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NONNEGATIVELY CURVED MANIFOLDS WITH SOULS OF CODIMENSION 2

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J. Cheeger and D. Gromoll have classified the complete noncompact manifolds of nonnegative curvature in dimensions ≤ 3 up to isometry (cf. [3]). This classification is partly based on the fact that for souls S of dimension 1 (respectively codimension 1), the manifold M is a locally isometrically trivial bundle over S (respectively a flat line bundle over S).

In dimension 4, an additional case may arise, namely $\dim S = \operatorname{codim} S = 2$. This situation is analyzed in §1, where it is shown that when S has codimension 2, there is a Riemannian submersion $\pi: M \to S$, or else the normal bundle $\nu(S)$ of S in M is flat with respect to the induced connection. Those M for which both conditions occur at the same time are the ones that split locally isometrically. Some results on total curvature follow. It turns out that the case where $\nu(S)$ is not flat is not as rigid as might be expected: in §2, the standard submersion metric on $S^3 \times_{S^1} \mathbf{R}^2$ is rather arbitrarily deformed while still retaining its nonnegative curvature. Finally, we show that given a metric of positive curvature on the n-sphere S, any 2-dimensional vector bundle over S admits a metric of nonnegative curvature with soul isometric to S.

1. Basic results

M will denote a complete noncompact manifold of nonnegative curvature with soul S. The reader is referred to [3] for the basic construction and main properties of souls, and to [6] for some facts about Riemannian submersions.

Lemma 1.1. *Let c: $[0, a] \to S$ be a piecewise smooth curve joining p and q in S, and suppose γ: $[0, \infty) \to M$ is a ray originating at p. If $u \in M_q$ denotes the parallel translate of $\dot\gamma(0) \in M_p$ along c, then $t \mapsto \exp_q(tu)$ is a ray originating at q.*

Proof. Since any piecewise smooth curve is a limit of broken geodesics, we may assume that c is a geodesic, and thus extendable to c: $\mathbf{R} \to S$. Carry

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out the basic soul construction at p, so that $M = \bigcup_{t\geq 0} C_t$, with $\gamma(t) \in \partial C_t$. Now $c(\mathbf{R})$ lies in the compact set S, and is therefore contained in some C_{t_0}, hence in every C_t for $t \geq t_0$. By [3, Theorem 1.10], the distance function $s \mapsto d(c(s), \partial C_t)$ is concave. Being bounded from below and defined on all of $\mathbf{R}$, it must be constant. Consider the parallel field X along c with $X(0) = \dot\gamma(0)$, and set $c_s(t) := \exp_{c(s)} tX(s)$. Again by [3, Theorem 1.10], c_s is a minimal geodesic from $c(s)$ to ∂C_t. Since this is true for all $t \geq t_0$, c_s is a ray. q.e.d.

Recall that M is diffeomorphic to the normal bundle $\nu(S)$ of S in M. The following result was already known to D. Gromoll in the case $\dim M = 4$.

Theorem 1.2. *Suppose* $\operatorname{codim} S = 2$. *Then one of the following holds*:

(a) *The normal bundle of S is flat* (*with respect to the induced connection*).

(b) *There is a Riemannian submersion* $\pi\colon M \to S$.

Remark. (a) and (b) are not mutually exclusive. In fact, their intersection consists precisely of those M which are locally isometrically trivial bundles over S (cf. 1.4).

Proof of 1.2. Since the fibers of ν are 2-dimensional, the reduced holonomy group $\Phi_0(p)$ of the connection is either trivial or isomorphic to $\mathrm{SO}(2) \cong S^1$. The trivial case corresponds to (a). Assume then that $\Phi_0(p)$ is isomorphic to S^1 for each $p \in S$. The remaining part of the proof is divided into several steps. First, notice that every direction in the normal bundle yields a ray, i.e. given $v \in \nu(S)$, $\|v\| = 1$, $t \mapsto \exp(tv)$ is a ray. Indeed, since M is noncompact, there is at least one ray emanating from any one point of M. Fix $p \in S$, and choose $v \in M_p$ so that $t \mapsto \exp_p(tv)$ is a ray. By [3, Theorem 5.1], $v \in \nu(S)$. Since S is totally geodesic in M, a parallel section of ν along a curve will be parallel in M. By 1.1, $t \mapsto \exp(tu)$ is a ray for any u in $\Phi_0(p)v$. Since $\Phi_0(p)$ is S^1, the result follows. Next, let $p \in S$, and carry out the basic soul construction at p. Then $S = C_0 = \partial C_0$, and the closure of $B_t(S)$ equals C_t, where $B_t(S) := \{q \in M | d(q, S) < t\}$. To see this, consider a minimal connection γ' from a given $q \in M - S$ to S. Then $\gamma := -\gamma'$ is a ray with $\gamma(t_0) = q$, where $t_0 := d(q, S)$. Let X denote the parallel vector field along some minimal geodesic $c\colon [0, a] \to S$ from $\gamma(0)$ to p, with $X(0) = \dot\gamma(0)$. Then $t \mapsto \tilde\gamma(t) := \exp tX(a)$ is a ray at p, $\tilde\gamma(t_0) \in \partial C_{t_0}$, and by [3, 1.10], $s \mapsto \exp_{c(s)} t_0 X(s)$ is a curve in ∂C_{t_0} from q to $\tilde\gamma(t_0)$. In particular, $q \in \partial C_{t_0}$. Thus $\partial \overline{B}_{t_0}(S) \subset \partial C_{t_0}$, $t_0 > 0$. This also shows that $C_0 \subset S$. Now assume q is in S, and choose a minimal geodesic c from p to q. By the argument in 1.1, $c(\mathbf{R})$ is contained in some ∂C_t. Then $p = c(0)$ belongs to $\partial C_0 \cap \partial C_t$, so $t = 0$. Hence $S \subset \partial C_0$. The inclusion $\partial C_t \subset \partial \overline{B_t}(S)$ now follows easily.

Finally, we show that $\exp_\nu\colon \nu(S) \to M$ is a diffeomorphism. Since every q in M has a minimal connection to S, $\exp_\nu$ is onto. Suppose there are two minimal connections $\gamma_i\colon [0, t_0] \to M$ from S to q, $i = 1, 2$. This would contradict $\gamma_1(t_0+\delta) \in \partial C_{t_0+\delta}$, since the composite curve $\gamma_2|_{[0,t_0]} * \gamma_1|_{[t_0,t_0+\delta]}$ is a connection of length $t_0+\delta$ from S to $\gamma_1(t_0+\delta)$ which can be shortened. Thus $\exp_\nu$ is 1-1.

To complete the proof of 1.2, recall that if K denotes the connection map of $\nu(S)$, then

$$\langle\langle a, b\rangle\rangle := \langle Ka, Kb\rangle + \langle \pi_{\nu *}a, \pi_{\nu *}b\rangle, \qquad a, b \in (T\nu)_v,$$

defines a metric on $\nu(S)$, called the connection metric, such that the projection $\pi_\nu\colon \nu(S) \to S$ becomes a Riemannian submersion.

Define $\pi := \pi_\nu \circ \exp_\nu^{-1}\colon M \to S$. Then π is a submersion, and to show π is Riemannian, it suffices to establish the following:

(1) $\exp_{\nu *}$ maps the horizontal and vertical subspaces of π_ν onto mutually orthogonal subspaces.

(2) $\exp_{\nu *}$ is isometric on the horizontal subspaces.

So let $0 \neq z \in \nu(S)$, $\pi_\nu(z) =: p$, $a \in (T\nu)_z$ horizontal, $b \in (T\nu)_z$ vertical. Since exp is radially isometric, we may assume $\langle\langle b, A_z z\rangle\rangle = 0$, where $A_z\colon \nu_p \to (\nu_p)_z$ denotes the canonical isomorphism between the fiber through p and its tangent space at z. Set $u := \exp_{\nu *} b$, $w := \exp_{\nu *} a$, and let γ denote the ray $\gamma(t) = \exp(tz/\|z\|)$. u determines a variation of γ through rays emanating from p, and thus a Jacobi field X along γ, with $X(0) = 0$, $X'(0) = (A_z^{-1}b)/\|z\|$, and $X(\|z\|) = u$. Consider the geodesic $c\colon \mathbf{R} \to S$ with $\dot{c}(0) = \pi_{\nu *}a = \pi_* w$. c and γ determine a flat totally geodesic rectangle $V(t,s) = \exp_{c(t)} sW(t)$, where W is the parallel vector field along c with $W(0) = z/\|z\|$. Thus the Jacobi field Y along γ, $Y(s) := V_*\partial_{t|0,s}$ is parallel along γ. Moreover, by uniqueness of horizontal lifts, $\|z\|\dot{W}(0) = a$, so that $w = \exp_* a = Y(\|z\|)$. Then $\|w\| = \|Y(\|z\|)\| = \|Y(0)\| = \|\pi_{\nu *}a\| = \|a\|$, which proves (2).

Finally, since X and Y are Jacobi and Y is parallel, $\langle X', Y\rangle - \langle Y', X\rangle = \langle X', Y\rangle$ is constant, and $\langle X', Y\rangle = \langle X', Y\rangle|_0 = \langle A_z^{-1}b, \pi_{\nu *}a\rangle/\|z\| = 0$. Therefore, $\langle X, Y\rangle$ is constant, and $\langle u, w\rangle = \langle X, Y\rangle|_{\|z\|} = \langle X, Y\rangle|_0 = 0$, which proves (1). q.e.d.

We now examine the submersion case in more detail. For the sake of simplicity, M and S will be assumed oriented, even though this hypothesis is often unnecessary. In any case, local results carry through to nonorientable M, while similar global results can be obtained by considering the orientation covering.

Denote by J the canonical complex structure on $\nu(S)$, i.e., $JU = V$ for (local) oriented orthonormal sections $\{U, V\}$ of ν. Define vector fields $\tilde{\partial}_r, \tilde{\partial}_\vartheta$ on $\nu(S) - S$ as follows:

$$\tilde{\partial}_r|_z := A_z z/\|z\|, \quad \tilde{\partial}_\vartheta|_z := A_z Jz, \qquad z \in \nu(S) - S,$$

where A is the isomorphism defined in 1.2. ($\tilde{\partial}_r, \tilde{\partial}_\vartheta$, when restricted to a fiber, are just the standard polar coordinates vector fields.) Let ∂_r and ∂_ϑ denote the corresponding $\exp_\nu$-related vector fields on $M - S$, with dual 1-forms dr and $d\vartheta$. Observe that $\partial_r = \nabla d_S$, where d_S is the distance function from the soul, while ∂_ϑ, when restricted to a ray originating at S, is a Jacobi field Y with initial conditions $Y(0) = 0, \|Y'(0)\| = 1$. Moreover, $[\partial_r, \partial_\vartheta] = 0$, and if $\overline{X}$ is the horizontal lift of $X \in \mathfrak{X}S$, then $[\overline{X}, \partial_r] = [\overline{X}, \partial_\vartheta] = 0$, since $[\tilde{X}, \tilde{\partial}_r] = [\tilde{X}, \tilde{\partial}_\vartheta] = 0$ in $\nu(S)$, for the horizontal lift $\tilde{X}$ of X to $\nu(S)$. Write $Z = Z^h + Z^v$ for the orthogonal splitting of $Z \in \mathfrak{X}M$ induced by the Riemannian submersion $\pi\colon M \to S$, with Z^v tangent to the fiber.

Proposition 1.3. (i) *Let Ω denote the curvature form of $\nu(S)$, viewed as a 2-form on S, i.e., $\Omega(X,Y) := \langle R(X,Y)U, JU\rangle$ for $X, Y \in \mathfrak{X}S$, $U \in \Gamma\nu$ of norm* 1. *If $\overline{X}, \overline{Y} \in \mathfrak{X}M$ are the horizontal lifts of $X, Y \in \mathfrak{X}S$, then*

$$[\overline{X}, \overline{Y}]^v = -\Omega(X,Y)\partial_\vartheta.$$

In particular, if the O'Neill tensor is zero (resp. nonzero) at some point q, then it is identically zero (resp. nowhere zero) on the fiber through q.

(ii) *Set $G^2 := \langle \partial_\vartheta, \partial_\vartheta\rangle$, so that the fiber metric is $dr^2 + G^2\, d\vartheta^2$. If the O'Neill tensor is nonzero on a fiber, then G is bounded on that fiber. The intrinsic sectional curvature of a fiber equals the one induced by M,*

$$K_{\text{fiber}} = -G^{-1}G_{rr}.$$

(iii) *Consider $\nu(S)$ with the connection metric, and replace the standard flat fiber metric $dr^2 + r^2\, d\vartheta^2$ by $dr^2 + (G \circ \exp_\nu)^2\, d\vartheta^2$. Then $\exp_\nu\colon \nu(S) \to M$ is an isometry.*

Proof. As before, $\tilde{X}$ and $\overline{X}$ are the horizontal lifts of $X \in \mathfrak{X}S$ to $\nu(S)$ and M respectively. Since $\exp_\nu$ preserves the orthogonal splitting,

$$[\overline{X}, \overline{Y}]^v|_{\exp z} = \exp_*[\tilde{X}, \tilde{Y}]^v|_z, \quad z \in \nu(S).$$

If R and K denote the curvature tensor and the connection map of $\nu(S)$, then

$$R(X,Y)z = -K[\tilde{X}, \tilde{Y}]_z,$$

or equivalently,

$$[\tilde{X}, \tilde{Y}]^v|_z = -A_z R(X,Y)z = -\Omega(X,Y)A_z Jz = -\Omega(X,Y)\tilde{\partial}_\vartheta|_z.$$

Applying $\exp_{\nu*}$ to the last equation now yields (i).

By O'Neill's formula and (i),

$$\tfrac{3}{4}\Omega^2(X,Y)G^2 = \tfrac{3}{4}\|[\overline{X}, \overline{Y}]^v\|^2 = K_{X,Y} - K_{\overline{X},\overline{Y}} \le K_{X,Y},$$

hence G is bounded if Ω is nonzero.

Consider a horizontal $u \in TM$. Since $\nabla_u \partial_r = 0$, we have $l_u(\partial_r, \partial_r) = l_u(\partial_r, \partial_\vartheta) = 0$, where l_u is the second fundamental form of the fiber with respect to u; the statement about the curvature of the fiber now follows from the Gauss equations. Finally, (iii) is implicitly contained in the proof of 1.2. q.e.d.

For any horizontal unit-speed geodesic $c\colon \mathbf{R} \to M$, $T := \partial_\vartheta \circ c$ is a Jacobi field along c. Let $\mu(t)$ denote the principal curvature of the fiber through $c(t)$ with corresponding principal curvature direction $G^{-1}T$. Thus $S_{\dot{c}}\partial_\vartheta = \mu T$ (S is the second fundamental tensor of the fiber), and

$$\mu = (G \circ c)^{-1}(G \circ c)' = (G \circ c)^{-2}\langle T', T\rangle.$$

Differentiating this equation yields:

$$\mu' = -K_{\dot{c},T} + (G \circ c)^{-2}\|T'\|^2 - 2\mu^2.$$

Suppose now that $\nu(S)$ is flat, or equivalently, that the O'Neill tensor is identically zero. Then $T'^h = 0$, and since $T'^v = S_{\dot{c}}\partial_\vartheta$, we obtain

$$\mu' = -\mu^2 - K_{\dot{c},T}.$$

This in turn implies that $\mu \equiv 0$. For if φ is an antiderivative of μ, then

$$(e^\varphi)'' = e^\varphi(\mu' + \mu^2) \leq 0.$$

Thus e^φ is concave and bounded from below, hence constant, and $\mu \equiv 0$. Therefore, the fibers are totally geodesic. Together with the fact that ν is flat, this implies (cf. [9]):

Theorem 1.4. *Assume S has codimension 2. If $\nu(S)$ is flat and if every normal direction represents a ray, then M is locally isometrically a product.*

One should take care, when dealing with flat normal bundles, to distinguish them from trivial ones. Of course, if S is topologically a 2-sphere, then $\nu(S)$ is trivial whenever it is flat. The converse is not true in general. Consider for example the free $\mathbf{R}$-action Γ on $S^2 \times \mathbf{R}^2 \times \mathbf{R}$ given by $(q, u, t_0) \mapsto (\varphi_t q, e^{it}u, t_0 + t)$, where φ_t denotes rotation by angle t in S^2 about the z-axis, and e^{it} is rotation by angle t in $\mathbf{R}^2$ around the origin. Γ acts freely by isometries on the Riemannian product $S^2 \times \mathbf{R}^2 \times \mathbf{R}$, and there is a unique metric of nonnegative curvature on $M = S^2 \times \mathbf{R}^2 \times \mathbf{R}/\Gamma$ for which the projection $\rho\colon S^2 \times \mathbf{R}^2 \times \mathbf{R} \to M$ becomes a Riemannian submersion (cf. §2). M is diffeomorphic to $S^2 \times \mathbf{R}^2$, and under this identification, the soul S turns out to be $S^2 \times 0$, while the submersion $\pi\colon M \to S$ becomes the projection $\pi_1\colon S^2 \times \mathbf{R}^2 \to S^2 \times 0$. Nevertheless, the metric on M is not a Riemannian product, hence the normal bundle of S is not flat even though it is trivial. The key obstruction here is that the fibers are not totally geodesic, as one can easily check. Indeed, one has

Theorem 1.5. *If M^4 is a trivial bundle over S, and $\pi\colon M \to S$ has totally geodesic fibers, then π is a locally isometrically trivial fibration.*

Together with 1.4, this result immediately implies

Corollary 1.6. *Suppose M^4 has soul S diffeomorphic to a 2-sphere, and every direction in $\nu(S)$ is a ray direction. Then the following statements are equivalent:*

(i) *$\nu(S)$ is flat.*

(ii) *M is diffeomorphic to $S \times \mathbf{R}^2$ and $\pi\colon M \to S$ has totally geodesic fibers.*

(iii) *$M = S \times P_2$ isometrically, where P_2 is $\mathbf{R}^2$ together with some metric of nonnegative curvature.*

To prove 1.5, we need

Lemma 1.7.

(i) $\operatorname{div} \partial_\vartheta = \partial_\vartheta \ln G$. *If ∂_ϑ is divergence-free, then it is a Killing field on M.*

(ii) *If $\nu(S)$ is not flat and $\pi\colon M \to S$ has totally geodesic fibers, then ∂_ϑ is a Killing field.*

Proof of 1.7. If $\{X_i\}$ is a local orthonormal basis of basic vectors fields, then

$$\begin{aligned}\operatorname{div} \partial_\vartheta &= G^{-2}\langle \nabla_{\partial_\vartheta}\partial_\vartheta, \partial_\vartheta\rangle + \langle \nabla_{\partial_r}\partial_\vartheta, \partial_r\rangle + \sum_i \langle \nabla_{X_i}\partial_\vartheta, X_i\rangle \\ &= \partial_\vartheta \ln G - \langle \partial_\vartheta, \nabla_{\partial_r}\partial_r\rangle - \sum_i \langle \partial_\vartheta, (\nabla_{X_i}X_i)^v\rangle \\ &= \partial_\vartheta \ln G.\end{aligned}$$

Assume $\operatorname{div} \partial_\vartheta = 0$. Then

$$\begin{gathered}\langle \nabla_{X_i}\partial_\vartheta, X_j\rangle + \langle \nabla_{X_j}\partial_\vartheta, X_i\rangle = -\langle \partial_\vartheta, (\nabla_{X_i}X_j)^v + (\nabla_{X_j}X_i)^v\rangle = 0, \\ \langle \nabla_{X_i}\partial_\vartheta, \partial_\vartheta\rangle + \langle \nabla_{\partial_\vartheta}\partial_\vartheta, X_i\rangle = \langle [X_i, \partial_\vartheta], \partial_\vartheta\rangle = 0, \\ \langle \nabla_{X_i}\partial_\vartheta, \partial_r\rangle + \langle \nabla_{\partial_r}\partial_\vartheta, X_i\rangle = -\langle \partial_\vartheta, \nabla_{X_i}\partial_r + \nabla_{\partial_r}X_i\rangle = 0, \\ \langle \nabla_{\partial_\vartheta}\partial_\vartheta, \partial_r\rangle + \langle \nabla_{\partial_r}\partial_\vartheta, \partial_\vartheta\rangle = \langle [\partial_r, \partial_\vartheta], \partial_\vartheta\rangle = 0.\end{gathered}$$

Thus ∂_ϑ is a Killing field. To prove (ii), choose $p \in S$ so that $\Omega_p \neq 0$. Since the fibers are totally geodesic, $[\overline{X}, \overline{Y}]^v = -\Omega(X,Y)\partial_\vartheta$ is Killing on the fiber through p, implying $\partial_\vartheta G = 0$ on this fiber. But for any basic X, $X\partial_\vartheta G = \partial_\vartheta XG = 0$, so that $\partial_\vartheta G \equiv 0$ on M. By (i), ∂_ϑ is Killing on M.

Proof of 1.5. If π is not locally isometrically trivial, then $\nu(S)$ cannot be flat by 1.4. By 1.7, ∂_ϑ is a Killing field. Fix some positive r, and consider the set N of points of M at distance r from S. N has nonnegative curvature by the Gauss equations, is diffeomorphic to $S \times S^1$, and thus admits a parallel vector field Z by basic harmonic theory or [3]. Then $\langle Z, \partial_\vartheta\rangle$ is constant, and since $G = \|\partial_\vartheta\|$ is also constant on N, the same must be true for the angle between Z and ∂_ϑ. Choose $p \in S$ so that $\Omega_p \neq 0$, and let $q \in N \cap \pi^{-1}(p)$. If $\overline{X}, \overline{Y}$ are basic orthonormal, equation (2.2) in §2 yields:

$$\nabla_{\overline{X}_q}\partial_\vartheta = \tfrac{1}{2}\Omega_p(X,Y)G^2\overline{Y}_q \neq 0, \qquad \nabla_{\overline{Y}_q}\partial_\vartheta = -\tfrac{1}{2}\Omega_p(X,Y)G^2\overline{X}_q \neq 0.$$

But $0 = \overline{X}\langle Z, \partial_\vartheta\rangle = \frac{1}{2}\Omega(X,Y)G^2\langle\overline{Y}, Z\rangle$, so that $Z \perp \overline{Y}$ on the fiber over p. Similarly $Z \perp \overline{X}$, and Z is then vertical on this fiber. Hence Z is vertical everywhere, and so ∂_ϑ, being a constant multiple of Z, is a parallel vector field, contradicting $\nabla_{\overline{X}_q}\partial_\vartheta \neq 0$. Thus π is locally isometrically trivial. q.e.d.

Recall that the total curvature of an oriented complete even-dimensional manifold M is defined as $\int_M \chi$ (if it exists), where χ is the Chern-Euler form of M. When $\dim M = 2$, $\chi = (1/2\pi)K$ (K is the sectional curvature), and for $K \geq 0$, it is known that the total curvature is bounded between 0 and 1 (cf. [4]).

Lemma 1.8. *Suppose $\Omega \neq 0$ at some $p \in S$. Then the fiber through p has total curvature 1. In particular, if $\pi\colon M \to S$ has totally geodesic fibers and is not locally a Riemannian product, then every fiber has total curvature 1.*

Proof. By 1.3(ii), G is bounded on the fiber through p. Since $r \mapsto G(r,\vartheta)$ is concave and positive, $G_r \to 0$ as $r \to \infty$. Thus

$$\int_0^\infty -G_{rr|_{r,\vartheta}}\, dr = \lim_{r\to 0} G_{r|_{r,\vartheta}} = 1,$$

and the total curvature of the fiber through p is:

$$\frac{1}{2\pi}\int_{\text{fiber}} K_{\text{fiber}} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty -G_{rr}\, dr\, d\vartheta = 1, \quad \text{by 1.3(ii)}.$$

If M is not locally isometrically a product, then Ω is nonzero at some $p \in S$ by 1.4. Thus the fiber through p has total curvature 1. Since the fibers are totally geodesic, they are all isometric to one another (cf. [5]), and the statement follows. q.e.d.

It is known that the total curvature of any 4-dimensional oriented manifold of nonnegative curvature exists, and is bounded between 0 and the Euler characteristic of M (cf. [7]). Assume $\dim M = 4$. Under our additional assumptions, namely $\dim S = 2$ and every normal direction represents a ray, we can prove a stronger result:

Theorem 1.9. *Let $\kappa(p)$ denote the total curvature of the fiber through $p \in S$, $\kappa\colon S \to [0,1]$. Then the total curvature of M^4 equals*

$$\frac{1}{2\pi}\int_S \kappa K_S,$$

where K_S is the sectional curvature of S.

Assume furthermore that S is diffeomorphic to the 2-sphere (the only other possibility is S = flat torus, in which case the total curvature of M is 0), and that $\pi\colon M \to S$ has totally geodesic fibers. Then the total curvature of M is 2, unless $M = S \times P_2$ isometrically, in which case it is 2κ.

Proof. Let $M^r := \{q \in M | d(q, S) \leq r\}$. Thus each ∂M^r is diffeomorphic via $\exp_\nu^{-1}$ to the sphere bundle of radius r over S, and admits the restriction of $\nabla d_S = \partial_r$ as unit normal vector field. ω_r and ω_s will denote the volume forms of ∂M^r and S respectively. The Gauss-Bonnet theorem for manifolds with boundary then yields:

$$\int_{M^r} \chi = \chi(S) + \int_{\partial M^r} g_r \omega_r,$$

where $\chi(S)$ is the Euler characteristic of S, and

$$g_r(q) = (-1/4\pi^2)\{\lambda_1 K_{23} + \lambda_2 K_{13} + \lambda_3 K_{12} + \lambda_1\lambda_2\lambda_3\},$$

(cf. [7]). Here the λ_i are the principal curvatures of ∂M^r at q, with principal curvature direction u_i, and K_{ij} is the sectional curvature of the plane spanned by u_i and u_j. Now $\nabla_u \partial_r = 0$ for horizontal u, and $\nabla_{(1/G)\partial_\vartheta} \partial_r = G^{-2} G_r \partial_\vartheta$. Thus

$$\int_{M^r} \chi = \chi(S) - \frac{1}{4\pi^2} \int_{\partial M^r} K_h G^{-1} G_r \omega_r,$$

where $K_h(q)$ is the sectional curvature of the unique horizontal 2-plane contained in $(\partial M^r)_q$. Since the restriction of π to ∂M^r is a Riemannian submersion, Fubini's theorem yields:

$$\begin{aligned}\int_{\partial M^r} K_h G^{-1} G_r \omega_r &= \int_{\partial M^r} \left\{K_S - \frac{3}{4} f^2 G^2\right\} G^{-1} G_r \omega_r \\ &= \int_S K_S \left(\int_0^{2\pi} G_r \, d\vartheta\right) \omega_s - \frac{3}{4} \int_S f^2 \left(\int_0^{2\pi} G^2 G_r \, d\vartheta\right) \omega_s.\end{aligned}$$

Here, f is defined by the equation $\Omega = f\omega_s$. Now

$$\begin{aligned}\int_S f^2 \left(\int_0^{2\pi} G^2 G_r \, d\vartheta\right) \omega_s &= \int_{\{f \neq 0\}} f^2 \left(\int_0^{2\pi} G^2 G_r \, d\vartheta\right) \omega_s \\ &\leq \frac{4}{3} \int_{\{f \neq 0\}} K_S \left(\int_0^{2\pi} G_r \, d\vartheta\right) \omega_s,\end{aligned}$$

by 1.3(ii). Thus

$$\lim_{r \to \infty} \int_S f^2 \left(\int_0^{2\pi} G^2 G_r \, d\vartheta\right) \omega_s = 0,$$

and

$$\begin{aligned}\int_M \chi = \lim_{r\to\infty}\int_{M^r}\chi = \chi(S) - \frac{1}{4\pi^2}\lim_{r\to\infty}\int_S K_s\left(\int_0^{2\pi} G_r\, d\vartheta\right)\omega_s \\ = \frac{1}{4\pi^2}\lim_{r\to\infty}\int_S\left(\int_0^{2\pi} 1 - G_r\, d\vartheta\right)K_s\omega_s = \frac{1}{2\pi}\int_S \kappa K_S\omega_s.\end{aligned}$$

The last statement of the theorem now follows from 1.6.

2. Some metrics on vector bundles over spheres

Theorem 1.4 shows that the flat bundle case is rigid. The standard examples of nonnegative curvature in the nonflat case are found in [2] and [3]. We briefly recall this construction for fibers diffeomorphic to $\mathbf{R}^2$:

Let G be a Lie group with bi-invariant metric, and let P_2 denote $\mathbf{R}^2$ together with a metric of nonnegative curvature. Suppose H is a closed subgroup of G which acts on P_2 by isometries. Then H acts freely on the Riemannian product $G\times P_2$ via $(g,m)\mapsto(gh,h^{-1}m)$, and there is a metric of nonnegative curvature on the quotient $M = G\times_H P_2$ with respect to which the projection $\pi\colon G\times P_2\to M$ becomes a Riemannian submersion. For example, let $G = S^3$, and $H = S^1$ acting on $\mathbf{R}^2$ by rotations around the origin, so that M is topologically the 2-dimensional vector bundle over S^2 associated with the Hopf fibration. It is straightforward to check that with the above metric, the soul (= the zero section) of M is isometric to the 2-sphere of constant curvature 4. The fibers are totally geodesic, and with the notation of §1, $G = r/(1+r^2)^{1/2}$, while $f\equiv 2$.

In contrast to the rigidity when $\nu(S)$ is flat, one has

Theorem 2.1. *Consider $M = S^3\times_{S^1}\mathbf{R}^2$ with the standard submersion metric. Let h denote an arbitrary real valued function with compact support in $M - S$ and with bounded derivatives up to order 2. Then for small enough $\varepsilon > 0$, the metric on M obtained by deforming G to $\tilde G = G + \varepsilon h$ has nonnegative sectional curvature.*

Notice that if one chooses h so that $h_\vartheta\neq 0$, then the resulting metric on M cannot originate from the construction described above, i.e., M is not isometrically a quotient $S^3\times_{S^1}\mathbf{R}^2$ for any metrics on S^3 and $\mathbf{R}^2$, since in such a quotient, ∂_ϑ must be a Killing field, implying $G_\vartheta = 0$.

Before proceeding to the proof of the theorem, we include for future reference some results that are valid for any 4-dimensional manifold M in the context of 1.2(b). X, Y will denote a local oriented orthonormal basis of vector fields on S, as well as their horizontal lifts. $\mu := (XG)/G$ and $\lambda := (YG)/G$ are the principal curvatures of the fibers of $\pi\colon M\to S$ in directions X and Y

respectively. Then straightforward computations yield

$$
(2.2)\qquad
\begin{aligned}
&\nabla_X \partial_r = \nabla_{\partial_r} X = 0;\\
&\nabla_X \partial_\vartheta = \nabla_{\partial_\vartheta} X = \mu \partial_\vartheta + \tfrac{1}{2} f G^2 Y;\\
&\nabla_Y \partial_\vartheta = \nabla_{\partial_\vartheta} Y = \lambda \partial_\vartheta - \tfrac{1}{2} f G^2 X;\\
&\nabla_{\partial_r} \partial_\vartheta = \nabla_{\partial_\vartheta} \partial_r = G^{-1} G_r \partial_\vartheta;\\
&\nabla_{\partial_r} \partial_r = 0;\\
&\nabla_{\partial_\vartheta} \partial_\vartheta = G^{-1} G_\vartheta \partial_\vartheta - G G_r \partial_r - G^2 \mu X - G^2 \lambda Y.
\end{aligned}
$$

These equalities in turn imply

$$
(2.3)\qquad
\begin{aligned}
&R(X, \partial_r)\partial_r = R(\partial_r, X)X = 0;\\
&2R(X, \partial_r)Y = R(X, Y)\partial_r = f G^{-1} G_r \partial_\vartheta;\\
&K_{X,\partial_\vartheta} = G^{-1}(\nabla_X X - XX)G + \tfrac{1}{4} f^2 G^2;\\
&K_{Y,\partial_\vartheta} = G^{-1}(\nabla_Y Y - YY)G + \tfrac{1}{4} f^2 G^2;\\
&K_{\text{fiber}} = -G^{-1} G_{rr};\\
&\langle R(\partial_\vartheta, X)X, Y\rangle = -\tfrac{1}{2}\{(Xf)G^2 + 3fG(XG)\};\\
&\langle R(\partial_\vartheta, Y)Y, X\rangle = \tfrac{1}{2}\{(Yf)G^2 + 3fG(YG)\};\\
&\langle R(\partial_r, \partial_\vartheta)\partial_\vartheta, X\rangle = -G\partial_r XG;\\
&\langle R(\partial_r, \partial_\vartheta), \partial_\vartheta, Y\rangle = -G\partial_r YG;\\
&\langle R(X, \partial_\vartheta)\partial_\vartheta, Y\rangle = G(\nabla_X Y - XY)G;\\
&K_{X,Y} = K_S - \tfrac{3}{4} f^2 G^2.
\end{aligned}
$$

Proof of 2.1. It is not hard to see that the only planes of zero curvature in the G-metric are those spanned by ∂_r and a horizontal vector. Thus, by choosing ε small enough, we need only consider expressions of the form $\langle R(U,V)V,U\rangle$, where $U = X + \alpha\partial_r + \beta\partial_\vartheta$, $V = \partial_r + \gamma X + \delta Y + \varsigma\partial_\vartheta$, $\alpha, \beta, \gamma, \delta, \varsigma \in \mathbf{R}$.

Then

$$
\begin{aligned}
\langle R(U,V)V,U\rangle = {}& \delta^2 K_{X,Y} + (\beta\gamma - \varsigma)^2 \tilde{G}^2 K_{X,\partial_\vartheta} + (\beta\delta)^2 \tilde{G}^2 K_{Y,\partial_\vartheta}\\
&+ (\beta - \alpha\varsigma)^2 \tilde{G}^2 K_{\text{fiber}} + 2\delta(\varsigma - \beta\gamma)\langle R(\partial_\vartheta, X)X, Y\rangle\\
&+ 3\delta(\beta - \alpha\varsigma)\langle R(X,Y)\partial_r, \partial_\vartheta\rangle\\
&+ 2(\beta\gamma - \varsigma)(\beta - \alpha\varsigma)\langle R(\partial_r, \partial_\vartheta)\partial_\vartheta, X\rangle\\
&+ 2\beta\delta(\beta - \alpha\varsigma)\langle R(\partial_r, \partial_\vartheta)\partial_\vartheta, Y\rangle\\
&+ 2\beta\delta(\beta\gamma - \varsigma)\langle R(X, \partial_\vartheta)\partial_\vartheta, Y\rangle + 2\beta\delta^2\langle R(\partial_\vartheta, Y)Y, X\rangle.
\end{aligned}
$$

Set $x_1 := \delta$, $x_2 := \beta - \alpha\varsigma$, $x_3 := \beta\gamma - \varsigma$, $x_4 := \beta\delta$. Then the above expression is a quadratic function of $x = (x_1, x_2, x_3, x_4)$, which by repeated use of (2.3) can be written as $Q_1(x) + \varepsilon Q_2(X)$, where the matrix of Q_1 is

$$\begin{bmatrix} (4+r^2)/(1+r^2) & 3r/(1+r^2)^2 & 0 & 0 \\ 3r/(1+r^2)^2 & 3r^2/(1+r^2)^3 & 0 & 0 \\ 0 & 0 & r^4/(1+r^2)^2 & 0 \\ 0 & 0 & 0 & r^4/(1+r^2)^2 \end{bmatrix}.$$

Q_1 is positive definite for $r \neq 0$, since the upper left corner matrix has positive trace and determinant. Let $\Theta > 0$ be a lower bound for the eigenvalues of Q_1 on $C := \operatorname{supp} h$. The hypotheses on h imply that there is an $\eta > 0$ such that $|Q_2(x)| \leq \eta \|x\|^2$ on C for all ε say, less than 1. Choose $0 < \varepsilon < \min\{\Theta/\eta, 1\}$. Then $(Q_1 + \varepsilon Q_2)(x) \geq Q_1(x) - \varepsilon |Q_2(x)| \geq 0$. Thus the $\tilde{G}$-metric has nonnegative curvature. Uniform boundedness in ε is crucial here, and the reader may want to compare this construction with the one given in [1]. q.e.d.

The associated bundle construction in [2] shows that any $\mathbf{R}^2$-bundle over S^n admits a metric of nonnegative curvature. Actually, a somewhat stronger result is true:

Theorem 2.4. *Let S denote the n-sphere together with some metric of positive curvature, and let $\pi: E \to S$ be a 2-dimensional vector bundle over S. Then there exists a family of metrics of nonnegative curvature on E, each of which has soul isometric to S, with totally geodesic fibers.*

Proof. For $n > 2$, E is a trivial bundle (cf. [8]), and one then takes the isometric product $S \times P_2$, where P_2 is $\mathbf{R}^2$ together with any metric of nonnegative curvature. Assume then that $n = 2$ and that E is nontrivial. By the classification theorem of bundles over spheres, every vector bundle over the 2-sphere is orientable (cf. [8]). Choose an orientation of E. As before, given a Riemannian connection on E with curvature tensor R, the corresponding curvature form Ω will be identified with $f\omega$ where ω is the volume form of S, and $f: S \to R$ is given locally by $f = \Omega(X, Y)(U, JU)$, X, Y local oriented orthonormal vector fields on S, U local section of E with $\|U\| = 1$.

Fix any Riemannian connection on E, and let Ω denote its curvature form. Set

$$c := \frac{1}{\operatorname{vol} S} \int_S \Omega = \left(\int_S \Omega\right) \Big/ \left(\int_S \omega\right),$$

and $\tilde{\Omega} = c\omega$. $c \neq 0$ since E is nontrivial.

We claim there exists a Riemannian connection $\tilde{\nabla}$ on E with curvature form $= \tilde{\Omega}$. To see this, notice that $\int_S (\tilde{\Omega} - \Omega) = 0$, so that $\tilde{\Omega} = \Omega + d\Theta$, for some 1-form Θ on S.

Now define $\tilde{\nabla}$ by $\tilde{\nabla}_X U = \nabla_X U + \Theta(X) JU$, $X \in \mathfrak{X}S$, $U \in \Gamma E$, where J is the canonical complex structure on E. J is parallel with respect to ∇, and it is easily verified that $\tilde{\nabla}$ is a Riemannian connection. If $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$, then

$$\begin{aligned}\tilde{R}(X,Y)U &= \tilde{\nabla}_X(\nabla_Y U + \Theta(Y)JU) - \tilde{\nabla}_Y(\nabla_X U + \Theta(X)JU) - \tilde{\nabla}_{[X,Y]}U \\ &= R(X,Y)U + \Theta(X)J\nabla_Y U + \nabla_X(\Theta(Y)JU) \\ &\quad + \Theta(X)\Theta(Y)JU - \Theta(Y)J\nabla_X U - \nabla_Y(\Theta(X)JU) \\ &\quad - \Theta(Y)\Theta(X)JU - \Theta([X,Y])JU \\ &= R(X,Y)U + d\Theta(X,Y)JU.\end{aligned}$$

Thus the curvature form of $\tilde{\nabla}$ is $\tilde{\Omega}$. Now choose a Riemannian connection ∇ as above, so that $\Omega = c\omega$. Given $u \in E$, let A_u denote the canonical vector space isomorphism between the fiber through u and its tangent space at u. One has the vector fields $\partial_r, \partial_\vartheta$ on $E - S$ given by

$$\partial_r|_u = A_u u/\|u\|, \quad \partial_\vartheta|_u = A_u Ju, \qquad u \in E - S.$$

Next define a Riemannian metric on E as follows: $\pi\colon E \to S$ is to be a Riemannian submersion, where the horizontal subspaces are those determined by the connection ∇, and the metric on the fibers is taken to be $dr^2 + G^2\,d\vartheta^2$, with $G := \varepsilon r/(\varepsilon^2 + r^2)^{1/2}$ for some fixed $\varepsilon > 0$ satisfying $\varepsilon^2 < (4/3c^2)\min K_s$ (K_S = sectional curvature of S). Notice that replacing the connection ∇ by $\tilde{\nabla}$, $\tilde{\nabla}_X U := \nabla_X U + dh(X)JU$, for $h\colon S \to \mathbf{R}$, changes the horizontal distribution and therefore the metric, even though the curvature form remains unchanged. Thus $\Omega = c\omega$ actually determines a family of metrics on E. A standard argument shows that (2.2) and (2.3) remain valid, with G as above, $f \equiv c$, $\mu = \lambda \equiv 0$. In particular, the fibers of E are totally geodesic. To see that E has nonnegative curvature, consider $u, w \in E_q$. If $q \in E - S$, then there exist local basic X, Y, $\{X, Y\}$ oriented orthonormal, such that

$$u = (\eta X + \alpha\partial_r + \beta\partial_\vartheta)|_q,\ w = (\Theta\partial_r + \gamma X + \delta Y + \varsigma\partial_\vartheta)|_q,\ \alpha, \beta, \gamma, \delta, \varsigma, \eta, \Theta \in \mathbf{R}.$$

Simplifying and grouping terms,

$$\begin{aligned}\langle R(u,w)w,u\rangle &= (\eta\delta)^2 K_{X,Y} + (\beta\Theta - \alpha\varsigma)^2 G^2 K_{\partial_r,\partial_\vartheta} + (\beta\gamma - \eta\varsigma)^2 G^2 K_{X,\partial_\vartheta} \\ &\quad + (\beta\delta)^2 G^2 K_{Y,\partial_\vartheta} + 3\eta\delta(\beta\Theta - \alpha\varsigma)\langle R(X,Y)\partial_r, \partial_\vartheta\rangle,\end{aligned}$$

where the right side is evaluated at q.

Thus $\langle R(u,w)w,u\rangle = Q(\eta\delta, \beta\Theta - \alpha\varsigma, \beta\gamma - \eta\varsigma, \beta\delta)$, where $Q\colon \mathbf{R}^4 \to \mathbf{R}$ is the quadratic function with matrix

$$A = \begin{bmatrix} K_{X,Y} & \frac{3}{2}\langle R(X,Y)\partial_r, \partial_\vartheta\rangle & 0 & 0 \\ \frac{3}{2}\langle R(X,Y)\partial_r, \partial_\vartheta\rangle & G^2 K_{\partial_r,\partial_\vartheta} & 0 & 0 \\ 0 & 0 & G^2 K_{X,\partial_\vartheta} & 0 \\ 0 & 0 & 0 & G^2 K_{Y,\partial_\vartheta} \end{bmatrix}.$$

Now $K_{X,\partial_\vartheta} = K_{Y,\partial_\vartheta} = \frac{1}{4}c^2G^2 > 0$, so A is positive definite iff its upper left corner

$$B = \begin{bmatrix} K_S - \frac{3}{4}c^2G^2 & \frac{3}{2}cGG_r \\ \frac{3}{2}cGG_r & -GG_{rr} \end{bmatrix}$$

is positive definite. But $K_S - \frac{3}{4}c^2G^2 > K_S - \frac{3}{4}c^2\varepsilon^2 > 0$ by choice of ε, while $-GG_{rr} = 3\varepsilon^2G^2/(\varepsilon^2+r^2)^2 > 0$. Thus the trace of B is positive. Finally,

$$\begin{aligned}\frac{\det B}{G^2} &= \left(K_s - \frac{3}{4}\frac{c^2\varepsilon^2r^2}{\varepsilon^2+r^2}\right)\cdot\frac{3\varepsilon^2}{(\varepsilon^2+r^2)^2} - \frac{9}{4}\frac{c^2\varepsilon^6}{(\varepsilon^2+r^2)^3} \\ &= 3\varepsilon^2(K_S - \tfrac{3}{4}c^2\varepsilon^2)/(\varepsilon^2+r^2)^2 > 0.\end{aligned}$$

Therefore B is positive definite, and $\langle R(u,w)w,u\rangle \geq 0$. It is worth mentioning that the only nontrivial solutions for $\langle R(u,w)w,u\rangle = 0$ are $\operatorname{span}\{u,w\} = \operatorname{span}\{\text{horizontal vector}, \partial_r\}$.

When $q \in S$, one replaces $\partial_r, \partial_\vartheta$ by an orthonormal basis of $S_q^\perp$. The matrix A then becomes

$$\begin{bmatrix} K_S & \frac{3}{2}c & 0 & 0 \\ \frac{3}{2}c & 3\varepsilon^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which again is nonnegative.

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