underline{\mathscr{C}}^{\mathrm{per}})$ where "Tot" means "total complex of a double complex," which send $1 \in \underline{\mathbf{R}}_M$ to $v_M = \omega^m/m! \in \mathscr{C}^{\mathrm{per}}_{k,2m+k}(M) = \Omega^{2m}(M)$. (The point is that $\delta(v_M) = 0$, by Proposition 2.2.6.) Here $\mathscr{C}_{..}$ is the double complex of sheaves on M, with $\underline{\mathscr{C}}^{\mathrm{per}}_{p,q} = \Omega^{q-p}$. It is clear that $\bigoplus \varphi_k$ induces a quasi-isomorphism of complexes of sheaves from $\bigoplus_{k\in\mathbf{Z}} \mathbf{R}_M[-2m-2k]$ to $\mathrm{Tot}(\underline{\mathscr{C}}^{\mathrm{per}}_{..})$, i.e., it induces an isomorphism on cohomology sheaves since the sheaves $\underline{\mathscr{C}}^{\mathrm{per}}_{p,q} = \Omega^{q-p}$ are fine, the hypercohomology of the complex of sheaves $\mathrm{Tot}(\underline{\mathscr{C}}^{\mathrm{per}}_{..})$ is equal to the hyperhomology of $\mathscr{C}^{\mathrm{per}}_{..}(M)$. On the other hand, the hypercohomology of $\mathbf{R}_M[-2m-2k]$ is equal to $H^i(M, \mathbf{C})$ in bi-degree $(k, 2m+k-i)$.

Remark. For $A^{\cdot}$ a complex of sheaves and $m \in \mathbf{Z}$, we have denoted by $A^{\cdot}[m]$ the same complex, shifted m steps to the left.

Observe the following:

Lemma 2.3.2. *If X is a compact space and $\mathscr{F} = \varinjlim_\lambda \mathscr{F}\lambda$ is a direct limit of sheaves of abelian groups on X, then $H^i(X, \mathscr{F}) = \varinjlim_\lambda H^i(X, \mathscr{F}_\lambda)$.*

For $i = 0$, this is proven in [10, Théorème 3.10.1]. For all i, one computes $H^i(X, \mathscr{F}_\lambda)$ using Godement's canonical resolution of $\mathscr{F}_\lambda$, and one observes that a direct limit of flasque sheaves on X is soft (see loc. cit.).

This lemma implies that the cohomology of a direct sum of sheaves is the direct sum of their cohomologies. The same holds for hypercohomology of a direct sum of bounded complexes of sheaves (using the spectral sequence for hypercohomology).

In our case, we deduce that the hypercohomology of

$$\bigotimes_{k\in\mathbf{Z}} \mathbf{R}_M[-2m-2k]$$

is equal in bi-degree (p,q) to $H^{2m-q+p}(M)$. So we conclude $E^\infty_{p,q} = H^{2m-q+p}(M) = E^1_{p,q}$.

2.4. We consider here a Kähler manifold M. So M is a complex manifold endowed with an hermitian metric H. We denote by ω the imaginary part of H, which is a real 2-form on M. Since M is Kähler, $d\omega = 0$, so (M, ω) is a symplectic manifold. We denote by g the real part of H, which is a Riemannian metric on M. Classically, g determines a $*$ operator $*: \Omega^k(M) \to \Omega^{2n-k}(M)$ determined by the equality (cf. [21])

$$\beta \wedge (*g\alpha) = \Lambda^k g^{-1}(\beta, \alpha) v_g,$$

where v_g is the volume form associated to g. In fact $v_g = v_M$ with the notations of 2.1. On the other hand, ω determines an operator $*: \Omega^k(M) \to \Omega^{2n-k}(M)$ as in 2.1.

Theorem 2.4.1. *For $\alpha \in \Omega^{p,q}(M)$, we have*

$$*_\omega(\alpha) = (\sqrt{-1})^{p-q} *_g (\alpha).$$

Proof. We take local complex coordinates $(z_1, \cdots, z_m)$ such that

$$\omega = \frac{\sqrt{-1}}{2} \cdot \sum_1^m dz_i \wedge d\overline{z}_i,$$

(modulo terms which vanish to order 2 for $z_1 = \cdots = z_m = 0$). Putting $z_i = x_i + \sqrt{-1}y_i$, we have $\omega = \sum_1^m dx_i \wedge dy_i$, hence $(\partial/\partial x_i, \partial/\partial y_j)$ form an orthonormal basis with respect to g. Hence the dual form G of ω is given by $G = \sum_1^m \partial/\partial y_i \wedge \partial/\partial x_i$, and the dual form g^{-1} of g is given by

$$g^{-1} = \sum_1^m \left(\frac{\partial}{\partial x_i}\right)^2 + \sum_1^m \left(\frac{\partial}{\partial y_i}\right)^2 .$$

Hence $G(dz_i, d\overline{z}_i) = 2 \cdot \sqrt{-1}$, $g^{-1}(dz_i, d\overline{z}_i) = 2$ and $G(dz_i, d\overline{z}_j) = g^{-1}(dz_i, d\overline{z}_j) = 0$ for $i \neq j$. Hence $G(\alpha, \overline{\beta}) = \sqrt{-1}g^{-1}(\alpha, \overline{\beta})$ if α, β are differential forms of type $(1,0)$ and $G(\alpha, \overline{\beta}) = -\sqrt{-1}g^{-1}(\alpha, \overline{\beta})$ if α, β are of type $(0,1)$. If α, β are 1-forms of different types, both $G(\alpha, \overline{\beta})$ and $g^{-1}(\alpha, \overline{\beta})$ are 0.

It follows that if α is a differential form of type (p,q) and γ a differential form of type (q,p), then for $k = p + q$

$$\Lambda^k G(\alpha, \gamma) = (\sqrt{-1})^{p-q} \Lambda^k g^{-1}(\alpha, \gamma),$$

from which the theorem follows.

Corollary 2.4.2. *Let α be a harmonic form on the Kähler manifold M, if pure type (p,q). Then $\delta\alpha = 0$, where δ is the operator of* §1.

Proof. If α is harmonic, then $*_g d *_g (\alpha) = 0$. It follows from Theorem 2.4.1, and the fact that α is of pure type, that $\delta(\alpha) = *_\omega d *_\omega (\alpha)$ is also 0.

Corollary 2.4.3. *If M is a compact Kähler manifold, then any cohomology class of M has a representative α such that $d\alpha = \delta\alpha = 0$.*

Proof. The Hodge theorem for compact Kähler manifolds says that the cohomology of M is generated by the classes of harmonic forms of some pure type (see [22]).

It remains to apply Corollary 2.4.2.

Remark 2.4.4. 2.4.1 and 2.4.2 still hold for an "almost Kähler" manifold M. I thank the referee for this observation.

3. Application to the Hochschild homology of noncommutative algebras

3.1. We work here in the following purely algebraic context. We are given some noetherian ring k of characteristic 0, and some k-algebra A which is equipped with a filtration $(A_n)_{n\in\mathbf{Z}}$ which is a ring filtration (we have $A_n \subset A_{n+1}$, $A = \bigcup_n A_n$, $\bigcap_n A_n = 0$, $A_n \cdot A_m \subset A_{n+m}$). We assume that the graded ring $\mathrm{Gr}(A) = \bigoplus_{n\in\mathbf{Z}} A_n/A_{n-1}$ is a *commutative* k-algebra, which is *smooth over* k. We denote by $\Omega^m_{\mathrm{Gr}(A)/k}$ the relative differential forms of Grothendieck [12, Chapter II, §8] $\Omega^*_{\mathrm{Gr}(A)/k}$ is the quotient of the exterior algebra generated over A by symbols da ($a \in A$), by relations $d(x,y) = x\,dy + y\,dx$, and $d\lambda = 0$ for $\lambda \in k$.

Then the Hochschild complex $C.(A) = C.(A,A)$ has an increasing filtration F_k such that

$$F_k(C_n(A)) = F_k(A^{\otimes(n+1)}) = \sum_{k_1+k_2+\cdots+k_{n+1}\leq k} [A_{k_1} \otimes A_{k_2} \otimes \cdots \otimes A_{k_{n+1}}].$$

Hence we get a spectral sequence with $E^1_{p,q} = H_{p+q}(\mathrm{Gr}(A),\mathrm{Gr}(A))_p$ the homogeneous part of degree p of the Hochschild homology group $H_{p+q}(\mathrm{Gr}(A),\mathrm{Gr}(A))$, and $E^\infty_n = H_n(A,A)$.

Now a slight generalization of a theorem of Hochschild, Kostant and Rosenberg [16] (they treat the case where k is a field) asserts that the natural map $\beta\colon H_m(\mathrm{Gr}(A),\mathrm{Gr}(A)) \to \Omega^m_{\mathrm{Gr}(A)/k}$ given by

$$\beta(a_0 \otimes \cdots \otimes a_m) = \frac{1}{m!} a_0 da_1 \wedge da_2 \wedge \cdots \wedge da_m$$

is an isomorphism, with inverse given by γ:

$$\gamma(a_0 da_1 \wedge \cdots \wedge da_m) = \text{class of} \sum_{\sigma\in\mathfrak{S}_m} \varepsilon(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(m)},$$

where $\mathfrak{S}_m$ is the group of permutations of $\{1,2,\cdots,m\}$ (see also [19]).

Now on $\mathrm{Gr}(A)$ we have a k-linear bracket operation; for $f \in \mathrm{Gr}(A)_j$, $g \in \mathrm{Gr}(A)_l$ choose $P \in A_j$ which maps to f under the canonical map $A_j \to \mathrm{Gr}(A)_j$, and $Q \in A_l$ which maps to g; then $PQ - QP$ belongs to A_{j+l-1}, and $\{f,g\}$ is the class of $PQ - QP$ in $\mathrm{Gr}(A)_{j+l-1}$.

This satisfies all properties of Poisson brackets listed in §1.1. Since here we are considering an algebraic scheme over k, rather than a C^∞-manifold, (i) simply means that for $f \in \mathrm{Gr}(A)$ fixed, the map $g \mapsto \{f,g\}$ is a k-linear derivation of $\mathrm{Gr}(A)$. In short, $\mathrm{Gr}(A)$ is endowed with a Poisson bracket, hence we have the complex $\cdots \to \Omega^i_{\mathrm{Gr}(A)/k} \xrightarrow{\delta} \Omega^{i-1}_{\mathrm{Gr}(A)/k} \to \cdots$ introduced in §1.2. All the results are constructions in §1 and §2, up to Theorem 2.2.1, are still valid in this new context.

Theorem 3.1.1. *For any $n \geq 0$, we have a commutative diagram*

$$\begin{tikzcd} E_n^1 = H_n(\mathrm{Gr}(A), \mathrm{Gr}(A)) \arrow[r, "\beta", "\sim"'] \arrow[d, "d_1"] & \Omega^n_{\mathrm{Gr}(A)/k} \arrow[d, "\delta"] \\ E_{n-1}^1 = H_{n-1}(\mathrm{Gr}(A), \mathrm{Gr}(A)) \arrow[r, "\beta", "\sim"'] & \Omega^{n-1}_{\mathrm{Gr}(A)/k} \end{tikzcd}$$

where d_1 is the differential in the spectral sequence.

Proof. It is enough to prove that $\delta = \beta \circ d_1 \circ \gamma$. Now $\Omega^n_{\mathrm{Gr}(A)/k}$ is generated by elements of the form $x_0 dx_1 \wedge \cdots \wedge dx_n$, where $x_i \in \mathrm{Gr}(A)$ is homogeneous, say of degree m_i. Let $m = m_0 + m_1 + \cdots + m_n$. Choose a_i in A_m, mapping to x_i under $A_{m_i} \to \mathrm{Gr}(A)_{m_i}$. First we have

$$\gamma(x_0 dx_1 \wedge \cdots \wedge dx_n) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x_0 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

This lifts to the chain $\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$ of $F_m(A^{\otimes(n+1)})$. Its Hochschild boundary is the sum of three terms (I), (II), (III), with

$$(\mathrm{I}) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) a_0 a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(n)},$$

$$(\mathrm{II}) = \sum_{\sigma} \sum_{1 \leq i \leq n-1} \varepsilon(\sigma)(-1)^i a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(i)} a_{\sigma(i+1)} \otimes \cdots \otimes a_{\sigma(n)},$$

$$(\mathrm{III}) = \sum_{\sigma} \varepsilon(\sigma)(-1)^n a_{\sigma(n)} a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n-1)}.$$

This Hochschild boundary lives in $F_{m-1}(A^{\otimes n})$ and we want to find its image in $F_{m-1}(A^{\otimes n})/F_{m-2}(A^{\otimes n}) = [\mathrm{Gr}(A)^{\otimes n}]_{m-1}$. To this purpose, we first rewrite (I) as follows (transforming $\sigma \in \mathfrak{S}_n$ to $\sigma\tau$ where τ is a cyclic permutation):

$$(\mathrm{I}) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{n+1} \varepsilon(\sigma) a_0 a_{\sigma(n)} \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n-1)}.$$

Since

$$(\mathrm{I}) + (\mathrm{III}) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)(-1)^{n+1} [a_0, a_{\sigma(n)}] \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n-1)},$$

it belongs to $F_{m-1}(A^{\otimes n})$; its image in $[\mathrm{Gr}(A)^{\otimes n}]_{m-1}$ is equal to

$$\sum_{\sigma} \varepsilon(\sigma)(-1)^{n+1} \{x_0, x_{\sigma(n)}\} \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n-1)}.$$

Now using the transformation $\sigma \mapsto \sigma s_i$ of $\mathfrak{S}_n$ to itself, where s_i is the transposition which exchanges i and $i+1$, we get

$$(\mathrm{II}) = \sum_{\sigma} \frac{1}{2} \sum_{1 \leq i \leq n-1} \varepsilon(\sigma)(-1)^i a_i \otimes \cdots \otimes [a_{\sigma(i)}, a_{\sigma(i+1)}] \otimes \cdots \otimes a_{\sigma(n)}.$$

So (II) belongs to $F_{m-1}(A^{\otimes n})$, and its image in $[\mathrm{Gr}(A)^{\otimes n}]_{m-1}$ is

$$\sum_{\sigma}\frac{1}{2}\sum_{1\leq i\leq n-1}\varepsilon(\sigma)(-1)^{i}x_0\otimes\cdots\otimes\{x_{\sigma(i)},x_{\sigma(i+1)}\}\otimes\cdots\otimes x_{\sigma(n)}.$$

So we have computed $d_1\circ\gamma(x_0dx_1\wedge\cdots\wedge dx_n)$. It remains to apply β to this. Now, for the sum

$$\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)(-1)^{n+1}\{x_0,x_{\sigma(n)}\}\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n-1)}$$

notice that all σ with $\sigma(n)=i$ fixed give the same value for $\beta(\{x_0,x_{\sigma(n)}\}\otimes x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n-1)})$. This value is

$$\frac{1}{(n-1)!}\cdot(-1)^{n-i}\{x_0,x_i\}\,dx_1\wedge\cdots\wedge d\hat{x}_i\wedge\cdots\wedge dx_n;$$

since there are $(n-1)!$ such permutations, we find for each i the expression

$$(\mathrm{A})=(-1)^{i+1}\{x_0,x_i\}\,dx_1\wedge\cdots\wedge d\hat{x}_i\wedge\cdots\wedge dx_n.$$

We have to sum these for $1\leq i\leq n$.

Next, all pairs (σ,i) such that the set $\{\sigma(i),\sigma(i+1)\}$ is equal to some fixed set $\{j,k\}$ (say $j<k$) give the same value for $\beta(\varepsilon(\sigma)(-1)^{i}x_0\otimes\cdots\otimes\{x_{\sigma(i)},x_{\sigma(i+1)}\}\otimes\cdots\otimes x_{\sigma(n)})$, namely,

$$\frac{(-1)^{j+k}}{(n-1)!}x_0d\{x_j,x_k\}\wedge dx_1\wedge\cdots\wedge d\hat{x}_j\wedge\cdots\wedge d\hat{x}_k\wedge\cdots\wedge dx_n.$$

There are $2(n-1)!$ such pairs (σ,i). Therefore the second term in $\beta d_1\gamma(x_0dx_1\wedge\cdots\wedge dx_n)$ is equal to

$$(\mathrm{B})=\sum_{1\leq j<k\leq n}(-1)^{j+k}x_0d\{x_j,x_k\}\wedge dx_1\wedge\cdots\wedge d\hat{x}_j\wedge\cdots\wedge d\hat{x}_k\wedge\cdots\wedge dx_n.$$

Now (A) + (B) coincides with the formula for $\delta(x_0dx_1\wedge\cdots\wedge dx_n)$ given in §1.2. This proves the theorem.

Example 3.1.2. Let L be a commutative $\mathbf{Q}$-algebra and let B be a smooth L-algebra endowed with a Poisson bracket $\{\ ,\ \}$. Then one may construct an algebra A over $L[\varepsilon]/(\varepsilon^2)=k$ as follows: A is equal to $B\otimes_L k$ as a k-module and the multiplication in A is given by $(x_0+\varepsilon x_1)\cdot(y_0+\varepsilon y_1)=x_0y_0+\varepsilon(\{x_0,y_0\}+x_1y_0+x_0y_1)$ (for $x_i,y_i\in B$). We filter A by $F_0(A)=A$, $F_{-1}(A)=B\subset A$, $F_{-2}(A)=0$. Then A is a filtered k-algebra, and $\mathrm{Gr}(A)$ is isomorphic, as a k-algebra, to $B\otimes_L k$. In our spectral sequence, we have therefore

$$E_n^1=\Omega^n_{B\underset{L}{\otimes}k/k}=(\Omega^n_{B/L})\underset{L}{\otimes}k=\Omega^n_{B/L}\oplus\Omega^n_{B/L}\cdot\varepsilon.$$

According to Theorem 3.1.1, the differential d_1 induces the map $\varepsilon\cdot\delta\colon \Omega^n_{B/L} \to \Omega^{n-1}_{B/L}\cdot\varepsilon$.

We could have used this construction in §1.2 to define δ. However, it is not clear from this approach that $\delta\circ\delta = 0$, unless of course there is an algebra $\tilde{A}$ over $L[\varepsilon]/(\varepsilon^3)$ such that $\tilde{A}/(\varepsilon^2)$ is isomorphic to A.

Deformations of smooth commutative algebras (often called "star-products") have been intensively studied by mathematicians and mathematical physicists in the last few years (see [2] for example).

3.2. We keep the notations of §3.1. We also assume that the unit 1 in A belongs to A_0. Recall the double complex $C_{\cdot\cdot}(A)$ of Connes [9], with $C_{i,j}(A) = A^{\otimes(j-i+1)}$ for k, $l \geq 0$, with horizontal differential B and vertical-differential b.

We filter this double complex in a somewhat strange way, which will be convenient for our purposes. We let $F_k(C_{\cdot\cdot}(A))$ be the sub-double complex such that $F_k(C_{i,j}(A))$ is the subspace of elements of $C_{i,j}(A) = A^{\otimes(j-i+1)}$ of filtration $\leq k-i$, i.e. $F_k(C_{i,j}(A)) = F_{k-i}(A^{\otimes(j-i+1)})$ in the notations of §3.1.

The quotient double complex F_k/F_{k-1} has (i,j)-component equal to $[\mathrm{Gr}(A)^{\otimes(j-i+1)}]_{k-i}$. Because B is homogeneous of degree 0, the horizontal differential is 0; the vertical differential is b. Therefore this double complex is quasi-isomorphic to the complex which has Ω_A^{j-i} in bi-degree (i,j) with $i,j \geq 0$, and which has zero differentials. This gives the E^1 term of the spectral sequence. Then the complex (E^1, d_1) is the total complex associated to the double complex $\mathscr{C}_{\cdot\cdot}(M)$ of §1.3 (where M is the *Poisson scheme* $\mathrm{Spec}(\mathrm{Gr}(A))$).

Therefore the E^2 term of the spectral sequence is the hyperhomology of this Poisson double complex (with horizontal differential d, vertical differential δ).

Similar considerations apply to the periodic Connes complex of A, filtered in the same manner. The E^2 term of the spectral sequence is then the hyperhomology of $\mathscr{C}^{\mathrm{per}}_{\cdot\cdot}(M)$.

3.3. In this paragraph, M is either a C^∞-manifold, or a Stein complex manifold, or an affine algebraic variety over a field k of characteristic 0. We let $D(M)$ be the algebra of C^∞ (resp. complex-analytic, resp. algebraic) globally defined differential operators. We are interested in the Hochschild homology of $D(M)$. Of course, in the first two cases, we endow $D(M)$ with its natural structure of locally convex topological algebra, for which $D(M)$ is complete (if M is connected, $D(M) = \varinjlim_m D_m(M)$, where $D_m(M)$ is the subspace of differential operators of order $\leq m$, so $D(M)$ is given the inductive limit topology). Then the Hochschild homology is defined in [9] using the complex $C(D(M)) = D(M)^{\hat{\otimes}(n+1)}$ (where $\hat{\otimes}$ denotes Grothendieck's completed projective tensor product, as in [9]).

Since $D(M)$ is filtered by the $D_m(M)$, we may apply §3.1 to get a spectral sequence with $E_n^1 = H_n(\operatorname{gr} D(M), \operatorname{gr} D(M))$. This spectral sequence converges to $E_n^\infty = H_n(D(M), D(M))$. Let us first examine the algebraic case, which is easier. We then have $H_n(\operatorname{gr} D(M), \operatorname{gr} D(M)) = H_n(\mathscr{O}(T^*M), \mathscr{O}(T^*M)) = \Omega^n_{T^*(M)/k}$ where $T^*(M)$ is the cotangent bundle of M, and $\mathscr{O}(T^*M)$ the algebra of regular functions on $T^*(M)$. Using Theorem 3.1.1, the E^2 term is the cohomology of the complex

$$\cdots \longrightarrow \Omega^n_{T^*(M)} \xrightarrow{\delta} \Omega^{n-1}_{T^*(M)} \longrightarrow \cdots .$$

If $\dim(M) = m$, Corollary 2.2.2 tells us that E_n^2 is equal to $H_{\mathrm{DR}}^{2m-n}(T^*M)$ (de Rham cohomology). This is equal to $H_{\mathrm{DR}}^{2m-n}(M)$; if $k = \mathbf{C}$, this is isomorphic to the ordinary de Rham cohomology of the C^∞-manifold M.

On the other hand, Kassel and Mitschi [15] prove that $H_n(D(M), D(M)) = H_{\mathrm{DR}}^{2m-n}(M)$. We therefore conclude

Theorem 3.3.1. *If M is a smooth algebraic variety over a field k of characteristic 0, then $E_n^2 = E_n^\infty = H_{\mathrm{DR}}^{2m-n}(M)$.*

Now the above considerations easily generalize to the case where M is a Stein complex manifold, except that $\Omega^n_{T^*(M)}$ is replaced by the space of holomorphic differential forms on $T^*(M)$ which are *algebraic* along the fibers of $T^*M \to M$. Anyway, we still get $E_n^2 = E_n^\infty = H_{\mathrm{DR}}^{2m-n}(M)$.

In the C^∞-case, we can only conclude that $E_n^2 = H_{\mathrm{DR}}^{2m-n}(M)$ (where m is now the real dimension of M). We do not have the result of Kassel and Mitschi in this case, but the degeneracy of the spectral sequence is proven in [5] and [24].

Let us remark that for M a complex-analytic manifold, $U \subset T^*M$ a Stein conical open set, we may filter the algebra $\mathscr{P}(U)$ of holomorphic pseudo-differential operators defined on U (see [3]) by their order, and obtain a spectral sequence with $E_n^2 = H_{\mathrm{DR}}^{2m-n}(U)$ and $E_n^\infty = H_n(\mathscr{P}(U), \mathscr{P}(U))$. It is not hard to generalize the result of Kassel and Mitschi, and obtain the degeneracy of this spectral sequence.

Now consider the cyclic homology of $D(M)$, say for M an affine algebraic variety. The term E of the spectral sequence constructed in §3.2 is the hyperhomology of the double complex of loc. cit. According to Corollary 2.2.13, the first spectral sequence for this double complex degenerates and we obtain

$$E_n^2 = H_{\mathrm{DR}}^{2m-n}(M) \oplus H_{\mathrm{DR}}^{2m-n+2}(M) \oplus \cdots .$$

From the results of Kassel and Mitschi [15], and from the above, it is reasonable to imagine that this is exactly $HC_n(D(M))$. This is actually proven in [5], [6] and [24].

3.4. Here k is a field of characteristic 0, $\mathfrak{g}$ is a finite-dimensional Lie algebra over k, and $A = U(\mathfrak{g})$ is the universal enveloping algebra filtered by $A_n = U_n(\mathfrak{g})$ (elements of order $\leq n$). The associated graded algebra $\mathrm{gr}(A)$ is the symmetric algebra $S(\mathfrak{g})$, which is the algebra of regular functions on the dual space $\mathfrak{g}^*$. The Poisson complex (with $\Omega^i_{\mathfrak{g}^*} = \Omega^i_{S(\mathfrak{g})|k}$)

$$\cdots \longrightarrow \Omega^i_{\mathfrak{g}^*} \xrightarrow{\delta} \Omega^{i-1}_{\mathfrak{g}^*} \longrightarrow \cdots$$

is very hard to work with, however what we are really interested in is the Hochschild homology $H_*(A, A)$. Let us note the following elementary lemma, valid for any associative k-algebra A. For any A-bimodule M, the standard Hochschild complex $C.(A, M)$

$$\cdots \longrightarrow M \underset{k}{\otimes} A^{\otimes n} \xrightarrow{b} M \underset{k}{\otimes} A^{\otimes(n-1)} \cdots$$

admits, for any $a \in A$, an endomorphism L_a defined by

$$L_a(m \otimes x_1 \otimes \cdots \otimes x_n) = [a, m] \otimes x_1 \otimes \cdots \otimes x_n + \sum_{i=1}^{n} x_1 \otimes \cdots \otimes [a, x_i] \otimes \cdots \otimes x_n$$

where we put $[a, m] = (a \otimes 1 - 1 \otimes a).m$.

Lemma 3.4.1. *For any A-bimodule M, the endomorphism of $H_*(A, M)$ induced by L_a is zero for any $a \in A$.*

Proof. L_a defines an endomorphism of the homological functor $M \mapsto (H_i(A, M))_{i \geq 0}$. We prove, by induction on i, that L_a acts trivially on $H_i(A, M)$. For $i = 0$, it is clear, since $H_0(A, M)$ is a quotient of M by $\sum_{b \in A} L_b(M)$. Now if we know the result for i (and all bimodules A), to prove it for $i + 1$, choose an exact sequence $0 \to N \to F \to M \to 0$ of $A \otimes A^0$-modules, where F is free. Then $H_{i+1}(A, M)$ injects in $H_i(A, N)$; since L_a acts trivially on $H_i(A, N)$, it acts trivially on $H_{i+1}(A, M)$.

Let us return to $A = U(\mathfrak{g})$. The lemma implies that the adjoint action of $\mathfrak{g}$ on $H_n(A, A)$ is trivial. Now assume $\mathfrak{g}$ is *reductive*; then the spectral sequence of §3.1, with $E^1_n = \Omega^n_{\mathfrak{g}^*}$, has the same E^∞ term as the $\mathfrak{g}$-invariant part of the spectral sequence. So we may just work with this smaller spectral sequence, whose E^2 term is the homology of the $\mathfrak{g}$-invariant part of the Poisson complex

$$\cdots \longrightarrow \Omega^n_{\mathfrak{g}^*} \xrightarrow{\delta} \Omega^{n-1}_{\mathfrak{g}^*} \longrightarrow \cdots .$$

Let us illustrate this in the case $\mathfrak{g} = \mathfrak{sl}(2)$. We take a basis (H, X, Y) of $\mathfrak{g}$ such that $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. The Poisson bracket on

$s(\mathfrak{g})$ is given by the following formulas:

$$\{a,X\} = 2X\frac{\partial a}{\partial H} - H\frac{\partial a}{\partial Y}, \quad \{a,Y\} = -2Y\frac{\partial a}{\partial H} + H\frac{\partial a}{\partial X},$$
$$\{a,H\} = -2X\frac{\partial a}{\partial X} + 2Y\frac{\partial a}{\partial Y}.$$

Let us explain how one describes $[\Omega^n_{\mathfrak{g}^*}]^{\mathfrak{g}}$, the space of $\mathfrak{g}$-invariant differential forms of degree n on $\mathfrak{g}^*$. We have

$$[\Omega^n_{\mathfrak{g}^*}]^{\mathfrak{g}} = [\Lambda^n(\mathfrak{g}) \underset{k}{\otimes} S(\mathfrak{g})]^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(\Lambda^n(\mathfrak{g}^*), S(\mathfrak{g})).$$

For a general reductive Lie algebra $\mathfrak{g}$, and an arbitrary finite-dimensional representation V of $\mathfrak{g}$, a theorem of Kostant [16] states that $\mathrm{Hom}_{\mathfrak{g}}(V, S(\mathfrak{g}))$ is a free module over $S(\mathfrak{g})^{\mathfrak{g}}$ of rank equal to the dimension of the zero weight space of V. The degrees of the generators, in case V is irreducible, are called the generalized exponents for V, and are described in [16].

In our case, all $\Lambda^n(\mathfrak{g}^*)$ are irreducible and have a one-dimensional zero weight space. Therefore, for $n = 0,1,2,3$, $[\Omega^n_{\mathfrak{g}^*}]^{\mathfrak{g}}$ is a free $S(\mathfrak{g})^{\mathfrak{g}}$-module of rank 1. The algebra $s(\mathfrak{g})^{\mathfrak{g}}$ is the polynomial algebra in the variable $u = 2XY + H^2/2$ (Casimir element). From this we easily obtain $[\Omega^0_{\mathfrak{g}^*}]^{\mathfrak{g}} = k[u]$, $[\Omega^1_{\mathfrak{g}^*}]^{\mathfrak{g}} = k[u]\cdot du$, $[\Omega^2_{\mathfrak{g}^*}]^{\mathfrak{g}} = k[u]\cdot\alpha$, with $\alpha = HdX\wedge dY + YdH\wedge dX - XdH\wedge dY$ and $[\Omega^3_{\mathfrak{g}^*}]^{\mathfrak{g}} = k[u]\cdot\beta$, with $\beta = dX\wedge dY\wedge dH$. The differential δ is $k[u]$-linear, since $\{u,f\} = 0$ for any $f \in S(\mathfrak{g})$. We have $\delta(du) = 0$, $\delta(\alpha) = du$, $\delta(\beta) = 0$. Therefore, our complex is:

$$\begin{array}{ccccccc}
k[u]\cdot\beta & \xrightarrow{0} & k[u]\cdot\alpha & \xrightarrow{\sim} & k[u]\cdot du & \xrightarrow{0} & k[u] \\
| & & | & & | & & | \\
\text{degree } 3 & & \text{degree } 2 & & \text{degree } 1 & & \text{degree } 0
\end{array}$$

Therefore its homology is $k[u]$ in degree 0, $k[u]\cdot\beta$ in degree 3, and 0 in all other degrees.

This fits very well with the results of Masuda, who computed the Hochschild homology of A [20]. For the Poisson double complex considered in §3.2, the homology is:

$k[u]$ in degree 0;

0 in degree 1;

k in degrees $2,4,\cdots$;

$k[u]\cdot\beta$ in degrees $3,5,\cdots$.

This is the E^2 term of the spectral sequence which converges to $HC_*(A)$. This spectral sequence cannot degenerate at E^2 because Masuda [20] shows that $HC_3(A) = 0$.

To conclude, let us remark that the Poisson structure on $\mathfrak{g}^*$ has a very rich geometric structure (the leaves are the $\mathfrak{g}$-orbits, with the symplectic structure of Kirillov), and one might wish to relate the Poisson homology to this geometry (which plays an important rôle in the beautiful work of Kostant).

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