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EXISTENCE AND REGULARITY OF EMBEDDED DISKS

F. H. LIN

0. Introduction

Given a smooth Jordan curve in $\mathbb{R}^3$, one would like to find a disk type surface spanning the given curve, and which minimizes an elliptic parametric integral. C. Morrey has formulated a problem in the mapping setting similar to the classical Plateau's problem. He proved the existence of a map in Morrey Space which minimizes an elliptic parametric integral (see [11, Chapter 9]). It remains, however, unknown whether or not the solution of Morrey represents a branched, immersed surface in $\mathbb{R}^3$.

One of the aims of this paper is to prove the following existence and strong uniqueness theorem.

Theorem 1. *Let Ψ be a $C^{2,\alpha}$ parametric elliptic even integral with constant coefficient, and let Γ be a $C^{2,\alpha}$-extreme Jordan curve in $\mathbb{R}^3$. Then either the only Ψ-stationary surfaces (including surfaces of higher genus) bounded by Γ are the unique Ψ-minimizing embedded disk or Γ bounds two distinct Ψ-stable embedded disks $\Sigma^\pm$ (they are one-sided Ψ-minimizing in the sense of geometric measure theory). Moreover, for the latter case, any other Ψ-stationary surfaces (including surfaces of higher genus) are supported in the region bounded by Σ^+ and Σ^-.*

For the case of area integral, the above theorem was proved in an earlier work of the author [8], which combined a modified Tomi-Tromba argument with a geometric maximum principle. Such a result for the area integral can also be deduced from a deep existence theorem of Meeks-Yau [10]. Other interesting applications of degree theory, such as in the Tomi-Tromba approach, have recently been obtained in [20].

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The key step to generalize the proof [8] to the case of general elliptic parametric integrands Ψ with constant coefficient is to prove a Hölder estimate up to boundary for the unit normal of Ψ-stationary embedded disks with smooth boundaries. For this we have the following regularity theorem.

Theorem 2. *Let M be a Ψ-stationary embedded disk bounded by a $C^{2,\alpha}$-Jordan curve in $\mathbb{R}^3$. Then M is a $C^{2,\alpha}$-embedded surface with boundary. Moreover, its $C^{2,\alpha}$-norm can be estimated uniformly in terms of the boundary curve.*

For embedded disks in $\mathbb{R}^3$ with quasiconformal Gauss maps, a uniform interior Hölder estimate of the unit normal was proved by Schoen-Simon [13]. Our proof is different from theirs. We use a generalized Lebesgue Lemma on such surfaces. Lebesgue's Lemma is the basic tool for proving the continuity at the boundary of solutions to the classical Plateau's problem or conformal mappings (see [2]). It is also natural for our problem.

Our proof can be generalized to the case that the embedded disks have quasiconformal Gauss-maps. It also seems possible that a similar estimate may be valid for the immersed surfaces or surfaces of higher genus (but given) in terms of the geometry of the boundaries. An estimate as given in [13], without concerning the behavior of the boundary, is easily seen to be impossible for such surfaces. These problems have recently been considered in [9] and [19].

The author was informed earlier by B. White that he was able to generalize the proof of Schoen-Simon [13] to the boundary. Thus he obtained the similar estimate.

1. Notation and Preliminaries

Throughout this paper we shall adopt the following notation

M = a $C^{2,\alpha}$-embedded disk in $\mathbb{R}^3$,

$\Gamma = \partial M$ = a $C^{2,\alpha}$-Jordan curve,

Ψ = a 2-dimensional elliptic parametric even integral with constant coefficient, and of class $C^{3,\alpha}$ (see [11, Chapter 9] or [14] for the discussion),

$B_\rho(\xi) = \{x \in \mathbb{R}^3 \colon |x - \xi| < \rho\}$,

$M_\rho(\xi) = M \cap B_\rho(\xi)$, for $\xi \in M$,

$M_\rho^*(\xi)$ = component of $M_\rho(\xi)$ containing ξ,

v = any continuous choice of unit normal of M,

$|A|$ = the length of the second fundamental form of M,

K = the Gauss curvature of M,

H = the mean curvature of M with respect to v, hence $|H| \leqslant |A|$.

We note that M is a Ψ-stationary surface; then the Gauss map of M is $(\Lambda_1, 0)$ quasiconformal, i.e.,

$$|A|^2(x) \leqslant -\Lambda_1 K(x), \qquad x \in M, \tag{1}$$

for a constant $\Lambda_1 > 0$ depending only on Ψ (see [14]).

By the Gauss-Bonnet Formula, we have

$$\begin{aligned}\int_M |H|^2 \leqslant \int_M |A|^2 \leqslant -\Lambda_1 \int_M K = \Lambda_1\left(\int_{\partial M} K_g - 2\pi\right) \\ \leqslant \Lambda_1(K(\Gamma) - 2\pi),\end{aligned} \tag{2}$$

for a Ψ-stationary disk M with boundary Γ. Here $K(\Gamma)$ is the total curvature of Γ.

Let $\xi \in \mathbb{R}^3$ be such that $\int_\Gamma (x - \xi) = 0$. Since

$$\Delta_M |x - \xi|^2 = 4 + 2H(x - \xi) \cdot v$$

and the divergence theorem implies

$$4\,\mathrm{Area}(M) + 2\int_M Hv \cdot (x - \xi) = 2\int_\Gamma (x - \xi) \cdot \frac{\partial x}{\partial n}, \tag{3}$$

we obtain

$$\begin{aligned}\mathrm{Area}(M) &\leqslant \frac{1}{2}\int_\Gamma |x - \xi| + \frac{1}{2}\int_M |H|\,|(x - \xi) \cdot v| \\ &\leqslant \frac{1}{4\pi} l^2(\Gamma) + l(\Gamma)\left(\int_M |H|^2\right)^{1/2} (\mathrm{Area}\, M)^{1/2},\end{aligned} \tag{4}$$

where $l(\Gamma)$ denotes the length of Γ.

By (2) we have

$$\mathrm{Area}(M) \leqslant l^2(\Gamma)\left[\frac{1}{2\pi} + \Lambda_1(K(\Gamma) - 2\pi)\right]. \tag{5}$$

By the monotonicity formula (see [15]),

$$\frac{|M_\rho|}{\rho^2} = \frac{|M_R^*|}{R^2} - \int_{M_R^* \sim M_\rho} \frac{|D^\perp r|^2}{r^2} - \int_{M_R^*} Hv \cdot (x - \xi)\left(\frac{1}{r_\rho^2} - \frac{1}{R^2}\right), \tag{6}$$

where $r_\rho = \max\{r = |x - \xi|, \rho\}$, $0 < \rho < R$, $|D^\perp r|^2 = |v(x - \xi)|^2/r^2$, and $\partial M \cap B_\rho(\xi) = \varnothing$, we obtain

$$\sup_{0<\rho<R} \frac{|M_\rho|}{\rho^2} \leqslant 2\left(\frac{|M_R^*|}{R^2} + \int_{M_R^*} |H|^2\right). \tag{7}$$

Since $\partial M = \Gamma$ is a $C^{2,\alpha}$-Jordan curve, we may sue the monotonicity formula at the boundary (see [5]), and the estimates (2), (5) to conclude that

$$\text{(8)} \qquad \sup_{0<\rho\leqslant l(\Gamma)} \frac{|M_\rho|}{\rho^2} \leqslant C(l(\Gamma), K(\Gamma), \Lambda_1).$$

2. Proof of Theorem 2

We begin with the proof of the following:

Generalized Lebesgue Lemma. *Let M be a C^2-immersed surface in $\mathbb{R}^3$ such that*

(i) $|M_\rho^*| \leqslant D_0^2\rho^2$ *for all* $0 < \rho < 1$;

(ii) $\int_{M_1^*} |A|^2 \leqslant K_0^2$.

Then, for each $\delta \in (0,1)$, *there is a* $\rho \in (\delta, \sqrt{\delta})$ *so that the following conclusions are valid*:

(a) $\partial M_\rho^* \sim \partial M = \bigcup_i \Gamma_i$, *where the* Γ_i*'s are either immersed arcs with end points on* ∂M *or immersed circles*;

(b) $\sup_i \operatorname{osc}_{\Gamma_i} v < 100 D_0 K_0 / \sqrt{\log(1/\delta)}$.

Proof. By Sard's theorem, (i) is valid for almost all $\rho \in (0,1)$. We want to show there is a $\rho \in (\delta, \sqrt{\delta})$ so that (b) is also valid.

Let $\rho \in (\delta, \sqrt{\delta})$ so that (a) is true. Let Γ_i belong to the decomposition of $\partial M_\rho^* \sim \partial M$, and be such that

$$\operatorname*{osc}_{\Gamma_i} v \geqslant \tfrac{1}{2} C(\rho) = \sup_i \operatorname*{osc}_{\Gamma_i} v.$$

Since $(\operatorname{osc}_{\Gamma_i} v)^2 \leqslant (\int_{\Gamma_i} |A|)^2 \leqslant |\Gamma_i| \int_{\Gamma_i} |A|^2$, we have

$$K_0^2 \geqslant \int_\delta^{\sqrt{\delta}} \int_{\partial M_\rho} |A|^2 \, d\rho = \int_\delta^{\sqrt{\delta}} \int_{\partial M_\rho} |A|^2 |\partial M_\rho^*| \cdot \frac{1}{|\partial M_\rho^*|} \, d\rho$$

$$\geqslant \frac{1}{4} \int_\delta^{\sqrt{\delta}} \frac{C^2(\rho)}{|\partial M_\rho^*|} \, d\rho.$$

Suppose $C(\rho) \geqslant a$ for all $\rho \in (\delta, \sqrt{\delta})$; then $a^2 \leqslant 4K_0^2 / \int_\delta^{\sqrt{\delta}} d\rho / |\partial M_\rho^*|$. We may choose $\delta = 2^{-2m}$ for some positive integer m, so that $\sqrt{\delta} = 2^{-m}$. On each interval, $[2^{-j-1}, 2^{-j}] = I_j$ for $j = m, m+1, \cdots, 2m-1$. We have, by hypothesis (i), that the set $\{\rho \in I_j \colon |\partial M_\rho^*| \leqslant 8 D_0^2 \rho\}$ has measure not less than $\frac{1}{2}|I_j|$. Therefore

$$\int_\delta^{\sqrt{\delta}} \frac{d\rho}{|\partial M_\rho^*|} = \sum_{j=m}^{2m-1} \int_{I_j} \frac{d\rho}{|\partial M_\rho^*|} \geqslant \sum_{j=m}^{2m-1} \frac{1}{2} |I_j| \frac{1}{8 D_0^2 2^{-j}} \geqslant \frac{m}{32 D_0^2}.$$

Hence $a \leqslant 50D_0K_0/\sqrt{\log(1/\delta)}$. Thus for all $\delta \in (0,1)$, there is $\rho \in (\delta, \sqrt{\delta})$ such that

$$C(\rho) = \sup_i \operatorname*{osc}_{\Gamma_i} v \leqslant 100D_0K_0/\sqrt{\log(1/\delta)}. \qquad \text{q.e.d.}$$

The next Lemma is a local pointwise bound on the curvature under an integral smallness assumption which is analogous to [1, Proposition 2].

Lemma 2. *Let M be a C^2-immersed surface in $\mathbb{R}^3$ which is stationary for an elliptic parametric integral Ψ of class $C^{3,\alpha}$. Suppose that either $0 \in M$, $\partial MLB_r(0) = \varnothing$ or $0 \in \Gamma_r = \partial MLB_r$, where Γ_r is a $C^{2,\alpha}$-Jordan arc. There are two positive constants C_0, ε_0 depending only on Ψ (also on Γ_r in the second case) such that*

$$\max_{\sigma \in (0,r)} \sigma^2 \sup_{B_{r-\sigma}(0)} |A|^2 \leqslant C_0, \tag{9}$$

provided that

$$\int_{M_r} |A|^2 \leqslant \varepsilon \leqslant \varepsilon_0. \tag{10}$$

If Ψ, in addition, is of constant coefficient, then, for the first case, we have

$$\max_{\sigma \in (0,r)} \sigma^2 \sup_{B_{r-\sigma}(0)} |A|^2 \leqslant C_0\varepsilon_0^{-1}\varepsilon. \tag{11}$$

Here $0 < r \leqslant 1$.

Proof. (11) will follows (9) by a suitable scaling in $\mathbb{R}^3$. We consider first the case that $0 \in M$, $\partial MLB_r(0) = \varnothing$. Let $K_0^2 = \max_{\sigma \in (0,r)} \sigma^2 \sup_{B_{r-\sigma}(0)} |A|^2$. If $K_0 \leqslant 4$, then we are done. Suppose $K_0 > 4$, and let $\xi \in B_{r-\sigma_0}(0)$ be such that $K_0^2 = \sigma_0^2|A|^2(\xi)$. One notices that $M_{\sigma_0}(\xi) = MLB_{\sigma_0}(\xi) \subset M_r(0)$. Let us consider $\tilde{M}_{K_0}(\xi) = \mu_{K_0/\sigma_0}\# M_{\sigma_0}(\xi)$ (scale M_{σ_0} by a factor K_0/σ_0). We have that

$$|\tilde{A}(\xi)| = 1, \qquad \sup_{\tilde{M}_{K_0/2}(\xi)} |\tilde{A}|^2 \leqslant 4, \tag{12}$$

$$\operatorname*{dist}_{\tilde{M}}\left(\xi, \partial \tilde{M}_{K_0/2}\right) \geqslant K_0/2 > 2. \tag{13}$$

The curvature bound (12) implies, in particular, that the intrinsic ball $S_{\pi/8}(\xi)$ in $\tilde{M}$ is a Lipschitz graph over the plane with the normal $\nu(\xi)$. Moreover, the Lipschitz norm $\leqslant 1$.

Therefore, by the nonparametric partial differential equation estimate (see [3, Chapter 12]), we obtain

$$|\tilde{A}(\xi)|^2 \leqslant C(\Psi) \int_{S_{\pi/8}(\xi)} |\tilde{A}|^2 \leqslant C(\Psi)\varepsilon. \tag{14}$$

If we choose $\varepsilon_0 < C(\Psi)^{-1}$, then (14) contradicts (12). Thus the proof for the first case is complete.

The proof of the second case follows in exactly the same manner. One may need also the nonparametric partial differential equation estimate near the smooth boundary.

Remark. For the case that Ψ is a constant coefficient, Lemma 2 can also be deduced from the Generalized Lebesgue Lemma and the fact that the Gauss map is a branched covering.

To prove Theorem 2, we note that, by (2) and (8), there are two constants D_0 and K_0 depending only on Γ and Ψ such that, for all $\xi \in M$,

$$\sup_{0<\rho\leqslant l(\Gamma)} \frac{|M_\rho^*|}{\rho^2} \leqslant D_0^2 \quad \text{and} \quad \int_M |A|^2 \leqslant K_0^2.$$

Consider a point $\xi \in \partial M = \Gamma$. Since M is an embedded Ψ-stationary disk, the convex hull property of M combined with Sard's Theorem imply that, for almost all $\rho \in (0, \rho_0)$, M_ρ^* is an embedded disk with $\partial M_\rho^* = \Gamma_\rho \cup \gamma_\rho$, where ρ_0 is a positive constant depending only on Γ. Here $\Gamma_\rho = \Gamma LB_\rho(\xi)$ and γ_ρ is a C^1-embedded arc linking two end points of Γ_ρ. Let N_0 be such an integer that $K_0^2/N_0 < \varepsilon_0$, and assume $\rho_0 < 1$. There is an interval of the form $I_k = [\rho_0^{4^{k+1}}\rho_0^{4^k}]$ for some $N_0 \leqslant k \leqslant 2N_0$, such that $\int_{I_k}|A|^2 < \varepsilon_0$. Thus

$$\sup_{M^*_{\rho_0^{4^k\cdot 3}} \sim M_{\rho_0^{4^k\cdot 3/2}}} |A|^2 \rho_0^{6\cdot 4^k} \leqslant C_0, \tag{15}$$

by Lemma 2. Apply the Generalized Lebesgue lemma for $\delta = \rho_0^{4^k\cdot 3}$ to obtain $\rho \in (\rho_0^{4^k\cdot 3}, \rho_0^{4^k\cdot 3/2})$ such that

$$\operatorname*{osc}_{\gamma_\rho}(v) \leqslant 100 D_0 K_0 \sqrt{\log(1/\delta)}\,. \tag{16}$$

Suppose we scale $M_\rho^*(\xi)$ by a factor ρ^{-1} to $\tilde{M}_1^*(\xi)$. Then for $\tilde{M}_1^*(\xi)$ we will have

(i) $\partial\tilde{M}_1^*(\xi)$ is a Lipschitz graph (with very small Lipschitz norm) over a planar domain Ω;

(ii) $\partial\tilde{M}_1^*(\xi) = \tilde{\Gamma}_1 \cup \tilde{\gamma}_1$, and $\tilde{\gamma}_1$ is uniformly $C^{2,\alpha}$ by (15) and the interior partial differential equation estimate;

(iii) Ω satisfies the uniform exterior ball condition along $\tilde{\Gamma}_1$.

Then [6], [7, §4] imply that one can solve nonparametric partial differential equation problems over Ω with boundary $\partial\tilde{M}_1^*$, and which is the unique Ψ-stationary surface bounded by ∂M_1^*. The conclusion of Theorem 2 follows from the nonparametric partial differential equation estimate, since δ here can be chosen so that $\delta \geqslant \delta_0(\Gamma, \Psi) > 0$.

3. Concluding remarks

Theorem 1 can be proved in the same way as for the area integral case (see [8]). Here we replace the Tomi-Tromba perturbation theory by the perturbation theory for Ψ-stationary surfaces due to B. White [18]. The rest of the proof allows from the estimate in Theorem 2 and nonlinear functional analysis (for the details see [8] and [17]).

As a consequence of Theorem 2 and a well-known Tomi-type argument, we have the following:

Corollary. *The number of Ψ-stable embedded disks bounded by a $C^{2,\alpha}$-Jordan curve is finite provided that Ψ is analytic.*

Finally, we would like to point out that Theorem 1 can be used to settle the following open question which was posed by J.C.C. Nitsche [12, §884]:

Question. The Jordan curve Γ possesses prescribed regularity properties and bounds a solution S of Plateau's problem. Let $\{\Gamma_n\}$ be a convergent sequence of a Jordan curve to Γ in a certain sense, which satisfies the same or more regularity assumptions. Is it possible to find a solution S_n of Plateau's problem for the curves Γ_n so that S_n converges to S in a determined sense?

The answer to the above question is no. To see this, we let Γ be an arbitrary C^∞-Jordan curve in S^2 which does not possess the strong uniqueness property, i.e., it bounds at least two distinct branched minimal surfaces (which may have higher genus). Let $\{\Gamma_t\}$, $0 \leqslant t \leqslant 1$, be a C^∞-family of C^∞-Jordan curves so that Γ_0 and Γ_1 are planar curves, $\Gamma_{1/2} = \Gamma$, and $\Gamma_{t_1} \cap \Gamma_{t_2} = \varnothing$ whenever $t_1 \neq t_2$. Consider $\{\Sigma_t^+\}$, the "top" solutions bounded by Γ_t's (for convenience, we assume that Γ_1 is in the "top" position). We claim that $\{\Sigma_t^+\}$ cannot be a continuous family of minimal disks. For otherwise, one would conclude by the maximum principle that $\Gamma = \Gamma_{1/2}$ bounds a unique embedded disks. Since $\lim_{t \downarrow t_0} \Sigma_t^+ = \Sigma_{t_0}^+$ is always true, we can assume for some $t_0 \in (0,1)$ that $\lim_{t \uparrow t_0} \Sigma_t^+ \neq \Sigma_{t_0}^+$. Thus $\{\Gamma_t\}$, $0 \leqslant t < t_0$, form a continuous family of C^∞-Jordan curves which converge to Γ_{t_0} in any C^k-topology. Nevertheless, any minimally immersed, branched surfaces bounded by Γ_t's for $0 \leqslant t < t_0$ cannot be C^0-close to $\Sigma_{t_0}^+$, which is an embedded stable minimal disk bounded by Γ_{t_0}.

It should also be noted that if S_{t_0} is a least area disk bounded by Γ_{t_0}, then, for t sufficiently close to t_0, Γ_t will bound a minimal disk S_t which is close to S_{t_0}. The latter statement is a special case of the perturbation theorem of F. Tomi [16]. A consequence of this fact is that, for $t > t_0$ and t sufficiently close to t_0, Σ_t^- will intersect $\Sigma_{t_0}^+$. Now fix such a $t > t_0$; since both Σ_t^- and $\Sigma_{t_0}^+$ are stable and $\Gamma_t \cap \Gamma_{t_0} = \varnothing$, we can connect Γ_t and Γ_{t_0} by a bridge to form a new Jordan curve in S^2. The bridge-principle (see [10]) implies that this Jordan curve will bound a stable immersed disk (not embedded). This solves a question posed by W. H. Meeks, and which was solved earlier by a different argument by P. Hall [4].

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