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ON THE HEAT OPERATORS OF NORMAL SINGULAR ALGEBRAIC SURFACES

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1. Introduction

Let X be a normal singular algebraic surface (over $\mathbf{C}$) embedded in the projective space $\mathbf{P}^N(\mathbf{C})$. The singularity set S of X is a finite set of isolated points. By restricting the Fubini-Study metric of $\mathbf{P}^N(\mathbf{C})$ to $\mathscr{X} = X - S$, we obtain an incomplete Riemannian manifold $(\mathscr{X}, g)$. Now consider the Laplacian $\Delta = \bar{\delta}\bar{d}$ acting on square-integrable functions on $(\mathscr{X}, g)$. Here $\bar{d}$ means the closure of the exterior derivative d acting on the smooth functions which are square-integrable, and whose images by d are square-integrable too. Also $\bar{\delta}$ means the closure of its formal adjoint δ acting on the smooth 1-forms which are square-integrable, and whose images by δ are square-integrable too. Then the purpose of this paper can be said to show the following.

Main Theorem. (1) *The Laplacian Δ is self-adjoint.*

(2) *The heat operator $e^{-\Delta t}$ is of trace class, and there exists a constant $K > 0$ such that*

$$\operatorname{Tr} e^{-\Delta t} \leq K t^{-2}, \qquad 0 < t \leq t_0. \tag{1.1}$$

Defining d_0 to be the exterior derivative d restricted to the subspace of smooth functions with compact supports, we have $\bar{\delta}^* = \bar{d}_0$ [4]. Hence (1) can be rewritten in the following way.

Assertion A. $\bar{d} = \bar{d}_0$.

In §5 we intend to prove this assertion, which is equivalent to (1). Thereby, we will prove (2) with $\Delta = \bar{\delta}\bar{d}_0$, the (self-adjoint) Laplacian of the (generalized) Dirichlet type (§§2–4).

In general, if a certain self-adjoint Laplacian on a certain Riemannian manifold has the *basic property* mentioned in (2), but replacing the 2 of t^{-2} by half of the real dimension of the manifold, then we say that the Laplacian has the *property* (BP). In using this expression, what we want to prove is stated as follows: $\Delta = \bar{\delta}\bar{d}_0$ has the property (BP). Let us transform this assertion (2)′ into a more convenient one.

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Let $\mathrm{dis}(x)$, $x \in X$, be the distance from x to the singularity set S (induced by the given metric g) and set, for sufficiently small $\varepsilon > 0$,

$$X_\varepsilon = \{x \in X \mid \mathrm{dis}(x) \geq \varepsilon\}. \tag{1.2}$$

Then the Laplacian Δ_ε of the Dirichlet type on $(X_\varepsilon, g \mid X_\varepsilon)$ obviously has the property (BP). Also the trace of the heat operator $e^{-\Delta_\varepsilon t}$ increases monotonically when ε decreases. Moreover, provided we define $\mathrm{Tr}\, e^{-\Delta t} = \infty$ when $e^{-\Delta t}$ is not of trace class, [1, VIII, Theorem 4] generally says

$$\mathrm{Tr}\, e^{-\Delta t} = \lim_{\varepsilon \to 0} \mathrm{Tr}\, e^{-\Delta_\varepsilon t}. \tag{1.3}$$

Note that $\Delta = \bar{\delta}\bar{d}_0$. Hence, in order to prove $(2)'$, we have only to prove

Assertion B. *There exists a constant $K > 0$ such that*

$$\mathrm{Tr}\, e^{-\Delta_\varepsilon t} \leq K t^{-2}, \qquad 0 < t \leq t_0,\ 0 < \varepsilon \leq \varepsilon_0. \tag{1.4}$$

We intend to prove this assertion in §§2–4. Let us introduce the principle on which our discussion is based.

Principle (*Cheeger* [3, *Lemma* 7.1]). *The property (BP) is of quasi-isometric invariant.*

Recall that a diffeomorphism $f\colon (Y_1, g_1) \to (Y_2, g_2)$ is called a *quasi-isometry* if there exists a constant $C > 0$ such that $C^{-1} g_1 \leq f^* g_2 \leq C g_1$. Our principle asserts that, as long as the object under consideration is of the property (BP), we have only to discuss it on a Riemannian manifold less complicated than and quasi-isometric to the given one. We will start by decomposing a neighborhood ($\subset \mathscr{X}$) of the singularity set S into certain less-complicated parts (§2). Precisely "less-complicated" means that each part (with the given metric g) is quasi-isometric to one of the Riemannian manifolds W of the following Types $(\pm)$.

Type $(-)$: Fix $c \geq 1$. Let Y be a compact polygon in $\mathbf{R}^2$ and $\tilde{g}$ be the standard metric on Y Then we set

$$W = \text{“}(0,1] \times [0,1] \times Y \ (\ni (r, \theta, y)) \quad \text{with metric } dr^2 + r^2 d\theta^2 + r^{2c}\tilde{g}(y).\text{”}$$

Type $(+)$: Fix $b > 0$ and $c \geq 1$. Let $f(r)$ be a smooth function on $(0,1]$ satisfying $f'(r) \geq 0$ for any $r > 0$, $f(r) = r^b$ for small $r > 0$ and $f(r) = 1/2$ for large $r \leq 1$. Also, let $l(x)$ be a smooth function on $[0, \infty)$ satisfying $l'(x) \geq 0$ and $l''(x) \geq 0$ for any $x \geq 0$, $l(x) = 1$ for $0 \leq x \leq 1 - \varepsilon$ and $l(x) = x$ for $x \geq 1 + \varepsilon$. Set $h(r,s) = f(r) l(s/f(r))$. Then we set

$$W = \text{“}(0,1] \times [0,1]^3 (\ni (r, \theta, s, \Theta)) \quad \text{with metric } dr^2 + r^2 d\theta^2 + r^{2c}(ds^2 + h^2(r,s) d\Theta^2).\text{”}$$

Finally we shall put $S = \{p\}$, the one-point set, which obviously causes no loss of generality.

2. Decomposition of $\mathscr{X}$

The purpose of this section is to decompose $\mathscr{X}$ into less-complicated finite parts, nonoverlapping, except on the boundaries. The parts which cover near the singular point p must be quasi-isometric to the W's of Types $(\pm)$.

We start, according to Hsiang and Pati [6], by looking over the metric g near p through a resolution of X.

Let the singular point p be at $[(1, 0, \cdots, 0)] \in \mathbf{P}^N(\mathbf{C})$. Using the local coordinates around the point,

$$[(w_0, w_1, \cdots, w_N)] \mapsto (z_1, \cdots, z_N) = (w_1/w_0, \cdots, w_N/w_0),$$

regard X as a normal surface which is contained in $\mathbf{C}^N$ and has the singularity at the origin. Then we must make a good resolution $\pi\colon \tilde{X} \to X$ at the origin to satisfy the condition that, through the resolution, the local parametrizations of the standard form can be taken [6, III]. That is, near an arbitrary point of $\pi^{-1}(0)$, take a suitable pair, a permutation $\sigma \in \mathfrak{S}_N$ and local coordinates (u, v); then the map π can be written on the coordinates as follows:

$$
\begin{aligned}
z_{\sigma(1)} &= u^{n_1} v^{m_1}, && (n_1, m_1) \neq (0,0),\\
z_{\sigma(2)} &= f_2(z_{\sigma(1)}) + u^{n_2} v^{m_2} g_2(u,v), && \det\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} \neq 0,\\
& && g_2(0,0) \neq 0,\\
&\vdots && \vdots \\
z_{\sigma(l)} &= f_l(z_{\sigma(1)}) + u^{n_l} v^{m_l} g_l(u,v), && \det\begin{pmatrix} n_1 & m_1 \\ n_l & m_l \end{pmatrix} \neq 0,\\
z_{\sigma(l+1)} &= f_{l+1}(z_{\sigma(1)}), && g_l(0,0) \neq 0,\\
&\vdots \\
z_{\sigma(N)} &= f_N(z_{\sigma(1)}),
\end{aligned}
\tag{2.1}
$$

satisfying that $f_j(z) = \sum a_{jn} z^{\varepsilon_n}$ with $\varepsilon_n \geq 1$ for $2 \leq j \leq N$, and moreover, $n_1 \leq n_2 \leq \cdots \leq n_l$ and $m_1 \leq m_2 \leq \cdots \leq m_l$. Such a resolution can be constructed by first resolving the singularity and (if necessary) next performing the quadratic transformations [6, II and III].

Now consider a sufficiently small local coordinate neighborhood $(U, (u, v)$, $|u| \leq \rho_0$, $|v| \leq \tau_0)$ with the standard local parametrization (2.1). Set

$V = U - \pi^{-1}(0)$. Then let us find the metric less-complicated than and quasi-isometric to π^*g on V. We set

$$
\begin{aligned}
&c = \min\left\{\frac{n_2}{n_1}, \frac{m_2}{m_1}\right\} \quad (\geq 1), && d = \det\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} (\neq 0),\\
&r_1 = |u^{n_1}v^{m_1}|, && \theta_1 = \arg u^{n_1}v^{m_1},\\
&r_2 = \begin{cases} |v|^{d/n_1}, & d > 0, \\ |u|^{|d|/m_1}, & d < 0, \end{cases} && \theta_2 = \arg u^{n_2}v^{m_2}.
\end{aligned}
\tag{2.2}
$$

With the definition of c, we consider $+/0 = \infty$.

Proposition 2.1 (*Hsiang and Pati* [6, *Lemma* 3.2]). *On V the metric π^*g is quasi-isometric to the metric*

$$
dr_1^2 + r_1^2 d\theta_1^2 + r_1^{2c}(dr_2^2 + r_2^2 d\theta_2^2). \tag{2.3}
$$

Further we will search for additionally less-complicated ones. Let us make some preparations. Because (2.3) is a metric, we have,

$$
\begin{aligned}
&\text{if } n_1 = 0, \quad \text{then } n_2 = 1,\\
&\text{if } m_1 = 0, \quad \text{then } m_2 = 1.
\end{aligned}
\tag{2.4}
$$

In fact, for example, if $m_1 = 0$, then $\pi^{-1}(0) =$ "v-axis" in U and the norm (defined by (2.3)) of the tangent vector $\partial/\partial|v|$ at $(u, 0) \in U$, $u \neq 0$, is equal to $|u|^{n_2} \lim_{|v|\to 0} m_2|v|^{m_2-1}$. Hence $m_2 \neq 1$ leads to a contradiction, that is, its norm is equal to 0. Next, setting

$$
\begin{cases} \rho = |u|, \\ \phi = \arg u, \end{cases} \qquad \begin{cases} \tau = |v|, \\ \psi = \arg v, \end{cases} \tag{2.5}
$$

we have the following:

(2.6) Proposition 2.1 holds even if Θ_2 is replaced by

$$
\Theta = \begin{cases} \psi, & d > 0, \\ \phi, & d < 0. \end{cases}
$$

(2.7) In particular, if $n_1 m_1 \neq 0$, then Proposition 2.1 still holds even if Θ_2 is replaced by

$$
\Theta = \begin{cases} \phi, & d > 0, \\ \psi, & d < 0. \end{cases}
$$

Let us prove only (2.6) with $d > 0$. Setting $\tilde{\Theta} = (d/n_1)\psi$, (2.3) can be rewritten as follows:

$$
dr_1^2 + (1 + c^2 r_1^{2(c-1)} r_2^2) r_1^2 d\theta_1^2 + r_1^{2c} dr_2^2 + r_1^{2c} r_2^2 d\tilde{\Theta}^2 + 2c r_1^{2c} r_2^2 d\theta_1 d\tilde{\Theta}. \tag{2.8}
$$

Since we have, for a sufficiently small $\varepsilon > 0$,

$$|2cr_1^{2c}r_2^2 d\theta_1 d\tilde{\Theta}| \leq \frac{c^2}{\varepsilon^2} r_1^{2c} r_2^2 d\theta_1^2 + \varepsilon^2 r_1^{2c} r_2^2 d\tilde{\Theta}^2, \tag{2.9}$$

(2.8) can be dominated from above and below by

(2.10)
$$dr_1^2 + \left\{1 + \left(1 \pm \frac{1}{\varepsilon^2}\right) c^2 r_1^{2(c-1)} r_2^2 \right\} r_1^2 d\theta_1^2 + r_1^{2c} \left\{ dr_2^2 + (1 \pm \varepsilon^2) r_2^2 d\tilde{\Theta}^2 \right\}.$$

This implies (2.6) with $d > 0$ for sufficiently small $r_1 > 0$ and $r_2 > 0$, which is obviously sufficient for the proof of (2.6) with $d > 0$ itself.

Now, observing (2.4)–(2.7), we can obtain the following corollary. That is, with the definitions:

(2.11) in the case where either the v-axis or the u-axis is the divisor contained in U, we set

$$r = \begin{cases} \rho^{n_1} \\ \tau^{m_1} \end{cases} \quad \theta = \begin{cases} \phi \\ \psi \end{cases} \quad s = \begin{cases} \tau \\ \rho \end{cases} \quad \Theta = \begin{cases} \psi; & d > 0, \\ \phi; & d < 0, \end{cases}$$

(2.12) in the case where both the v-axis and the u-axis are the divisors contained in U, we set

$$\begin{cases} r = \rho^{n_1}\tau^{m_1} \\ \theta = n_1\phi + m_1\psi \end{cases} \quad s = \begin{cases} \tau^{d/n_1} \\ \rho^{|d|/m_1} \end{cases} \quad \Theta = \begin{cases} \phi; & d > 0, \\ \psi; & d < 0, \end{cases}$$

we get

Corollary 2.2. *On V the metric $\pi^* g$ is quasi-isometric to*

$$dr^2 + r^2 d\theta^2 + r^{2c}(ds^2 + s^2 d\Theta^2). \tag{2.13}$$

As for $\pi^{-1}(0)$ from which certain neighborhoods of the intersection points of the (irreducible) divisors are deleted, the corollary (in the case (2.11)) says that we can decompose its neighborhood ($\subset \pi^{-1}(\mathscr{X})$) into the parts quasi-isometric to the W's of Type $(-)$. Note that the indices c fixed in Type $(-)$ are those of (2.13); they depend only on divisors (not on the choice of U) and are called the *exponents* of the divisors [6, III]. On the other hand, the corollary does not give the desired decomposition of the neighborhoods of the intersection points; the map $(u, v) \mapsto (r, \theta, s, \Theta)$ does not introduce the desired product structure into them, despite the fact that the metric is of Type $(-)$. In the following, we will show that they are quasi-isometric to the W's of Type $(+)$.

As they are treated similarly, we need only treat the case (2.12) with $d > 0$. That is, on our U, both the v-axis and the u-axis are divisors and the index d is positive. We have $c = n_2/n_1$, which is the exponent of the v-axis; note that m_2/m_1 is that of the u-axis. Set $b = m_2/m_1 - n_2/n_1$. Now (performing the rescale if necessary) we put $\rho_0 = 2$, and fix a smooth function $\tilde{f}(\rho)$ on $(0, 2]$ satisfying $\tilde{f}'(\rho) \leq 0$ for any $\rho > 0$, $\tilde{f}(\rho) = 1$ on $(0, 1]$ and $\tilde{f}(\rho) = (2-\rho)\rho^{d/m_1}$ near $\rho = 2$. Then, replacing s by $\tilde{s} = r^{d/n_1}\tilde{f}(\rho)$, the map

$$\pi_V\colon (u, v) \mapsto (r, \tilde{s}, \theta, \Theta) \tag{2.14}$$

induces a diffeomorphism from V to

$$\left\{(r, \tilde{s}) | 0 < r \leq 2^{n_1}\tau_0^{m_1},\ 0 \leq \tilde{s} \leq \tau_0^{d/n_1}\tilde{f}\left(\tau_0^{-m_1/n_1}r^{1/n_1}\right)\right\} \times T, \tag{2.15}$$

where $T = \mathbf{R}^2/\{(2n_1\pi, 2\pi), (2m_1\pi, 0)\}$. Regarding $\tau_0 > 0$ as sufficiently large (by rescaling), we set

$$\tilde{V} = \pi_V^{-1}((0, 1] \times [0, 1] \times T). \tag{2.16}$$

Then, using the function $h(r, s)$ defined in Type (+), we have

Corollary 2.3. *On $\tilde{V}$ the metric π^*g is quasi-isometric to*

$$dr^2 + r^2 d\theta^2 + r^{2c}(d\tilde{s}^2 + h^2(r, \tilde{s})d\Theta^2). \tag{2.17}$$

Proof. It suffices to prove the corollary for small $r > 0$, so that $f(r) = r^b$. Let $\varepsilon > 0$ be the one given in Type (+):

(i) *On the part $\tilde{s} \geq (1+\varepsilon)r^b$.* Since $\rho \leq (1+\varepsilon)^{-m_1/d} < 1$, we have $\tilde{s} = s$ (given in (2.12)) and $h(r, \tilde{s}) = h(r, s) = r^b l(sr^{-b}) = s$. Hence the corollary restricted to the part is guaranteed by Corollary 2.2.

(ii) *On the part $\tilde{s} \leq (1+\varepsilon)r^b$.* We have $\rho \geq (1+\varepsilon)^{-m_1/d} > 0$. Therefore Corollary 2.2 with (2.11), corresponding to Type (−), asserts that the metric π^*g on the part is quasi-isometric to the metric associated to the divisor "u-axis," that is,

$$dr^2 + r^2 d\theta^2 + r^{2\tilde{c}}(d\rho^2 + d\Theta^2), \qquad \tilde{c} = m_2/m_1. \tag{2.18}$$

Here r, θ and Θ are those defined in (2.12). Hence it suffices to prove that (2.17) is quasi-isometric to (2.18). Let us rewrite (2.17):

$$\begin{aligned} \tilde{s} &= r^{d/n_1 m_1}\rho^{-d/m_1}\tilde{f}(\rho) = r^b\tilde{F}(\rho), \\ d\tilde{s}^2 &= b^2 r^{2(b-1)}\tilde{F}^2 dr^2 + r^{2b}(\tilde{F}')^2 d\rho^2 + 2br^{2b-1}\tilde{F}\tilde{F}' dr d\rho. \end{aligned} \tag{2.19}$$

Since, for a sufficiently small $\xi > 0$, we have

$$|2br^{2b-1}\tilde{F}\tilde{F}' dr d\rho| \leq \frac{b^2}{\xi^2}r^{2(b-1)}\tilde{F}^2 dr^2 + \xi^2 r^{2b}(\tilde{F}')^2 d\rho^2, \tag{2.20}$$

(2.17) can be dominated from above and below by

$$\begin{aligned} &\left\{1 + \left(1 \pm \frac{1}{\xi^2}\right) b^2 r^{2(\tilde{c}-1)}\tilde{F}^2\right\} dr^2 + r^2 d\theta^2 \\ &\quad + r^{2\tilde{c}}\left\{(1 \pm \xi^2)(\tilde{F}')^2 d\rho^2 + r^{-2b}h^2(r, r^b\tilde{F}(\rho))d\Theta^2\right\}. \end{aligned} \tag{2.21}$$

Here we know that $1 \leq r^{-b}h(r, r^b\tilde{F}(\rho)) \leq 2$ and there exists a constant $C > 0$ such that $-C \leq \tilde{F}' \leq -C^{-1}$. Thus (when $r > 0$ is sufficiently small) (2.17) is quasi-isometric to (2.18).

Now we can decompose a neighborhood ($\subset \mathscr{X}$) of p into the desired parts. That is, observing (2.16), first, decompose T into $\bigcup_j [\theta_{-j}, \theta_{+j}] \times [\Theta_{-j}, \Theta_{+j}]$ and next decompose $\tilde{V}$ compatibly. Each part is quasi-isometric to the W of Type (+). Second, decompose the closure of $\pi^{-1}(0) - \overline{\tilde{V}}$ into polygons Y_j and we get the decomposition of a neighborhod $\tilde{W}$ ($\subset \pi^{-1}(\mathscr{X})$) of the closure, each part of which is quasi-isometric to the $(0,1] \times S^1 \times Y_j$ with metric (2.13). By decomposing S^1, we get the decomposition of $\tilde{W}$, each part of which is quasi-isometric to the W of Type (−). Thus, a neighborhood ($\subset \mathscr{X}$) of p, which is diffeomorphic to a neighborhood of $\pi^{-1}(0)$, can be decomposed desirably into $\bigcup_\alpha W_\alpha$. Adding the part $M =$ "the closure of $\mathscr{X} - \bigcup_\alpha W_\alpha$", we get the desired decomposition

$$\mathscr{X} = M \cup \left(\bigcup_\alpha W_\alpha \right). \tag{2.22}$$

3. Proof of Assertion B

In this section, we prove Assertion B assuming that the following proposition is true. The proof of the proposition will be given in the next section.

On each (W_α, g) given in (2.22), consider the self-adjoint Laplacian

$$\Delta_\alpha = \bar{d}_\alpha^* \bar{d}_\alpha. \tag{3.1}$$

Here d_α is the exterior derivative acting on functions, smooth on W_α (up to the boundary ∂W_α), with compact supports. Also $\bar{d}_\alpha^*$ means the dual of $\bar{d}_\alpha$. Note that ∂W_α does not contain the singular point p.

Proposition 3.1. *Each Δ_α has the property (BP).*

Now, set $X_{\alpha\varepsilon} = W_\alpha \cap X_\varepsilon$ (see (1.2)). Decompose its boundary into

$$\begin{aligned} \partial X_{\alpha\varepsilon} &= \partial_0 X_{\alpha\varepsilon} \cup \partial_1 X_{\alpha\varepsilon}, \\ \partial_0 X_{\alpha\varepsilon} &= \partial X_{\alpha\varepsilon} \cap \partial X_\varepsilon, \\ \partial_1 X_{\alpha\varepsilon} &= \partial X_{\alpha\varepsilon} \cap \partial W_\alpha, \end{aligned} \tag{3.2}$$

and, on $X_{\alpha\varepsilon}$, consider the self-adjoint Laplacian $\Delta_{\alpha\varepsilon}$, together with the boundary conditions of the Dirichlet type on $\partial_0 X_{\alpha\varepsilon}$ and of the Neumann type on $\partial_1 X_{\alpha\varepsilon}$. If we denote by $d_{\alpha\varepsilon}$ the exterior derivative acting on smooth functions f on $X_{\alpha\varepsilon}$ satisfying $f|\partial_0 X_{\alpha\varepsilon} = 0$, we can also write

$$\Delta_{\alpha\varepsilon} = \bar{d}_{\alpha\varepsilon}^* \bar{d}_{\alpha\varepsilon}. \tag{3.3}$$

This obviously has the property (BP). Also, on M, consider the self-adjoint Laplacian Δ_M of the Neumann type, which has the property (BP) as well. Then, combined with Proposition 3.1, the following proposition justifies Assertion B.

Proposition 3.2. Suppose $0 < \varepsilon \leq \varepsilon_0$. Then

(1) $\operatorname{Tr} e^{-\Delta_{\alpha\varepsilon} t} \leq \operatorname{Tr} e^{-\Delta_\alpha t}$,

(2) $\operatorname{Tr} e^{-\Delta_\varepsilon t} \leq \operatorname{Tr} e^{-\Delta_M t} + \sum_\alpha \operatorname{Tr} e^{-\Delta_{\alpha\varepsilon} t}$.

The proof follows the argument given in [10, Chapter XIV, 14.5 and 14.6].

Proof of (1). Let

$$(0 \leq)\ \lambda_0 \leq \lambda_1 \leq \cdots \uparrow \infty, \qquad (0 \leq)\ \mu_0 \leq \mu_1 \leq \cdots \uparrow \infty \tag{3.4}$$

be the eigenvalues (with multiplicities) of Δ_α and $\Delta_{\alpha\varepsilon}$ respectively. Then we have only to prove

$$\lambda_n \leq \mu_n \tag{3.5}$$

for any n. Let $\{\phi_m\}$ and $\{\psi_m\}$ be the sequences of the orthonormal eigenfunctions corresponding to (3.4) respectively. Moreover, consider the (energy) integrals,

$$\begin{aligned} D(f,g) &= \langle df, dg\rangle_{W_\alpha}, \qquad f,g \in \operatorname{dom} \bar{d}_\alpha, \\ D_\varepsilon(f,g) &= \langle df, dg\rangle_{X_{\alpha\varepsilon}}, \qquad f,g \in \operatorname{dom} \bar{d}_{\alpha\varepsilon}, \end{aligned} \tag{3.6}$$

where $\langle df, dg\rangle_{W_\alpha} = \int_{W_\alpha} df \wedge *dg$ etc. We set $D(f) = D(f,f)$ etc., for short. The integral $D(f)$ has the following inequality: for $f \in \operatorname{dom} \bar{d}_\alpha$, expanding $f = \sum_{m=0}^\infty c_m\phi_m$, $c_m = \langle f, \phi_m\rangle$, we have

$$\sum_{m=0}^{\infty} \lambda_m c_m^2 \leq D(f). \tag{3.7}$$

In fact, since $D(f,\phi_m) = \langle f, \delta d\phi_m\rangle = \lambda_m c_m$, we have, for any n,

$$\begin{aligned} 0 \leq D\left(f - \sum_{m=0}^{n} c_m\phi_m\right) &= D(f) + \sum_{l=0}^{n}\sum_{m=0}^{n} c_l c_m D(\phi_l, \phi_m) \\ &\quad - 2\sum_{m=0}^{n} c_m D(f, \phi_m) \\ &= D(f) - \sum_{m=0}^{n} \lambda_m c_m^2. \end{aligned} \tag{3.8}$$

On the other hand, $f \in \operatorname{dom} \bar{d}_{\alpha\varepsilon}$ can be regarded as $f \in \operatorname{dom} \bar{d}_\alpha$ provided we define $f = 0$ on $W_\alpha - X_{\alpha\varepsilon}$. In this sense, we have the implication that $\operatorname{dom} \bar{d}_{\alpha\varepsilon} \subset \operatorname{dom} \bar{d}_\alpha$.

Now we can prove (3.5). In setting $c_m = \langle \psi_0, \phi_m \rangle$, (3.7) says

$$\lambda_0 = \lambda_0 \sum_{m=0}^{\infty} c_m^2 \leq \sum_{m=0}^{\infty} \lambda_m c_m^2 \leq D(\psi_0) = D_\varepsilon(\psi_0) = \mu_0. \tag{3.9}$$

Thus (3.5) with $n = 0$ was proved. Next, take

$$f = a_0\psi_0 + a_1\psi_1, \quad a_0^2 + a_1^2 = 1, \quad \langle f, \phi_0 \rangle = 0. \tag{3.10}$$

That is, setting $A = \langle \psi_0, \phi_0 \rangle$ and $B = \langle \psi_1, \phi_0 \rangle$, we put $a_0 = B(A^2 + B^2)^{-1/2}$ and $a_1 = -A(A^2 + B^2)^{-1/2}$. (If $A = B = 0$, a_0 and a_1 can be chosen clearly to satisfy (3.10).) Then, setting $c_m = \langle f, \phi_m \rangle$, we have

$$\begin{gathered} \sum_{m=1}^{\infty} c_m^2 = \sum_{m=0}^{\infty} c_m^2 = \langle f, f \rangle = a_0^2 + a_1^2 = 1, \\ \lambda_1 = \lambda_1 \sum_{m=1}^{\infty} c_m^2 \leq \sum_{m=1}^{\infty} \lambda_m c_m^2 \leq D(f) = D_\varepsilon(f) = \mu_0 a_0^2 + \mu_1 a_1^2 \leq \mu_1. \end{gathered} \tag{3.11}$$

That is, (3.5) with $n = 1$ was also proved. In order to prove (3.5) generally, it suffices to find $f = a_0\psi_0 + \cdots + a_n\psi_n$ satisfying $a_0^2 + \cdots + a_n^2 = 1$ and $\langle f, \phi_m \rangle = 0$ for $0 \leq m \leq n - 1$. It is obviously possible.

Proof of (2). Let us gather all of the eigenvalues (with multiplicities) of $\Delta_M, \Delta_{\alpha\varepsilon}$ (any α and fixed ε) and arrange them in nondecreasing order, $(0 \leq)\ \lambda_0 \leq \lambda_1 \leq \cdots \uparrow \infty$. Also arrange the eigenvalues of Δ_ε in nondecreasing order, $(0 \leq)\ \mu_0 \leq \mu_1 \leq \cdots \uparrow \infty$. Then it suffices to prove

$$\lambda_n \leq \mu_n \tag{3.12}$$

for any n. Let $\{\phi_m\}$ and $\{\psi_m\}$ be the corresponding orthonormal eigenfunctions respectively. Here ϕ_m, which is a function on one of M or the $X_{\alpha\varepsilon}$, must be regarded as a functions on $M \sqcup (\bigsqcup_\alpha X_{\alpha\varepsilon})$ by setting $\phi_m = 0$ elsewhere. Next consider the (energy) integrals,

$$\begin{aligned} D_\varepsilon(f, g) &= \langle df, dg \rangle_{X_\varepsilon}, & f, g &\in \{f \in C^\infty(X_\varepsilon) | f | \partial X_\varepsilon = 0\}, \\ D_M(f, g) &= \langle df, dg \rangle_M, & f, g &\in C^\infty(M), \\ D_{\alpha\varepsilon}(f, g) &= \langle df, dg \rangle_{X_{\alpha\varepsilon}}, & f, g &\in \{f \in C^\infty(X_{\alpha\varepsilon}) |\ f | \partial_0 X_{\alpha\varepsilon} = 0\}. \end{aligned} \tag{3.13}$$

Set $D_\varepsilon(f) = D_\varepsilon(f, f)$, etc. Then, for $f \in C^\infty(X_\varepsilon)$ with $f | \partial X_\varepsilon = 0$, we have

$$\sum_{m=0}^{\infty} \lambda_m c_m^2 \leq D_M(f) + \sum_\alpha D_{\alpha\varepsilon}(f), \qquad c_m = \langle f, \phi_m \rangle. \tag{3.14}$$

The proof is similar to that of (3.7). Now setting $c_m = \langle \psi_0, \phi_m \rangle$, we have

$$(3.15)\qquad \begin{aligned} \lambda_0 = \lambda_0 \langle \psi_0, \psi_0 \rangle &= \lambda_0 \left\{ \langle \psi_0, \psi_0 \rangle_M + \sum_\alpha \langle \psi_0, \psi_0 \rangle_{X_{\alpha\varepsilon}} \right\} \\ &\leq \sum_{m=0}^{\infty} \lambda_m c_m^2 \leq D_M(\psi_0) + D_{\alpha\varepsilon}(\psi_0) = D_\varepsilon(\psi_0) = \mu_0. \end{aligned}$$

That is, (3.12) with $n = 0$ was proved. As for the general case, it can be proved with a discussion similar to the one following (3.9).

4. Proof of Proposition 3.1

In this section we prove Proposition 3.1. According to (2.22) and our principle, it suffices to prove the following. The self-adjoint Laplacians Δ on the W's of Types $(\pm)$ are defined similarly to (3.1).

Proposition 4.1. *Each Δ on W has the property (BP).*

Now, fix $0 < R < 1$ and consider the self-adjoint Laplacian Δ_R on $W_R = \{(r, \cdots) \in W \mid R \leq r \leq 1\}$ defined in the same way as (3.3). It has the property (BP) and the trace $\operatorname{Tr} e^{-\Delta_R t}$ increases monotonically when R decreases, and, moreover, we have $\operatorname{Tr} e^{-\Delta t} = \lim_{R \to 0} \operatorname{Tr} e^{-\Delta_R t}$ (see (1.3)). Hence Proposition 4.1 can be reduced to the following.

Assertion 4.2. *There exists a constant $K > 0$ such that*

$$(4.1)\qquad \operatorname{Tr} e^{-\Delta_R t} \leq K t^{-2}, \qquad 0 < t \leq t_0,\ 0 < R \leq R_0.$$

We shall introduce a certain lemma. If we assume that it is true, we can prove the above assertion. First consider the ordinary differential equation $w''(x) + \lambda w(x) = 0$, $0 \leq x \leq 1$, with the boundary conditions of the three types; the sequences on the right-hand sides denote the eigenvalues in the cases respectively:

$$(4.2)\qquad \begin{aligned} &w'(0) = w'(1) = 0; \quad (0 =)\ \mu_0 < \mu_1 < \mu_2 < \cdots \uparrow \infty,\ \mu_n = (n\pi)^2, \\ &w(0) = \frac{d}{dx}(x^{-(2c+1)/2} w)(1) = 0; \quad \nu_0 < \nu_1 < \nu_2 < \cdots \uparrow \infty, \\ &w'(0) = \frac{d}{dx}(x^{-1/2} w)(1) = 0; \quad \xi_0 < \xi_1 < \xi_2 < \cdots \uparrow \infty. \end{aligned}$$

Here $c \geq 1$ is the one fixed in Types $(\pm)$. Note that $\nu_0 < 0 < \nu_1$ and $\xi_0 < 0 < \xi_1$. Second, let $(0 =)\ \eta_0 \leq \eta_1 \leq \eta_2 \leq \cdots \uparrow \infty$ be the eigenvalues of the Laplacian of the Neumann type on Y given in Type $(-)$. Third, adding

$\tilde{\xi}_0 = 0$ to the $\tilde{\xi}_j$, $j \geq 1$, the zero points of $\sqrt{\lambda} J_0'(\sqrt{\lambda})$, we make the sequence, $(0 =)\ \tilde{\xi}_0 = \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots \uparrow \infty$. Here $J_0(x)$ is the Bessel function of order 0. Note that $\sqrt{\lambda} J_0'(\sqrt{\lambda})$ has its zero points only on the nonnegative half line on the real axis [11].

Let us explain briefly why we consider its zero points $\tilde{\xi}_j$, $j \geq 1$. It is because we must later consider the boundary value problem,

$$(4.3)_{x_0} \quad \begin{cases} (4.3)'_{x_0} & w''(x) + \left(\lambda + \dfrac{1}{4x^2}\right) w(x) = 0, \quad (0 <)\ x_0 \leq x \leq 1, \\ (4.3)''_{x_0} & w'(x_0) = \dfrac{d}{dx}(x^{-1/2} w)(1) = 0, \end{cases}$$

and the singular boundary value problem $(4.3)_0 = \lim_{x_0 \to 0} (4.3)_{x_0}$, that is, the problem obtained by making $x_0 \downarrow 0$; in the latter problem we are in the limit-circle case. The $\tilde{\xi}_j$, $j \geq 1$, are the eigenvalues of the latter problem $(4.3)_0$. According to the general expansion theory [10, Chapter III], we will explain the above somewhat further. The differential equation $(4.3)'_{x_0}$, with the conditions $w_1(1, \lambda) = 1$, $w_1'(1, \lambda) = 0$, $w_2(1, \lambda) = 0$, $w_2'(1, \lambda) = 1$ added, has the following solutions ($\lambda \neq 0$, Lommel's formulas):

$$(4.4) \quad \begin{aligned} w_1(x, \lambda) &= \frac{\pi}{2} \sqrt{\lambda x} \{ N_0'(\sqrt{\lambda}) J_0(\sqrt{\lambda} x) - J_0'(\sqrt{\lambda}) N_0(\sqrt{\lambda} x) \} - \frac{1}{2} w_2(x, \lambda), \\ w_2(x, \lambda) &= -\frac{\pi}{2} \sqrt{x} \{ N_0(\sqrt{\lambda}) J_0(\sqrt{\lambda} x) - J_0(\sqrt{\lambda}) N_0(\sqrt{\lambda} x) \}, \end{aligned}$$

where J_0, N_0 are the Bessel and Neumann functions of order 0. Therefore, if we define, according to [10, (2.1.5)],

$$(4.5) \quad \begin{aligned} m_1(\lambda) &= l_1\left(\lambda, -\frac{1}{2}\right) = -\frac{-\frac{1}{2} w_1(1, \lambda) + w_1'(1, \lambda)}{-\frac{1}{2} w_2(1, \lambda) + w_2'(1, \lambda)} = \frac{1}{2}, \\ m_0(\lambda) &= \lim_{x_0 \to 0} l_{x_0}(\lambda, 0) = -\lim_{x_0 \to 0} \frac{w_1'(x_0, \lambda)}{w_2'(x_0, \lambda)} \\ &= \lim_{x_0 \to 0} \left[\frac{1}{2} + \sqrt{\lambda} \{ M(x_0, \lambda) N_0'(\sqrt{\lambda}) - J_0'(\sqrt{\lambda}) \} \right. \\ &\qquad \left. \times \{ M(x_0, \lambda) N_0(\sqrt{\lambda}) - J_0(\sqrt{\lambda}) \}^{-1} \right] \\ &= \frac{1}{2} + \sqrt{\lambda} J_0'(\sqrt{\lambda}) (J_0(\sqrt{\lambda}))^{-1}, \\ M(x_0, \lambda) &= \frac{J_0(\sqrt{\lambda} x_0) + 2\sqrt{\lambda} x_0 J_0'(\sqrt{\lambda} x_0)}{N_0(\sqrt{\lambda} x_0) + 2\sqrt{\lambda} x_0 N_0'(\sqrt{\lambda} x_0)}, \end{aligned}$$

then the eigenvalues of the problems $(4.3)_{x_0}$ and $(4.3)_0$ are given as the poles of the meromorphic functions $(l_{x_0}(\lambda,0) - l_1(\lambda,-1/2))^{-1}$ and

$$(m_0(\lambda) - m_1(\lambda))^{-1} = J_0(\sqrt{\lambda})\{\sqrt{\lambda}J_0'(\sqrt{\lambda})\}^{-1}$$

respectively. Thus the $\tilde{\xi}_j$, $j \geq 1$, are the eigenvalues of $(4.3)_0$.

Also, the general expansion theory says that, if the eigenvalues of $(4.3)_{x_0}$ are denoted by $\tilde{\xi}_1(x_0) < \tilde{\xi}_2(x_0) < \cdots$, then we have

$$\lim_{x_0 \to 0} \tilde{\xi}_j(x_0) = \tilde{\xi}_j, \qquad j \geq 1, \tag{4.6}$$

which will also be a key point of our later discussion.

Now, arranging the eigenvalues of Δ_R in nondecreasing order,

$$(0 <)\ \lambda_1(R) \leq \lambda_2(R) \leq \cdots \uparrow \infty, \tag{4.7}$$

we get

Lemma 4.3. (1) *In the case of Type* $(-)$. *Rearranging the elements of the set* $\{\mu_i + \eta_j + \mu_k\}$ *in nondecreasing order,* $\lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$, *we have* $\lambda_n \leq \lambda_n(R)$ *for any* n.

(2) *In the case of Type* $(+)$. *Fix a (possibly negative) constant* a. *Then, rearranging the elements of the set* $\{\mu_i+\mu_j+\xi_k+\nu_l+a; j>0\} \cup \{\mu_i+\tilde{\xi}_k+\nu_l+a\}$ *in nondecreasing order,* $\lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$, *we have* $\lambda_n \leq \lambda_n(R)$ *for any* n.

If we assume the above, we can prove Assertion 4.2 as follows.

Proof of Assertion 4.2. There exists a constant $K_1 > 0$ such that

$$\mu_n, \nu_n \xi_n, \tilde{\xi}_n \geq K_1(n-1)^2, \qquad n \geq 1. \tag{4.8}$$

This implies that there exists a constant $K_2 > 0$ such that

$$\sum e^{-\mu_n t}, \sum e^{-\nu_n t}, \sum e^{-\xi_n t}, \sum e^{-\tilde{\xi}_n t} \leq K_2 t^{-1/2}, \qquad 0 < t \leq t_0. \tag{4.9}$$

Also we have $K_3 > 0$ satisfying $\sum e^{-\eta_n t} \leq K_3 t^{-1}$, $0 < t \leq t_0$. These facts combined with Lemma 4.3 say that $\operatorname{Tr} e^{-\Delta_R t} = \sum e^{-\lambda_n(R)t}$ has the estimate (4.1).

Thus only the proof of Lemma 4.3 remains. Let $\{\phi_m\}$ and $\{\psi_m\}$ be the orthonormal eigenfunctions corresponding to $\{\mu_m\}$ and $\{\eta_m\}$ respectively.

4.1. *Proof of Lemma* 4.3(1). Let us consider the differential equation $\Delta_R F = \lambda F$, $F \in \operatorname{dom} \Delta_R$, on W of Type $(-)$. Its solutions are given as the linear combinations of the functions $G(r)\phi_i(\theta)\psi_j(y)$, where the G are the solutions of the boundary value problem,

$$\begin{aligned} &G'' + (2c+1)r^{-1}G' + (\lambda - \mu_i r^{-2} - \eta_j r^{-2c})G = 0, \\ &G(R) = G'(1) = 0. \end{aligned} \tag{4.10}$$

Performing the normalization $G(r) = r^{-(2c+1)/2}H(r)$, we get

$$\tag{4.11} \begin{aligned} &H'' + (\lambda - q_{ij}(r))H = 0, \qquad R \leq r \leq 1, \\ &H(R) = \frac{d}{dr}(r^{-(2c+1)/2}H)(1) = 0, \\ &q_{ij}(r) = (\mu_i + c^2 + \tfrac{1}{4})r^{-2} + \eta_j r^{-2c}. \end{aligned}$$

Let $\lambda_{ij0}(R) \leq \lambda_{ij1}(R) \leq \cdots$ be the eigenvalues of the problem (4.11). Rearrange $\{\lambda_{ijk}(R)\}_{i,j,k}$ in nondecreasing order and we get the sequence (4.7). Now, let us consider the problem reduced from (4.11) by replacing $q_{ij}(r)$ by $p_{ij} = \mu_i + \eta_j - \nu_0$; the eigenvalues of this problem, denoted by (4.11)$'$, are written as $(0 \leq)\ \tilde{\lambda}_{ij0}(R) \leq \tilde{\lambda}_{ij1}(R) \leq \cdots$. Then we have $\tilde{\lambda}_{ijk}(R) \leq \lambda_{ijk}(R) - \nu_0$ for any i, j and k. The proof is similar to that of Proposition 3.2(1), however, instead of (3.6), we use

$$\tag{4.12} \begin{aligned} &(4.12)' \qquad \tilde{D}(f,g) = \int_R^1 \{f'g' + p_{ij}fg\}\,dr - f'(1)g(1), \\ &(4.12)'' \qquad D(f,g) = \int_R^1 \{f'g' + (q_{ij} - \nu_0)fg\}\,dr - f'(1)g(1), \end{aligned}$$

for $f, g \in \{f \in C^1([R,1]) \mid f(R) = (d/dr)(r^{-(2c+1)/2}f)(1) = 0\}$. Notice that both $\tilde{D}$ and D are symmetric with respect to f, g and we have $0 \leq \tilde{D}(f) \leq D(f)$.

Further we have the problem reduced from (4.11)$'$ by replacing $R \leq r \leq 1$ by $0 \leq r \leq 1$ (and, of course, $H(R) = 0$ by $H(0) = 0$); its eigenvalues are the $\mu_i + \eta_j + \nu_k - \nu_0$, $0 \leq k < \infty$. We have $\mu_i + \eta_j + \nu_k - \nu_0 \leq \tilde{\lambda}_{ijk}(R)$ for any i, j and k. The proof is similar to that of Proposition 3.2(1), however, instead of (3.6) we use both (4.12)$'$ and (4.12)$'$ with $\int_R^1$ replaced by $\int_0^1$.

Thus the proof of Lemma 4.3(1) is complete.

4.2. *Proof of Lemma* 4.3(2). Let us consider the differential equation $\Delta_R F = \lambda F$, $F \in \operatorname{dom}\Delta_R$, on W of Type (+). Its solutions are given as the linear combinations of the functions $G(r,s)\phi_i(\theta)\phi_j(\Theta)$, where the G are the solutions of the boundary value problem,

$$\tag{4.13} \begin{aligned} &\frac{\partial^2 G}{\partial r^2} + \frac{1}{r^{2c}}\frac{\partial^2 G}{\partial s^2} + \left\{\frac{2c+1}{r} + \frac{1}{h}\frac{\partial h}{\partial r}\right\}\frac{\partial G}{\partial r} + \frac{1}{r^{2c}h}\frac{\partial h}{\partial s}\frac{\partial G}{\partial s} \\ &\qquad\qquad + (\lambda - \mu_i r^{-2} - \mu_j h^{-2} r^{-2c})G = 0, \\ &G(R,s) = \frac{\partial G}{\partial r}(1,s) = \frac{\partial G}{\partial s}(r,0) = \frac{\partial G}{\partial s}(r,1) = 0. \end{aligned}$$

Performing the normalization $G(r,s) = r^{-(2c+1)/2}h^{-1/2}H(r,s)$, we get

$$\frac{\partial^2 H}{\partial r^2} + r^{-2c}\frac{\partial^2 H}{\partial s^2} + (\lambda - q_{ij}(r,s))H = 0, \qquad R \le r \le 1,\ 0 \le s \le 1,$$

$$\begin{aligned} H(R,s) &= \frac{\partial}{\partial r}(r^{-(2c+1)/2}H)(1,s) \\ &= \frac{\partial H}{\partial s}(r,0) = \frac{\partial}{\partial s}(s^{-1/2}H)(r,1) = 0, \end{aligned} \tag{4.14}$$

$$q_{ij}(r,s) = \left\{\mu_i + c^2 - \frac{1}{4} + \frac{2c+1}{2}\frac{r}{h}\frac{\partial h}{\partial r} - \frac{r^2}{4h^2}\left(\frac{\partial h}{\partial r}\right)^2 + \frac{r^2}{2h}\frac{\partial^2 h}{\partial r^2}\right\}\frac{1}{r^2}$$
$$+ \left\{\mu_j - \frac{1}{4}\left(\frac{\partial h}{\partial s}\right)^2 + \frac{1}{2}h\frac{\partial^2 h}{\partial s^2}\right\}h^{-2}r^{-2c}.$$

Let $\lambda_{ij0}(R) \le \lambda_{ij1}(R) \le \cdots$ be the eigenvalues of the problem (4.14). Rearrange $\{\lambda_{ijm}(R)\}_{i,j,m}$ in nondecreasing order and we get the sequence (4.7).

4.4. (1) *There exists a constant* $\tilde{a}$ *such that, for* $0 < r \le 1$ *and* $0 \le s \le 1$,

$$\left\{c^2 - \frac{1}{4} + \frac{2c+1}{2}\frac{r}{h}\frac{\partial h}{\partial r} - \frac{r^2}{4h^2}\left(\frac{\partial h}{\partial r}\right)^2 + \frac{r^2}{2h}\frac{\partial^2 h}{\partial r^2}\right\}\frac{1}{r^2} \ge \tilde{a}. \tag{4.15}$$

(2)

$$-\frac{1}{4}\left(\frac{\partial h}{\partial s}\right)^2 + \frac{1}{2}h\frac{\partial^2 h}{\partial s^2} \ge -\frac{1}{4}.$$

Proof. As for (1), it suffices to show that the left side of (4.15) is nonnegative, that is, bounded from below, for small $r > 0$. Hence we may assume $r > 0$ is small, so that $f(r) = r^b$. By noticing $h(r,s) = r^b l(sr^{-b})$, the left side of (4.15) can be rewritten as follows:

$$\left\{c^2 - \frac{1}{4} + bc\left(1 - \frac{sl'}{fl}\right) + \frac{b^2}{4}\left(1 - \left(\frac{sl'}{fl}\right)^2\right) + \frac{b^2}{2}\frac{s^2 l''}{f^2 l}\right\}r^{-2}.$$

Since $sl'/fl \le s/fl = x/l(x) \le 1$, (1) was proved. On the other hand, (2) can be shown by using the facts $\partial h/\partial s = l'(sf^{-1}) \le 1$ and $\partial^2 h/\partial s^2 = f^{-1}l''(sf^{-1}) \ge 0$.

From here, we divide our discussion into two cases, i.e., the case $j > 0$ ($\mu_j > 1/4$) and the case $j = 0$ ($\mu_0 = 0$). Set $a = \tilde{a} - 1/4$.

(I) *The case* $j > 0$ ($\mu_j > 1/4$).

Let us consider the problem reduced from (4.14) by replacing $q_{ij}(r,s)$ by $\tilde{q}_{ij}(r) = \mu_i + (\mu_j - 1/4)r^{-2c} + 1/4 - \xi_0 - \nu_0$; denote its eigenvalues by

$\tilde{\lambda}_{ij0}(R) \le \tilde{\lambda}_{ij1}(R) \le \cdots$. Next, consider the boundary value problem,

$$(4.16)\qquad \begin{aligned} &z'' + (\lambda - \tilde{p}_{ijk}(r))z = 0, \qquad R \le r \le 1, \\ &z(R) = \frac{d}{dr}(r^{-(2c+1)/2}z)(1) = 0, \\ &\tilde{P}_{ijk}(r) = \mu_i + \left(\mu_j + \xi_k - \frac{1}{4}\right) r^{-2c} + \frac{1}{4} - \xi_0 - \nu_0. \end{aligned}$$

Denote its eigenvalues by $\tilde{\lambda}_{ijk0}(R) < \tilde{\lambda}_{ijk1}(R) < \cdots$; if we rearrange $\{\tilde{\lambda}_{ijkl}(R)\}_{k,l}$ in nondecreasing order, the sequence thus obtained is $\{\tilde{\lambda}_{ijm}(R)\}_m$. Moreover, let us consider the problem (4.16)′ reduced from (4.16) by replacing $\tilde{p}_{ijk}(r)$ by $\tilde{p}_{ijk}(1)$; denote its eigenvalues by $\lambda_{ijk0}(R) < \lambda_{ijk1}(R) < \cdots$. Finally, let us consider the problem (4.16)″ reduced from (4.16)′ by replacing $R \le r \le 1$ by $0 \le r \le 1$ (and, of course, $z(R) = 0$ by $z(0) = 0$); its eigenvalues are precisely the $\mu_i + \mu_j + \xi_k + \nu_l - \xi_0 - \nu_0$ with l arbitrary.

These eigenvalues have the following relation:

$$(4.17)\qquad \begin{aligned} &(4.17)' \quad (0 \le)\ \mu_i + \mu_j + \xi_k + \nu_l - \xi_0 - \nu_0 \le \lambda_{ijkl}(R) \le \tilde{\lambda}_{ijkl}(R), \\ &(4.17)'' \quad \tilde{\lambda}_{ijm}(R) \le \lambda_{ijm}(R) - a - \xi_0 - \nu_0. \end{aligned}$$

Each inequality can be proved similarly to that of Proposition 3.2(1). The proof of Lemma 4.3(1) is a more direct model for that of (4.17)′; notice that $\mu_j + \xi_k - 1/4 \ge \mu_1 + \xi_0 - 1/4 = \pi^2 + \xi_0 - 1/4 \ge 0$. To prove (4.17)″, we use the following integrals instead of (3.6):

$$(4.18)\qquad \begin{aligned} \tilde{D}(f,g) &= \int_0^1 \int_R^1 \left\{ \frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + r^{-2c}\frac{\partial f}{\partial s}\frac{\partial g}{\partial s} + \tilde{q}_{ij} f g \right\} dr\,ds \\ &\quad - \int_0^1 \left(\frac{\partial f}{\partial r} g\right)(1,s)\,ds - \int_R^1 r^{-2c}\left(\frac{\partial f}{\partial s} g\right)(r,1)\,dr, \\ D(f,g) &= \int_0^1 \int_R^1 \left\{ \frac{\partial f}{\partial r}\frac{\partial g}{\partial r} + r^{-2c}\frac{\partial f}{\partial s}\frac{\partial g}{\partial s} + (q_{ij} - a - \xi_0 - \nu_0) f g \right\} dr\,ds \\ &\quad - \int_0^1 \left(\frac{\partial f}{\partial r} g\right)(1,s)\,ds - \int_R^1 r^{-2c}\left(\frac{\partial f}{\partial s} g\right)(r,1)\,dr, \end{aligned}$$

for $f, g \in \{f \in C^1([R,1] \times [0,1]) \mid f$ satisfies the condition $(4.12)''\}$. Both $\tilde{D}$ and D are symmetric with respect to f and g. Observing Lemma 4.4, we have $0 \le \tilde{D}(f) \le D(f)$.

(II) *The case* $j = 0$ $(\mu_0 = 0)$. Let us consider the problem reduced from (4.14) by replacing $q_{i0}(r,s)$ by $q_i(r,s) = \mu_i - \frac{1}{4}(\partial h/\partial s)^2 h^{-2} r^{-2c} - \nu_0$; denote its eigenvalues by $\tilde{\lambda}_{i0}(R) \le \tilde{\lambda}_{i1}(R) \le \cdots$.

In order to estimate the sequence $\{\tilde{\lambda}_{im}(R)\}_m$, we consider the following. Take $s_0 > 0$ small, so that $h(r,s) = f(r)$ for $R \le r \le 1$ and $0 \le s \le s_0$. Giving our attention to this $s_0 > 0$, we consider the boundary value problems:

$$(4.19)\qquad \begin{aligned} &\frac{\partial^2 H}{\partial r^2} + \left[\lambda - \left\{\mu_i - r^{-2c}\left(\frac{\partial^2}{\partial s^2} + \frac{1}{4s^2}\right) - \nu_0\right\}\right] H = 0, \\ &\qquad\qquad\qquad R \le r \le 1,\ s_0 \le s \le 1, \\ &H(R,s) = \frac{\partial}{\partial r}\left(\frac{H}{r^{(2c+1)/2}}\right)(1,s) = \frac{\partial H}{\partial s}(r,s_0) \\ &\qquad = \frac{\partial}{\partial s}\left(\frac{H}{s^{1/2}}\right)(r,1) = 0, \end{aligned}$$

and

$$(4.20)\qquad \begin{aligned} &\frac{\partial^2 H}{\partial r^2} + \left[\lambda - \left\{\mu_i - r^{-2c}\frac{\partial^2}{\partial s^2} - \nu_0\right\}\right] H = 0, \\ &\qquad\qquad\qquad R \le r \le 1,\ 0 \le s \le s_0, \\ &H(R,s) = \frac{\partial}{\partial r}(r^{-(2c+1)/2}H)(1,s) \\ &\qquad = \frac{\partial H}{\partial s}(r,0) = \frac{\partial H}{\partial s}(r,s_0) = 0. \end{aligned}$$

Gather their eigenvalues and rearrange them in nondecreasing order; denote the sequence by $(0 <)\ \tilde{\mu}_{i0}(s_0,R) \le \tilde{\mu}_{i1}(s_0,R) \le \cdots$. Then the facts that $(\partial h/\partial s)^2 h^{-2} \le s^{-2}$ for $s \ge s_0$ and $(\partial h/\partial s)^2 h^{-2} = 0$ for $0 \le s \le s_0$ demand $\tilde{\mu}_{im}(s_0,R) \le \tilde{\lambda}_{im}(R)$ for any m. The proof is similar to that of Proposition 3.2(2). Hence, setting $\tilde{\mu}_{im}(R) = \lim_{s_0\to 0}\tilde{\mu}_{im}(s_0,R)$, we get

$$(4.21)\qquad \tilde{\mu}_{im}(R) \le \tilde{\lambda}_{im}(R)$$

for any m. Moreover, when we consider the problems, $k \ge 0$,

$$(4.22)\qquad \begin{aligned} &w'' + \{\lambda - (\mu_i + \tilde{\xi}_k r^{-2c} - \nu_0)\}w = 0, \qquad R \le r \le 1, \\ &w(R) = \frac{d}{dr}(r^{-(2c+1)/2}w)(1) = 0, \end{aligned}$$

their eigenvalues $\{\tilde{\mu}_{ikl}(R)\}_{k,l}$ are rearranged into the nondecreasing sequence $\{\tilde{\mu}_{im}(R)\}_m$. This fact is obviously deduced from (4.6) and the fact that the eigenvalues of the problem $y'' + \lambda y = 0,\ 0 \le x \le s_0$, with $y'(0) = y'(s_0) = 0$ are the $\mu_j(s_0) = \mu_j s_0^{-2},\ j \ge 0$.

Next, let us consider the problem (4.22)$'$ reduced from (4.22) by replacing $\tilde{\xi}_k r^{-2c}$ by $\tilde{\xi}_k$; denote its eigenvalues by $\mu_{ik0}(R) < \mu_{ik1}(R) < \cdots$. Finally, consider the problem reduced from (4.22)$'$ by replacing $R \le r \le 1$ by $0 \le r \le 1$ (and, of course, $w(R) = 0$ by $w(0) = 0$); its eigenvalues are exactly the $\mu_i + \tilde{\xi}_k + \nu_l - \nu_0$ with l arbitrary.

These eigenvalues have the following relation:

$$(0 \le)\ \mu_i + \tilde{\xi}_k + \nu_l - \nu_0 \le \mu_{ikl}(R) \le \tilde{\mu}_{ikl}(R), \qquad (4.23)$$
$$\tilde{\lambda}_{im}(R) \le \lambda_{im}(R) - a - \nu_0.$$

Each proof is similar to that of Proposition 3.2(1).

Now Lemma 4.3(2) is a natural consequence following from (4.17), (4.21) and (4.23).

5. Proof of Assertion A

First of all, according to Hsiang and Pati [6, IV], we will review the method of introducing a product structure into a neighborhood ($\subset \mathscr{X}$) of the singular point p. Observing (2.22), we take the $\tilde{W}_\alpha$ of Types ($\pm$) corresponding to the W_α and take the quasi-isometries $\iota_\alpha; W_\alpha \cong \tilde{W}_\alpha$. Let us make the vector field $\tilde{\xi}_\alpha$ on the $\tilde{W}_\alpha$ by rescaling and perturbing the vector field $\partial/\partial r$ (however, the $\tilde{\xi}_\alpha$ and the $\partial/\partial r$ must be quasi-isometric), so that the $\iota_\alpha^* \tilde{\xi}_\alpha$ together produce a smooth vector field $\tilde{\xi}$ on $Y = \bigcup W_\alpha$. Moreover, let us denote by $R(x) > 0$, $x \in Y$, the distance along the flow line of $\tilde{\xi}$ from x to the singular point p. We may assume (by performing the rescale) that each flow line extends to the point where $R > 1$. Then the flow lines and the function $R\colon Y \to (0, \infty)$ produce a product structure

$$R^{-1}(0,1] = (0,1] \times R^{-1}(1). \qquad (5.1)$$

Here the decomposition

$$R^{-1}(0,1] = \bigcup_\alpha (W_\alpha \cap R^{-1}(0,1]) \qquad (5.2)$$

is compatible with the structure (5.1), that is, each $W_\alpha \cap R^{-1}(0,1]$ has a natural product structure induced from (5.1). Moreover, by replacing $\iota_\alpha(x) = (r, \cdots)$ by $I_\alpha(x) = (R, \cdots)$, that is, by replacing only the r by the R, we get the quasi-isometries

$$I_\alpha\colon W_\alpha \cap R^{-1}(0,1] \simeq \tilde{W}_\alpha. \qquad (5.3)$$

Now, let us start the proof of Assertion A. Because of the Stokes' theorem and the fact $\bar{\delta}^* = \bar{d}_0$, it suffices to prove the following.

Proposition 5.1. *For any $F \in \operatorname{dom} d$ and any $G \in \operatorname{dom} \delta$, there exists a sequence $\varepsilon_n \downarrow 0$ such that*

$$\lim_{n\to\infty} \int_{R^{-1}(\varepsilon_n)} F \wedge *G = 0.$$

Proof. Considering the structure (5.1), we can write $*G = A + dR \wedge B$, where A and B do not involve dR. Hence we have $\int_{R^{-1}(\varepsilon)} F \wedge *G = \int_{R^{-1}(\varepsilon)} F \wedge A$. Now, let $\tilde{*}_\eta$ be the $*$-operator on $R^{-1}(\eta)$ with the metric g restricted and let us define

$$\|F\|^2_{\{\eta,\varepsilon\}} = \int_{R^{-1}(\varepsilon)} F \wedge \tilde{*}_\eta F, \qquad \|A\|^2_{\{\eta,\varepsilon\}} = \int_{R^{-1}(\varepsilon)} A \wedge \tilde{*}_\eta A. \tag{5.4}$$

For example, regard $F \,|\, R^{-1}(\varepsilon)$ as a function on $R^{-1}(\eta)$ naturally; then $\|F\|_{\{\eta,\varepsilon\}}$ is precisely its L^2-norm. Using (5.4), we have

$$\left| \int_{R^{-1}(\varepsilon)} F \wedge A \right| \leq \|F\|_{\{\varepsilon,\varepsilon\}} \|A\|_{\{\varepsilon,\varepsilon\}}. \tag{5.5}$$

And, because of [2, Lemma 1.2] and the fact $\|A\|^2_{\{\varepsilon,\varepsilon\}} \in L^1(0,1)$, there exists a sequence $\varepsilon_n \downarrow 0$ such that

$$\|A\|_{\{\varepsilon_n,\varepsilon_n\}} = o(\varepsilon_n^{-1/2} |\log \varepsilon_n|^{-1/2}). \tag{5.6}$$

Hence the following lemma asserts that the proposition is true.

Lemma 5.2. *There exists a constant $K > 0$ such that*

$$\|F\|_{\{\varepsilon,\varepsilon\}} \leq K\{\|F\| + \|dF\|\}\varepsilon^{1/2}$$

for $0 < \varepsilon < 1/2$ and $F \in \operatorname{dom} d$.

Here $\|F\|$ and $\|dF\|$ are the L^2-norms of F and dF on $\mathscr{X}$ respectively. Considering the quasi-isometries (5.3) and providing $F_\alpha = I_{\alpha *}F$ for $F \in C^\infty(\mathscr{X})$, there exists a constant $K_1 > 0$ such that

$$\|F\|_{\{\varepsilon,\varepsilon\}} \leq K_1 \sum_\alpha \|F_\alpha\|_{\{\varepsilon,\varepsilon\}} \tag{5.7}$$

for $0 < \varepsilon < 1$ and $F \in C^\infty(\mathscr{X})$. Here we define $\|F_\alpha\|_{\{\varepsilon,\varepsilon\}}$ in the same way as (5.4). Hence we have only to prove the following.

Lemma 5.3. *There exists a constant $K > 0$ such that*

$$\|F_\alpha\|_{\{\varepsilon,\varepsilon\}} \leq K\{\|F_\alpha\| + \|dF_\alpha\|\}\varepsilon^{1/2}$$

for $0 < \varepsilon < 1/2$ and $F \in \operatorname{dom} d$.

Here $\|F_\alpha\|$ and $\|dF_\alpha\|$ mean the L^2-norms of F_α and dF_α on $\tilde{W}_\alpha$ respectively. The proof follows the argument given in [2, Lemma 2.3].

Proof in the case $\tilde{W}_\alpha$ is of Type $(-)$. To simplify the description, we use F and W instead of F_α and $\tilde{W}_\alpha$. Here W is of Type $(-)$. Fix $0 < \varepsilon < a$.

Then we have

$$
\begin{aligned}
\left\| \int_\varepsilon^a \frac{\partial F}{\partial r}\, dr \right\|_{\{\varepsilon,\varepsilon\}} &= \varepsilon^{(2c+1)/2} \left\| \int_\varepsilon^a \frac{\partial F}{\partial r}\, dr \right\|_{\{1,\varepsilon\}} \\
&\leq \varepsilon^{(2c+1)/2} \int_\varepsilon^a \left\| \frac{\partial F}{\partial r} \right\|_{\{1,r\}} dr \\
&= \varepsilon^{(2c+1)/2} \int_\varepsilon^a r^{-(2c+1)/2} \left\| \frac{\partial F}{\partial r} \right\|_{\{r,r\}} dr \\
&\leq \varepsilon^{(2c+1)/2} \left\{ \int_\varepsilon^a r^{-(2c+1)}\, dr \right\}^{1/2} \|dF\| \\
&\leq \frac{1}{\sqrt{2c}} \varepsilon^{1/2} \|dF\|.
\end{aligned} \tag{5.8}
$$

On the other hand, assuming that the function $\|F\|_{\{1,r\}}$, $1/2 \leq r \leq 1$, takes the minimum at $r = a$, we have

$$
\begin{aligned}
\|F(a)\|_{\{\varepsilon,\varepsilon\}} &= \varepsilon^{(2c+1)/2}\|F\|_{\{1,a\}} = 2\varepsilon^{(2c+1)/2} \int_{1/2}^1 \|F\|_{\{1,a\}}\, dr \\
&\leq 2\varepsilon^{(2c+1)/2} \int_{1/2}^1 \|F\|_{\{1,r\}}\, dr \\
&= 2\varepsilon^{(2c+1)/2} \int_{1/2}^1 r^{-(2c+1)/2} \|F\|_{\{r,r\}}\, dr \\
&\leq \frac{1}{\sqrt{2c}} 2^{c+1} \varepsilon^{(2c+1)/2} \|F\|.
\end{aligned} \tag{5.9}
$$

Hence the following inequality implies the lemma:

$$
\|F\|_{\{\varepsilon,\varepsilon\}} \leq \left\| \int_\varepsilon^a \frac{\partial F}{\partial r} dr \right\|_{\{\varepsilon,\varepsilon\}} + \|F(a)\|_{\{\varepsilon,\varepsilon\}}. \tag{5.10}
$$

Proof in the case $\tilde{W}_\alpha$ is of Type (+). We use F and W instead of F_α and $\tilde{W}_\alpha$. Here W is of Type (+). First of all, we know that the metric on W is quasi-isometric to

$$
dr^2 + r^2 d\theta^2 + r^{2c}\{ds^2 + (r^b + s)^2 d\Theta^2\}. \tag{5.11}
$$

In fact, there exists a constant $C > 0$ such that $C^{-1}(r^b + s) \leq h(r,s) \leq C(r^b + s)$ for $0 < r \leq 1$ and $0 \leq s \leq 1$. This is a consequence of a straightforward computation; take $r_0 > 0$ so small that $f(r) = r^b$ for $0 < r \leq r_0$, decompose the region defined by $0 < r \leq r_0$ into three parts, $sf^{-1} \leq 1 - \varepsilon$, $1 - \varepsilon \leq sf^{-1} \leq 1 + \varepsilon$ and $1 + \varepsilon \leq sf^{-1}$, and then estimate $(r^b + s)(h(r,s))^{-1}$ on each part.

Therefore it suffices to prove the lemma on W with metric (5.11). Set $b = \tilde{c} - c$ (> 0). Decompose $\|F\|^2_{\{\eta,\varepsilon\}}$ in the following way:

$$
\begin{aligned}
\|F\|^2_{\{\eta,\varepsilon\}} &= \int \eta^{2c+1}(\eta^b + s)F^2(\varepsilon,\theta,s,\Theta)\,d\theta\,ds\,d\Theta \\
&= \int \eta^{\tilde{c}+c+1}F^2(\varepsilon,\theta,s,\Theta)\,d\theta\,ds\,d\Theta \\
&\quad + \int \eta^{2c+1}sF^2(\varepsilon,\theta,s,\Theta)\,d\theta\,ds\,d\Theta \\
&= \|F\|^2_{1,\{\eta,\varepsilon\}} + \|F\|^2_{2,\{\eta,\varepsilon\}}.
\end{aligned} \tag{5.12}
$$

Then, similarly to (5.8), we have

$$
\begin{aligned}
\left\|\int_\varepsilon^a \frac{\partial F}{\partial r}dr\right\|^2_{\{\varepsilon,\varepsilon\}} &= \left\|\int_\varepsilon^a \frac{\partial F}{\partial r}dr\right\|^2_{1,\{\varepsilon,\varepsilon\}} + \left\|\int_\varepsilon^a \frac{\partial F}{\partial r}dr\right\|^2_{2,\{\varepsilon,\varepsilon\}} \\
&\le \varepsilon\left\{\frac{1}{c+c}\int_\varepsilon^a \left\|\frac{\partial F}{\partial r}\right\|^2_{1,\{r,r\}} dr + \frac{1}{2c}\int_\varepsilon^a \left\|\frac{\partial F}{\partial r}\right\|^2_{2,\{r,r\}} dr\right\} \\
&\le \frac{1}{2c}\varepsilon\int_\varepsilon^a \left\|\frac{\partial F}{\partial r}\right\|^2_{\{r,r\}} dr \le \frac{1}{2c}\varepsilon\|dF\|^2.
\end{aligned} \tag{5.13}
$$

Also, similarly to (5.9), assuming that the function $\|F\|_{\{1,r\}}$, $1/2 \le r \le 1$, takes the minimum at $r = a$, we have

$$
\begin{aligned}
\|F(a)\|^2_{\{\varepsilon,\varepsilon\}} &= \varepsilon^{c+\tilde{c}+1}\|F\|^2_{1,\{1,a\}} + \varepsilon^{2c+1}\|F\|^2_{2,\{1,a\}} \\
&\le \varepsilon^{2c+1}\|F\|^2_{\{1,a\}} \le 2\varepsilon^{2c+1}\int_{1/2}^1 \|F\|^2_{\{1,r\}}\,dr \\
&\le 2\varepsilon^{(2c+1)/2}\int_{1/2}^1 r^{-(c+\tilde{c}+1)}\|F\|^2_{\{r,r\}}\,dr \\
&\le 2^{c+\tilde{c}+2}\varepsilon^{2c+1}\|F\|^2.
\end{aligned} \tag{5.14}
$$

Hence the same inequality as (5.10) implies the lemma.

The author would like to thank the referee, who gave him the comment, "It shouldn't be too difficult to prove $\bar{d} = \bar{d}_0$ for i-forms rather than functions, as in Cheeger [2]. Since Hsiang and Pati [6] use Cheeger's argument without verifying $\bar{d} = \bar{d}_0$, your result seems to partially fill that gap in their proof." However, the assertion $\bar{d} = \bar{d}_0$ for forms (which must be true) seems to be difficult to prove in the same way as in §5 (or as in [2]). The readers may have already noticed that [6] did not treat Type(+), which is certainly a gap of [6]. Because of the complexity of Type(+), the above assertion has a subtle problem and also Hsiang-Pati's argument in [6] needs to be revised (at least we must treat Type(+)), which will be discussed elsewhere. Finally the author would like to thank the Institute for Advanced Study for their hospitality during the author's stay there (in 1985–86).

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