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ON ISOMORPHIC CLASSICAL DIFFEOMORPHISM GROUPS. II

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Abstract

We prove that the geometric structures defined by a volume form or a symplectic form (with a mild additional condition) on smooth manifolds are determined by their automorphism groups. This is a contribution to the Erlangen Program of Klein.

1. Introduction

The goal of this paper is to prove that the geometric structures defined by a volume form or a symplectic form (with a mild additional hypothesis) on smooth manifolds are determined by their automorphism groups. This is a contribution to the Erlangen Program of Klein [7].

We draw heavily on Filipkiewicz's paper [6] in which he proves that the differentiable C^r structures on manifolds are determined by their automorphism groups, i.e., the group $\mathrm{Diff}^r(M)$ of C^r diffeomorphisms of M for $1 \leq r \leq \infty$, Filipkiewicz in turn draws heavily on Whittaker's paper [12], where the above result is established for $r = 0$, i.e., for homeomorphisms. In order to handle the differentiable case, Filipkiewicz had to avoid the infinite patching methods used by Whittaker in [12]: he developed the necessary new machinery essentially in §2 of his paper.

Unfortunately, his new arguments fail in the volume preserving case and, a fortiori, in the symplectic case. Indeed, a key lemma (his Lemma 2.1) seeks, $\forall a \in (0, 1]$, a diffeomorphism (which is a product of commutators) which has the property to take a ball of radius 1 into a ball of radius a. Obviously, this cannot happen in our case. Besides, all the results of §2 are based on this lemma and on Epstein's theorem [5]. However, it is clear that Epstein's axioms are not satisfied in our cases.

The starting point of the investigations reported here was the observation that, nevertheless, some of his conclusions (for instance the conclusion of

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Theorem 2.2) are still true in the symplectic and volume preserving cases: due to deep theorems on the structure of volume preserving diffeomorphisms [11] and symplectic diffeomorphisms [1]. We call herein "condition L" the conclusions of Theorem 2.2 of [6].

Our second main observation was that "condition L" is the crucial condition to impose on groups of diffeomorphisms $G_i(M_i)$, $i = 1, 2$, of smooth manifolds M_i for a group isomorphism (of abstract groups) $\phi\colon G_1(M_1) \to G_2(M_2)$ to be induced by a bijection $w\colon M_1 \to M_2$, i.e. so that $\phi(h)(y) = whw^{-1}(y)$ $\forall h \in G_1(M_1),\ y \in M_2$.

We have put this result in a general setting (Theorem 2) and proved it, mainly using a judicious reorganization of Filipliewicz's and Whittaker's ideas. The main point is that using condition L and some standard facts, we recover the useful facts of §2 of Filipkiewicz, without using the "bad" Lemma 2.1.

It follows from the first part of this paper [2] that if $G_i(M_i)$ are the automorphism groups of the geometric structures considered above, the bijection $w\colon M_1 \to M_2$ is a C^∞-diffeomorphism which exchanges the structures. The main results announced then follow.

It would be interesting to point out more structures determined by their automorphism groups. For instance we "believe" that a contact structure is determined by its automorphism group; we cannot prove it at present due to our ignorance of its algebraic structure: more precisely, we are not able to verify "condition L".

2. The main results

Recall that a *volume form* on a smooth n-dimensional manifold M is a nowhere vanishing n-form on M, and a *symplectic form* on a smooth $2n$-dimensional manifold M is a closed 2-form α such that $\alpha^n = \alpha \wedge \cdots \wedge \alpha$ (n times) is a volume form on M. To formulate the result on symplectic diffeomorphisms, we need to recall the definition of the *symplectic pairing* [4]. If α is a symplectic form on a smooth $2n$-dimensional manifold M, the symplectic pairing (determined by α)

$$\langle\ ,\ \rangle\colon H^1_c(M, \mathbf{R}) \times H^1_c(M, \mathbf{R}) \to \mathbf{R}$$

(where $H^1_c(M, \mathbf{R})$ is the first de Rham cohomology of M with compact supports) is defined as follows: if $a, b \in H^1_c(M, \mathbf{R})$ are represented by two closed 1-forms with compact support ω_1, ω_2, then

$$\langle a, b\rangle = \int_M \omega_1 \wedge \omega_2 \wedge \alpha^{n-1}.$$

If α is a smooth p-form on a smooth manifold M, we denote by $\mathrm{Diff}_\alpha^\infty(M)$ the group of all C^∞-diffeomorphisms $h\colon M \to M$ such that $h^*\alpha = \alpha$. Let $\mathrm{Diff}_\alpha^\infty(M)_c$ be the subgroup formed by those $h \in \mathrm{Diff}_\alpha^\infty(M)$ with compact supports and $\mathrm{Diff}_\alpha^\infty(M)_0$ the subgroup of $\mathrm{Diff}_\alpha^\infty(M)_c$ formed elements which are isotopic to the identity through $\mathrm{Diff}_\alpha^\infty(M)_c$.

The following is our main result.

Theorem 1. *Let (M_i, α_i), $i = 1, 2$, be two paracompact connected smooth manifolds of dimension n equipped with volume forms or symplectic forms α_i. Assume furthermore that if α_i are volume forms, $n > 2$, and if α_i are symplectic forms, either M_i are compact or the symplectic pairings of α_i are identically zero. Let $G(M_i)$ be either* $\mathrm{Diff}_{\alpha_i}^\infty(M_i), \mathrm{Diff}_{\alpha_i}^\infty(M_i)_0$. *If $\phi\colon G(M_1) \to G(M_2)$ is a group isomorphism, then there exists a unique C^∞-diffeomorphism $w\colon M_1 \to M_2$ such that $\phi(f) = wfw^{-1}$ $\forall f \in G(M_1)$ and $w^*\alpha_2 = \lambda\alpha_1$ for some constant number λ.*

Corollary. *Let (M, α) be a compact connected smooth manifold of dimension $n \geq 2$ equipped with a symplectic form or a volume form α. If ϕ is an automorphism of* $\mathrm{Diff}_\alpha^\infty(M)$*, then ϕ^2 is an inner automorphism.*

Proof. By Theorem 1, there is a C^∞-diffeomorphism $w\colon M \to M$ and a constant number λ such that $w^*\alpha = \lambda\alpha$. Consider first the case where α is a volume form on M. If the diffeomorphism w is orientation preserving, the formula of change of variables gives

$$\int_M w^*\alpha = \int_M \alpha.$$

Since $\int_M w^*\alpha = \lambda \int_M \alpha$ and $\int_M \alpha \neq 0$, $\lambda = 1$, i.e., $w \in \mathrm{Diff}_\alpha^\infty(M)$ and $\phi(f) = wfw^{-1}$ $\forall f \in \mathrm{Diff}_\alpha^\infty(M)$. Therefore ϕ, and hence ϕ^2, are inner automorphisms. If w is orientation reversing, then $\int_M w^*\alpha = -\int_M \alpha$. In this case $\lambda = -1$, i.e., $w^*\alpha = -\alpha$. Therefore $(w^2)^*\alpha = \alpha$, i.e., $w^2 = w \cdot w \in \mathrm{Diff}_\alpha^\infty(M)$ and $\phi^2(f) = w^2 f (w^2)^{-1}$, i.e., ϕ^2 is an inner automorphism. Suppose now α is a symplectic form on the $2n$-dimensional manifold M, then α^n is a volume form and $w^*\alpha^n = (w^*\alpha)^n = \lambda^n\alpha^n$. The argument above shows that λ^n is 1 or -1, hence λ is 1 or -1. This implies that ϕ^2 is an inner automorphism.

3. The general theory

1. Definitions and examples. By a *smooth manifold*, we will mean a paracompact connected finite dimensional C^∞ manifold without boundary.

Let $\mathrm{Diff}^r(M)$, $1 \leq r \leq \infty$, be the group of all C^r-diffeomorphisms of a smooth manifold M. Any subgroup $G(M)$ of $\mathrm{Diff}^r(M)$ is called *a group* *of* C^r-diffeomorphisms of M. Here we list a few examples of groups of C^r-diffeomorphisms.

(i) The group $\mathrm{Diff}^r_c(M)$ of C^r-diffeomorphisms with compact supports. Recall that the *support* of a transformation $h\colon M \to M$ is the closure of the subset $\{x \in M \mid h(x) \neq x\}$.

(ii) If $G(M)$ is a group of C^r-diffeomorphisms of M, we can consider the following subgroups:

(a) $G_c(M) = G(M) \cap \mathrm{Diff}^r_c(M)$.

(b) Let $U \subseteq M$ be an open set of M, we can consider the subgroup $G_U(M)$ of $G(M)$ formed by elements $h \in G(M)$ which have compact supports in U.

(c) For each $x \in M$, we denote by $S_xG(M)$ the isotropy subgroup of the point x, i.e., $\{h \in G(M)$ such that $h(x) = x\}$.

(d) $G(M)$*-isotopies*: A $G(M)$-isotopy is a map $h\colon [0,1] \to G(M)$ with $h(0) = e$ (identity diffeomorphism) and such that the associated evaluation map $H\colon [0,1] \times M \to M\colon (t,x) \mapsto h(t)(x)$ is C^r. An element $h \in G(M)$ is said to be $G(M)$-isotopic to the identity if there is a $G(M)$-isotopy $h\colon [0,1] \to G(M)$ with $h(1) = h$. We will consider the group $G(M)_0$ formed by all elements in $G(M)$ that there are $G(M)$-isotopic to the identity.

(e) The commutator subgroup $[G(M), G(M)]$ of a group of C^r-diffeomorphisms is the subgroup of $G(M)$ generated by commutators $fgf^{-1}g^{-1}$, $f, g \in G(M)$.

The class $\mathscr{D}_M$ *and the general result.* A group of C^r-diffeomorphisms $G(M)$ of a smooth manifold M belongs to *the class* $\mathscr{D}_M$ if the following two conditions are satisfied.

Condition A (*the path transitivity*). *Given* $x, y \in M$, $x \neq y$, *and a path* $c\colon [0,1] \to M$ *joining* x *to* y, *i.e.* $c(0) = x, c(1) = y$, *there exists* $h \in G(M)$ *such that* $h(x) = y$ *and* $\mathrm{supp}(h)$ *is contained in an arbitrarily small open neighborhood of* $\bigcup_{t\in[0,1]} c(t)$.

An open subset $U \subseteq M$ is called an *open ball* centered at $x_0 \in U$ if there is an embedding $e\colon D^n_\rho \to M$ of the open n-disk in $\mathbf{R}^n$, centered at $0 \in \mathbf{R}^n$ with radius $\rho > 0$, into M and $U = e(D^n_\sigma)$ for some $\sigma \leq \rho$ and $e(0) = x_0$.

Condition B. *For any small open ball* U *in* M *centered at* $x_0 \in U$ *there is an* $h \in G(M)$ *such that*

$$\mathrm{Fix}(h) = (M - U) \cup \{x_0\}.$$

Here $\mathrm{Fix}(h) = \{x \in M \mid h(x) = x\}$ is the fixed set of h.

Definition. Let $G(M)$ be a group of C^r-diffeomorphisms of a smooth manifold M. A subgroup $F \subseteq G(M)$ is said to have the *property L* (locality) if whenever $[G_{U_i}(M)_0, G_{U_i}(M)_0] \subseteq F$ for every open ball U_i belonging to an open cover $\mathscr{U} = (U_i)_{i\in I}$ of M, by balls U_i, then $[G_c(M)_0, G_c(M)_0] \subseteq F$.

The main result of this section is the following.

Theorem 2. *Let $\phi\colon G(M) \to G(N)$ be a group isomorphism between two groups of diffeomorphisms of smooth manifolds M and N. If $G(M)$ and $G(N)$ are nonabelian and belong to the classes $\mathscr{D}_M$ and $\mathscr{D}_N$, and $F_n = \phi^{-1}(S_n G(N))$ and $F_m = \phi(S_m G(M))$ have the property L for all $m \in M$, $n \in N$, then there exists a unique homeomorphism $w\colon M \to N$ with $\phi(f) = wfw^{-1}$ $\forall f \in G(M)$.*

2. More conditions on diffeomorphism groups.

Definition. A group of diffeomorphisms $G(M)$ of a smooth manifold M satisfies *condition* C if for any nonempty open connected subset $U \subseteq M$ and $x \in U$ there exists $h \in G(M) h \neq e$ with $\mathrm{supp}(h) \subset U$, $x \in \mathrm{Int}(\mathrm{supp}\,(h))$.

Definition. $G(M)$ is said to be n-fold transitive or to satisfy the condition $T(n)$, $n \geq 1$, if given any two sets $\{x_1, x_2, \cdots, x_n\}, \{y_1, y_2, \cdots, y_n\}$ of nonrepeating points, there is an $h \in G(M)$ such that $h(x_i) = y_i$, $i = 1, \cdots, n$. $G(M)$ is said to be ω-transitive if $T(n)$ is satisfied for each $n \in \mathbf{N}$. The following fact is obvious.

Proposition 1. *If a group of diffeomorphisms satisfies the Condition* A *(path transitivity), then it satisfies C and $T(n)$ $\forall n$ provided* $\dim M > 1$.

In the subsequent sections, the property $T(n)$ (n-fold transitivity) will play a central role.

Lemma 1. *Let $G(M)$ and $G(N)$ be two groups of diffeomorphisms satisfying $T(1)$ and condition C. If $\phi\colon G(M) \to G(N)$ is a group isomorphism such that there exists $x_0 \in M$ and $y_0 \in N$ with $\phi(S_{x_0}G(M)) = S_{y_0}G(N)$, then there exists a homeomorphism $w\colon M \to N$ and $\phi(f) = wfw^{-1}$. Moreover if $G(M)$ and $G(N)$ satisfy $T(2)$, then w is unique.*

Proof. The condition $T(1)$ implies that the maps

$$E_{x_0} : G(M) \to M : E_{x_0}(g) = g(x_0), \qquad g \in G(M),$$
$$E_{y_0} : G(N) \to N : E_{y_0}(h) = h(y_0), \qquad h \in G(N),$$

are onto with "kernels" $S_{x_0} = S_{x_0}G(M)$ and $S_{y_0} = S_{y_0}G(N)$. Therefore if ϕ takes S_{x_0} into S_{y_0}, it induces a well-defined map $w\colon M \to N$. We may define it as follows: for $x \in M$, choose $g \in G(M)$ with $g(x_0) = x$. Define $w(x) = \phi(g)(y_0) \in N$. This is a good definition, since if $g' \in G(M)$ and $g'(x_0) = x = g(x_0)$, then $g'^{-1}g \in S_{x_0}$ and $\phi(g'^{-1}g) = \phi(g'^{-1})\phi(g) \in S_{y_0}G(N)$, i.e.,

$$\phi(g')(y_0) = \phi(y)(y_0).$$

Clearly w is one-to-one. Since $x_1, x_2 \in M$ and $w(x_1) = w(x_2)$ there are $g_1, g_2 \in G(M)$ with $g_1(x_0) = x_1$, $g_2(x_0) = x_2$, $\phi(g_1)(y_0) = \phi(g_2)(y_0)$, i.e. $\phi(g_1)\phi(g_2)^{-1} = \phi(g_1 g_2^{-1})$ belongs to S_{y_0}, i.e. $g_1 g_2^{-1} \in S_{x_0}$, hence $g_1(x_0) = g_2(x_0)$ so $x_1 = x_2$. The map is onto: Let $y \in N$, choose $h \in G(N)$ with

$h(y_0) = y$ and set $x = \phi^{-1}(h)(x_0)$. Then $w(x) = \phi(\phi^{-1}(h))(y_0) = h(y_0) = y$. Therefore w is a bijection.

Let us show that it induces ϕ: Let $y \in N$ and $h \in G(N)$ with $h(y_0) = y$. Let $x = \phi^{-1}(h)(x_0)$. Then $w(x) = y$. Let $f \in G(M)$ and choose $g \in G(M)$ with $g(x_0) = f(x)$. Then $f^{-1}g(x_0) = x = \phi^{-1}(h)(x_0)$. Thus

$$g^{-1}f\phi^{-1}(h) \in S_{x_0}.$$

Hence $\phi(g)^{-1}\cdot\phi(f)\cdot h \in S_{y_0}$, i.e. $\phi(f)\cdot h(y_0) = \phi(g)(y_0)$. But $h(y_0) = y = w(x)$ and $\phi(g)(y_0) = w(f(x))$ since $g(x_0) = f(x)$. So

$$\phi(f)(w(x)) = w(f(x)), \quad \text{i.e., } \phi(f) \circ w = w \circ f,$$

since w is a bijection,

$$\phi(f) = w \circ f \circ w^{-1}.$$

We now show that w is a homeomorphism: Let $\mathscr{A} = \{\text{Fix}(f) \mid f \in G(M)\}$, where $\text{Fix}(f) = \{x \in M \mid f(x) = x\}$ and $\mathscr{B} = \{B = M - A,\ A \in \mathscr{A}\}$.

The condition C implies that $\mathscr{B}$ is a basis for a topology on M which is easily seen to coincide with the usual topology of M. Since $\phi(f) = wfw^{-1}$, $\text{Fix}(\phi(g)) = w(\text{Fix}(g))$. This implies that w and w^{-1} take basic open sets into basic open sets, therefore they are both continuous, i.e., w is a homeomorphism (see also [2], [6]).

We now prove the uniqueness of w: Suppose there is another homeomorphism $w'\colon M \to N$ inducing ϕ, i.e.,

$$\phi(f) = w \cdot f \cdot w^{-1} = w' \cdot f \cdot w'^{-1} \quad \forall f \in G(M).$$

Setting $\rho = w'^{-1} \cdot w$, we see

$$\rho \cdot f \cdot \rho^{-1} = f \quad \forall f \in G(M).$$

Suppose $\rho \neq e$. Then there exists $x \in M$ with $y = \rho(x) \neq x$. Let $z \in M, z \neq x, z \neq y$. Since $G(M)$ is 2-fold transitive, there is $f \in G(M)$ with $f(x) = x$ and $f(y) = z$. Therefore

$$\rho f \rho^{-1}(y) = \rho f(x) = \rho(x) = y, \quad f(y) = z \neq y.$$

Hence $\rho f \rho^{-1} \neq f$, a contradiction. Therefore $\rho = w'^{-1} \cdot w$ is the identity map, i.e., $w' = w$.

3. Characterization of isotropy subgroups of points. The following result is due to [6]. Its proof uses only the 3-fold transitivity and the condition C.

Lemma 2 (*Filipkiewicz* [6, *Lemma* 3.2]). *Let $G(M), G(N)$ be two groups of diffeomorphisms of smooth manifolds M and N, satisfying the conditions $T(3)$ and C, and let $\phi\colon G(M) \to G(N)$ be a group isomorphism. Let*

$F = \phi^{-1}(S_y) \subseteq G(M)$ $(S_y = S_yG(N)$ *for some* $y \in N)$. *Suppose there exists a nonempty proper closed subset* $A \subset M$ *such that* $f(A) = A$ $\forall f \in F$. *Then* $A = \{x\}$ *and* $F = S_x = S_x(G(M))$.

Remark. This lemma and Lemma 1 imply that to prove the main theorem, we need only construct a nonempty proper closed subset invariant under $F = \phi^{-1}(s_y)$.

Existence of a proper closed subset invariant under F. Let $\phi\colon G(M) \to G(N)$ be an isomorphism between $G(M)$ and $G(N)$. Following Filipkiewicz, for $n \in N$, we let $\mathscr{E}_n$ be the set of all open balls U of M with

$$[G_U(M)_0, G_U(M)_0] \subseteq F_n = \phi^{-1}(S_n), \qquad S_n = S_nG(N).$$

Likewise we define the set $\mathscr{D}_m$ for $m \in M$, as the set of all open balls V in N such that

$$[G_V(N)_0, G_V(N)_0] \subseteq F'_m = \phi(S_m), \qquad S_m = S_mG(M).$$

A priori, the sets $\mathscr{E}_n$ and $\mathscr{D}_m$ may be empty. Let $C_n = M - \bigcup_{v\in\mathscr{E}_n} v, D_m = N - \bigcup_{v\in\mathscr{D}_n} v$.

Proposition 2. *The subsets* C_n *and* D_m *are closed subsets of* M *and* N, *respectively. Moreover* $f(C_n) = C_n$ $\forall f \in F_n$, $g(D_m) = D_m$ $\forall g \in F'_m$. *Assuming that* $G(M)$ *and* $G(N)$ *are nonabelian groups, transitive* (*i.e. have the property* $T(1)$) *and that* $F_{y'}$ *resp.* F'_x, *have property* L $\forall x, y$, *then* C_n, *resp.* D_m, *are nonempty.*

Proof. (a) It is clear that C_n and D_m are closed subsets.

(b) Let $U \in \mathscr{E}_n$ and $f \in F_n$. Set $V = f(U)$. Clearly, $G_V(M) = fG_U(M)f^{-1}$. Therefore

$$[G_V(M)_0, G_V(M)_0] = f[G_U(M)_0, G_U(M)_0]f^{-1} \subseteq fF_nf^{-1} \subseteq F_n,$$

i.e. $V \in \mathscr{E}_n$. It follows that $f(C_n) = C_n$. The same considerations show that $g(D_m) = D_m$ $\forall g \in F'_m$.

(c) Suppose now C_n is empty. This means that $\mathscr{E}_n$ is an open cover of M by balls $(U_\alpha)_{\alpha\in A}$ such that

$$[G_{U_\alpha}(M)_0, G_{U_\alpha}(M)_0] \subseteq F_n \quad \forall\alpha \in A.$$

Let y be an arbitrary point in N. Since $G(N)$ is transitive, there is an $f \in G(N)$ such that $f(n) = y$. Then

$$S_y = S_yG(N) = f \cdot S_n \cdot f^{-1},$$

$$F_y = \phi^{-1}(S_y) = \phi^{-1}(f \cdot S_n \cdot f^{-1}) = \phi^{-1}(f) \cdot \phi^{-1}(S_n) \cdot \phi^{-1}(f^{-1}) = \rho F_n \rho^{-1},$$

where $F_n = \phi^{-1}(S_n)$ and $\rho = \phi^{-1}(f)$. Let $\mathscr{V} = \{V_\alpha = \rho(U_\alpha), U_\alpha \in \mathscr{E}_n\}$. This is an open cover of M by balls V_α and

$$[G_{V_\alpha}(M)_0, G_{V_\alpha}(M)_0] = \rho[G_{U_\alpha}(M)_0, G_{U_\alpha}(M)_0]\rho^{-1} \subseteq \rho F_n \rho^{-1}.$$

Hence

$$[G_{V_\alpha}(M)_0, G_{V_\alpha}(M)_0] \subseteq F_y.$$

For each $V_\alpha \in \mathscr{V}$. By property L,

$$[G_c(M)_0, G_c(M)_0] \subseteq F_y.$$

Therefore $\phi([G_c(M)_0, G_c(M)_0]) \subseteq S_y$ $\forall y \in N$. Hence

$$\phi([G_c(M)_0, G_c(M)_0]) \subseteq \bigcap_{y\in N} S_y = \{\mathrm{id}_M\}.$$

Since ϕ is one-to-one

$$[G_c(M)_0, G_c(M)_0] = \{\mathrm{id}_M\}.$$

This is impossible since $G_c(M)_0$ is nonabelian. Thus C_n is nonempty. The same argument shows that D_m is nonempty. q.e.d.

The next result shows when C_n, resp. D_m, are proper subsets.

Lemma 3. *Suppose $G(M)$ and $G(N)$ satisfy $T(3)$ and Condition* B. *Then either C_n is a proper subset or there exists $m \in M$ such that D_m is a proper subset.*

Conclusion. Given $\phi\colon G(M) \to G(N)$ satisfying the hypothesis of the main theorem, starting with any point $y_0 \in N$, then either $A = C_{y_0}$ is a proper nonempty closed subset invariant by $F_{y_0} = \phi^{-1}(S_{y_0}G(N))$ (by Lemma 2 then $C_{y_0} = \{x_0\}$ and $F_{y_0} = S_{x_0}G(M)$), or there exists $z_0 \in M$ such that D_{z_0} is a nonempty closed subset invariant by $\phi(G_{z_0}(M)) = F'_{z_0}$. Lemma 2, applied to $\psi = \phi^{-1}\colon G(N) \to G(M)$ shows that D_{z_1} is a single point $\{u_0\}$, $u_0 \in N$, and $F'_{z_0} = S_{u_0}G(N)$. In any case ϕ takes the isotropy subgroups $S_{m_0}G(M)$ of some point $m_0 \in M$ into the isotropy subgroups $S_{n_0}(G(N))$ of some point $n_0 \in N$. Therefore the hypothesis of Lemma 1 is satisfied. Hence, Lemmas 1, 2 and 3 yield a complete proof of the main theorem.

The following result is proved by Filipkiewicz [6, Lemma 3.3].

Sublemma. *Let $G(M), G(N)$ be two groups of diffeomorphisms of smooth manifolds M and N and $\phi\colon G(M) \to G(N)$ is a group isomorphism. Suppose that $G(M)$ satisfies Condition* B. *Let $F = \phi^{-1}(S_yG(N))$, $y \in N$. There exists $f \in F, f \neq e$ such that* $\mathrm{Int}(\mathrm{Fix}(f)) \neq \varnothing$.

Proof of Lemma 3. Our proof follows Filipkiewicz closely but we carefully avoid the "bad" result of his §2. Since Condition B holds, we may apply the sublemma: $\exists g_0 \neq e, g_0 \in F_n = \phi^{-1}(S_nG(N))$ with

$$A = \mathrm{Int}(\mathrm{Fix}(g_0)) \neq \varnothing.$$

The set $B = \mathrm{Fix}(\phi(g_0)) \neq \varnothing$ since $n \in B$. Let

$$H = \phi^{-1}\{h \in G(N) \text{ with } h(B) = B\},$$
$$K = \phi^{-1}\{h \in G(N) \text{ with } B \subset \mathrm{Fix}(h)\}.$$

Then K is a normal subgroup of H. Since K contains g_0, H and K are nontrivial groups. If $h \in \phi(K)$, $h(n) = n$ since $n \in B \subset \mathrm{Fix}(h)$: this means that $\phi(K) \subset S_n = S_n G(N)$, i.e. $K \subseteq F_n$. We now analyze these two possibilities:

(a) Either, $\forall x \in A, \forall k \in K, k(x) = x$, i.e. $\forall x \in K,\ A \subseteq \mathrm{Fix}(k)$,

(b) or, $\exists x_0 \in A, k_0 \in K$ such that $k_0(x_0) \neq x_0$.

Case (a). Let $h \in G_{N-B}(N)$ (h has compact support in the open set $N - B$). Then $B \subset \mathrm{Fix}(h)$, i.e. $\phi^{-1}(h) \in K$. By (a), $\forall x \in A, \phi^{-1}(h)(x) = x$, i.e. $\phi^{-1}(h) \in S_x$ or $h \in \phi(S_x)$. We thus have shown

$$G_{N-B}(N) \subseteq \phi(S_x) \quad \forall x \in A.$$

Let V be any open ball $V \subset N - B$. Then

$$[G_V(N)_0, G_V(N)_0] \subseteq G_V(N)_0 \subseteq G_{N-B}(N)_0 \subseteq \phi(S_x) \quad \forall x \in A.$$

By the definition of $\mathscr{D}_x$, we have therefore shown that $\forall x \in A$, any open ball $V \subset N - B$ belongs to $\mathscr{D}_x$. Therefore if (a) holds, $\mathscr{D}_x \neq \varnothing\ \forall x \in A$. We want now to show that under Condition (b) (the negative of (a)), then $\mathscr{C}_n \neq \varnothing$. We will then have proved: If D_y is not a proper subset for some $y \in A$ then C_n is a proper subset. This proves the lemma.

Let us investigate (b): First, show (exactly like in [6] that $G_A(M) \subseteq H$. Indeed let $g \in G_A(M)$, $g \neq e$. (This exists by condition C.) Then $g g_0 g^{-1} = g_0$. Indeed if $x \in \mathrm{Fix}(g_0) - A = \partial(\mathrm{Fix}(g_0))$, then $g^{-1}(x) = x$, $g_0(x) = x \Rightarrow g g_0 g^{-1}(x) = g_0(x) = x$. If $x \notin \mathrm{Fix}(g_0)$, then $g(x_0) \notin \mathrm{Fix}(g_0)$. Since $\mathrm{supp}(g) \subset A$, $g^{-1}(x) = x$, $g_0 g^{-1}(x) = g_0(x)$ and $g g_0 g^{-1}(x) = g(g_0(x)) = g_0(x)$. For $x \in A$, $g g_0 g^{-1}(x) = x = g_0(x)$. Therefore

$$\phi(g)\phi(g_0)\phi(g)^{-1} = \phi(g_0),$$

$$\begin{aligned} B = \mathrm{Fix}(\phi(g_0)) &= \mathrm{Fix}(\phi(g) \cdot \phi(g_0) \cdot \phi(g)^{-1}, \\ &= \phi(g)\mathrm{Fix}(\phi(g_0)) = \phi(g)(B), \end{aligned}$$

i.e. $g = \phi^{-1}(\phi(g)) \in H$.

We have shown that $G_A(M) \subseteq H$. Now (b) tells us there exists $x_0 \in A$ and $k_0 \in K$ such that $k_0(x_0) \neq x_0$. Let $U \subseteq M$ be an open ball contained in A with $x_0 \in U$. We may assume that $k_0(x_0) \in U$. Indeed, if not, choose any $f \in G_U(M) \subseteq G_A(M) \subseteq H$ with $y_0 = f(x_0) \neq x_0$ and $y_0 \in U$ (use transitivity for x_0 and y_0 given in U), $f^{-1}k_0(x_0) = k_0(x_0)$ since $k_0(x_0) \notin U$ and $\mathrm{supp}(F) = \mathrm{supp}(f^{-1}) \subset U$. Hence

$$\tilde{k}_0(x_0) = f k_0^{-1} f^{-1} k_0(x_0) = f(x_0) = y_0.$$

Then $\tilde{k}_0(x_0) \neq x_0$, $\tilde{k}_0(x_0) = y_0 \in U$ and $\tilde{k}_0 = (f k_0^{-1} f^{-1}) \cdot k_0 \in K$ since k is normal in H and $f \in H$.

We now want to show that K acts transitively on U. Indeed, let $y \in U$; by Condition A, $\exists \rho \in G_U(M)$ with $\rho(y) = x_0$, $\rho(y_0) = y_0$. Remember that $y_0\tilde{k}_0(x_0) \in U$. Then

$$\rho\tilde{k}_0(x_0) = \tilde{k}_0(x_0), \quad \tilde{k}_0^{-1}\rho\tilde{k}_0(x_0) = x_0 = \rho(y),$$

i.e.,

$$\rho^{-1}\tilde{k}_0^{-1}\rho\tilde{k}_0(x_0) = y.$$

$\hat{g} = (\rho^{-1}k_0^{-1}\rho)\tilde{k}_0 \in K$ since $\tilde{k}_0 \in K$, $\rho \in G_U(A) \subseteq G_A(M) \subseteq H$, and $\hat{g}(x_0) = y$, i.e., K acts transitively on U. Therefore given $x_1, x_2 \in U$, $x_1 \neq x_2$, $x_0 \neq x_1$, $x_0 \neq x_2$ there are $g_1, g_2 \in K$ with $g_i(x_0) = x_i$, $i = 1, 2$.

Let U_0 be a small open ball containing x_0 with U_0, g_1U_0, g_2U_0, $g_1^{-1}U_0$, $g_2^{-1}U_0$ all mutually disjoint, and $U_0 \cup g_1U_0 \cup g_2U_0 \cup g_1^{-1}U_0 \cup g_2^{-1}U_0$ contained in U.

An easy argument of Thurston shows that if $h_1, h_2 \in G_{U_0}(M)$, then $[h_1, h_2] = [C_1, C_2]$, where $C_i = [h_i, g_i] = (h_ig_ih_i^{-1})g_i^{-1} \in K$. This proves that

$$[G_{U_0}(M)_0, G_{U_0}(M)_0] \subseteq K \subseteq F_n = \phi^{-1}(S_n),$$

i.e., $U_0 \in \mathscr{C}_n$, and hence $\mathscr{C}_n \neq \emptyset$.

4. Classical diffeomorphism groups

1. Properties A and B. In this section, we show that the classical diffeomorphism groups considered in the main theorem satisfy the hypothesis of Theorem 2.

It is a classical fact that $\mathrm{Diff}_c^r(M)_0$ has the path transitivity property (see for instance Milnor [9]). The following result is also well known.

Theorem (*Boothby* [3], *see also* [8]). *If α is a volume form or a symplectic form and β is a contact form on a smooth manifold M, then $(\mathrm{Diff}_\alpha^\infty(M)_c)_0$ and $\mathrm{Diff}_c^\infty(M, \beta)_0$ have the path transitivity property, where $\mathrm{Diff}_c^\infty(M, \beta)$ is the group of C^∞-diffeomorphisms of M which preserves β up to a function.*

Remark. The statement above of Boothby's theorem is in the proof of the n-fold transitivity (see [3] or [8]).

We now investigate Condition B for volume or symplectic forms. Condition B holds also for contact form, but we will not present it here.

By Darboux's theorem, there is an open neighborhood V_x of each point $x \in M$ and a diffeomorphism $\phi_x\colon D_\varepsilon^n \to V_x$, where D_ε^n is the open disk in $\mathbf{R}^n$ (n = dimension of M) with radius $\varepsilon > 0$ with $\phi_x(0) = x$ and $\phi_n^*(\alpha\,|\,_{V_x})$ is the standard volume or symplectic form on $\mathbf{R}^n$ ($n = 2m$ in the symplectic case), restricted to D_ε^n. For $0 < \rho \leq \varepsilon$, let $U = \phi_x(D_\rho^n) \subseteq V_x$.

In $\mathbf{R}^n$, $n \geq 2$, introduce the coordinates $(r, \theta, x_3, x_4, \cdots, x_n)$, where (r, θ) are the polar coordinates in the plane (x_1, x_2). The standard symplectic form on $\mathbf{R}^{2m}$ $(2m = n)$ takes the aspect:

$$\underline{\alpha} = rdr \wedge d\theta + dx_3 \wedge dx_4 + \cdots + dx_{2m-1} \wedge dx_{2m},$$

and the standard volume form on $\mathbf{R}^n$ becomes

$$\underline{\alpha} = rdr \wedge d\theta \wedge dx_3 \wedge dx_4 \wedge \cdots \wedge dx_n.$$

Let $\lambda\colon \mathbf{R} \to \mathbf{R}$ be a smooth function with

$$\lambda(x) \geq 0, \quad \lambda(x) > 0 \quad \text{for } 0 < x < \rho,$$
$$\lambda(x) = 0 \quad for x \leq 0,\ x \geq \rho.$$

Define

$$h(r, \theta, x_3, \cdots, x_n) = (r, \theta + \lambda(r), x_3, \cdots, x_n).$$

It is easy to see that $h^*\underline{\alpha} = \underline{\alpha}$, where $\underline{\alpha}$ is one of the standard form above. The diffeomorphism h moves the points of $D^n - \{0\}$ and fixes the boundary of $\overline{D^n_\rho}$. Using the chart ϕ_x, we make this h into a diffeomorphism $\bar{h} \in \mathrm{Diff}^\infty_\alpha(M)$ such that $\mathrm{Fix}(\bar{h}) = (M - U) \cup \{x\}$. Hence we have proved that volume preserving diffeomorphisms and symplectic diffeomorphisms belong to the class $\mathscr{D}_M$.

Perhaps the most subtle property is the property L. Filipkiewicz handled it in the differentiable case $\mathrm{Diff}^r_c(M)$, $1 \leq r \leq \infty$, using Epstein's theorem [4] and a "compression" lemma (Lemma 2.1 or p. 160) which obviously cannot be true in the volume and symplectic cases. However, we still have the following:

Lemma 4. *Let (M, α) be a smooth manifold of dimension $n > 2$ equipped with a volume form α. Then any subgroup F of $\mathrm{Diff}^\infty_\alpha(M)$ has the property L.*

Proof. Recall that the statement of the lemma means the following: Given an open cover $\mathscr{U} = (U_i)_{i \in I}$ of M by balls and if $G_\alpha(U_i)$ is the set of $h \in \mathrm{Diff}^\infty_\alpha(M)$ with compact supports in U_i and if $[G_\alpha(U_i)_0, G_\alpha(U_i)_0] \subseteq F\ \forall U_i \in \mathscr{U}$, then we must have

$$[(\mathrm{Diff}^\infty_\alpha(M)_c)_0, (\mathrm{Diff}^\infty_\alpha(M)_c)_\alpha] \subseteq F.$$

There is a homomorphism V, called the flux homomorphism from $(\mathrm{Diff}^\infty_\alpha(M)_c)_0$ into a quotient of $H^{n-1}_c(M, \mathbf{R})$ (see for instance [1]). A deep theorem due to Thurston [11] says that the kernel of the flux homomorphism $\mathrm{Ker}\, V$ is a simple group. In particular

$$[(\mathrm{Diff}^\infty_\alpha(M)_c)_0, (\mathrm{Diff}^\infty_\alpha(M)_c)_0] = \mathrm{Ker}\, V.$$

One step in the proof of this result is the fact that $\mathrm{Ker}\, V$ has the "fragmentation" property, i.e. if $h \in \mathrm{Ker}\, V$ and $\mathscr{U} = (U_i)$ is any open cover of M, then $h = h_1 \cdots h_r$ where $h_i \in \operatorname{Ker} V$, $\operatorname{supp}(h_i) \subseteq U_i$, in fact $h_i \in G_\alpha(U_i)$. Moreover if U_i are balls, then $G_\alpha(U_i)_0$ are simple; in particular, $G_\alpha(U_i)_0 = [G_\alpha(U_i)_0, G_\alpha(U_i)_0]$. Therefore each

$$h \in \operatorname{Ker} V = [(\operatorname{Diff}^\infty_\alpha(M)_c)_0, (\operatorname{Diff}^\infty_\alpha(M)_c)_0]$$

can be written $h = h_1 \cdots h_r$, where $h_i \in G_\alpha(U_i)_0 = [G_\alpha(U_i)_0, G_\alpha(U_i)_0]$ which is contained in F by hypothesis. Hence each $h_i \in F$, therefore $f \in F$ and the lemma is proved.

The symplectic case is more subtle.

Lemma 5. *Let (M, α) be a smooth manifold equipped with a symplectic form α. Assume that either M is compact or the symplectic pairing is identically zero. Then any subgroup F of* $\operatorname{Diff}^\infty_\alpha(M)$ *has the property L.*

Proof. There is an analogous homomorphism S from $(\operatorname{Diff}^\infty_\alpha(M)_c)_0$ into a quotient of $H^1_c(M, \mathbf{R})$ (see [1] or [4]). If M is compact, $\operatorname{Ker} S$ is a simple group and it is equal to $[(\operatorname{Diff}^\infty_\alpha(M)_c)_0, (\operatorname{Diff}^\infty_\alpha(M)_c)_0]$. If M is not compact, $\operatorname{Ker} S$ is no longer simple and there is a surjective homomorphism $\mathscr{R}$ from $\operatorname{Ker} S$ to a quotient of $\mathbf{R}$. The group $\operatorname{Ker} \mathscr{R}$ is a simple group. These facts constitute the main theorem proved in [1]. If U is a contractible open subset of M, we let $\mathscr{R}_U$ be the restriction of $\mathscr{R}$ to $G_\alpha(U) \subseteq \operatorname{Ker} S$. The main step in proving the results of [1] mentioned above is to show that $\operatorname{Ker} \mathscr{R}_U$ is perfect, i.e. $\operatorname{Ker} \mathscr{R}_U = [\operatorname{Ker} \mathscr{R}_U, \operatorname{Ker} \mathscr{R}_u]$ where U is an open ball in M. Another important fact (much easier to prove than the later) is that $\operatorname{Ker} S$ (in case M is compact) and $\operatorname{Ker} \mathscr{R}$ have the following fragmentation property: if $\mathscr{U} = (V_i)_{i \in I}$ is an open cover of M by balls, then any $h \in \operatorname{Ker} S$ (if M is compact) or $h \in \operatorname{Ker} \mathscr{R}$ (if M is not compact) can be written $h = h_1 \cdots h_r$ where $h_i \in \operatorname{Ker} \mathbf{R}_{U_i}$. If M is compact, then $[(\operatorname{Diff}^\infty_\alpha(M)_c)_0, (\operatorname{Diff}^\infty_\alpha(M)_c)_0] = \operatorname{Ker} S$. Therefore, if $h \in [(\operatorname{Diff}^\infty_\alpha(M)_c)_0, (\operatorname{Diff}^\infty_\alpha(M)_c)_0]$, $h = h_1 \cdots h_r$ with $h_i \in \operatorname{Ker} \mathscr{R}_{U_i} = [\operatorname{Ker} \mathscr{R}_{U_i}, \operatorname{Ker} \mathscr{R}_{U_i}]$, i.e., each $h_i \in [G_\alpha(U_i), G_\alpha(U_i)] \subseteq F$, then $h \in F$. If M is noncompact and the symplectic pairing is trivial, then $[(\operatorname{Diff}^\infty_\alpha(M)_c)_0, (\operatorname{Diff}^\infty_\alpha(M)_c)_0] = \operatorname{Ker} \mathscr{R}$ [10] and each $h \in \operatorname{Ker} \mathscr{R}$ can be written $h = h_1 \cdots h_r$, with $h_i \in \operatorname{Ker} \mathscr{R}_{U_i} = [\operatorname{Ker} \mathscr{R}_{U_i}, \operatorname{Ker} \mathscr{R}_{U_i}]$. As above, $h \in F$. We have completed the proof of the lemma.

Remark. If the symplectic pairing is nontrivial, $\operatorname{Ker} \mathscr{R}$ is an example of a subgroup of $\operatorname{Diff}^\infty_\alpha(M)$ which does not have the property L. Let $\mathscr{U} = (U_i)_{i \in I}$ be an open cover of M by open balls. Then $[G_\alpha(U_i)_0, G_\alpha(U_i)_0] \subseteq \operatorname{Ker} \mathscr{R}$ for each i. Indeed, if $h, g \in (\operatorname{Diff}^\infty_\alpha(M)_c)_0$, one has the following formula [10] (see also [1]):

$$\mathscr{R}([h, g]) = n\langle S(h), S(g)\rangle,$$

where $\langle \ , \ \rangle$ is the symplectic pairing and $[h, g]$ is the commutator $hgh^{-1}g^{-1}$. $[G_\alpha(U_i)_0, G_\alpha(U_i)_0] \subseteq \operatorname{Ker} \mathscr{R}$ since $\forall f \in G_\alpha(V_i)_0, S(f) = 0$. Since $\langle \ , \ \rangle$ is nontrival, let $a, b \in H_c^1(M, \mathbf{R})$ with $\langle a, b\rangle \neq 0$. Since S is surjective, let $h, g \in (\mathrm{Diff}_\alpha^\infty(M)_c)_0$ with $S(h) = a$, $S(g) = b$. Then $\mathscr{R}([h, g]) = n\langle a, b\rangle \neq 0$, i.e. $[(\mathrm{Diff}_\alpha^\infty(M)_c)_0, (\mathrm{Diff}_\alpha^\infty(M)_c)_0]$ is not contained in $\mathrm{Ker}\,\mathscr{R}$. As an example consider the torus T^2 with one point removed and the natural symplectic form. The pairing is nontrivial (see [10]).

5. End of the proof of Theorem 1

Let $\phi\colon G(M_1) \to G(M_2)$ be an isomorphism between the groups of volume preserving diffeomorphisms and symplectic diffeomorphisms in the statement of Theorem 1.

We have just shown that the hypothesis of Theorem 2 is satisfied. Therefore there is a bijection $w\colon M_1 \to M_2$ such that $\phi(f) = wfw^{-1}$ $\forall f \in G(M_1)$. By Theorem 1 of [2], w must be a C^∞-diffeomorphism which exchanges the structures.

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